ISRAEL JOURNAL OF MATHEMATICS **244** (2021), 245–271
DOI: 10.1007/s11856-021-2179-0

WEAK $(1,1)$ ESTIMATES FOR MULTIPLE OPERATOR INTEGRALS AND GENERALIZED ABSOLUTE VALUE FUNCTIONS

BY

MARTIJN CASPERS*

Institute of Applied Mathematics
Faculty of Electrical Engineering, Mathematics & Computer Science
Technische Universiteit Delft
P.O. Box 5031, 2600 GA Delft, The Netherlands
e-mail: m.p.t.caspers@tudelft.nl

AND

FEDOR SUKOCHEV** AND DMITRIY ZANIN

School of Mathematics and Statistics
University of New South Wales
Kensington 2052, NSW, Australia
e-mail: f.sukochev@unsw.edu.au, d.zanin@unsw.edu.au

* MC is supported by the NWO Vidi grant 'Non-commutative harmonic analysis and rigidity of operator algebras', VI.Vidi.192.018.

** FS is supported by the ARC Laureate Fellowship.

Received April 5, 2020 and in revised form June 30, 2020

ABSTRACT

Consider the generalized absolute value function defined by

$$a(t) = |t|t^{n-1}, \quad t \in \mathbb{R}, n \in \mathbb{N}_{\geq 1}.$$

Further, consider the n-th order divided difference function $a^{[n]} : \mathbb{R}^{n+1} \to \mathbb{C}$ and let $1 < p_1, \ldots, p_n < \infty$ be such that $\sum_{l=1}^n p_l^{-1} = 1$. Let $\mathcal{S}_{p_l}$ denote the Schatten–von Neumann ideals and let $\mathcal{S}_{1,\infty}$ denote the weak trace class ideal. We show that for any $(n+1)$-tuple $\mathbf{A}$ of bounded self-adjoint operators the multiple operator integral $T^{\mathbf{A}}_{a^{[n]}}$ maps $\mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n}$ to $\mathcal{S}_{1,\infty}$ boundedly with uniform bound in $\mathbf{A}$. The same is true for the class of C^{n+1}-functions that outside the interval $[-1, 1]$ equal a. In [CLPST16] it was proved that for a function f in this class such boundedness of $T^{\mathbf{A}}_{f^{[n]}}$ from $\mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n}$ to $\mathcal{S}_1$ may fail, resolving a problem by V. Peller. This shows that the estimates in the current paper are optimal. The proof is based on a new reduction method for arbitrary multiple operator integrals of divided differences.

1. Introduction

This paper is concerned with the following problem. Consider a Borel function $f : \mathbb{R} \to \mathbb{C}$. Consider the divided difference function $f^{[n]} : \mathbb{R}^{n+1} \to \mathbb{C}$ and assume it is bounded. For an $(n+1)$-tuple $\mathbf{A} = (A_0, \ldots, A_n)$ of bounded self-adjoint operators, consider the multiple operator integral

$$T^{\mathbf{A}}_{f^{[n]}} : \mathcal{S}_2 \times \cdots \times \mathcal{S}_2 \to \mathcal{S}_2. \tag{1.1}$$

Here $\mathcal{S}_2$ is the Hilbert–Schmidt ideal and by [CLPST16] the map (1.1) is well-defined. We now ask for an extension of the multi-linear map (1.1) to other Schatten $\mathcal{S}_p$-spaces. Such extensions have several important applications to differentiability properties of functions on non-commutative spaces; see, e.g., [PSS13], [ST].

Problem 1: For which class of functions $f : \mathbb{R} \to \mathbb{C}$ and which values of $1 \leq p_1, \ldots, p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l} = 1$ does $T^{\mathbf{A}}_{f^{[n]}}$ for every $\mathbf{A} \in B(H)^{\times n+1}_{sa}$ extend to a bounded map

$$T^{\mathbf{A}}_{f^{[n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}? \tag{1.2}$$

In case $n = 1$ a complete solution to this problem was found by D. Potapov and the authors in [CPSZ19]. In this case (1.2) concerns boundeness of

$$T^{\mathbf{A}}_{f^{[1]}} : \mathcal{S}_1 \to \mathcal{S}_{1,\infty}. \tag{1.3}$$

The main result of [CPSZ19] yields that (1.3) is bounded uniformly in $\mathbf{A} \in B(H)_{sa}^{\times 2}$ if and only if f is Lipschitz. Preliminary results on this problem have been obtained by Nazarov and Peller [NaPe09] for rank 1 operators and by the authors [CPSZ15] for f the absolute value map. Through interpolation [CPSZ19] (see also [CSZ18]) it implies the main results of [PS11] and [CMPS14] as well as many previous results on perturbation of commutators and non-commutative Lipschitz properties. In this sense the so-called weak type $(1,1)$ estimate (1.3) is the optimal one. Crucial in the proof of [CPSZ19] is the connection to non-commutative Calderón–Zygmund theory and the results by Parcet [Par09] and Cadilhac [Cad18].

That Problem 1 is the right question to pose is further witnessed by the fact that there is no uniform bound in $\mathbf{A} \in B(H)_{sa}^{n+1}$ of the map

$$T_{f^{[n]}}^{\mathbf{A}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_1. \tag{1.4}$$

For $n = 1$ counterexamples were (in different but related contexts) obtained by Farforovskaya [Far67], [Far68], [Far72], Kato [Kat73] and Davies [Dav88]. Most notably Davies proves in [Dav88] that the estimate (1.4) fails for $n = 1$ and for the absolute value map f. For $n \geq 2$ negative results were obtained much more recently in [CLPST16]. The functions that are used in [CLPST16] to show failure of a uniform bound in $\mathbf{A} \in B(H)_{sa}^{n+1}$ of (1.4) are variations of a generalized (higher order) absolute value map

$$a(t) = |t|t^{n-1}, \quad t \in \mathbb{R}. \tag{1.5}$$

This class of functions is exactly the object of study of the current paper, see the final Remark 5.3. Further negative results for $n \geq 2$ can be found in [PSST17].

The results so far naturally motivate a study of Problem 1 for $n \geq 2$. Moreover, affirmative answers to Problem 1 for classes of functions give optimal solutions to some of the main results in [PSS13] where it was proved that for any $f \in C^n(\mathbb{R})$ with bounded n-th order derivative $f^{(n)}$ we have

$$T_{f^{[n]}}^{\mathbf{A}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_p, \quad 1 < p, p_1, \ldots, p_n < \infty, \text{ with } \sum_{l=1}^{n} \frac{1}{p_l} = \frac{1}{p}.$$

Despite its importance up until now for $n \geq 2$ nothing is known about the boundedness of (1.2) for any class of functions f unless already the stronger estimate (1.4) holds. This paper is the first attempt to fill in a void in this area. Namely, we give an affirmative answer to Problem 1 for the generalized

absolute value function a as well as for a class of related functions. Note that these examples are historically the most natural ones, since several results have been obtained in the past for absolute value maps. In particular, we show that the class of counterexamples obtained in [CLPST16] to the estimate (1.4) does satisfy the weak $(1,1)$ estimate (1.2). For other results on absolute value maps in this context we refer to [McI71], [Kat73], [Dav88], [Kos92], [DDPS97], [DDPS99], [APS05], [CPSZ15], [CLPST16], [PSST17].

Here is our main theorem. We draw the reader's attention that our assumption on the indices $p_1, \ldots, p_n$ below are wider than those in [PSS13]. This explains a serious difference between our method and that of [PSS13].

THEOREM 1.1: *Let $n \in \mathbb{N}_{\geq 1}$ and $1 \leq p_1, \ldots, p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l} = 1$. There exists a constant $D(p_1, \ldots, p_n) > 0$ such that for every $\mathbf{A} \in B(H)_{sa}^{\times n+1}$ we have*

$$\|T^{\mathbf{A}}_{f^{[n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq D(p_1, \ldots, p_n), \tag{1.6}$$

where $f = a$ as defined in (1.5). *Moreover, the same result holds for any function $f \in C^{n+1}(\mathbb{R})$ such that $f(t) = a(t)$ for $t \in \mathbb{R}\backslash[-1,1]$.*

Note that condition $\sum_{l=1}^n \frac{1}{p_l} = 1$ implies that $n = 1$ if and only if for some $1 \leq k \leq n$ we have $p_k = 1$. Further, if $n = 1$ then Theorem 1.1 is the main result of [CPSZ15]. Therefore this paper mainly deals with the case $n \geq 2$ and $p_k > 1$ for all $1 \leq k \leq n$.

Let us comment on the proof. In contrast to [CPSZ19], which covers the case $n = 1$, we do not rely on Calderón–Zygmund theory but rather rely on the key results from [CPSZ15] together with a new reduction technique. Theorem 4.5 shows that the problem of finding weak type $(1,1)$ estimates of a double operator integral of divided differences is concentrated on the case that $\mathbf{A} = (A, \ldots, A)$ with either $A \geq 0$ or $A \leq 0$. To prove this we use reductions from multiple operator integrals to double operator integrals.

STRUCTURE. Section 2 settles all notation and preliminaries on divided differences and multiple operator integrals. Section 3 introduces several reduction techniques for multiple operator integrals. Of crucial importance is Lemma 3.3. Section 4 proves a reduction theorem which is fundamental to our paper. Then in Section 5 we present the main results. In particular we prove Theorem 1.1.

2. Preliminaries

Throughout the entire paper n is a fixed number in $\mathbb{N}_{\geq 1}$. For sets $B_0 \subseteq B_1$ we write $B_1 \backslash B_0$ for the set of all elements in B_1 that are not in B_0. We write χ_G for the indicator function of a set $G \subseteq \mathbb{R}^{n+1}$ and χ_0 when $G = \{0\}$. For $p \in [1, \infty]$ we denote the conjugate exponent by $p' \in [1, \infty]$ which is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

For normed spaces X and Y we denote $X \times Y$ for the Cartesian product equipped with the max norm $\|(x, y)\| = \max(\|x\|, \|y\|)$.

Inner products are linear in the first argument and anti-linear in the second one. In this paper H is an infinite dimensional separable Hilbert space, $B(H)$ is the algebra of all bounded operators on H and $B(H)_{sa}$ stands for the set of all bounded self-adjoint operators. Note that the separability of H is used in [CLS17]. We write Tr for the trace on $B(H)$. For $A \in B(H)_{sa}$ we denote $\sigma(A)$ for the spectrum of A and $\sigma_p(A)$ for the point spectrum of A. So $\sigma_p(A)$ consists of all eigenvalues of A. Let E_A be the spectral measure of A (see [Rud91]). A scalar valued spectral measure of A is a positive scalar valued finite measure λ_A on the Borel sets of $\sigma(A)$ having the same sets of measure 0 as E_A. As observed in [CLS17, Preliminaries] such a measure always exists and the constructions below are independent of the choice of λ_A. See also [Con90, Section IX.8].

2.1. Schatten spaces $\mathcal{S}_p$ and operator ideals. For additional information concerning material reviewed in this subsection, we refer to [LSZ]. We let $\mathcal{S}_p(H), 1 \leq p < \infty$ be the Schatten–von Neumann non-commutative L_p-spaces associated with $B(H)$. We simply write $\mathcal{S}_p$ for $\mathcal{S}_p(H)$ and omit H in the notation; $\mathcal{S}_p$ is the Banach space consisting of all $x \in B(H)$ such that $\|x\|_p := \mathrm{Tr}(|x|^p)^{1/p} < \infty$; $\mathcal{S}_\infty$ denotes the compact operators. The Hölder inequality holds $\|xy\|_p \leq \|x\|_q \|y\|_r$ whenever $x \in \mathcal{S}_q, y \in \mathcal{S}_r$ and $p^{-1} = q^{-1} + r^{-1}$.

For $x \in B(H)$ we set the singular value sequence

$$\mu_k(x) = \inf\{\|x(1-p)\| \mid p \in B(H) \text{ projection}, \mathrm{Tr}(p) \leq k\}, \quad k \in \mathbb{N}_{\geq 0}.$$

We let $\mathcal{S}_{1,\infty}$ be the space of $x \in B(H)$ for which $(\mu_k(x))_{k \in \mathbb{N}_{\geq 0}}$ is in $\ell_{1,\infty}$, e.g.

$$\|x\|_{1,\infty} := \sup_{k \in \mathbb{N}_{\geq 0}} (k+1)\mu_k(x) < \infty.$$

Then $\mathcal{S}_{1,\infty}$ is a quasi-Banach space with quasi-triangle inequality

$$\|x + y\|_{1,\infty} \leq 2\|x\|_{1,\infty} + 2\|y\|_{1,\infty}, \quad x, y \in \mathcal{S}_{1,\infty}.$$

Further $\mathcal{S}_1 \subseteq \mathcal{S}_{1,\infty} \subseteq \mathcal{S}_p, 1 < p \leq \infty$.

2.2. Multiple operator integrals. Fix a separable Hilbert space H. Let $\mathbf{A} = (A_0, \dots, A_n)$ be an $(n+1)$-tuple of self-adjoint operators $A_i \in B(H)$. We shall write this as $\mathbf{A} \in B(H)_{sa}^{\times n+1}$. Let λ_{A_i} be the scalar valued spectral measure of A_i. For functions $\phi_i \in L_\infty(\sigma(A_i), \lambda_{A_i})$, $i = 0, \dots, n$ set $\phi = \phi_0 \otimes \cdots \otimes \phi_n$ and define

$$T_\phi^{\mathbf{A}} : \overbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}^{n \text{ copies}} \to \mathcal{S}_2,$$
$$(x_0, \dots, x_n) \mapsto \phi_0(A_0) x_1 \phi_1(A_1) x_1 \cdots \phi_{n-1}(A_{n-1}) x_n \phi_n(A_n).$$

We have

$$L_\infty(\sigma(A_0), \lambda_{A_0}) \otimes \cdots \otimes L_\infty(\sigma(A_n), \lambda_{A_n}) \subseteq L_\infty(\sigma(A_0) \times \cdots \times \sigma(A_n), \lambda_{A_0} \times \cdots \times \lambda_{A_n}),$$

by identifying

$$(\phi_0 \otimes \cdots \otimes \phi_n)(t_0, \dots, t_n) = \phi_0(t_0) \dots \phi_n(t_n), t_i \in \sigma(A_i).$$

Next, the space $L_\infty(\sigma(A_0) \times \cdots \times \sigma(A_n), \lambda_{A_0} \times \cdots \times \lambda_{A_n})$ is equipped with the weak-$*$ topology induced by the predual $L_1(\sigma(A_0) \times \cdots \times \sigma(A_n), \lambda_{A_0} \times \cdots \times \lambda_{A_n})$ and the linear span of the elementary tensor products is weak-$*$ dense in this space. In [CLS17] it is explained that also the space of bounded multi-linear maps $\mathcal{S}_2 \times \cdots \times \mathcal{S}_2 \to \mathcal{S}_2$ is canonically a dual space and therefore carries the weak-$*$ topology. More precisely, for $\xi_0 \otimes \cdots \otimes \xi_n \otimes \eta \in \mathcal{S}_2 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{S}_2$ ($n+2$ projective tensor products) define the multi-linear map

$$\overbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}^{n+1 \text{ copies}} \to \mathcal{S}_2,$$
$$\mu_0 \otimes \cdots \otimes \mu_n \mapsto \langle \mu_0, \xi_0^* \rangle \cdots \langle \mu_n, \xi_n^* \rangle \eta.$$

By [CLS17] this isomorphism complex linearly identifies the space of multi-linear maps $\mathcal{S}_2 \times \cdots \times \mathcal{S}_2 \to \mathcal{S}_2$ as the dual of $\mathcal{S}_2 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{S}_2$ ($n+2$ tensors). This isomorphism defines the weak-$*$ topology on $\mathcal{S}_2 \times \cdots \times \mathcal{S}_2 \to \mathcal{S}_2$.

By [CLS17, Proposition 5] (see also [ST, Section 4.2] and [Pav69], [SoSt69]) the assignment $\phi \mapsto T_\phi^{\mathbf{A}}$ extends uniquely to a linear weak-$*$ continuous map:

$$L_\infty(\sigma(A_0) \times \cdots \times \sigma(A_n), \lambda_{A_0} \times \cdots \times \lambda_{A_n}) \to B(\mathcal{S}_2 \times \cdots \times \mathcal{S}_2, \mathcal{S}_2).$$

This defines $T_\phi^{\mathbf{A}}$ for ϕ in this domain. By [CLS17, Remarks after Corollary 10] we have for such ϕ that

$$\|\phi\|_\infty = \|T_\phi^{\mathbf{A}} : \mathcal{S}_2 \times \cdots \times \mathcal{S}_2 \to \mathcal{S}_2\|.$$

Note that if $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ is a bounded Borel function, then it defines an equivalence class $[\phi]$ in $L_\infty(\sigma(A_0) \times \cdots \times \sigma(A_n), \lambda_{A_0} \times \cdots \times \lambda_{A_n})$ and we keep denoting $T_\phi^{\mathbf{A}}$ for $T_{[\phi]}^{\mathbf{A}}$.

Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a bounded Borel function and $\mathbf{A} \in B(H)_{sa}^{\times n+1}$. In this paper we shall be interested in extensions of $T_\phi^{\mathbf{A}}$ to various Schatten classes. Let $1 \leq p, p_1, \ldots, p_n < \infty$. We denote

$$\|T_\phi^{\mathbf{A}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_p\| \tag{2.1}$$

for the infimum of all constants $C > 0$ such that for every

$$(x_1, \ldots, x_n) \in (\mathcal{S}_2 \cap \mathcal{S}_{p_1}) \times \cdots \times (\mathcal{S}_2 \cap \mathcal{S}_{p_n})$$

we have $T_\phi^{\mathbf{A}}(x_1, \ldots, x_n) \in \mathcal{S}_p$ and moreover,

$$\|T_\phi^{\mathbf{A}}(x_1, \ldots, x_n)\|_{\mathcal{S}_p} \leq C \prod_{l=1,\ldots,n} \|x_l\|_{\mathcal{S}_{p_l}}.$$

In case (2.1) is finite, $T_\phi^{\mathbf{A}}$ extends to a bounded map $\mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n}$ to $\mathcal{S}_p$ still denoted by $T_\phi^{\mathbf{A}}$. Analogously we can replace the target space $\mathcal{S}_p$ by $\mathcal{S}_{1,\infty}$ in this terminology. We shall also say that $T_\phi^{\mathbf{A}}$ is bounded from $\mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n}$ to $\mathcal{S}_p$ or $\mathcal{S}_{1,\infty}$.

LEMMA 2.1: *Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a bounded Borel function. Let χ_+ (resp. χ_-) be the indicator function on $\mathbb{R}_{\geq 0}^{\times n+1}$ (resp. $-\mathbb{R}_{\geq 0}^{\times n+1}$). Let $A \in B(H)_{sa}$.*

(1) *If $A \geq 0$ we have $T_\phi^{(A,\ldots,A)} = T_{\phi\chi_+}^{(A,\ldots,A)}$ and if $A \leq 0$ we have*

$$T_\phi^{(A,\ldots,A)} = T_{\phi\chi_-}^{(A,\ldots,A)}.$$

(2) *$T_{\chi_\pm}^{(A,\ldots,A)} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_1$ is a contraction for every $1 \leq p_1, \ldots, p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l} = 1$.*

Proof. Set the projection $P := \chi_{[0,\infty)}(A)$. (1) First assume that

$$\phi = \phi_0 \otimes \cdots \otimes \phi_n \in L_\infty(\sigma(A_0), \lambda_{A_0}) \otimes \cdots \otimes L_\infty(\sigma(A_n), \lambda_{A_n})$$

is an elementary tensor product. If $A \geq 0$, then $P = 1$. Then for $x_i \in \mathcal{S}_2$,

$$\begin{aligned} T_\phi^{(A,\ldots,A)}(x_1, \ldots, x_n) &= \phi_0(A) x_1 \phi_1(A) \ldots \phi_{n-1}(A) x_n \phi_n(A) \\ &= \phi_0(A) P x_1 \phi_1(A) P \ldots \phi_{n-1}(A) P x_n \phi_n(A) P \\ &= T_{\phi\chi_+}^{(A,\ldots,A)}(x_1, \ldots, x_n). \end{aligned}$$

By weak-$*$ density of the linear span of elementary products we conclude the lemma for every bounded Borel function $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$. The statement for $A \leq 0$ follows similarly. (2) We find for $x_l \in \mathcal{S}_{p_l} \cap \mathcal{S}_2$ that

$$T_{\chi_+}^{(A,\dots,A)}(x_1,\dots,x_n) = Px_1P\cdots Px_nP,$$

which defines a contraction by the Hölder inequality. The case for χ_- is treated similarly. ∎

Remark 2.2: Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a bounded Borel function and let

$$\phi_m = \sum_{(l_0,\dots,l_n)\in\mathbb{Z}^{n+1}} \phi\Big(\frac{l_0}{m},\dots,\frac{l_n}{m}\Big)\chi_{\prod_{i=0}^n [\frac{l_i}{m},\frac{l_i+1}{m})}.$$

Let $A \in B(H)_{sa}$ with spectral measure E and set $E_{l,m} = E([\frac{l}{m},\frac{l+1}{m}))$. Let $1 \leq p, p_1,\dots,p_n < \infty$ be such that $\frac{1}{p} = \sum_{k=1}^n \frac{1}{p_k}$. Then for $x_k \in \mathcal{S}_2 \cap \mathcal{S}_{p_k}$ we have

$$\begin{aligned} &T_{\phi_m}^{(A,\dots,A)}(x_1,\dots,x_n) \\ &\quad = \sum_{(l_0,\dots,l_n)\in\mathbb{Z}^{n+1}} \phi\Big(\frac{l_0}{m},\dots,\frac{l_n}{m}\Big)E_{l_0,m}x_1E_{l_1,m}x_2\cdots E_{l_{n-1},m}x_nE_{l_n,m}. \end{aligned} \tag{2.2}$$

Assume that $\phi_m \to \phi$ pointwise (which holds true in particular if ϕ is continuous on $\mathbb{R}^{n+1}\backslash\{0\}$). Then by the Lebesgue dominated convergence theorem $\phi_m \to \phi$ in the weak-$*$ topology of $L_\infty(\sigma(A),\lambda_A)^{\otimes n+1}$. By weak-$*$ continuity we have a convergence in $\mathcal{S}_2$

$$T_{\phi_m}^{(A,\dots,A)}(x_1,\dots,x_n) \to T_{\phi}^{(A,\dots,A)}(x_1,\dots,x_n).$$

On the other hand assume that ϕ is in the class $\mathfrak{C}_n$ from [PSS13, Eqn. (3.1)]. If the maps $T_{\phi_m}^{(A,\dots,A)}$ are bounded $\mathcal{S}_{p_1}\times\cdots\times\mathcal{S}_{p_n} \to \mathcal{S}_p$ uniformly in m and if (2.2) converges in $\mathcal{S}_p$ for every $x_i \in \mathcal{S}_2\cap\mathcal{S}_{p_i}$, then we see that the limiting operator equals the one from [PSS13, Definition 3.1]. In particular, this applies to the class of polynomial integral momenta [PSS13, Theorem 5.3] (where $1 < p < \infty$) and the multiple operator integrals appearing in Proposition 2.6 below.

We conclude that the multiple operator integrals that occur in this paper coincide with those defined in [PSS13, Definition 3.1]. Further, it was already observed in [PSS13, p. 510] that these multiple operator integrals also agree with Peller's definition and approach [Pel06] (see also [ACDS09]).

2.3. Reduction to the case $\mathbf{A} = A^{\times n+1}$. Let H be a Hilbert space. Consider $H^{(n+1)} = H \oplus \cdots \oplus H$ the $(n+1)$-fold direct sum and identify

$$B(H^{(n+1)}) \simeq M_{n+1}(\mathbb{C}) \otimes B(H)$$

and

$$\mathcal{S}_p(H^{(n+1)}) \simeq \mathcal{S}_p(\mathbb{C}^{n+1}) \otimes \mathcal{S}_p(H), \quad 1 \leq p \leq \infty.$$

Let $E_{i,j} \in M_{n+1}(\mathbb{C})$ denote the matrix unit with zero entries everywhere except for a 1 at the i-th row and j-th column. We have the following matrix amplification trick.

Proposition 2.3: *Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a bounded Borel function. Let $\mathbf{A} = (A_0, \ldots, A_n) \in B(H)^{\times n+1}_{sa}$ and let $x_1, \ldots, x_n \in \mathcal{S}_2$. Define elements of $M_{n+1}(\mathbb{C}) \otimes B(H)$ by*

$$\overrightarrow{A} = \sum_{l=0}^{n} E_{l,l} \otimes A_l, \quad \text{and} \quad z_l = E_{l-1,l} \otimes x_l, \ l = 1, \ldots, n.$$

Note that $z_l \in \mathcal{S}_2(\mathbb{C}^{n+1}) \otimes \mathcal{S}_2(H)$ with $\|z_l\|_2 = \|x_l\|_2$. Further, set the $(n+1)$-tuple $\overrightarrow{\mathbf{A}} = (\overrightarrow{A}, \ldots, \overrightarrow{A})$. Then

$$T_\phi^{\overrightarrow{\mathbf{A}}}(z_1, \ldots, z_n) = E_{0,n} \otimes T_\phi^{\mathbf{A}}(x_1, \ldots, x_n). \tag{2.3}$$

Proof. By linearity and weak-$*$ continuity of the maps

$$\phi \to T_\phi^{\mathbf{A}}, \quad \phi \to T_\phi^{\overrightarrow{\mathbf{A}}},$$

it suffices to check (2.3) for $\phi = \phi_0 \otimes \cdots \otimes \phi_n$ an elementary tensor product of bounded Borel functions $\phi_i : \mathbb{R} \to \mathbb{C}$. We have

$$\phi_k(\overrightarrow{A}) = \sum_{l=0}^{n} E_{l,l} \otimes \phi_k(A_l).$$

Thus,

$$\phi_k(\overrightarrow{A}) z_{k+1} = (E_{k,k} \otimes \phi_k(A_k))(E_{k,k+1} \otimes x_{k+1}).$$

Therefore,

$$\begin{aligned} T_\phi^{\overrightarrow{\mathbf{A}}}(z_1, \ldots, z_n) =& \phi_0(\overrightarrow{A}) z_1 \phi_1(\overrightarrow{A}) \cdots \phi_{n-1}(\overrightarrow{A}) z_n \phi_n(\overrightarrow{A}) \\ =& (E_{0,0} \otimes \phi_0(A_0))(E_{0,1} \otimes x_1)(E_{1,1} \otimes \phi_1(A_1))(E_{1,2} \otimes x_2) \\ & \times \cdots (E_{n-1,n-1} \otimes \phi_{n-1}(A_{n-1}))(E_{n-1,n} \otimes x_n)(E_{n,n} \otimes \phi_n(A_n)) \\ =& E_{0,n} \otimes (\phi_0(A_0) x_1 \phi_1(A_1) \cdots \phi_{n-1}(A_{n-1}) x_n \phi_n(A_n)) \\ =& E_{0,n} \otimes T_\phi^{\mathbf{A}}(x_1, \ldots, x_n). \end{aligned}$$

This concludes the proof. ∎

COROLLARY 2.4: *Recall that H is an infinite dimensional separable Hilbert space. Let $1 \leq p_1, \ldots, p_n < \infty$. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a Borel function. Suppose that there exists a constant $D > 0$ such that for all $A \in B(H)_{sa}$ we have*

$$\|T_\phi^{(A,\ldots,A)} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq D.$$

Then in fact for all $A_0, \ldots, A_n \in B(H)_{sa}$ we have

$$\|T_\phi^{(A_0,\ldots,A_n)} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq D.$$

The same statement holds with the target space $\mathcal{S}_{1,\infty}$ replaced by $\mathcal{S}_r$ for $1 \leq r \leq \infty$.

Proof. We use the notation of Proposition 2.3 and the fact that H is isomorphic to $H^{(n+1)}$ because the dimension of H is infinite. Note that for $l = 1, \ldots, n$ we have

$$\|x_l\|_{p_l} = \|z_l\|_{p_l}.$$

We thus have

$$\begin{aligned}\|T_\phi^{(A_0,\ldots,A_n)}(x_1, \ldots, x_n)\|_{\mathcal{S}_{1,\infty}} &= \|E_{0,n} \otimes T_\phi^{(A_0,\ldots,A_n)}(x_1, \ldots, x_n)\|_{\mathcal{S}_{1,\infty}} \\ &= \|T_\phi^{(\overrightarrow{A},\ldots,\overrightarrow{A})}(z_1, \ldots, z_n)\|_{\mathcal{S}_{1,\infty}} \\ &\leq D \prod_{l=1}^n \|z_l\|_{\mathcal{S}_{p_l}} = D \prod_{l=1}^n \|x_l\|_{\mathcal{S}_{p_l}}. \qquad \blacksquare\end{aligned}$$

2.4. REDUCTION TO THE CASE $0 \notin \sigma_p(A)$. For $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ bounded Borel and $\delta \in \mathbb{R}$ let

$$\tau_\delta(\phi)(t_0, \ldots, t_n) = \phi(t_0 + \delta, \ldots, t_n + \delta).$$

Recall that $\chi_0 := \chi_{\{0\}}$. For $A \in B(H)_{sa}$ we have

$$T_{\tau_\delta(\phi)}^{(A,\ldots,A)} = T_\phi^{(A-\delta,\ldots,A-\delta)}, \quad T_{c\chi_0}^{(A,\ldots,A)} = T_c^{(\chi_0(A),\ldots,\chi_0(A))}, \quad c \in \mathbb{C}. \tag{2.4}$$

Indeed, one first verifies the first equality of (2.4) on elementary tensors

$$\phi = \phi_0 \otimes \cdots \otimes \phi_n$$

and then uses weak-$*$ density. The second equality of (2.4) follows straight from the definitions.

PROPOSITION 2.5: *Let $1 \leq p_1, \ldots, p_n < \infty$ be such that $\sum_{l=1}^n \frac{1}{p_l} = 1$. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be a function that is continuous on $\mathbb{R}^{n+1}\backslash\{0\}$ and with $\phi(0) = 0$ and $\|\phi\|_\infty \leq 1$. Suppose that there exists a constant $D > 0$ such that for all $A \in B(H)_{sa}$ with $0 \notin \sigma_p(A)$ we have that*

$$\|T_\phi^{(A,\ldots,A)} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq D. \tag{2.5}$$

Then for all $A \in B(H)_{sa}$ we have

$$\|T_\phi^{(A,\ldots,A)} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq 2D + 2.$$

Proof. For $\delta > 0$ let $\phi_\delta(t_0, \ldots, t_n) = \phi(t_0+\delta, \ldots, t_n+\delta)$ if $(t_0, \ldots, t_n)$ is non-zero and $\phi_\delta(0, \ldots, 0) = 0$. So

$$\phi_\delta = \tau_\delta(\phi) - \phi(\delta, \ldots, \delta)\chi_0. \tag{2.6}$$

We have that $\phi_\delta \to \phi$ uniformly on compact sets in $\mathbb{R}^{n+1}\backslash\{0\}$ as $\delta \to 0$. Assume first that $A \in B(H)_{sa}$ is such that for some $\alpha > 0$ we have spectral gap

$$\sigma(|A|) \subseteq \{0\} \cup (\alpha, \infty).$$

Then $\phi_\delta \to \phi$ uniformly on $\sigma(A)^{\times n+1}$. Therefore (see [CLS17, Remark after Corollary 10]) for $x_l \in \mathcal{S}_2 \cap \mathcal{S}_{p_l}$ with $\|x_l\|_{\mathcal{S}_{p_l}} \leq 1$ we have

$$T_{\phi_\delta}^{(A,\ldots,A)}(x_1, \ldots, x_n) \to T_\phi^{(A,\ldots,A)}(x_1, \ldots, x_n)$$

in norm of $\mathcal{S}_2$. Further, for $\delta > 0$ we have

$$T_{\phi_\delta}^{(A,\ldots,A)} \overset{(2.6)}{=} T_{\tau_\delta(\phi)}^{(A,\ldots,A)} - T_{\phi(\delta,\ldots,\delta)\chi_0}^{(A,\ldots,A)} \overset{(2.4)}{=} T_\phi^{(A-\delta,\ldots,A-\delta)} - T_{\phi(\delta,\ldots,\delta)\chi_0}^{(A,\ldots,A)}.$$

If $0 < \delta < \alpha$ then $0 \notin \sigma_p(A - \delta)$, so that by assumption (2.4) and the quasi-triangle inequality,

$$\begin{aligned}\|T_{\phi_\delta}^{(A,\ldots,A)}(x_1, \ldots, x_n)\|_{\mathcal{S}_{1,\infty}} \leq & 2\|T_\phi^{(A-\delta,\ldots,A-\delta)}(x_1, \ldots, x_n)\|_{\mathcal{S}_{1,\infty}} \\ & + 2\|T_{\phi(\delta,\ldots,\delta)}^{(\chi_0(A),\ldots,\chi_0(A))}(x_1, \ldots, x_n)\|_{\mathcal{S}_{1,\infty}} \\ \leq & 2(D + \phi(\delta, \ldots, \delta)) \leq (2D + 2).\end{aligned}$$

By the Fatou property [DDPS97] we have $T_\phi^{(A,\ldots,A)}(x_1, \ldots, x_n) \in \mathcal{S}_{1,\infty}$ with norm majorized by $2D + 2$.

Now take general $A \in B(H)_{sa}$, not necessarily with spectral gap. Take again $x_l \in \mathcal{S}_2 \cap \mathcal{S}_{p_l}$ with $\|x_l\|_{\mathcal{S}_{p_l}} \leq 1$. For $\alpha > 0$ set

$$P_\alpha := \chi_{(-\infty,-\alpha)\cup\{0\}\cup(\alpha,\infty)}(A).$$

We have as $\alpha \searrow 0$ that $P_\alpha x_l P_\alpha \to x_l$ both in the norm of $\mathcal{S}_2$ and $\mathcal{S}_{p_l}$; see [ChSu94]. It follows that for $\alpha \searrow 0$,

$$T_\phi^{(A,\dots,A)}(P_\alpha x_1 P_\alpha, \dots, P_\alpha x_n P_\alpha) \to T_\phi^{(A,\dots,A)}(x_1, \dots, x_n),$$

in the norm of $\mathcal{S}_2$. Further,

$$T_\phi^{(A,\dots,A)}(P_\alpha x_1 P_\alpha, \dots, P_\alpha x_n P_\alpha) = T_\phi^{(P_\alpha A,\dots,P_\alpha A)}(P_\alpha x_1 P_\alpha, \dots, P_\alpha x_n P_\alpha),$$

and the right hand side of this expression is in $\mathcal{S}_{1,\infty}$ with norm majorized by $2D + 2$. By the Fatou property [DDPS97] we conclude that

$$T_\phi^{(A,\dots,A)}(x_1, \dots, x_n) \in \mathcal{S}_{1,\infty}$$

with norm majorized by $2D + 2$. ∎

2.5. Divided differences. Let $C^n(I)$ be the set of all n times continuously differentiable functions on I. For $g \in C^n(\mathbb{R})$ let $g^{(n)}$ be the n-th order derivative of g. Let $C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$ be the space of functions $f \in C^{n-1}(\mathbb{R})$ whose restriction to $\mathbb{R}\backslash\{0\}$ is in $C^n(\mathbb{R}\backslash\{0\})$. For $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$ we set the n-th order divided difference function $f^{[k,n]} : \mathbb{R}^{k+1} \to \mathbb{C}$ by inductively defining for $0 \leq k \leq n$ the function (here, $f^{[0,n]} = f$)

$$f^{[k,n]}(t_0, \dots, t_k) = \begin{cases} \frac{f^{[k-1,n]}(t_0,t_2,t_3,\dots,t_k) - f^{[k-1,n]}(t_1,t_2,t_3\dots,t_k)}{t_0 - t_1} & t_0 \neq t_1, \\ 0 & k = n \text{ and } 0 = t_0 = t_1 = \cdots = t_n, \\ \frac{d}{dt}|_{t=t_0} f^{[k-1,n]}(t, t_2, \dots, t_k) & \text{otherwise.} \end{cases} \tag{2.7}$$

Since f is $n-1$ times differentiable on $\mathbb{R}$ and n times differentiable on $\mathbb{R}\backslash\{0\}$ the formulae (2.7) are well-defined. Further, $f^{[k,n]}, 0 \leq k < n$ is continuous on $\mathbb{R}^k$ and $f^{[n,n]}$ is continuous on $\mathbb{R}^{n+1}\backslash\{0\}$. For $k = n$ this definition of the divided difference function differs from the usual one (as in [PSS13]) in the point $0 \in \mathbb{R}^{n+1}$; the conventional definition in our current notation would be

$$f^{[n,n]} + f^{(n)}(0)\chi_0$$

(which requires f to be in $C^n(\mathbb{R})$). We have that $f^{[k,n]}$ is symmetric under permutation of the variables (see [DeLo93]); i.e., for any permutation σ of $\{0,\dots,k\}$ we have

$$f^{[k,n]}(t_0,\dots,t_k) = f^{[k,n]}(t_{\sigma(0)},\dots,t_{\sigma(k)}). \tag{2.8}$$

In this paper we shall fix n and write

$$f^{[k]} := f^{[k,n]}, \quad 0 \le k \le n.$$

The following result follows from proofs and observations that were made in [PSS13].

PROPOSITION 2.6: *Let $g \in C^{n+1}(\mathbb{R})$ have compact support. Then there exists a constant $D > 0$ such that for every $\mathbf{A} \in B(H)^{\times n+1}_{sa}$ we have*

$$\|T^{\mathbf{A}}_{g^{[n,n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_1\| \le D.$$

Proof. The conditions on g imply that the Fourier transform of the n-th order derivative $g^{(n)}$ is integrable [PS09, Lemma 7] or [PSS13, Top of p. 503]. By [ACDS09, Lemma 2.3] (see [PSS13, Top of p. 512]) $g^{[n,n]} + g^{(n)}\chi_0$ and hence $g^{[n,n]}$ belong to the class $\mathfrak{C}_n$ defined in [PSS13]. So by [PSS13, Lemma 3.5] we conclude the argument. ∎

3. A reduction formula for divided differences

The aim of this section is to demonstrate reduction techniques for multiple operator integrals. In particular, Lemma 3.3 is crucial in this paper.

3.1. A SPECIAL DOUBLE OPERATOR INTEGRAL. Define the auxiliary functions

$$\rho(s) = \frac{|s_0|}{|s_0| + |s_1|}, \quad \psi(s) = \frac{|s_1|}{|s_0| + |s_1|}, \quad s = (s_0, s_1) \in \mathbb{R}^2 \backslash \{(0,0)\}.$$

Further,

$$\rho(0,0) = \psi(0,0) = 0.$$

The following lemma is the main tool behind the paper [CPSZ15] and it is implicitly stated and proved there. We show how to derive it from [CPSZ15] in the discrete case and then refer to [CPSZ19] for an approximation argument. The lemma can also be derived from the much stronger result [CPSZ19, Theorem 1.2].

LEMMA 3.1 ([CPSZ15]): *There exists $C > 0$ such that for every $A_0, A_1 \in B(H)_{sa}$ we have,*

$$\|T_\rho^{A_0,A_1} : \mathcal{S}_1 \to \mathcal{S}_{1,\infty}\| \leq C. \tag{3.1}$$

The same statement is true with ρ replaced by ψ.

Proof sketch. By Corollary 2.4 we may assume that $A_0 = A_1 = A \in B(H)_{sa}$. By Proposition 2.5 we may assume that $0 \notin \sigma_p(A)$. For $\epsilon_1, \epsilon_2 \in \{-, +\}$ let $\chi_{\epsilon_1\epsilon_2}$ be the indicator function of $\epsilon_1\mathbb{R}_{\geq 0} \times \epsilon_2\mathbb{R}_{\geq 0}$. Under these assumptions it follows from the definition of the double/multiple operator integral that

$$T_\rho^{A,A} = T_{\chi_{++}\rho}^{A,A} + T_{\chi_{+-}\rho}^{A,A} + T_{\chi_{-+}\rho}^{A,A} + T_{\chi_{--}\rho}^{A,A}.$$

Hence it suffices to estimate the norm of each of the latter four summands. Note that $\rho(s_0, s_1) = \rho(|s_0|, |s_1|)$. We have

$$T_{\chi_{--}\rho}^{-A,-A} = T_{\chi_{-+}\rho}^{-A,A} = T_{\chi_{+-}\rho}^{A,-A} = T_{\chi_{++}\rho}^{A,A},$$

and so it suffices to estimate $T_{\chi_{++}\rho}^{A,A}$. Setting $A_+ = \chi_{[0,\infty)}(A)$ we have that

$$T_{\chi_{++}\rho}^{A,A} = T_{\chi_{++}\rho}^{A_+,A_+}$$

so that we may assume without loss of generality that A has non-negative spectrum.

Assume further that A has finite spectrum and $0 \notin \sigma(A)$. So

$$A = \sum_{k=1}^{K} \lambda_k q_k$$

with $\lambda_k > 0$ and q_k the spectral projections. Then

$$T_\rho^{A,A}(x) = \sum_{k,l=1}^{K} \frac{\lambda_k}{\lambda_k + \lambda_l} q_k x q_l = \frac{1}{2} \sum_{k,l=1}^{K} \Big(1 + \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l}\Big) q_k x q_l.$$

Then [CPSZ15, Lemma 3.2] shows that $T_\rho^{A,A}$ is bounded $\mathcal{S}_1 \to \mathcal{S}_{1,\infty}$ with bound uniform in $A \in B(H)_{sa}$. For $A \geq 0$ arbitrary we have that $T_\rho^{A,A} : \mathcal{S}_1 \to \mathcal{S}_{1,\infty}$ uniformly boundedly in A by approximation (see [CPSZ19, Section 5]). Since $\psi = 1 - \phi$ the last statement of the lemma follows from the others. ∎

3.2. A reduction formula for divided differences.

Lemma 3.2: *Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$. We have*

$$f^{[n]}(0, t_1, \ldots, t_n) = g^{[n-1]}(t_1, t_2, \ldots, t_n), \tag{3.2}$$

where $g(t) = f^{[1]}(t, 0)$ and $t, t_1, \ldots, t_n \in \mathbb{R}$. Here

$$f^{[n]} = f^{[n,n]} \quad \text{and} \quad g^{[n-1]} = g^{[n-1,n-1]}.$$

Proof. If all t_i's are 0, then (3.2) follows from (2.7). So assume not all t_i's are 0. For $n = 1$ we obtain

$$f^{[1]}(0, t_1) = \frac{f(0) - f(t_1)}{0 - t_1} = g(t_1).$$

We proceed by induction and suppose that the assertion holds for n. We prove it for $n + 1$. Assume $t_n \neq t_{n+1}$. We have by definition (2.7) and permutation invariance of the variables (2.8) that

$$\begin{aligned} f^{[n+1]}(0, t_1, \ldots, t_{n+1}) &\overset{(2.8)}{=} f^{[n+1]}(t_n, t_{n+1}, 0, t_1, \ldots, t_{n-1}) \\ &\overset{(2.7)}{=} \frac{f^{[n]}(t_n, 0, t_1, \ldots, t_{n-1}) - f^{[n]}(t_{n+1}, 0, t_1, \ldots, t_{n-1})}{t_n - t_{n+1}} \\ &\overset{(2.8)}{=} \frac{f^{[n]}(0, t_1, \ldots, t_{n-1}, t_n) - f^{[n]}(0, t_1, \ldots, t_{n-1}, t_{n+1})}{t_n - t_{n+1}}. \end{aligned}$$

By induction

$$\begin{aligned} f^{[n]}(0, t_1, \ldots, t_{n-1}, t_n) &= g^{[n-1]}(t_1, \ldots, t_{n-1}, t_n), \\ f^{[n]}(0, t_1, \ldots, t_{n-1}, t_{n+1}) &= g^{[n-1]}(t_1, \ldots, t_{n-1}, t_{n+1}). \end{aligned}$$

Hence

$$\begin{aligned} f^{[n+1]}(0, t_1, \ldots, t_{n+1}) &= \frac{g^{[n-1]}(t_1, \ldots, t_{n-1}, t_n) - g^{[n-1]}(t_1, \ldots, t_{n-1}, t_{n+1})}{t_n - t_{n+1}} \\ &= g^{[n]}(t_1, \ldots, t_{n+1}). \end{aligned}$$

Finally, if $t_n = t_{n+1}$ and not all t_i's are 0, then (3.2) follows by continuity from the previous cases. ∎

The following formula shall be crucial in the proof of our main theorem. It contains a new decomposition of $f^{[n]}$ as a linear combination of a product of a function of 2 variables and a function of n variables.

LEMMA 3.3: *Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$. We have for every $0 \le i, j \le n, i \ne j$ and every $t \in \mathbb{R}^{n+1}\backslash\{0\}$ with $t_i \ne t_j$ and $t_i \ne 0, t_j \ne 0$ that*

$$\begin{aligned} f^{[n]}(t_0, \dots, t_n) =& \frac{t_i}{t_i - t_j} g^{[n-1]}(t_0, \dots, t_{j-1}, t_{j+1}, \dots, t_n) \\ &- \frac{t_j}{t_i - t_j} g^{[n-1]}(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_n). \end{aligned} \tag{3.3}$$

Here $g(t) = f^{[1]}(t, 0), t \in \mathbb{R}$, where $f^{[n]} = f^{[n,n]}$ and $g^{[n-1]} = g^{[n-1,n-1]}$.

Proof. Since $t_i \ne t_j$ not all variables are equal, so we are not in the second case of the defining relation for $f^{[n]}$; see (2.7). By (2.8) we have

$$\begin{aligned} f^{[n-1]}(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_{j-1}, 0, t_{j+1} \dots, t_n) \\ = f^{[n-1]}(t_0, \dots, t_{i-1}, 0, t_{i+1} \dots, t_{j-1}, t_{j+1}, \dots, t_n). \end{aligned}$$

We have, by using (2.7) and (2.8) for the first and last equality,

$$\begin{aligned} & f^{[n]}(t_0, \dots, t_n) \\ =& \frac{f^{[n-1]}(t_0, \dots, t_{j-1}, t_{j+1} \dots t_n) - f^{[n-1]}(t_0, \dots, t_{i-1}, t_{i+1} \dots t_n)}{t_i - t_j} \\ =& \frac{t_i}{t_i - t_j} \frac{f^{[n-1]}(t_0, \dots, t_{j-1}, t_{j+1} \dots t_n)}{t_i} \\ & - \frac{t_i}{t_i - t_j} \frac{f^{[n-1]}(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_{j-1}, 0, t_{j+1} \dots, t_n)}{t_i} \\ & + \frac{t_j}{t_i - t_j} \frac{f^{[n-1]}(t_0, \dots, t_{i-1}, 0, t_{i+1} \dots, t_{j-1}, t_{j+1}, \dots, t_n)}{t_j} \\ & - \frac{t_j}{t_i - t_j} \frac{f^{[n-1]}(t_0, \dots, t_{i-1}, t_{i+1} \dots t_n)}{t_j} \\ =& \frac{t_i}{t_i - t_j} f^{[n]}(t_0, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_n) \\ & - \frac{t_j}{t_i - t_j} f^{[n]}(t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n). \end{aligned}$$

∎

Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$. Now define inductively $f_0 = f$ and then

$$f_l(t) = f_{l-1}^{[1]}(t, 0), \quad t \in \mathbb{R}, 1 \le l < n. \tag{3.4}$$

LEMMA 3.4: *Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$. We have, for $l = 0, 1, \ldots, n-1$,*

$$f^{[n]}(t_0, \ldots, t_{n-l}, 0, \ldots, 0) = f_l^{[n-l]}(t_0, \ldots, t_{n-l}), \tag{3.5}$$

where $t_0, \ldots, t_{n-l} \in \mathbb{R}$. Here $f^{[n]} = f^{[n,n]}$ and $f_l^{[n-l]} = f_l^{[n-l,n-l]}$.

Proof. The proof follows by induction on l. If $l = 0$ the statement is trivial. Suppose that the corollary is proved for l. We shall prove it for $l+1$. Indeed by induction, symmetry of the variables (2.8) and Lemma 3.2 applied to the function $f_l^{[n-l]}$ we have

$$\begin{aligned} f^{[n]}(t_0, \ldots, t_{n-l-1}, 0, \ldots, 0) &= f_l^{[n-l]}(t_0, \ldots, t_{n-l-1}, 0) \\ &= f_l^{[n-l]}(0, t_0, \ldots, t_{n-l-1}) \\ &= f_{l+1}^{[n-l-1]}(t_0, \ldots, t_{n-l-1}). \end{aligned}$$ ∎

4. Main results

The aim of this section is the proof of Theorem 4.5. The theorem remarkably reduces the problem of estimating multiple operator integrals of divided differences $T_{f^{[n]}}^{(A_0,\ldots,A_n)}$ to the case that $A := A_0 = \cdots = A_n$ and $A \geq 0$. We shall see in Section 5 that for functions that are close to the generalized absolute value map this reduction is sufficient to obtain weak type $(1,1)$ estimates.

4.1. MAIN THEOREM IN A SPECIAL CASE. We assume in this subsection that $A := A_0 = \cdots = A_n$ is in $B(H)_{sa}$ with $0 \notin \sigma_p(A)$ where $\sigma_p(A)$ is the point spectrum of A. Let $1 < p_1, \ldots, p_n < \infty$ such that $\sum_{k=1}^n \frac{1}{p_k} = 1$. Take $x_k \in \mathcal{S}_{p_k} \cap \mathcal{S}_2$. Let $\mathcal{A} \subseteq \{0, \ldots, n\}$ and set

$$x_k^{\mathcal{A}} = \begin{cases} \chi_{(0,\infty)}(A) x_k \chi_{(0,\infty)}(A), & k-1, k \in \mathcal{A}, \\ \chi_{(0,\infty)}(A) x_k \chi_{(-\infty,0)}(A), & k-1 \in \mathcal{A}, k \notin \mathcal{A}, \\ \chi_{(-\infty,0)}(A) x_k \chi_{(0,\infty)}(A), & k-1 \notin \mathcal{A}, k \in \mathcal{A}, \\ \chi_{(-\infty,0)}(A) x_k \chi_{(-\infty,0)}(A), & k-1, k \notin \mathcal{A}. \end{cases}$$

Since we assumed $0 \notin \sigma_p(A)$ we have $1 = P_- + P_+$ with $P_- = \chi_{-(\infty,0)}(A)$ and $P_+ = \chi_{(0,\infty)}(A)$. Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$ and assume that $f^{[n]}$ is

bounded. Since $0 \notin \sigma_p(A)$ multi-linearity of the multiple operator integral gives

$$\begin{aligned} &T_{f^{[n]}}^{(A,\dots,A)}(x_1,\dots,x_n)\\ &\quad=T_{f^{[n]}}^{(A,\dots,A)}((P_-+P_+)x_1(P_-+P_+),\dots,(P_-+P_+)x_n(P_-+P_+))\\ &\quad=\sum_{\mathcal{A}\subseteq\{0,\dots,n\}} T_{f^{[n]}}^{(A,\dots,A)}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}). \end{aligned} \tag{4.1}$$

The $\mathcal{S}_{1,\infty}$-norms of these summands where $\mathcal{A}\neq\emptyset, \mathcal{A}\neq\{0,\dots,n\}$ turn out to be much easier to estimate. Recall the auxiliary functions

$$\rho(s)=\frac{|s_0|}{|s_0|+|s_1|},\quad \psi(s)=\frac{|s_1|}{|s_0|+|s_1|},\quad s=(s_0,s_1)\in\mathbb{R}^2\backslash\{(0,0)\}.$$

Further $\rho(0,0)=\psi(0,0)=0$. Further, recall that f_l was defined in (3.4).

LEMMA 4.1: *Let $f\in C^{n-1}(\mathbb{R})\cap C^n(\mathbb{R}\backslash\{0\})$ with $f^{[n]}$ bounded, let $A\in B(H)_{sa}$, $0\notin\sigma_p(A)$ and let $1<p_1,\dots,p_n<\infty$ be such that $\sum_{k=1}^n\frac{1}{p_k}=1$. Suppose that $\mathcal{A}\neq\emptyset$, $\mathcal{A}\neq\{0,\dots,n\}$. Then there exists an absolute constant $C>0$ such that for $x_k\in\mathcal{S}_{p_k}\cap\mathcal{S}_2$,*

$$\|T_{f^{[n]}}^{(A,\dots,A)}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}}\leq C\max_{1\leq k\leq n}\{p_k,p_k'\}\cdot M(f;A)\cdot\prod_{k=1}^n\|x_k\|_{\mathcal{S}_{p_k}}, \tag{4.2}$$

where $M(f;A)$ is the maximum of the terms

$$\begin{aligned} &\max_{1\leq k\leq n-1}\|T_{f_1^{[n-1]}}^{(A,\dots,A)}:\mathcal{S}_{p_1}\times\cdots\times\mathcal{S}_{p_{k-1}}\times\mathcal{S}_{\frac{p_kp_{k+1}}{p_k+p_{k+1}}}\times\mathcal{S}_{p_{k+2}}\times\cdots\times\mathcal{S}_{p_n}\to\mathcal{S}_{1,\infty}\|,\\ &\qquad\|T_{f_1^{[n-1]}}^{(A,\dots,A)}:\mathcal{S}_{p_1}\times\ldots\times\mathcal{S}_{p_{n-1}}\to\mathcal{S}_{q_1}\|,\\ &\qquad\|T_{f_1^{[n-1]}}^{(A,\dots,A)}:\mathcal{S}_{p_2}\times\ldots\times\mathcal{S}_{p_n}\to\mathcal{S}_{q_2}\|, \end{aligned} \tag{4.3}$$

with $\frac{1}{q_1}=\sum_{i=1}^{n-1}\frac{1}{p_i}$, $\frac{1}{q_2}=\sum_{i=2}^{n}\frac{1}{p_i}$.

Proof. Since we may multiply f with a positive scalar we may assume without loss of generality that $M(f;A)=1$.

Throughout the entire proof, fix $0\leq k<n$ such that $k\in\mathcal{A}$ and $k+1\notin\mathcal{A}$ (or $k+1\in\mathcal{A}$ and $k\notin\mathcal{A}$). We assume the first case, the second case can be proved similarly. We then have

$$x_{k+1}^{\mathcal{A}}=\chi_{(0,\infty)}(A)x_{k+1}^{\mathcal{A}}\chi_{(-\infty,0)}(A)$$

and, therefore,

$$\begin{aligned} &T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) \\ &\quad= T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_k^{\mathcal{A}},\chi_{(0,\infty)}(A)x_{k+1}^{\mathcal{A}}\chi_{(-\infty,0)}(A),x_{k+2}^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) \\ &\quad= T^{(\overbrace{A,\dots,A}^{k\text{ terms}},\chi_{(0,\infty)}(A)A,\chi_{(-\infty,0)}(A)A,\overbrace{A,\dots,A}^{n-k-1\text{ terms}})}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}). \end{aligned}$$

Consequently, this expression only depends on the values $f^{[n]}(t_0,\dots,t_n)$ with $t_k>0$ and $t_{k+1}<0$. For $t_k>0$ and $t_{k+1}<0$ we have by Lemma 3.3 that

$$\begin{aligned} f^{[n]}(t_0,\dots,t_n) =&\rho(t_k,t_{k+1})f_1^{[n-1]}(t_0,\dots,t_{k-1},t_k,t_{k+2},\dots,t_n) \\ &+\psi(t_k,t_{k+1})f_1^{[n-1]}(t_0,\dots,t_{k-1},t_{k+1},t_{k+2},\dots,t_n). \end{aligned}$$

Therefore if $0<k<n-1$ we have[1] that

$$\begin{aligned} &T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) \\ &\quad=T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{k-1}^{\mathcal{A}},x_k^{\mathcal{A}},T_\rho^{A,A}(x_{k+1}^{\mathcal{A}})\cdot x_{k+2}^{\mathcal{A}},x_{k+3}^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) \\ &\quad\quad+T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{k-1}^{\mathcal{A}},x_k^{\mathcal{A}}\cdot T_\psi^{A,A}(x_{k+1}^{\mathcal{A}}),x_{k+2}^{\mathcal{A}},\dots,x_n^{\mathcal{A}}). \end{aligned}$$

In case $k=0$ we have

$$\begin{aligned} T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) =&T^{(A,\dots,A)}_{f_1^{[n-1]}}(T_\rho^A(x_1^{\mathcal{A}})\cdot x_2^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) \\ &+T_\psi^{A,A}(x_1^{\mathcal{A}})\cdot T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_2^{\mathcal{A}},\dots,x_n^{\mathcal{A}}). \end{aligned} \tag{4.4}$$

[1] Here, we are using the equality

$$T^{(A,\dots,A)}_{h_1}(V_1,\dots,V_n)=T^{(A,\dots,A)}_{h_2}(V_1,\dots,V_k,T^{(A,A)}_{h_3}(V_{k+1}),V_{k+2},\dots,V_n),$$

which is valid whenever

$$h_1(t_0,\dots,t_n)=h_3(t_k,t_{k+1})\cdot h_2(t_0,\dots,t_n),$$

and the observation

$$T^{(\overbrace{A,\dots,A}^{n+1\text{ terms}})}_h(V_1,\dots,V_n)=T^{(\overbrace{A,\dots,A}^{n\text{ terms}})}_h(V_1,\dots,V_k,V_{k+1}\cdot V_{k+2},V_{k+3},\dots,V_n),$$

which is valid whenever h does not depend on the $(k+1)$-st variable. These equalities can be verified directly when $h=h_0\otimes\cdots\otimes h_n$ is an elementary tensor product; then the general case follows from weak-$*$ continuity just as in the proofs of Lemma 2.1 and Proposition 2.3. See also [PSS13, Lemma 3.2].

In case $k = n-1$ we have

$$\begin{aligned} T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}) =& T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{n-1}^{\mathcal{A}})\cdot T_\rho^{A,A}(x_n^{\mathcal{A}}) \\ &+ T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{n-2}^{\mathcal{A}},x_{n-1}^{\mathcal{A}}\cdot T_\psi^{A,A}(x_n^{\mathcal{A}})). \end{aligned} \tag{4.5}$$

Let us first consider the case $0 < k < n-1$; the cases $k = 0, n-1$ can be proved similarly. We find from the quasi-triangle inequality and the assumption $M(f;A) \le 1$ that

$$\begin{aligned} &\|T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}} \\ &\quad\le 2\|T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{k-1}^{\mathcal{A}},x_k^{\mathcal{A}},T_\rho^{A,A}(x_{k+1}^{\mathcal{A}})x_{k+2}^{\mathcal{A}},x_{k+3}^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}} \\ &\qquad + 2\|T^{(A,\dots,A)}_{f_1^{[n-1]}}(x_1^{\mathcal{A}},\dots,x_{k-1}^{\mathcal{A}},x_k^{\mathcal{A}}T_\psi^{A,A}(x_{k+1}^{\mathcal{A}}),x_{k+2}^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}} \\ &\quad\le 2\prod_{l=1}^{k}\|x_l^{\mathcal{A}}\|_{\mathcal{S}_{p_l}}\|T_\rho^{A,A}(x_{k+1}^{\mathcal{A}})x_{k+2}^{\mathcal{A}}\|_{\mathcal{S}_{\frac{p_{k+1}p_{k+2}}{p_{k+1}+p_{k+2}}}}\prod_{l=k+3}^{n}\|x_l^{\mathcal{A}}\|_{\mathcal{S}_{p_l}} \\ &\qquad + 2\prod_{l=1}^{k-1}\|x_l^{\mathcal{A}}\|_{\mathcal{S}_{p_l}}\|x_k^{\mathcal{A}}T_\psi^{A,A}(x_{k+1}^{\mathcal{A}})\|_{\mathcal{S}_{\frac{p_kp_{k+1}}{p_k+p_{k+1}}}}\prod_{l=k+2}^{n}\|x_l^{\mathcal{A}}\|_{\mathcal{S}_{p_l}} \\ &\quad\le 2\prod_{l=1}^{n}\|x_l\|_{\mathcal{S}_{p_l}}(\|T_\rho^{A,A}:\mathcal{S}_{p_{k+1}}\to\mathcal{S}_{p_{k+1}}\| + \|T_\psi^{A,A}:\mathcal{S}_{p_{k+1}}\to\mathcal{S}_{p_{k+1}}\|). \end{aligned}$$

By Lemma 3.1 and complex interpolation there is some absolute constant $C > 0$ such that

$$\|T_\rho^A:\mathcal{S}_{p_{k+1}}\to\mathcal{S}_{p_{k+1}}\|, \|T_\psi^A:\mathcal{S}_{p_{k+1}}\to\mathcal{S}_{p_{k+1}}\| \le C\max\{p_{k+1},p'_{k+1}\}.$$

This concludes the proof in case $0 < k < n-1$. The cases $k = 0, n-1$ can be proved completely analogously by estimating the expressions (4.4) and (4.5). ∎

Definition 4.2: We define, for $f \in C^{n-1}(\mathbb{R})\cap C^n(\mathbb{R}\setminus\{0\})$ with $f^{[n]}$ bounded and $1 \le p_1,\dots,p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l} = 1$,

$$\begin{aligned} M_n(f,p_1,\dots,p_n) &= \sup_{A\in B(H)_{sa}}\|T^{(A,\dots,A)}_{f^{[n]}}:\mathcal{S}_{p_1}\times\cdots\times\mathcal{S}_{p_n}\to\mathcal{S}_{1,\infty}\|, \\ M_n^+(f,p_1,\dots,p_n) &= \sup_{A\in B(H)_{sa},A\ge 0}\|T^{(A,\dots,A)}_{f^{[n]}}:\mathcal{S}_{p_1}\times\cdots\times\mathcal{S}_{p_n}\to\mathcal{S}_{1,\infty}\|, \\ M_n^-(f,p_1,\dots,p_n) &= \sup_{A\in B(H)_{sa},A\le 0}\|T^{(A,\dots,A)}_{f^{[n]}}:\mathcal{S}_{p_1}\times\cdots\times\mathcal{S}_{p_n}\to\mathcal{S}_{1,\infty}\|. \end{aligned}$$

PROPOSITION 4.3: *Using the assumptions of Lemma 4.1 there exists a constant* $C(p_1,\dots,p_n)>0$ *such that*

$$\begin{aligned}&M_n(f,p_1,\dots,p_n)\\&\quad\le 4M_n^+(f,p_1,\dots,p_n)+4M_n^-(f,p_1,\dots,p_n)+C(p_1,\dots,p_n)\\&\qquad\times\Big(\max_{1\le k\le n-1}M_{n-1}\Big(f_1,p_1,\dots,p_{k-1},\frac{p_kp_{k+1}}{p_k+p_{k+1}},p_{k+2},\dots,p_n\Big)+\|f^{[n]}\|_\infty\Big).\end{aligned}$$

Proof. Take $A\in B(H)_{sa}$ with $0\notin\sigma_p(A)$. By [PSS13, Remark 5.4] we have that there exists a constant $C_0(p_1,\dots,p_n)>0$ such that

$$\begin{aligned}\|T^{(A,\dots,A)}_{f_1^{[n-1]}}:\mathcal{S}_{p_1}\times\ldots\times\mathcal{S}_{p_{n-1}}\to\mathcal{S}_{q_1}\|<&C_0(p_1,\dots,p_n)\|f^{[n]}\|_\infty,\\\|T^{(A,\dots,A)}_{f_1^{[n-1]}}:\mathcal{S}_{p_2}\times\ldots\times\mathcal{S}_{p_n}\to\mathcal{S}_{q_2}\|<&C_0(p_1,\dots,p_n)\|f^{[n]}\|_\infty,\end{aligned}$$

where $\frac{1}{q_1}=\sum_{i=1}^{n-1}\frac{1}{p_i}$, $\frac{1}{q_2}=\sum_{i=2}^{n}\frac{1}{p_i}$. Therefore by Lemma 4.1 for $\mathcal{A}\neq\emptyset$ and $\mathcal{A}\neq\{0,\dots,n\}$ there exists a constant $C_1(p_1,\dots,p_n)>0$ such that for $x_k\in\mathcal{S}_{p_k}\cap\mathcal{S}_2$,

$$\begin{aligned}&\|T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}}\\&\quad\le C_1(p_1,\dots,p_n)\\&\qquad\times\Big(\max_{1\le k\le n-1}M_{n-1}\Big(f_1,p_1,\dots,p_{k-1},\frac{p_kp_{k+1}}{p_k+p_{k+1}},p_{k+2},\dots,p_n\Big)+\|f^{[n]}\|_\infty\Big)\\&\qquad\times\prod_{k=1}^{n}\|x_k\|_{\mathcal{S}_{p_k}}.\end{aligned}\tag{4.6}$$

Further, for $\mathcal{A}=\emptyset$ we have

$$T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})=T^{(A_-,\dots,A_-)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}),\quad A_-=\chi_{(-\infty,0]}(A).$$

And similarly for $\mathcal{A}=\{0,\dots,n\}$ we have

$$T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})=T^{(A_+,\dots,A_+)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}}),\quad A_+=\chi_{[0,\infty)}(A).$$

It follows that

$$\|T^{(A,\dots,A)}_{f^{[n]}}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}}\le M_n^{\pm}(f,p_1,\dots,p_n)\prod_{k=1}^{n}\|x_k\|_{\mathcal{S}_{p_k}},\tag{4.7}$$

where $\pm = -$ if $\mathcal{A} = \emptyset$ and $\pm = +$ if $\mathcal{A} = \{0, \dots, n\}$. Then we use (4.1) and the quasi-triangle inequality followed by estimates (4.6), (4.7) to get

$$\begin{aligned}
&\|T_{f^{[n]}}^{(A,\dots,A)}(x_1,\dots,x_n)\|_{\mathcal{S}_{1,\infty}} \\
&\quad\le 2^{n+1} \sum_{\substack{\mathcal{A}\subseteq\{1,\dots,n\},\\ \mathcal{A}\neq\emptyset,\mathcal{A}\neq\{1,\dots,n\}}} \|T_{f^{[n]}}^{(A,\dots,A)}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}} \\
&\qquad + 4 \sum_{\mathcal{A}=\emptyset,\{1,\dots,n\}} \|T_{f^{[n]}}^{(A,\dots,A)}(x_1^{\mathcal{A}},\dots,x_n^{\mathcal{A}})\|_{\mathcal{S}_{1,\infty}} \\
&\quad\le \Big(2^{2n+1} C_1(p_1,\dots,p_n) \\
&\qquad \times \Big(\max_{1\le k\le n-1} M_{n-1}\Big(f_1,p_1,\dots,p_{k-1},\frac{p_k p_{k+1}}{p_k+p_{k+1}},p_{k+2},\dots,p_n\Big) + \|f^{[n]}\|_\infty\Big) \\
&\qquad\quad + 4M_n^+(f,p_1,\dots,p_n) + 4M_n^-(f,p_1,\dots,p_n)\Big) \\
&\qquad \times \prod_{k=1}^n \|x_k\|_{\mathcal{S}_{p_k}}.
\end{aligned} \tag{4.8}$$

This estimate (4.8) is uniform in $A \in B(H)_{sa}$ with $0 \notin \sigma_p(A)$. Therefore by Proposition 2.5 the estimate (4.8) holds uniformly for every $A \in B(H)_{sa}$ which is exactly the desired estimate. ∎

For π_1, π_2 disjoint subsets of $\mathbb{N}$ we write $\pi_1 < \pi_2$ if every element in π_1 is (strictly) smaller than every element in π_2.

Definition 4.4: We say that $(q_1,\dots,q_k)$ is a **consummation** of $(p_1,\dots,p_n)$ if there exists a partition $\{\pi_1,\dots,\pi_k\}$ of $\{1,\dots,n\}$ with $\pi_1 < \pi_2 < \cdots < \pi_n$ such that

$$\frac{1}{q_k} = \sum_{l\in\pi_k} \frac{1}{p_l}.$$

For $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$ with $f^{[n]}$ bounded and $1 \le p_1,\dots,p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l}$ set

$$\begin{aligned}
L_n^+(f,p_1,\dots,p_n) &= \sup\{M_{n-k}^+(f_k,q_1,\dots,q_k) \mid 0 \le k < n\}, \\
L_n^-(f,p_1,\dots,p_n) &= \sup\{M_{n-k}^-(f_k,q_1,\dots,q_k) \mid 0 \le k < n\},
\end{aligned} \tag{4.9}$$

where the suprema are taken over all consummations $(q_1,\dots,q_k)$ of $(p_1,\dots,p_n)$.

THEOREM 4.5: *Let $f \in C^{n-1}(\mathbb{R}) \cap C^n(\mathbb{R}\backslash\{0\})$ with $f^{[n]}$ bounded and let $1 \le p_1, \dots, p_n < \infty$ with $\sum_{l=1}^n \frac{1}{p_l} = 1$. There exists a constant $C(p_1, \dots, p_n) > 0$ such that for every $\mathbf{A} \in B(H)_{sa}^{\times n+1}$ we have*

$$\begin{aligned} &\|T^{\mathbf{A}}_{f^{[n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n}\| \\ &\quad \le C(p_1, \dots, p_n)(\|f^{[n]}\|_\infty + L_n^+(f, p_1, \dots, p_n) + L_n^-(f, p_1, \dots, p_n)). \end{aligned} \tag{4.10}$$

Proof. Proposition 4.3 shows that for every $0 \le k < n-1$ and any consummation $(q_1, \dots, q_{n-k})$ of $(p_1, \dots, p_n)$ there exists a constant $C(q_1, \dots, q_{n-k})$ such that

(4.11)

$$\begin{aligned} &M_{n-k}(f_k, q_1, \dots, q_{n-k}) \\ &\quad \le 4L_n^+(f, p_1, \dots, p_n) + 4L_n^-(f, p_1, \dots, p_n) \\ &\qquad + C(q_1, \dots, q_{n-k}) \\ &\qquad \times \Big(\max_{1 \le k \le n-1} M_{n-k-1}\Big(f_{k+1}, q_1, \dots, q_{k-1}, \frac{q_k q_{k+1}}{q_k + q_{k+1}}, q_{k+2}, \dots, q_n\Big) + \|f^{[n]}\|_\infty \Big). \end{aligned}$$

If $k = n-1$, then by [CPSZ19, Theorem 1.2] there exists $C > 0$ such that

$$M_1(f_{n-1}, 1) \le C\|f^{[n]}\|_\infty. \tag{4.12}$$

Applying the estimate (4.11) inductively from $k = 0$ to $k = n-2$ and using (4.12) for $k = n-1$ we see that there is a constant $C(p_1, \dots, p_n) > 0$ such that

$$\begin{aligned} &M_n(f, p_1, \dots, p_n) \\ &\quad \le C(p_1, \dots, p_n)(L_n^+(f, p_1, \dots, p_n) + L_n^-(f, p_1, \dots, p_n) + \|f^{[n]}\|_\infty). \end{aligned}$$

This is the desired estimate (4.10) for $\mathbf{A} = (A, \dots, A), A \in B(H)_{sa}$. The general case follows from Corollary 2.4. ∎

5. Consequences of Theorem 4.5: Weak $(1,1)$ estimates for generalized absolute value functions

We now arrive at the applications of Theorem 4.5.

THEOREM 5.1: *Let $a(t) = |t|t^{n-1}, t \in \mathbb{R}$. Fix $1 < p_1, \dots, p_n < \infty$ such that $\sum_{l=1}^n \frac{1}{p_l} = 1$. Then there exists a constant $D > 0$ such that for every $\mathbf{A} \in B(H)_{sa}^{\times n+1}$ we have that*

$$\|T^{\mathbf{A}}_{a^{[n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \le D. \tag{5.1}$$

Proof. Let $\epsilon = \pm 1$. Set $b(t) = t^n$. Then $a(t) = \epsilon b(t)$ for every $t \in \mathbb{R}$ with $\epsilon t \geq 0$. Consequently, $a^{[n]}(t) = \epsilon b^{[n]}(t)$ for every $t \in \epsilon \cdot \mathbb{R}^{n+1}_{\geq 0}$. Recall that a_k and b_k are defined in (3.4) with f replaced by a and b respectively. So by Lemma 3.4 we certainly have

$$a_k^{[n-k]}(t) = \epsilon b_k^{[n-k]}(t), \quad t \in \epsilon \cdot \mathbb{R}^{n-k+1}_{\geq 0},\ 0 \leq k < n. \tag{5.2}$$

Further, for the n-th order derivative we have $b^{(n)}(t) = n!$ for $t \in \mathbb{R}$. By the integral expression for divided differences [PSS13, Lemma 5.1] we conclude that $b^{[n]}(t) = n!$ for $t \in \mathbb{R}^{n+1}\backslash\{0\}$ and so by (5.2) we find $a^{[n]}(t) = \epsilon n!$ for all $t \in \epsilon \cdot \mathbb{R}^{n+1}_{\geq 0}\backslash\{0\}$. So by Lemma 3.4 we certainly have

$$a_k^{[n-k]}(t) = \epsilon n!, \quad t \in \epsilon \cdot \mathbb{R}^{n-k+1}_{\geq 0}\backslash\{0\}, 0 \leq k < n.$$

Let $\chi_{\epsilon,k}$ be the indicator function of $\epsilon \cdot \mathbb{R}^{n-k+1}_{\geq 0}$. Then $\chi_{\epsilon,k} a_k^{[n-k]} = \epsilon n!(\chi_{\epsilon,k} - \chi_0)$. We conclude from Lemma 2.1 that for $\epsilon A \geq 0$,

$$T^{(A,\dots,A)}_{a_k^{[n-k]}} = T^{(A,\dots,A)}_{\chi_{\epsilon,k} a_k^{[n-k]}} = \epsilon n!(T^{(A,\dots,A)}_{\chi_{\epsilon,k}} - T^{(A,\dots,A)}_{\chi_0}).$$

By Lemma 2.1 the multiple operator integrals $T^{(A,\dots,A)}_{\chi_{\epsilon,k}}$ and $T^{(A,\dots,A)}_{\chi_0}$ are contractions $\mathcal{S}_{r_1} \times \cdots \times \mathcal{S}_{r_{n-k}} \to \mathcal{S}_1$ for any $(r_1, \dots, r_{n-k})$ with $\sum_{l=1}^{n-k} \frac{1}{r_l} = 1$. So certainly they are contractions $\mathcal{S}_{r_1} \times \cdots \times \mathcal{S}_{r_{n-k}} \to \mathcal{S}_{1,\infty}$. We conclude that

$$L_n^{\epsilon}(a, p_1, \dots, p_n) < 2n!.$$

Hence we conclude the theorem from Theorem 4.5. ∎

THEOREM 5.2: *Let $g \in C^{n+1}(\mathbb{R})$ be such that $g(t) = |t|t^{n-1}, t \in \mathbb{R}\backslash[-1,1]$. Fix $1 < p_1, \dots, p_n < \infty$ such that*

$$\sum_{l=1}^{n} \frac{1}{p_l} = 1.$$

Then there exists a constant $D > 0$ such that for every $\mathbf{A} \in B(H)^{\times n+1}_{sa}$ we have that

$$\|T^{\mathbf{A}}_{g^{[n]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}\| \leq D. \tag{5.3}$$

Proof. Set $c = g - a$, where a is defined in Theorem 5.1. From the definition of divided differences $c^{[n]} = g^{[n]} - a^{[n]}$. By Theorem 5.1 and the quasi-triangle inequality it suffices to show that $T^{\mathbf{A}}_{c^{[n]}}$ is bounded $\mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_{1,\infty}$ uniformly in $\mathbf{A} \in B(H)^{\times n+1}_{sa}$.

Note that by the assumptions c is supported on $[-1,1]$. Let $\epsilon = \pm 1$. Let $B : \mathbb{R} \to [0,1]$ be a smooth compactly supported function that is 1 on the interval $\epsilon[0,1]$. Set

$$c_\epsilon(t) = B(t)(g(t) - \epsilon t^n), \quad t \in \mathbb{R}.$$

Then c_ϵ is a compactly supported C^{n+1}-function and $c_\epsilon(t) = c(t), t \in \epsilon\mathbb{R}_{\geq 0}$. Therefore for all $t \in \epsilon \cdot \mathbb{R}_{\geq 0}^{n+1}$ we have $c_\epsilon^{[n]}(t) = c^{[n]}(t)$. By Lemma 3.4 for all $t \in \epsilon \cdot \mathbb{R}_{\geq 0}^{n-k+1}$ we have

$$c_{\epsilon,k}^{[n-k]}(t) = c_k^{[n-k]}(t).$$

In other words, if $\chi_{\epsilon,k}$ is the indicator function on $\epsilon \cdot \mathbb{R}_{\geq 0}^{n-k+1}$,

$$c_{\epsilon,k}^{[n-k]}\chi_{\epsilon,k} = c_k^{[n-k]}\chi_{\epsilon,k}. \tag{5.4}$$

By Proposition 2.6 there exists a constant $D > 0$ such that for all $A \in B(H)_{sa}$ we have

$$\|T^{(A,\ldots,A)}_{c_{\epsilon,k}^{[n-k]}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_1\| \leq D. \tag{5.5}$$

We see by (5.4), (5.5) and Lemma 2.1 that for all $A \in B(H)_{sa}$ with $\epsilon A \geq 0$,

$$\|T^{(A,\ldots,A)}_{c_k^{[n-k]}} = T^{(A,\ldots,A)}_{c_k^{[n-k]}\chi_{\epsilon,k}} = T^{(A,\ldots,A)}_{c_{\epsilon,k}^{[n-k]}\chi_{\epsilon,k}} : \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_n} \to \mathcal{S}_1\| \leq D.$$

So in Theorem 4.5 we have that $L_n^\epsilon(c, p_1, \ldots, p_n) < \infty$ and so by the same Theorem 4.5 we conclude the proof. ∎

Remark 5.3: In [CLPST16, Lemma 28], see in particular the line after equation (37) of [CLPST16], the following result was proved and is a key step in the resolution of Peller's problem as stated in [CLPST16]. Let $n = 2$ and let $g : \mathbb{R} \to \mathbb{C}$ be a function as in the statement of Theorem 5.2 with the additional assumption that $g(0) = g'(0) = g''(0) = 0$. There exists no constant $0 < D < \infty$ such that for all $\mathbf{A} \in B(H)_{sa}^{\times 3}$ we have

$$\|T_g^{\mathbf{A}} : \mathcal{S}_2 \times \mathcal{S}_2 \to \mathcal{S}_1\| \leq D.$$

This shows that Theorem 5.2 is optimal.

Acknowledgement. The authors are grateful to the referee for detailed comments which helped to improve the exposition.

References

[APS05] W. van Ackooij, B. de Pagter and F. Sukochev, *Domains of infinitesimal generators of automorphism flows*, Journal of Functional Analysis **218** (2005), 409–424.

[ACDS09] N. Azamov, A. Carey, P. Dodds and F. Sukochev, *Operator integrals, spectral shift, and spectral flow*, Canadian Journal of Mathematics **61** (2009), 241–263.

[Cad18] L. Cadilhac, *Weak boundedness of Calderón–Zygmund operators on noncommutative L_1-spaces*, Journal of Functional Analysis **274** (2018), 769–796.

[CSZ18] M. Caspers, F. Sukochev and D. Zanin, *Weak type operator Lipschitz and commutator estimates for commuting tuples*, Université de Grenoble. Annales de l'Institut Fourier **68** (2018), 1643–1669.

[CMPS14] M. Caspers, S. Montgomery-Smith, D. Potapov and F. Sukochev, *The best constants for operator Lipschitz functions on Schatten classes*, Journal of Functional Analysis **267** (2014), 3557–3579.

[CPSZ15] M. Caspers, D. Potapov, F. Sukochev and D. Zanin, *Weak type estimates for the absolute value mapping*, Journal of Operator Theory **73** (2015), 361–384.

[CPSZ19] M. Caspers, D. Potapov, F. Sukochev and D. Zanin, *Weak type commutator and Lipschitz estimates: resolution of the Nazarov–Peller conjecture*, American Journal of Mathematics **141** (2019), no. 3, 593–610.

[ChSu94] V. Chilin and F. Sukochev, *Weak convergence in non-commutative symmetric spaces*, Journal of Operator Theory **31** (1994), no. 1, 35–65.

[Con90] J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Vol. 96, Springer, New York, 1990.

[CLS17] C. Coine, C. Le Merdy and F. Sukochev, *When do triple operator integrals take value in the trace class?* Universitŕ de Grenoble. Annales de l'Institut Fourier, to appear.

[CLPST16] C. Coine, C. Le Merdy, D. Potapov, F. Sukochev and A. Tomskova, *Resolution of Peller's problem concerning Koplienko-Neidhardt trace formulae*, Proceedings of the London Mathematical Society **113** (2016), 113–139.

[Dav88] E. Davies, *Lipschitz continuity of functions of operators in the Schatten classes*, Journal of the London Mathematical Society **37** (1988), 148–157.

[DeLo93] R. Devore and G. Lorentz, *Constructive Approximation*, Grundlehren der Mathematischen Wissenschaften, Vol. 303, Springer, Berlin, 1993.

[DDPS97] P. Dodds, T. Dodds, B. de Pagter and F. Sukochev, *Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces*, Journal of Functional Analysis **148** (1997), 28–69.

[DDPS99] P. Dodds, T. Dodds, B. de Pagter and F. Sukochev, *Lipschitz continuity of the absolute value in preduals of semifinite factors*, Integral Equations and Operator Theory **34** (1999), 28–44.

[Far67] Y. Farforovskaya, *An estimate of the nearness of the spectral decompositions of self-adjoint operators in the Kantorovich–Rubinstein metric*, Vestnik Leningrad University. Mathematics **22** (1967), 155–156.

[Far68] Y. Farforovskaya, *The connection of the Kantorovich–Rubinstein metric for spectral resolutions of selfadjoint operators with functions of operators*, Vestnik Leningrad. University. Mathematics **23** (1968), 94–97.

[Far72] Y. Farforovskaya, *An example of a Lipschitz function of self-adjoint operators with non-nuclear difference under a nuclear perturbation*, Zapiski Naučnyh Seminarov Leningradskogo Otdelenija Matematičeskogo Instituta imena V. A. Steklova Akademii Nauk SSSR **30** (1972), 146–153.

[Kat73] T. Kato, *Continuity of the map $S \mapsto |S|$ for linear operators*, Proceedings of the Japan Academy **49** (1973), 157–160.

[Kos92] H. Kosaki, *Unitarily invariant norms under which the map $A \mapsto |A|$ is continuous*, Kyoto University. Research Institute for Mathematical Sciences. Publications **28** (1992), 299–313.

[LSZ] S. Lord, F. Sukochev and D. Zanin, *Singular Traces*, De Gruyter Studies in Mathematics, Vol. 46, de Gruyter, Berlin, 2013.

[McI71] A. McIntosh, *Counterexample to a question on commutators*, Proceedings of the American Mathematical Society **29** (1971), 337–340.

[NaPe09] F. Nazarov and V. Peller, *Lipschitz functions of perturbed operators*, Comptes Rendus Mathématique. Académie des Sciences. Paris **347** (2009), 857–862.

[Par09] J. Parcet, *Pseudo-localization of singular integrals and noncommutative Calderón–Zygmund theory*, Journal of Functional Analysis **256** (2009), 509–593.

[Pav69] B. S. Pavlov, *Multidimensional operator integrals*, in *Problems of Mathematical Analysis no. 2: Linear Operators and Operator Equations*, Izdat, Leningrad University, Leningrad, , pp. 99-122.

[Pel06] V. V. Peller, *Multiple operator integrals and higher operator derivatives*, Journal of Functional Analysis **233** (2006), 515–544.

[PSS13] D. Potapov, A. Skripka and F. Sukochev, *Spectral shift function of higher order*, Inventiones Mathematicae **193** (2013), 501–538.

[PS09] D. Potapov and F. Sukochev, *Unbounded Fredholm modules and double operator integrals*, Journal für die Reine und Angewandte Mathematik **626** (2009), 159–185.

[PS11] D. Potapov and F. Sukochev, *Operator-Lipschitz functions in Schatten–von Neumann classes*, Acta Mathematica **207** (2011), 375–389.

[PSST17] D. Potapov, A. Skripka, F. Sukochev and A. Tomskova, *Multilinear Schur multipliers and Schatten properties of operator Taylor remainders*, Advances in Mathematics **320** (2017), 1063–1098.

[Rud91] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.

[SoSt69] M. Solomjak and V. Sten'kin, *A certain class of multiple operator Stieltjes integrals*, in *Problems of Mathematical Analysis, No. 2: Linear Operators and Operator Equations*, Izdat, Leningrad University, Leningrad, 1969, pp. 122–134.

[ST] A. Skripka and A. Tomskova, *Multilinear Operator Integrals*, Lecture Notes in Mathematics, Vol. 2250, Springer, Cham, 2019.