

AN INVARIANT VERSION OF THE LITTLE
GROTHENDIECK THEOREM
FOR SOBOLEV SPACES

BY

KRYSTIAN KAZANIECKI

*Institute of Mathematics, Polish Academy of Sciences
ul. Śniadeckich 8, 00-656 Warszawa, Poland
and
Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
e-mail: kkazaniecki@impan.pl*

AND

PIOTR PAKOSZ

*Faculty of Mathematics, Informatics and Mechanics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
e-mail: pakosz100@gmail.com*

AND

MICHAŁ WOJCIECHOWSKI

*Institute of Mathematics, Polish Academy of Sciences
ul. Śniadeckich 8, 00-656 Warszawa, Poland
e-mail: miwoj@impan.pl*

ABSTRACT

We prove that every Hilbert space operator which factorizes invariantly through Sobolev space $W_1^1(\mathbb{T}^d)$ belongs to some non-trivial Schatten class.

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1. Introduction

One of the most important theorems of the operator ideals theory proved by Grothendieck in 1953 [5] states that any bounded linear operator $T : L_1 \rightarrow L_2$ is absolutely summing (for definitions and notation used in this paper see [3], [14]). This theorem, beginning with the Lindenstrauss–Pełczyński (cf. [8]) paper, inspired a great development of Banach space geometry and many other fields (cf. [12]) and still attracts wide attention (cf. [2], [10], [9], [1]).

A weaker (and simpler) version of Grothendieck’s theorem claims that such operators are 2-absolutely summing. This weaker statement is equivalent to the property that every operator between Hilbert spaces which factorizes through L_1 is a Hilbert–Schmidt operator (cf. [4], Prop. 16.3.2). As observed by Kislyakov (cf. [7]), one cannot replace L_1 by the Sobolev space W_1^1 in this theorem. Indeed, the classical embedding operator of $W_1^1(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$ is not 2-absolutely summing. However, as proved in [13], it is $(p, 1)$ -summing for every $p > 1$. This suggests the following conjecture.

CONJECTURE 1: *Any operator between Hilbert spaces which factorizes through the Sobolev space W_1^1 belongs to some non-trivial Schatten class.*

Not only are we unable to prove the conjecture, but we do not even know if there exists an infinitely dimensional complemented subspace of W_1^1 which is isomorphic to a Hilbert space. However, as proved (cf. [11]), there are no such spaces which are translation invariant. This suggests that assuming some additional structure may help to verify the conjecture and motivate us to consider its special case—translation invariant operators. We introduce the following definition

Definition 1: Let $X(\mathbb{T}^d), Y(\mathbb{T}^d), Z(\mathbb{T}^d)$ be translation invariant spaces on the d -dimensional torus. We say that a bounded linear operator $T : X(\mathbb{T}^d) \rightarrow Y(\mathbb{T}^d)$ admits **invariant factorization** through $Z(\mathbb{T}^d)$ provided there exist translation invariant bounded linear operators $A : X(\mathbb{T}^d) \rightarrow Z(\mathbb{T}^d)$ and $B : Z(\mathbb{T}^d) \rightarrow Y(\mathbb{T}^d)$ such that $T = B \circ A$.

Under this restriction we were able to prove our conjecture. The main result of this paper is

THEOREM 1: *Let $T : L_2(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ be a bounded linear operator which admits an invariant factorization through $W_1^1(\mathbb{T}^d)$. Then $\sigma_p(T) < \infty$ for every $p > 2d + 4$, where σ_p denotes the p -th Schatten norm.*

The main theorem is a direct consequence of the following estimate on the growth of the coefficients of Fourier multiplier operators.

THEOREM 2: *Let $T : W_1^1(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ be a translation invariant operator such that*

$$\widehat{T(f)}(n) = \lambda_n \cdot \widehat{f}(n).$$

Then for any $\varepsilon > 0$ the following inequality is satisfied:

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \left(\frac{|\lambda_n|}{|n|_2} \right)^{2d+4+\varepsilon} < \infty.$$

We show now how the main result follows from Theorem 2. Let T given by $T(e^{i\langle n,t \rangle}) = \lambda_n e^{i\langle n,t \rangle}$ factorize by A, B , where $A : W_1^1(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is given by $A(e^{i\langle n,t \rangle}) = \alpha_n e^{i\langle n,t \rangle}$ and $B : L_2(\mathbb{T}^d) \rightarrow W_1^1(\mathbb{T}^d)$ is given by $B(e^{i\langle n,t \rangle}) = \beta_n e^{i\langle n,t \rangle}$ for $n \in \mathbb{Z}^d$. Then using an obvious estimate $|\beta_n| |n|_2 \leq \|B\|$ we get

$$\sum |\lambda_n|^p = \sum |\alpha_n \cdot \beta_n|^p \leq \|B\|^p \sum \left(\frac{\alpha_n}{|n|_2} \right)^p \leq C \|A\|^p \|B\|^p.$$

Theorem 2 is a special case of the following result:

THEOREM 3: *Let $1 < p \leq 2$ and $T : W_1^1(\mathbb{T}^d) \rightarrow L_p(\mathbb{T}^d)$ be a translation invariant operator such that*

$$\widehat{T(f)}(n) = \lambda_n \cdot \widehat{f}(n).$$

Then for any $\varepsilon > 0$, the following inequality is satisfied:

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1} + \frac{p}{p-1}(d+1)+\varepsilon} < \infty.$$

The rest of the paper is devoted to the proof of Theorem 3.

Remark 1: We don't know whether the exponent in Theorem 1 is sharp. The most prominent example of an operator which is a subject of Theorem 1 is the classical Sobolev embedding operator $T : W_2^1(\mathbb{T}^2) \rightarrow L_2(\mathbb{T}^2)$. In this case we have $\lambda_n = 1$ and $\sigma_p(T) < \infty$ for $p > 2$ which is much stronger than the statement of Theorem 1 (which gives $p > 8$). Possibly the exponent in Theorem 1 could be improved.

Remark 2: As a matter of fact the result of this paper is a subject of harmonic analysis and concerns Fourier multipliers. We only indicated an interpretation for it in terms of operator ideals and used this interpretation to support Conjecture 1. Note that the invariant little Grothendieck theorem for L_1 is an obvious well known property (see [6] Theorem 1.4).

Remark 3: In the case when X and Y are Hilbert spaces we can formulate the definition of invariant factorization in a more abstract way. For a compact abelian group G and an invariant function space $Z(G)$ we say that an operator $T : H_1 \rightarrow H_2$ factorizes invariantly through $Z(G)$ if there are orthonormal bases $\{h_{n,1}\}$ of H_1 and $\{h_{n,2}\}$ of H_2 and operators $A : H_1 \rightarrow Z(G)$ and $B : Z(G) \rightarrow H_2$ such that $T = B \circ A$ and $A(h_{n,1}) \in \text{span } \gamma_n$ and $B(\gamma_n) \in \text{span } h_{n,2}$ for $n = 1, 2, \dots$, where (γ_n) is an enumeration of characters of the group G .

CONJECTURE 2: *If $Z(G)$ has no complemented, invariant infinitely dimensional subspaces isomorphic to a Hilbert space, then any Hilbert space operator T which factorizes invariantly through $Z(G)$ belongs to some nontrivial Schatten class.*

2. Summability of the multiplier

For any $k \in \mathbb{N}$ we denote by $R_k = \{(n_1, \dots, n_d) \in \mathbb{Z}^d : 3^k \leq \max_i \{|n_i|\} < 3^{k+1}\}$ a k -th triadic ring. For $n \in \mathbb{Z}^d$ by $n^{(i)}$ we denote the i -th coordinate of n and by $|n|_2$ its euclidean norm. Observe that for $n \in R_k$ we have

$$(1) \qquad 3^k \leq |n|_2 \leq \sqrt{d}3^{k+1}.$$

In order to prove Theorem 3 we will use two auxiliary lemmas on the growth of λ_n on triadic rings. In the first lemma we control the behavior of the sequence λ_n on a single triadic ring.

LEMMA 1: *There exists a constant $C > 0$ independent of k , such that for any $k \in \mathbb{N}$*

$$\sum_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}} < C.$$

The second lemma provides an estimate on the growth of the sequence of maximal elements of λ_n in the triadic rings.

LEMMA 2: For every $\varepsilon > 0$

$$\sum_{k \in \mathbb{N}} \max_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+1)+\varepsilon} < \infty.$$

We postpone the proofs of the lemmas to Section 3 and Section 4. Now we use them to prove the main theorem.

Proof of Theorem 3. For $n \in R_k$ we have the following estimate

$$\begin{aligned} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1} + \frac{p}{p-1}(d+1)+\varepsilon} &= \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}} \cdot \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+1)+\varepsilon} \\ &\leq \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}} \cdot \max_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+1)+\varepsilon}. \end{aligned}$$

From Lemma 1 and the above estimate we get

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+2)+\varepsilon} &= \sum_{k \in \mathbb{N}} \sum_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1} + \frac{p}{p-1}(d+1)+\varepsilon} \\ &\leq \sum_{k \in \mathbb{N}} \sum_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}} \cdot \max_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+1)+\varepsilon} \\ &< C \cdot \sum_{k \in \mathbb{N}} \max_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1}(d+1)+\varepsilon}. \end{aligned}$$

By Lemma 2, the right hand side of the above inequality is finite. Hence

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \left(\frac{|\lambda_n|}{|n|_2} \right)^{\frac{p}{p-1} + \frac{p}{p-1}(d+1)+\varepsilon} < \infty. \quad \blacksquare$$

3. Proof of Lemma 1

The proof of Lemma 1 is based on the well known properties of Fejer’s kernel. It is standard, however we present it here for the reader’s convenience.

We will denote by K_n the classical Fejer’s kernel:

$$\widehat{K}_n(k) = \begin{cases} 1 - \frac{|k|}{n} & \text{for } |k| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed $k \in \mathbb{N}$ we define $\phi(x_1, \dots, x_d): \mathbb{T}^d \rightarrow \mathbb{C}$ by the formula

$$\widehat{\phi}(m_1, \dots, m_d) = \prod_{j=1}^d \widehat{K}_{3^{k+2}}(m_j).$$

Since K_n is a trigonometric polynomial of degree n , by the classical Bernstein's inequality we have

$$\left\| \frac{\partial}{\partial x_j} \phi \right\|_1 = \left\| \frac{\partial}{\partial y} K_{3^{k+2}}(y) \right\|_1 \leq 3^{k+2} \|K_{3^{k+2}}(y)\|_1.$$

Therefore

$$\|\phi\|_{1,1} := \|\phi\|_1 + \sum_{j=1}^d \left\| \frac{\partial}{\partial x_j} f \right\|_1 \leq 1 + d \cdot 3^{k+2}.$$

Let us observe that for $m \in R_k$ we have $|\widehat{\phi}(m)| \geq (\frac{2}{3})^d$. Indeed, $|m_j| < 3^{k+1}$ and

$$1 - \frac{|m_j|}{3^{k+2}} > \frac{2}{3}.$$

By the Hausdorff–Young inequality, we get

$$\|T\phi\|_p \geq \|\lambda \cdot \widehat{\phi}\|_{\frac{p}{p-1}} \geq \left(\frac{2}{3}\right)^d \left(\sum_{n \in R_k} |\lambda_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Combining the above estimates with (1) completes the proof:

$$\begin{aligned} \sum_{n \in R_k} \left(\frac{|\lambda_n|}{|n|_2}\right)^{\frac{p}{p-1}} &\leq 3^{-\frac{pk}{p-1}} \cdot \sum_{n \in R_k} |\lambda_n|^{\frac{p}{p-1}} \\ &\leq 3^{-\frac{pk}{p-1}} \cdot \left(\frac{3}{2}\right)^{\frac{dp}{p-1}} \cdot \|T\phi\|_p^{\frac{p}{p-1}} \\ &\leq 3^{-\frac{pk}{p-1}} \cdot \left(\frac{3}{2}\right)^{\frac{dp}{p-1}} \cdot \|T\|_{\frac{p}{p-1}} \cdot \|\phi\|_{1,1}^{\frac{p}{p-1}} \\ &\leq 3^{-\frac{pk}{p-1}} \cdot \left(\frac{3}{2}\right)^{\frac{dp}{p-1}} \cdot \|T\|_{\frac{p}{p-1}} \cdot (1 + d \cdot 3^{k+2})^{\frac{p}{p-1}} \\ &\leq C(p, d) \|T\|_{\frac{p}{p-1}}. \end{aligned}$$

4. Proof of Lemma 2

For any fixed sequence $\{n_i\}_{i=1}^N \subset \mathbb{Z}^d \setminus \{0\}$ such that $\frac{|n_{i+1}|_2}{|n_i|_2} > 3$, we define a finite Riesz product corresponding to the sequence $\{n_j\}_{j=1}^N$, a trigonometric polynomial $R: \mathbb{T}^d \rightarrow \mathbb{R}$ given by a formula

$$R(t) = \prod_{j=1}^N (1 + \cos(2\pi \langle n_j, t \rangle)).$$

Let $X = \{-1, 0, 1\}^N$. For any $\xi \in X$ we define $L(\xi)$ as a number of non-zero coordinates of ξ , i.e., $L(\xi) = \sum_{j=1}^N |\xi_j|$ and $M(\xi) = \sum_{j=1}^N \xi_j \cdot n_j$. One can easily check that

$$(2) \quad R(t) = \sum_{\xi \in X} \frac{1}{2^{L(\xi)}} e^{2\pi i \langle M(\xi), t \rangle}$$

and

$$(3) \quad \|R(t)\|_1 = 1.$$

We introduce the auxiliary notion of the growth of the sequence $\{n_j\}_{j=1}^N$.

Definition 2: Let $\alpha > 1$. We will call a sequence $n_1, \dots, n_N \in \mathbb{Z}^d \setminus \{0\}$ **α -sparse** if $\frac{|n_{j+1}|_2}{|n_j|_2} \geq 3^\alpha$ for every $j = 1, \dots, N - 1$.

For $j \in 1, \dots, d$ we define the set

$$A_j = \{n \in \mathbb{Z}^d \setminus \{0\} : j \text{ is a smallest index such that } |n_j| = \max_{k \in \{1, \dots, d\}} |n_k|\}.$$

Clearly the sets A_j are pairwise disjoint and

$$\mathbb{Z}^d \setminus \{0\} = \bigcup_{j=1}^d A_j.$$

Let $j \in 1, \dots, d$. Then for $n \in A_j$ and $k \in \{1, \dots, d\} \setminus \{j\}$ we have $\frac{n_k}{n_j} \in [-1, 1]$. Fix a large number N . We want to further subdivide the sets A_j into pieces to control the value of quotients n_k/n_j up to $1/N$ for n in a single piece. For such N we define sets $\tilde{A}_{j,a} := \tilde{A}_{j,a}(N) \subseteq \mathbb{Z}^d \setminus \{0\}$ for $j \in \{1, \dots, d\}$, $a \in \{1, \dots, N\}^d \cap \{a_j = 1\}$:

$$\tilde{A}_{j,a} = \left\{ n \in \mathbb{Z}^d \setminus \{0\} : \left| \frac{n_k}{n_j} - \left(\frac{2a_k - 1}{N} - 1 \right) \right| \leq \frac{1}{N}, k = 1, \dots, d \right\}.$$

One can check that

$$A_j = \bigcup_a \tilde{A}_{j,a}$$

Since the sets $\tilde{A}_{j,a}$ are no longer pairwise disjoint, we choose symmetric and pairwise disjoint sets $A_{j,a} := A_{j,a}(N)$ such that $A_{j,a} \subset \tilde{A}_{j,a}$ and

$$\bigcup_a A_{j,a} = A_j.$$

Obviously

$$\bigcup_{j,a} A_{j,a} = \bigcup_j A_j = \mathbb{Z}^d \setminus \{0\}.$$

Definition 3: We call the set $A_{j,a}$ an N -**sector**.

Remark 4: We choose sets $A_{j,a}$ in such a way that vectors from the fixed set $A_{j,a}$ almost point in the same direction, up to some error which depends on N . This allows us to construct in the next section a test function ϕ , which behaves like a function whose Fourier spectrum is contained in a fixed line. Note that the function ψ whose Fourier spectrum is contained in a fixed line satisfies the equation

$$\nabla\psi = v\frac{\partial}{\partial x_j}\psi$$

for some fixed $v \in \mathbb{R}^d$. In fact, what we need from the sets $A_{j,a}$ is for them

- (1) to be symmetric,
- (2) to be pairwise disjoint,
- (3) $\bigcup_{j,a} A_{j,a} = \mathbb{Z}^d \setminus \{0\}$,
- (4) $|\tan \angle(v_{j,\alpha}, n)| < \frac{C}{N}$ for all $n \in A_{j,a}$, fixed $C > 0$ and fixed $v_{j,\alpha} \in A_{j,\alpha}$,
- (5) for $n \in A_{j,\alpha}$ the j -th coordinate dominates other coordinates.

CONSTRUCTION OF THE TEST FUNCTION. Let N be a fixed natural number and let a sequence $\{n_j\}_{j=1}^N \subset \mathbb{Z}^d \setminus \{0\}$ be an N -sparse sequence contained in the single N -sector $A_{j_0,a}$. Let $R(t)$ be a finite Riesz product corresponding to the sequence n_1, \dots, n_N . We define a function $\phi = \phi_{N;n_1, \dots, n_N} : \mathbb{T}^d \rightarrow \mathbb{C}$ by

$$\int_{\mathbb{T}^d} \phi(t) d\mu(t) = 0$$

and

$$(4) \quad \frac{\partial}{\partial x_{j_0}}\phi = R(t) - 1.$$

We will estimate the Sobolev norm of the function ϕ . Without loss of generality we can assume that the sequence is a subset of $A_{1,a}$. From the triangle inequality and (3) we get

$$\left\| \frac{\partial}{\partial x_1}\phi \right\|_1 = \|R(t) - 1\|_1 \leq \|R(t)\|_1 + \|1\|_1 = 2.$$

We will estimate the remaining derivatives.

LEMMA 3: *There exists a constant $C > 0$, such that for any number N , any $a \in [0, N]^d$ and any N -sparse sequence $n_1, \dots, n_N \in \mathbb{Z}^d \setminus \{0\}$ contained in the N -sector $A_{1,a}$, we have*

$$\left\| \frac{\partial}{\partial x_j} \phi \right\|_1 < C \quad \forall j \in \{2, \dots, d\}.$$

Proof. Note that

$$\phi = \sum_{\xi \in X \setminus \{0\}} \frac{1}{2^{L(\xi)}} \frac{1}{M(\xi)^{(1)}} e^{2\pi i \langle M(\xi), t \rangle}$$

and

$$(5) \quad \frac{\partial}{\partial x_j} \phi = \sum_{\xi \in X \setminus \{0\}} \frac{1}{2^{L(\xi)}} \frac{M(\xi)^{(j)}}{M(\xi)^{(1)}} e^{2\pi i \langle M(\xi), t \rangle}.$$

Let

$$X_l := \{\xi \in X : \forall_{k>l} \xi_k = 0\}.$$

We define ψ_l by the formula

$$\psi_l = \sum_{\xi \in X_{l-1}} \frac{1}{2^{L(\xi)}} e^{2\pi i \langle M(\xi), t \rangle}.$$

The function ψ_l is just a finite Riesz product corresponding to the sequence n_1, \dots, n_l . Hence $\|\psi_l\|_1 = 1$. We can rewrite equation (2) in the following way:

$$(6) \quad \begin{aligned} R(t) - 1 &= \sum_{\xi \in X \setminus \{0\}} \frac{1}{2^{L(\xi)}} e^{2\pi i \langle M(\xi), t \rangle} = \sum_{l=1}^N \sum_{\xi \in X_l \setminus X_{l-1}} \frac{1}{2^{L(\xi)}} e^{2\pi i \langle M(\xi), t \rangle} \\ &= \sum_{l=1}^N \frac{1}{2} (e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}) \psi_l. \end{aligned}$$

We define an auxiliary function $H_l(\xi)$ for $\xi \in X_{l-1}$ as follows:

$$\begin{aligned} H_l(\xi) &= \frac{1}{2} e^{2\pi i \langle n_l, t \rangle} \left(\frac{M(\xi)^{(j)} + n_l^{(j)}}{M(\xi)^{(1)} + n_l^{(1)}} - \frac{n_l^{(j)}}{n_l^{(1)}} \right) \\ &\quad + \frac{1}{2} e^{2\pi i \langle -n_l, t \rangle} \left(\frac{M(\xi)^{(j)} - n_l^{(j)}}{M(\xi)^{(1)} - n_l^{(1)}} - \frac{n_l^{(j)}}{n_l^{(1)}} \right). \end{aligned}$$

Similarly as in (6) we can rewrite equation (5) in terms of $H_l(\xi)$ and ψ_l :

$$(7) \quad \begin{aligned} \frac{\partial}{\partial x_j} \phi &= \sum_{l=1}^N \sum_{\xi \in X_{l-1}} \frac{1}{2^{L(\xi)}} H_l(\xi) e^{2\pi i \langle M(\xi), t \rangle} \\ &+ \sum_{l=1}^N \frac{n_l^{(j)}}{n_l^{(1)}} \frac{1}{2} (e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}) \psi_l. \end{aligned}$$

In order to estimate the norm of the first term on the right hand side we need the following lemma:

LEMMA 4: *There exists a constant $C > 0$ independent of N and a sequence $\{n_k\}$ such that for any $\xi \in X_l$ we have $\|H_l(\xi)\|_1 \leq \frac{C'}{3^N}$.*

Assuming Lemma 4 we get the following bound:

$$(8) \quad \begin{aligned} \left\| \sum_{l=1}^N \sum_{\xi \in X_l} \frac{1}{2^{L(\xi)}} H_l(\xi) e^{2\pi i \langle M(\xi), t \rangle} \right\|_1 &\leq \sum_{l=1}^N \sum_{\xi \in X_l} \|H_l(\xi)\|_1 \\ &\leq \sum_{l=1}^N \sum_{\xi \in X_l} \frac{C'}{3^N} \leq \sum_{l=1}^N \frac{C'' \cdot 3^{l-1}}{3^N} \leq C''. \end{aligned}$$

Now we estimate the second term on the right hand side of (7). Let

$$\theta = -1 + \frac{2a_j - 1}{N}$$

where $\{n_k\} \subset A_{i,a}$; obviously $|\theta| \leq 1$. Moreover, if $n \in A_{1,a}$ then $|\frac{n^{(j)}}{n^{(1)}} - \theta| \leq \frac{1}{N}$. From the triangle inequality and (6) we get that

$$\begin{aligned} &\left\| \sum_{l=1}^N \frac{n_l^{(j)}}{n_l^{(1)}} \frac{e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}}{2} \psi_l \right\|_1 \\ &\leq \left\| \sum_{l=1}^N \left(\frac{n_l^{(j)}}{n_l^{(1)}} - \theta \right) \frac{e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}}{2} \psi_l \right\|_1 \\ &\quad + |\theta| \cdot \left\| \sum_{l=1}^N \frac{e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}}{2} \psi_l \right\|_1 \\ &\leq \frac{1}{N} \sum_{l=1}^N \left\| \frac{e^{2\pi i \langle n_l, t \rangle} + e^{2\pi i \langle -n_l, t \rangle}}{2} \psi_l \right\|_1 + \|R(t) - 1\|_1 \\ &\leq 1 + 2 = 3. \end{aligned}$$

This together with (8) implies Lemma 3. ■

It remains to prove Lemma 4.

Proof. Since the sequence $\{n_k\}$ is N -sparse we know that for $k \in \{1, \dots, l\}$ we have

$$\max(|n_{l-j}^{(1)}|, |n_{l-j}^{(j)}|) \leq |n_{l-k}|_2 \leq 3^{-kN} |n_l|_2.$$

Hence for $\xi \in X_l$, from the triangle inequality we have the following bounds

$$\max(|M(\xi)^{(1)}|, |M(\xi)^{(j)}|) \leq |n_l|_2 \cdot \left(\frac{1}{3^N} + \frac{1}{3^{2N}} + \dots \right) = |n_l|_2 \frac{3}{2 \cdot 3^N}.$$

Since $n_i \in A_{1,a}$, we know that for $1 \leq i \leq l$ we have

$$|n_i^{(1)}| \geq |n_i^{(j)}| \quad \text{and} \quad \sqrt{d} \cdot |n_i^{(1)}| \geq |n_i|_2.$$

From the triangle inequality

$$\begin{aligned} \left| \frac{n_l^{(j)} \pm M(\xi)^{(j)}}{n_l^{(1)} \pm M(\xi)^{(1)}} - \frac{n_l^{(j)}}{n_l^{(1)}} \right| &= \left| \frac{n_l^{(1)} M(\xi)^{(j)} - n_l^{(j)} M(\xi)^{(1)}}{n_l^{(1)} (n_l^{(1)} \pm M(\xi)^{(1)})} \right| \\ &\leq C \frac{|n_l^{(1)} M(\xi)^{(j)} - n_l^{(j)} M(\xi)^{(1)}|}{|n_l|_2^2} \\ &\leq \frac{C'}{3^N} \frac{|n_l|_2^2}{|n_l|_2^2} = \frac{C'}{3^N}. \end{aligned}$$

Therefore we get

$$\|H_l(\xi)\|_1 \leq \frac{1}{2} \cdot 2 \cdot \left| \frac{n_l^{(j)} \pm M(\xi)^{(j)}}{n_l^{(1)} \pm M(\xi)^{(1)}} - \frac{n_l^{(j)}}{n_l^{(1)}} \right| \leq \frac{C'}{3^N}. \quad \blacksquare$$

From Lemma 3 and the Poincaré inequality we deduce a bound on a W_1^1 norm of the function ϕ ,

$$(9) \quad \|\phi\|_{1,1} \leq C \|\nabla \phi\|_1 \leq C_1,$$

where the constant C_1 depends only on d . This estimate is crucial in the proof of Lemma 2.

LEMMA 5: *There exists a constant $K > 0$ independent of N such that for any N -sparse sequence $n_1, \dots, n_N \in \mathbb{Z}^d \setminus \{0\}$ which is a subset of a single N -sector, we have*

$$\sum_{i=1}^N \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} \leq K.$$

Proof. Let ϕ be defined as in (4). Recall that

$$\phi = \sum_{\xi \in X \setminus \{0\}} \frac{1}{2^{L(\xi)}} \frac{1}{M(\xi)^{(1)}} e^{2\pi i \langle M(\xi), t \rangle}.$$

Hence

$$T\phi = \sum_{\xi \in X \setminus \{0\}} \lambda_{M(\xi)} \frac{1}{2^{L(\xi)}} \frac{1}{M(\xi)^{(1)}} e^{2\pi i \langle M(\xi), t \rangle}.$$

From the Hausdorff–Young inequality

$$\|T\phi\|_p^p \geq \left(\sum_{\xi \in X \setminus \{0\}} \left| \lambda_{M(\xi)} \frac{1}{2^{L(\xi)}} \frac{1}{M(\xi)^{(1)}} \right|^{\frac{p}{p-1}} \right)^{p-1}.$$

We estimate the right hand side summing only over ξ with $L(\xi) = 1$. We get

$$\|T\phi\|_p^p \geq C \left(\sum_{i=1}^N \left| \frac{\lambda_{n_i}}{n_i^{(1)}} \right|^{\frac{p}{p-1}} \right)^{p-1} = C \left(\sum_{i=1}^N \left| \frac{\lambda_{n_i}}{|n_i|_2} \cdot \frac{|n_i|_2}{n_i^{(1)}} \right|^{\frac{p}{p-1}} \right)^{p-1}.$$

Obviously $|n_i|_2 \geq |n_i^{(1)}|$. Hence

$$\|T\phi\|_p^p \geq C \left(\sum_{i=1}^N \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} \right)^{p-1}.$$

Boundedness of T and the inequality (9) yields the existence of a constant $C > 0$ independent of N such that

$$\sum_{i=1}^N \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} \leq C \|T\|_{\frac{p}{p-1}}. \quad \blacksquare$$

We will use the following simple property of the sum of monotonic sequences:

LEMMA 6: *Let $\{b_j\}_{j=1}^\infty$ be a non-negative, non-increasing sequence such that*

$$\sum_{j=1}^N b_j \leq O(N^\alpha),$$

where $0 < \alpha < 1$. Then the sequence $\{b_j\} \in \ell_q$ for any $q > \frac{1}{1-\alpha}$.

Proof. Since the sequence is non-increasing we have

$$N \cdot a_N \leq \sum_{j=1}^N a_j \leq C \cdot N^\alpha.$$

Therefore $a_N \leq CN^{\alpha-1}$. \blacksquare

The next lemma will be used only to justify the existence of a non-increasing rearrangement of the sequence.

LEMMA 7: *If $\{n_i\}$ is a sequence of points in $\mathbb{Z}^d \setminus \{0\}$ such that $\lim_{i \rightarrow \infty} |n_i|_2 = \infty$, then*

$$\lim_{i \rightarrow \infty} \frac{|\lambda_{n_i}|}{|n_i|_2} = 0.$$

Proof. Assume that there is a sequence $n_i \rightarrow \infty$ such that $\frac{|\lambda_{n_i}|}{|n_i|_2} > c > 0$. We fix N and we divide $\mathbb{Z}^d \setminus \{0\}$ into a finite number of N -sectors (see Definition 3). There exists an infinite subsequence $\{n_i\}$ contained in one of them. Passing again to the subsequence we can assume that $\{n_i\}$ is N -sparse. From the assumptions on n_i we have

$$\sum_{i=1}^N \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} > N c^{\frac{p}{p-1}}.$$

On the other hand, the sequence n_1, \dots, n_N satisfies assumptions of Lemma 5. Therefore

$$\sum_{i=1}^N \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} \leq K,$$

where K is independent of N . Hence for any $N \in \mathbb{N}$ we have $N c^{\frac{p}{p-1}} < K$. This is a contradiction. ■

Proof of Lemma 2. Let $\mu_k = \max_{n \in R_k} \frac{|\lambda_n|}{|n|_2}$. From Lemma 7 we deduce the existence of a bijection $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that $\mu_{\sigma(k)}$ is a non-increasing sequence. It is enough to show that

$$(10) \quad \sum_{j=1}^{N^{d+1}} \mu_{\sigma(j)}^{\frac{p}{p-1}} < C \cdot N^d.$$

Indeed assuming (10), for large enough N ,

$$\sum_{j=1}^N \mu_{\sigma(j)}^{\frac{p}{p-1}} \leq \sum_{j=1}^{\lceil d+1\sqrt{N} \rceil^{d+1}} \mu_{\sigma(j)}^{\frac{p}{p-1}} < C \cdot \lceil d+1\sqrt{N} \rceil^d \leq C' \cdot N^{\frac{d}{d+1}}.$$

Hence the assumptions of Lemma 6 are satisfied and Lemma 2 follows.

To obtain (10) we fix N . Let $n_k \in R_k$ be such that $\mu_k = \frac{|\lambda_{n_k}|}{|n_k|_2}$ for $k \in \mathbb{N}$. We divide $\mathbb{Z}^d \setminus \{0\}$ into N -sectors. We consider the sequence $n_{\sigma(1)}, \dots, n_{\sigma(N^{d+1})}$. Let S denote the set of all N -sectors. For $A \in S$ we denote by $I_A \subset \{1, 2, \dots, N^{d+1}\}$

the set of indices k such that $n_{\sigma(k)} \in A$. We can divide the set $\{n_{\sigma(i)} : i \in I_A\}$ into at most $(\frac{\#I_A}{N} + 2N + 1)$ different N -sparse sequences of length N (note that $\{n_{\sigma(i)} : i \in I_A\}$ can be ordered in such a way that every element is at least three times bigger than its predecessor). From Lemma 5 we get

$$\sum_{i \in I_A} \left(\frac{|\lambda_{n_i}|}{|n_i|_2} \right)^{\frac{p}{p-1}} \leq K \cdot \left(\frac{\#I_A}{N} + 2N + 1 \right).$$

Summing the above inequality over all N -sectors yields

$$\sum_{j=1}^{N^{d+1}} \mu_{\sigma(j)}^{\frac{p}{p-1}} \leq \sum_{A \in S} K \cdot \left(\frac{\#I_A}{N} + 2N + 1 \right).$$

Observe that $\#S = d \cdot N^{d-1}$ and $\sum_{A \in S} \#I_s = N^{d+1}$. In conclusion we obtain

$$\begin{aligned} \sum_{A \in S} K \cdot \left(\frac{\#I_A}{N} + 2N + 1 \right) &= K \cdot \frac{1}{N} \left(\sum_{A \in S} \#I_s \right) + K \cdot \sum_{A \in S} (1 + 2N) \\ &= K \cdot N^d + K \cdot (d \cdot N^d + d \cdot N^{d-1}) \\ &\leq CN^d. \quad \blacksquare \end{aligned}$$

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