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THE CANONICAL QUADRATIC PAIR ON A CLIFFORD ALGEBRA AND TRIALITY*

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ABSTRACT

We define a canonical quadratic pair on the Clifford algebra of an algebra with quadratic pair over a field. This allows us to extend to the characteristic 2 case the notion of trialitarian triples, from which we derive a characterization of totally decomposable quadratic pairs in degree 8. We also describe trialitarian triples involving algebras of small Schur index.

1. Introduction

Triality is a phenomenon that arises due to the high level of symmetry in the Dynkin diagram D_4. This symmetry is reflected in objects associated to groups of type D_4, such as 8-dimensional quadratic forms, and degree 8 central simple algebras with orthogonal involution. More precisely, consider a degree 8 central simple algebra A over a field F of characteristic different from 2. Assume A

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is endowed with an orthogonal involution σ_A with trivial discriminant. The Clifford algebra $\mathcal{C}(A, \sigma_A)$, with its canonical involution $\underline{\sigma_A}$, is a direct product of two central simple algebras with involution, which also have degree 8 and are of orthogonal type, so that we actually get a triple

$$((A, \sigma_A), (B, \sigma_B), (C, \sigma_C)),$$

called a trialitarian triple, and an isomorphism

$$(\star) \qquad (\mathcal{C}(A, \sigma_A), \underline{\sigma_A}) \simeq (B, \sigma_B) \times (C, \sigma_C).$$

By [17, §42], triality then permutes the algebras with involution in this expression. That is, $(\star)$ implies the existence of isomorphisms

$$(\mathcal{C}(B, \sigma_B), \underline{\sigma_B}) \simeq (C, \sigma_C) \times (A, \sigma_A),$$
$$(\mathcal{C}(C, \sigma_C), \underline{\sigma_C}) \simeq (A, \sigma_A) \times (B, \sigma_B).$$

In particular, it follows that the Clifford algebra, viewed as an algebra with involution, is a complete invariant for orthogonal involutions with trivial discriminant on a degree 8 algebra.

This trialitarian relation has proven to be extremely fruitful; roughly speaking, triality plays the same role in degree 8 as the so-called exceptional isomorphisms in smaller degree. For instance, it can be used to characterize totally decomposable orthogonal involutions on algebras of degree 8 (see [17, §42.B] and connected problems in [4]). It is related to the classification of groups of type D_4 (see [17, §44] and [13]). It makes the degree 8 case a crucial test case for some general questions on algebras with involution; see, for instance, [21, Thm. 5.2] and [22, §4]. Finally, this better understanding of the degree 8 case can in turn be used to answer questions in larger degree; for instance, it leads to an example of a degree 16 non-totally decomposable algebra with involution that is totally decomposable after generic splitting of the underlying algebra, see [23, Thm. 1.3].

For fields of characteristic 2, triality is not as well studied due to complications arising when studying quadratic forms and orthogonal groups over these fields. In particular, the notions of symmetric bilinear forms and quadratic forms are no longer equivalent. Over such a base, the automorphism group of a bilinear form is not semisimple anymore, so the corresponding twisted objects, in particular orthogonal involutions, cannot be used to describe such algebraic groups. Twisted groups of type D in characteristic 2 were initially studied by Tits, who used so-called generalized quadratic forms (see [26] or §5.2), which appear to be a good replacement for hermitian forms in this setting. Involution-like corresponding objects, namely quadratic pairs, were introduced later in [17, §5]. They are related to generalized quadratic forms by an adjunction process, and behave better than generalized quadratic forms, for instance under scalar extension. They provide an appropriate tool to describe groups of type D. This theory is developed in The Book of Involutions [17], where most of the material, about involutions and quadratic pairs, their invariants, and relations to algebraic groups, is developed over a field of arbitrary characteristic. However, Chapter X, about Trialitarian Central Simple Algebras, is one of the rare exceptions in [17]; the base field is assumed to be of characteristic not 2 in that section.

That the group Spin_8 has this exceptionally large group of outer automorphisms is true independent of the characteristic of the underlying field, and hence some trialitarian relation should hold for quadratic pairs in characteristic 2 also. One has a notion of a Clifford algebra of a quadratic pair, and we again have that the Clifford algebra of a quadratic pair with trivial discriminant is the direct product of two degree 8 central simple algebras with involution (see [17, §7 and §8]). However, in order to fully recapture the trialitarian relation, one also needs that the Clifford algebra be equipped with a canonical quadratic pair, not just a canonical involution. A definition of this canonical quadratic pair is briefly sketched out in [17, p. 149], in the particular case where A is split, of degree divisible by 8, and endowed with a hyperbolic quadratic pair. From this, one can define a canonical quadratic pair in the more general case via Galois descent. However, this definition is not easy to use, and the lack of a 'rational' definition, that is a definition that avoids the use of Galois descent, is one reason why the results in [17, Chapt. X] are restricted to fields of characteristic different from 2 (see [17, Chapt. X, Notes] for more details).

The main purpose of this paper is to provide a rational definition for the canonical quadratic pair of the Clifford algebra of an algebra with quadratic pair, see Section 3.1. We use as a crucial tool the Lie algebra structures described in [17, §8.C]. We also provide an explicit description of this canonical quadratic pair in the split case in Section 3.2. With this in hand, we extend to arbitrary fields the main results of [17, (§42)]. In particular, we define a notion of trialitarian triple, and describe the trialitarian action in Section 4, and we characterize totally decomposable algebras with quadratic pair in degree 8 in Theorem 5.1. The last section describes all trialitarian triples of small enough Schur index, see Section 5.2. Partial results in this direction were previously obtained by Knus and Villa [18, § 7].

We first recall some notation and basic results (§2.1 to 2.3), and make some preliminary observations on quadratic pairs and tensor products (§2.4).

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2. Preliminaries

Throughout the paper, F is a field. We refer the reader to [19] as a general reference on central simple algebras, [17] for involutions and quadratic pairs and [11] for hermitian, bilinear and quadratic forms. Most of our notations are borrowed from those books.

2.1. Algebras with involution. Let A be a central simple algebra of degree n over F. To all $a \in A$, we associate its reduced characteristic polynomial

$$\mathrm{Prd}_{A,a}(X) = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} - \cdots + (-1)^n s_n(a),$$

which is the characteristic polynomial of $a \otimes 1 \in A \otimes_F \Omega \simeq M_n(\Omega)$, where Ω is an algebraic closure of F, see [19, §16.1]. The coefficients of $\mathrm{Prd}_{A,a}$ belong to F; $s_1(a)$ and $s_n(a)$ are the reduced trace and the reduced norm of a, respectively denoted by $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$, and $s_2(a)$ is denoted by $\mathrm{Srd}_A(a)$.

All the involutions considered in this paper are F-linear. If the algebra A is split, that is, $A \simeq \mathrm{End}_F(V)$ where V is a finite dimensional F-vector space, an F-linear involution on A is the adjoint of a nondegenerate symmetric or skew-symmetric bilinear form $b : V \times V \to F$, uniquely defined up to a scalar factor. We denote this algebra with involution by Ad_b. The involution is symplectic if b is alternating, and orthogonal if b is symmetric and non-alternating.

Let σ be an F-linear involution on A. We use the same notation as in [17, §2.A] for the subvector spaces $\mathrm{Sym}(A,\sigma)$, $\mathrm{Symd}(A,\sigma)$, $\mathrm{Skew}(A,\sigma)$ and $\mathrm{Alt}(A,\sigma)$ of symmetric, symmetrized, skew-symmetric and alternating elements, respectively. Recall $\mathrm{Sym}(A,\sigma) = \mathrm{Symd}(A,\sigma)$ if the base field has characteristic different from 2, while in characteristic 2, $\mathrm{Symd}(A,\sigma) = \mathrm{Alt}(A,\sigma)$ is a strict subspace of $\mathrm{Sym}(A,\sigma) = \mathrm{Skew}(A,\sigma)$, and these spaces have dimension $\frac{n(n-1)}{2}$ and $\frac{n(n+1)}{2}$, respectively. Still assuming the base field has characteristic 2, one may prove that the involution σ is symplectic if and only if 1 is a symmetrized element, or equivalently all symmetric elements have reduced trace 0 [17, (2.6)]. In particular, in characteristic 2, a tensor product of involutions with at least one symplectic factor always is symplectic. In characteristic different from 2, a tensor product of involutions is symplectic if and only if there are an odd number of symplectic involutions in the product (see [17, (2.23)]).

Recall that in arbitrary characteristic, an F-quaternion algebra has a basis $(1, u, v, w)$ such that

$$u(1-u) = a, v^2 = b \quad \text{and} \quad w = uv = v(1-u)$$

for some $a \in F$ with $4a \neq -1$ and $b \in F^\times$ (see [1, Chap. IX, Thm. 26]); any such basis is called a quaternion basis throughout this paper. Conversely, for $a \in F$ and $b \in F^\times$ the above relations uniquely determine an F-quaternion algebra, which we denote by $H = [a, b)$. If the characteristic of F is different from 2, substituting $i = u - \frac{1}{2}$ and $j = v$ gives the more usual presentation of Q, that is a basis $\{1, i, j, ij\}$, with $i^2 = c$, $j^2 = d$ for $c, d \in F^\times$ and $ij = -ji$. In this case we denote Q by (c, d).

Recall $H = [a, b)$ has a unique symplectic involution, called the canonical involution, which is determined by the conditions that $\overline{u} = 1 - u$ and $\overline{v} = -v$. Considering H as a 4-dimensional vector space over F, we may view Nrd_H as a 4-dimensional quadratic form over F, which we call the norm form of H.

2.2. Quadratic forms and their Clifford algebras. We refer to [11] as a general reference on bilinear and quadratic forms. For $b \in F^\times$, we denote the 2-dimensional symmetric bilinear form

$$(x_1, x_2) \times (y_1, y_2) \mapsto x_1y_1 - bx_2y_2$$

by $\langle\langle b\rangle\rangle$. Such a form is called a 1-fold bilinear Pfister form. For a nonnegative integer m, by an m-fold bilinear Pfister form, we mean a nondegenerate symmetric bilinear form isometric to a tensor product of m 1-fold bilinear Pfister forms; we use the notation $\langle\langle b_1, \ldots, b_m\rangle\rangle \simeq \langle\langle b_1\rangle\rangle \otimes \cdots \otimes \langle\langle b_m\rangle\rangle$, where $\otimes$ denotes the usual tensor product on bilinear forms.

Let V be a finite dimensional F-vector space and $q : V \to F$ a quadratic form. The polar form of q is the symmetric bilinear form on V defined by

$$b_q(x, y) = q(x + y) - q(x) - q(y),$$

so that $b_q(x,x) = 2q(x)$ for all $x \in V$. The quadratic form q is called nonsingular if its polar form is nondegenerate. In characterictic 2, the polar form b_q is alternating; hence if q is nonsingular, then V has even dimension and b_q is hyperbolic [11, Prop. 1.8]. We denote the isometry of quadratic forms using $\simeq$ and orthogonal sum of quadratic forms using $\perp$. For all $b_1, b_2 \in F$, we let $[b_1, b_2]$ be the quadratic form $(x,y) \to b_1x^2 + xy + b_2y^2$. This form is nonsingular if and only if $-1 \neq 4b_1b_2$. Note that the hyperbolic plane $\mathbb{H} = [0,0]$ satisfies the following isometries $\mathbb{H} \simeq [a,0] \simeq [0,a]$ for all $a \in F$, so for every nonsingular 2-dimensional quadratic form we may choose a presentation $[a,b]$ with $a \neq 0$. Recall that any nonsingular quadratic form ϕ over F decomposes uniquely as the orthogonal sum of an anisotropic form called the anisotropic part of ϕ and denoted by ϕ_{an}, and a hyperbolic form. We thus have $\phi \simeq \phi_{an} \perp i_W(\phi) \times \mathbb{H}$, and the integer $i_W(\phi)$ is the so-called Witt index of ϕ.

To a quadratic form $q : V \to F$ and a symmetric bilinear form $b : W \times W \to F$, one associates the quadratic form, denoted by $b \otimes q$ and defined on $W \otimes V$ by

$$(b \otimes q)(w \otimes v) = b(w,w)q(v) \quad \text{for all } w \in W \text{ and } v \in V$$

(see [11, p. 51]). For any positive integer m, by an m-fold quadratic Pfister form we mean a quadratic form that is isometric to the tensor product of an $(m-1)$-fold bilinear Pfister form and a nonsingular binary quadratic form representing 1. We use the notation $\langle\langle b_1, \ldots, b_{m-1}, c]] = \langle\langle b_1, \ldots, b_{m-1}\rangle\rangle \otimes [1, c]$. In particular, the 1-fold quadratic Pfister form $\langle\langle c]]$ is the quadratic form $[1, c]$, regardless of the characteristic of the base field; we thus slightly depart here from the notations in [11, Example 9.4].

Recall that the Clifford algebra of a quadratic space (V, q) is a quotient of the tensor algebra $T(V)$ by the ideal $I(q)$ generated by elements of the form $v \otimes v - q(v) \cdot 1$ for $v \in V$. It has a natural $\mathbb{Z}/2\mathbb{Z}$-grading, and the subalgebra $\mathcal{C}_0(q)$ of degree 0 elements is called the even Clifford algebra. The identity map on V extends to an involution on $\mathcal{C}(q)$ and $\mathcal{C}_0(q)$ called the canonical involution and denoted $\underline{\sigma_q}$. If q is even-dimensional and nonsingular, the center of $\mathcal{C}_0(q)$ is a quadratic étale extension of F. It is determined by a class which belongs to the multiplicative group of square classes $F^\times/F^{\times 2}$ in characteristic different from 2 and the additive group $F/\wp(F)$ in characteristic 2, where

$$\wp(F) = \{a^2 + a \mid a \in F\}$$

is the image of the Artin–Schelter map. In both cases, we will refer to this class as the discriminant of q and denote it by $\operatorname{disc}(q)$. If F has characteristic 2 and $q \simeq [a_1, b_1]\perp \cdots \perp[a_n, b_n]$ for some a_i, b_i in F, then $\operatorname{disc}(q)$ is the class of $a_1b_1 + \cdots + a_nb_n$ in $F/\wp(F)$.

Example 2.1: For ease of reference in the sequel, we give an explicit description of the even Clifford algebra of a nonsingular quadratic form over a field of characteristic 2. Given such a form q, with polar form b, pick a decomposition

$$q \simeq [a_1, b_1] \perp \cdots \perp [a_m, b_m],$$

and let $e_1, \ldots, e_m, e'_1, \ldots, e'_m$ be a basis of the underlying vector space V such that for all i with $1 \leqslant i \leqslant m$, we have $q(e_i) = a_i$, $q(e'_i) = b_i$, $b(e_i, e'_i) = 1$ and all other pairs of basis vectors are orthogonal. We may assume $a_i \neq 0$ for all i.

The elements $u_i = e_ie'_i$ and $v_i = e_ie_m$, for $i \in \{1, \ldots, m-1\}$, belong to the even part $\mathcal{C}_0(q)$ of the Clifford algebra, and satisfy

$$u_i(1 + u_i) = a_ib_i, \quad v_i^2 = a_ia_m, \quad u_iv_i = v_i(1 + u_i).$$

They generate pairwise commuting quaternion subalgebras. Further, we have that

$$\underline{\sigma_q}(u_i) = 1 + u_i \quad \text{and} \quad \underline{\sigma_q}(v_i) = v_i.$$

Moreover, the element

$$\xi = \sum_{i=1}^{m} e_ie'_i$$

also belongs to $\mathcal{C}_0(q)$, commutes with u_i and v_i for all i, $1 \leqslant i \leqslant m-1$, and satisfies $\xi^2 = \xi + a_1b_1 + \cdots + a_nb_n$. Hence, $F[\xi]$ is a quadratic étale extension of F, central in $\mathcal{C}_0(q)$. We also have that

$$\underline{\sigma_q}(\xi) = \begin{cases} 1 + \xi & \text{if } m \text{ is odd,} \\ \xi & \text{if } m \text{ is even.} \end{cases}$$

So we finally get

$$(\mathcal{C}_0(q), \underline{\sigma_q}) \simeq \bigotimes_{i=1}^{m-1} (Q_i, \overline{}) \otimes (F[\xi], \gamma),$$

where $Q_i = [a_ib_i, a_ia_m)$, $\overline{}$ stands for the canonical involution, $F[\xi]$ is the center of $\mathcal{C}_0(q)$, and γ is the identity if m is even and the non-trivial F-automorphism of $F[\xi]$ if m is odd.

We finish this subsection with a characteristic free version of [15, (Example 9.12)] which we require in the sequel. The proof is similar, but we provide full details and relevant references in characteristic 2 for convenience.

PROPOSITION 2.2: *Let q be an 8-dimensional nonsingular quadratic form over F with trivial discriminant and Clifford algebra $\mathcal{C}(q)$ of Schur index at most 2. Then there exists a 4-dimensional symmetric bilinear form B and a 2-dimensional nonsingular form ϕ over F such that $q \simeq B \otimes \phi$.*

Proof. Over its function field, q is Witt equivalent to a 6-dimensional quadratic form with trivial discriminant, that is an Albert form. By [17, (16.5)], this Albert form is isotropic. Therefore, the Witt index of q over its function field is at least 2. Choose a quadratic separable extension $F(u)/F$ where $u - u^2 = a$ for some $a \in F$ with $-1 \neq 4a$ such that q becomes isotropic after extending scalars to $F(u)$. By [11, (25.1)], the Witt index of q over $F(u)$ also is at least 2. Hence by [3, Chapter V, (4.2)], we have that $q \simeq c\langle\langle b, a]] \perp q'$ for some $b, c \in F^\times$ and a 4-dimensional nonsingular quadratic form q' over F. Since q has trivial discriminant, it follows that q' also has trivial discriminant and hence q' is similar to a Pfister form, which we denote by π. The form $\langle\langle b, a]] \perp -\pi$ is Witt equivalent to an Albert form, and has the same Clifford invariant as q, of index $\leqslant 2$. Therefore, again its Witt index is at least 2, and by [11, (24.2)], there exist symmetric bilinear forms B' and B'' and $d \in F$ such that

$$\langle\langle b, a]] \simeq B' \otimes [1, d] \quad \text{and} \quad \pi \simeq B'' \otimes [1, d].$$

In particular, we have

$$q \simeq B \otimes [1, d]$$

for some symmetric bilinear form B over F. ∎

2.3. QUADRATIC PAIRS AND THEIR CLIFFORD ALGEBRAS. In arbitrary characteristic, algebraic groups of type D can be described in terms of quadratic pairs. For the reader's convenience, we recall here some basic facts on quadratic pairs which can be found in [17, §5,7.B, 8.B], and which are used throughout the paper.

Let A be a degree n central simple algebra over F. A quadratic pair on A is a couple (σ, f), where σ is an F-linear involution on A, with $\mathrm{Sym}(A, \sigma)$ of dimension $\frac{n(n+1)}{2}$, and f is a so-called semi-trace on (A, σ). That is, f is an F-linear map $f : \mathrm{Sym}(A,\sigma) \to F$ such that

$$f(x+\sigma(x)) = \mathrm{Trd}_A(x) \quad \text{for all } x \in A.$$

In particular, all semi-traces coincide on the subspace $\mathrm{Symd}(A,\sigma)$ of symmetrized elements. In characteristic different from 2, if follows from this definition that σ is of orthogonal type and there is a unique semi-trace on (A,σ) given by $f(x) = \frac{1}{2}\mathrm{Trd}_A(x)$ for all $x \in \mathrm{Sym}(A,\sigma)$. Therefore quadratic pairs and orthogonal involutions are equivalent notions when the characteristic is not 2. Conversely, in characteristic 2, the existence of a semi-trace implies σ is symplectic. Indeed, since

$$\mathrm{Trd}_A(c) = f(c+\sigma(c)) = f(2c) = 0$$

for all $c \in \mathrm{Sym}(A,\sigma)$, the reduced trace vanishes on $\mathrm{Sym}(A,\sigma)$, and this characterizes symplectic involutions by [17, (2.6)(2)].

One may easily check that for all $\ell \in A$ such that $\ell + \sigma(\ell) = 1$, the F-linear map defined by $f_\ell(s) = \mathrm{Trd}_A(\ell s)$ for all $s \in \mathrm{Sym}(A,\sigma)$ is a semi-trace on (A,σ). Conversely, it is proved in [17, (5.7)] that any semi-trace $f : \mathrm{Sym}(A,\sigma) \to F$ coincides with f_ℓ for some $\ell \in A$ satisfying $\ell + \sigma(\ell) = 1$. We say that the element ℓ gives or determines the semi-trace f_ℓ. Two distinct such elements ℓ and ℓ' determine the same semi-trace if and only if they differ by an alternating element, that is $\ell - \ell' = x - \sigma(x)$ for some x in A.

Let (V,q) be a nonsingular quadratic space over the field F. The polar form b_q of q induces an involution $\sigma_q = \mathrm{ad}_{b_q}$ on $A = \mathrm{End}_F(V)$. As explained in [17, p. 55], there is an isomorphism of algebras with involution

$$\varphi_b : (\mathrm{End}_F(V), \sigma_q) \xrightarrow{\sim} (V \otimes V, \varepsilon),$$

where the product in $V \otimes V$ is defined on elementary tensors by

$$(x \otimes y)(x' \otimes y') = x b_q(y,x') \otimes y' \quad \text{for all } x,y,x',y' \in V$$

and ε is the exchange involution, that is $\varepsilon(x \otimes y) = y \otimes x$. Moreover, by [17, (5.11)], there exists a unique semi-trace f defined on $\mathrm{Sym}(V \otimes V, \varepsilon)$ and satisfying $f(x \otimes x) = q(x)$ for all $x \in V$. Under the isomorphism above, f defines a semi-trace f_q on $(\mathrm{End}_F(V), \sigma_q)$. The quadratic pair $\mathrm{ad}_q = (\sigma_q, f_q)$ is called the adjoint of q, and we use the notation Ad_q for the algebra with quadratic pair $(\mathrm{End}_F(V), \sigma_q, f_q)$. As explained in loc. cit., any quadratic pair on a split algebra $\mathrm{End}_F(V)$ is the adjoint of a nonsingular quadratic form q on V.

Let (A,σ,f) be an F-algebra with quadratic pair. We assume in addition A has even degree $n=2m$. For further use, we briefly recall the definition of the discriminant and the Clifford algebra of (A,σ,f), as featured in [17]. Pick an element $\ell\in A$ which determines the semi-trace $f=f_\ell$. Consider the sandwich linear map $\mathrm{Sand}:\underline{A}\otimes\underline{A}\to\mathrm{End}_F(A)$, as defined in [17, (3.4)]), where $\underline{A}$ denotes A viewed as an F-vector-space. The Clifford algebra $\mathcal{C}(A,\sigma,f)$ of the algebra with quadratic pair (A,σ,f) is the quotient of the tensor algebra $T(\underline{A})$:

$$\mathcal{C}(A,\sigma,f)=\frac{T(\underline{A})}{J_1(\sigma,f)+J_2(\sigma,f)}$$

where

(1) $J_1(\sigma,f)$ is the ideal generated by all the elements of the form $s-f(s)\cdot 1$ for $s\in\underline{A}$ such that $\sigma(s)=s$;
(2) $J_2(\sigma,f)$ is the ideal generated by all elements of the form $u-\mathrm{Sand}(u)(\ell)$ for $u\in\underline{A}\otimes\underline{A}$ such that $\sigma_2(u)=u$ and where σ_2 is defined by the condition

$$\mathrm{Sand}(\sigma_2(u))(x)=\mathrm{Sand}(u)(\sigma(x))\quad\text{for } u\in\underline{A}\otimes\underline{A},\ x\in\underline{A}.$$

The involution on $T(\underline{A})$ acting as σ on $\underline{A}$ induces an involution $\underline{\sigma}$ of $\mathcal{C}(A,\sigma,f)$ called the canonical involution, and satisfying

$$\underline{\sigma}(a_1\otimes\ldots\otimes a_r)=\sigma(a_r)\otimes\ldots\otimes\sigma(a_1)\quad\text{for all } a_1,\ldots,a_r\in\underline{A}.$$

This construction extends the even Clifford algebra for quadratic spaces, that is if $(A,\sigma,f)\simeq\mathrm{Ad}_q$ for some even dimensional nonsingular quadratic form q, there is a canonical isomorphism between $\mathcal{C}(A,\sigma,f)$ and $\mathcal{C}_0(q)$, and the canonical involution $\underline{\sigma}$ corresponds to $\underline{\sigma_q}$ under this isomorphism. See [17, §8] for more details.

Similarly, one may extend the discriminant of even dimensional quadratic forms to quadratic pairs. More precisely, the center of the Clifford algebra $\mathcal{C}(A,\sigma,f)$ is a quadratic étale extension of F. It is determined by a class which belongs to the group of square classes $F^\times/F^{\times 2}$ in characteristic different from 2, and to the quotient $F/\wp(F)$ of F by the image of the Artin–Schreier map in characteristic 2. In both cases, this class is called the discriminant of the quadratic pair, denoted by $\mathrm{disc}(\sigma,f)$. If $(A,\sigma,f)\simeq\mathrm{Ad}_q$ for some nonsingular quadratic form q, then $\mathrm{disc}(\sigma,f)=\mathrm{disc}(q)$ in the relevant quotient.

The discriminant of a quadratic pair can also be explicitly computed as follows. If the characteristic of F is different from 2,

$$\operatorname{disc}(\sigma, f) = (-1)^m \operatorname{Nrd}_A(a) \in F^\times/F^{\times 2}$$

for any element $a \in \operatorname{Alt}(A, \sigma) \cap A^\times$, while in characteristic 2,

$$\operatorname{disc}(\sigma, f) = \operatorname{Srd}(\ell) + \frac{m(m-1)}{2} \in F/\wp(F),$$

where ℓ determines the semi-trace f, and A has degree $n = 2m$. See [17, §7.B] for more details.

2.4. Semi-traces and tensor products. Let A and B be two F-algebras, which are central simple over a finite field extension of F. Assume σ and ρ are symplectic if F has characteristic 2 and orthogonal otherwise, and consider an embedding of F-algebras with involution

$$i : (A, \sigma) \to (D, \rho).$$

Any element $\ell \in A$ such that $\ell + \sigma(\ell) = 1$ maps to an element $i(\ell) \in D$ such that $i(\ell) + \rho(i(\ell)) = 1$. In addition, alternating elements in (A, σ) map to alternating elements in (D, ρ). Therefore, to any semi-trace $f = f_\ell$ on (A, σ), we may associate a well defined semi-trace $g = f_{i(\ell)}$ on (D, ρ). Clearly, the semi-trace g depends not only on f, but also on the embedding i. When i is canonical, we forget the embedding and use the same notation f_ℓ for both semi-traces. This correspondence is not extending or restricting the semi-trace viewed as a map, even though i maps symmetric elements in (A, σ) to symmetric elements in (D, ρ). For instance, if D is F-central and A has center Z_A, then f is Z_A-linear with values in Z_A, while g is F-linear with values in F, and Z_A may be strictly larger than F. We will refer to g as the semi-trace induced by f on (D, ρ).

An important example of the situation above is when (D, ρ) is decomposable. Namely, let (B, τ) be an algebra with involution. We assume τ is orthogonal if F has characteristic different from 2, and either orthogonal or symplectic if F has characteristic 2. Let (A, σ, f) be an algebra with quadratic pair. The involution $\tau \otimes \sigma$ is then orthogonal if the characteristic of F is different from 2 and symplectic otherwise. Therefore, the construction above applies to the canonical embedding $(A, \sigma) \to (B \otimes A, \tau \otimes \sigma)$, so that f induces a semi-trace $f_\star$ on $(B \otimes A, \tau \otimes \sigma)$.

For all $b \in \mathrm{Sym}(B,\tau)$ and $a \in \mathrm{Sym}(A,\sigma)$, $b \otimes a \in \mathrm{Sym}(B\otimes A, \tau\otimes\sigma)$ and we have

$$f_\star(b\otimes a) = \mathrm{Trd}_{B\otimes A}((1\otimes \ell)(b\otimes a)) = \mathrm{Trd}_B(b)\,\mathrm{Trd}_A(\ell a) = \mathrm{Trd}_B(b) f(a),$$

where $\ell \in A$ is an element defining the semi-trace f. In [17, (5.18)], it is proved that the condition

$$f_\star(b\otimes a) = \mathrm{Trd}_B(b) f(a) \tag{1}$$

actually characterizes the semi-trace $f_\star$. This construction defines a tensor product

$$(B,\tau)\otimes(A,\sigma,f) = (B\otimes A, \tau\otimes\sigma, f_\star).$$

One may check that this tensor product corresponds to the usual one in the split case, that is $\mathrm{Ad}_b \otimes \mathrm{Ad}_\rho = \mathrm{Ad}_{b\otimes\rho}$ for all nondegenerate symmetric bilinear forms b and nonsingular quadratic forms ρ; see [17, (5.19)]. In addition, it is associative, that is

$$((C,\gamma)\otimes(B,\tau))\otimes(A,\sigma,f) \simeq (C,\gamma)\otimes((B,\tau)\otimes(A,\sigma,f)),$$

for any algebra with involution (C,γ), with γ orthogonal in characteristic different from 2; see [6, (5.3)]. In particular, we may write $(C,\gamma)\otimes(B,\tau)\otimes(A,\sigma,f)$ without any ambiguity. We say that (A,σ,f) is totally decomposable if there exist F-quaternion algebras with involution $(Q_i,\sigma_i)_{1\leqslant i\leqslant n-1}$, with $\bigotimes_i^{n-1}\sigma_i$ orthogonal in characteristic different from 2, and an F-quaternion algebra with quadratic pair (Q_n,σ_n,g) such that

$$(A,\sigma,f) \simeq \left(\bigotimes_{i=1}^{n-1}(Q_i,\sigma_i)\right)\otimes(Q_n,\sigma_n,g).$$

Assume now that F has characteristic 2 and (A,σ) and (B,τ) both are of symplectic type. We have $\mathrm{Trd}_B(b) = 0$ for all $b\in\mathrm{Sym}(B,\tau)$ (see [17, (2.6)(2)]) and formula (1) above shows that, given an arbitrary semi-trace f on $\mathrm{Sym}(A,\sigma)$, the induced semi-trace $f_\star$ on $\mathrm{Sym}(B\otimes A,\tau\otimes\sigma)$ vanishes on

$$\mathrm{Sym}(B,\tau)\otimes\mathrm{Sym}(A,\sigma) \subset \mathrm{Sym}(B\otimes A,\tau\otimes\sigma).$$

Again, this condition characterizes $f_\star$, see [17, (5.20)], and in particular, $f_\star$ does not depend on the choice of f. We now extend this result to a product with r symplectic factors.

Proposition 2.3: *Assume F has characteristic 2 and let $(A_i,\sigma_i)_{1\leqslant i\leqslant r}$ be r algebras with symplectic involution for some $r\geqslant 2$. There exists a unique semi-trace*

$$f_\otimes : \mathrm{Sym}\left(\bigotimes_{i=1}^r (A_i,\sigma_i)\right)\to F$$

such that

$$f_\otimes|_{\bigotimes_{i=1}^r \mathrm{Sym}(A_i,\sigma_i)}=0.$$

Proof. For any algebra with involution (A,σ), given two elements $a\in\mathrm{Symd}(A,\sigma)$ and $s\in\mathrm{Sym}(A,\sigma)$ such that s and a commute, the product as is in $\mathrm{Symd}(A,\sigma)$. From this and [17, (5.17)], an induction argument shows that

$$\mathrm{Sym}\left(\bigotimes_{i=1}^r (A_i,\sigma_i)\right)=\mathrm{Symd}\left(\bigotimes_{i=1}^r (A_i,\sigma_i)\right)+\bigotimes_{i=1}^r \mathrm{Sym}(A_i,\sigma_i).$$

The uniqueness of the semi-trace $f_\otimes$ follows as, by definition, all semi-traces coincide on the subspace of symmetrized elements. It remains to prove the existence of such a semi trace. Let

$$(B,\tau)=\bigotimes_{i=1}^{r-1}(A_i,\sigma_i).$$

Since F has characteristic 2, the involution τ is symplectic, and $\mathrm{Trd}_B(b)=0$ for all $b\in\mathrm{Sym}(B,\tau)$ by [17, (2.6)(2)]. Pick an arbitrary semi-trace f on (A_r,σ_r) and consider the tensor product

$$(B,\tau)\otimes(A_r,\sigma_r,f)=(B\otimes A_r,\tau\otimes\sigma_r,f_\star).$$

Formula (1) above shows that $f_\star$ vanishes on $\mathrm{Sym}(B,\tau)\otimes\mathrm{Sym}(A_r,\sigma_r)$ which contains $\bigotimes_{i=1}^r\mathrm{Sym}(A_i,\sigma_i)$, hence $f_\star$ satisfies the required condition. ∎

Remark 2.4: The proof actually shows that $f_\otimes$ is the semi-trace induced by an arbitrary semi-trace on one of the symplectic factors (A_i,σ_i), that is

$$(A_1\otimes\ldots\otimes A_r,\sigma_1\otimes\ldots\otimes\sigma_r,f_\otimes)=\bigotimes_{1\leqslant k\leqslant r,k\neq i}(A_k,\sigma_k)\otimes(A_i,\sigma_i,f_i),$$

for any choice of i and of a semi-trace f_i on (A_i,σ_i).

Remark 2.5: Note that the semi trace $f_\otimes$ on a totally decomposable algebra with involution $\bigotimes_{i=1}^r(A_i,\sigma_i)$ does depend on the choice of the F-algebras with involution (A_i,σ_i) in the decomposition. Indeed, consider two quaternion F-algebras Q_1 and Q_2. Since we are in characteristic 2, a tensor product of two symplectic involutions is symplectic. Therefore, the algebras with involution $(Q_1, \bar{\ })\otimes(Q_1, \bar{\ })$ and $(Q_2, \bar{\ })\otimes(Q_2, \bar{\ })$ are both split and symplectic, hence isomorphic by [11, Prop. 1.8]. However, for $i = 1, 2$ we have that

$$(Q_i \otimes Q_i, \bar{\ } \otimes \bar{\ }, f_\otimes) \simeq \mathrm{Ad}_{\mathrm{Nrd}_{Q_i}}$$

by [10, (2.9)], and $\mathrm{Ad}_{\mathrm{Nrd}_{Q_1}} \simeq \mathrm{Ad}_{\mathrm{Nrd}_{Q_2}}$ holds if and only if $Q_1 \simeq Q_2$.

To get an example in higher degree, consider two r-fold Pfister forms π_1 and π_2 over F. By [6, (6.2)], there exist two families of quaternion algebras $Q_{i,j}$ for $i = 1, 2$ and $1 \leqslant j \leqslant r$ such that

$$\mathrm{Ad}_{\pi_i} \simeq (Q_{i,1} \otimes \cdots \otimes Q_{i,r}, \bar{\ } \otimes \cdots \otimes \bar{\ }, f_\otimes).$$

In particular, again the algebras are split and the involutions symplectic hence hyperbolic, so that

$$\bigotimes_{j=1}^{r}(Q_{1,j}, \bar{\ }) \simeq \bigotimes_{j=1}^{r}(Q_{2,j}, \bar{\ }).$$

However, $\mathrm{Ad}_{\pi_1} \simeq \mathrm{Ad}_{\pi_2}$ if and only if $\pi_1 \simeq \pi_2$. Therefore the semi-trace $f_\otimes$ does depend on the choice of the quaternion algebras in the decomposition.

Notation 2.6: Let (A_i, σ_i) be r algebras with symplectic involutions. In characteristic different from 2, we assume in addition that r is even, so that $\bigotimes_{i=1}^{r} \sigma_i$ is orthogonal, and we denote by $f_\otimes$ the unique semi-trace on $\bigotimes_{i=1}^{r}(A_i, \sigma_i)$. In characteristic 2, $f_\otimes$ is as defined in Proposition 2.3. In both cases, we call $f_\otimes$ the canonical semi-trace on the tensor product of F-algebras with symplectic involution $\bigotimes_{i=1}^{r}(A_i, \sigma_i)$.

3. Canonical quadratic pair on a Clifford algebra

Throughout this section, (A, σ, f) is an algebra with quadratic pair. We assume A has degree $n = 2m$, with m even, and $m \equiv 0 \mod 4$ if F is of characteristic different from 2. Under this assumption on the degree of A, the canonical involution $\underline{\sigma}$ of the Clifford algebra $\mathcal{C} = \mathcal{C}(A, \sigma, f)$ has symplectic type in characteristic 2, and orthogonal type in characteristic different from 2; see [17, (8.12)]. If (σ, f) has trivial discriminant, so that $\mathcal{C}$ has center $F \times F$, this means as in [17] that the canonical involution $\underline{\sigma}$ of $\mathcal{C} = \mathcal{C}^+ \times \mathcal{C}^-$ is symplectic or orthogonal on each component, and by a semi-trace on $(\mathcal{C}, \underline{\sigma})$, we mean in this case a pair (f^+, f^-), where f^+ (respectively f^-) is a semi-trace on $(\mathcal{C}^+, \sigma^+)$ (respectively $(\mathcal{C}^-, \sigma^-)$). The purpose of this section is to define a semi-trace $\underline{f}$

on $(\mathcal{C}, \underline{\sigma})$, which we call the canonical semi-trace of the Clifford algebra, provided A satisfies the above conditions and has degree $2m \geqslant 8$. We also give an explicit description of $\underline{f}$ in the split case.

3.1. Definition of the canonical semi-trace. Consider the canonical F-linear map $c : A \to \mathcal{C}$ induced by $A \to \underline{A} \to T(\underline{A})$. By [17, (8.16)] we have

$$c(x) + \underline{\sigma}(c(x)) = \operatorname{Trd}_A(x).$$

Hence we have

$$\text{(2)} \qquad \text{for all } x \in A, \quad \begin{cases} c(x) \in \operatorname{Skew}(\mathcal{C}, \underline{\sigma}) & \text{if and only if } \operatorname{Trd}_A(x) = 0, \\ c(x) + \underline{\sigma}(c(x)) = 1 & \text{if and only if } \operatorname{Trd}_A(x) = 1. \end{cases}$$

The main result in this section is the following:

Proposition 3.1: *Assume A has degree $2m \geqslant 8$ with m even, and assume further that $m \equiv 0 \mod 4$ if the characteristic is different from 2. Then $\underline{\sigma}$ is symplectic in characteristic 2 and orthogonal otherwise. For any $\lambda \in A$ with $\operatorname{Trd}_A(\lambda) = 1$, the element $c(\lambda)$ defines a semi-trace on $(\mathcal{C}, \underline{\sigma})$, which does not depend on the choice of λ.*

Proof. Let $\lambda \in A$ be an element with reduced trace 1. By (2), we have that

$$c(\lambda) + \underline{\sigma}(c(\lambda)) = 1.$$

Hence the linear form which maps $s \in \operatorname{Sym}(\mathcal{C}, \underline{\sigma})$ to $\operatorname{Trd}_{\mathcal{C}}(c(\lambda)s)$ is a semi-trace on $(\mathcal{C}, \underline{\sigma})$. If (σ, f) has trivial discriminant, so that $(\mathcal{C}, \sigma) = (\mathcal{C}^+, \sigma^+) \times (\mathcal{C}^-, \sigma^-)$, we have $c(\lambda) = (c_\lambda^+, c_\lambda^-)$ with

$$c_\lambda^+ + \sigma^+(c_\lambda^+) = 1 \quad \text{and} \quad c_\lambda^- + \sigma^-(c_\lambda^-) = 1.$$

The semi-trace $\underline{f}$ consists in this case of the pair of semi-traces respectively induced by c_λ^+ and c_λ^-.

Given another element $\lambda' \in A$, also of reduced trace 1, the difference $\mu = \lambda - \lambda'$ has reduced trace 0. Hence, applying again (2), we get that

$$c(\mu) = c(\lambda) - c(\lambda') \in \operatorname{Skew}(\mathcal{C}, \underline{\sigma}).$$

On the other hand $c(\lambda')$ defines the same semi-trace as $c(\lambda)$ if and only if the difference $c(\mu) = c(\lambda) - c(\lambda')$ belongs to $\operatorname{Alt}(\mathcal{C}, \underline{\sigma})$ by [17, (5.7)]. Hence the following lemma finishes the proof:

Lemma 3.2: *Assume A has degree at least 6. In the Clifford algebra $\mathcal{C}$, we have*

$$c(A) \cap \mathrm{Skew}(\mathcal{C}, \underline{\sigma}) = c(A) \cap \mathrm{Alt}(\mathcal{C}, \underline{\sigma}).$$

The statement is trivial if the characteristic of F is different from 2, as in this case $\mathrm{Skew}(\mathcal{C}, \underline{\sigma}) = \mathrm{Alt}(\mathcal{C}, \underline{\sigma})$. Assume now that the characteristic of F is 2. The inclusion of the left hand side in the right is clear. We now prove the converse. It suffices to show the result in the case where A is split.

Assume that $(A, \sigma, f) = \mathrm{Ad}_q$, for some quadratic form $q : V \to F$, and $\mathcal{C} = \mathcal{C}_0(q)$. Let $q = [a_1, b_1] \perp \cdots \perp [a_m, b_m]$ be a decomposition of q and $e_1, \ldots, e_m, e_1', \ldots e_m'$ a basis of V as in Example 2.1. As explained in [17, Proof of (8.14)], the vector space $c(A)$ has dimension

$$\frac{n(n-1)}{2} + 1 = 2m^2 - m + 1,$$

and has a basis consisting of

$$\{1\} \cup \{e_i e_j, e_i' e_j' \mid 1 \leqslant i < j \leqslant m\} \cup \{e_i e_j' \mid 1 \leqslant i, j \leqslant m\}.$$

All elements in this basis are (skew-)symmetric except for the elements $e_i e_i'$ for all $i \in \{1, \ldots, m\}$. However, we have $e_i e_i' + \underline{\sigma}(e_i e_i') = 1$, hence $e_i e_i' + e_{i+1} e_{i+1}'$ is symmetric for all $1 \leqslant i \leqslant m-1$. By [17, (8.18)],

$$c(A)_0 = c(A) \cap \mathrm{Skew}(\mathcal{C}, \underline{\sigma})$$

has codimension 1 in $c(A)$. Therefore,

$$\{1\} \cup \{e_i e_j, e_i' e_j' \mid 1 \leqslant i < j \leqslant m\} \cup \{e_i e_j' \mid i \neq j\} \cup \{e_i e_i' + e_{i+1} e_{i+1}' \mid 1 \leqslant i \leqslant m-1\}$$

is a basis of $c(A) \cap \mathrm{Skew}(\mathcal{C}, \underline{\sigma})$. To complete the proof, we show that these basis elements lie in $\mathrm{Alt}(\mathcal{C}, \underline{\sigma})$.

Pick $i, j \in \{1, \ldots, m\}$ with $i \neq j$. As $m \geqslant 3$, there exists some

$$k \in \{1, \ldots, m\} \setminus \{i, j\}.$$

As $e_k e_k' + e_k' e_k = 1$, and both e_k and e_k' commute with e_i and e_j, and e_i and e_j commute, we have that

$$\begin{aligned} e_i e_j &= e_i e_j \cdot 1 = e_i e_j (e_k e_k' + e_k' e_k) \\ &= e_i e_j e_k e_k' + e_i e_j e_k' e_k = e_i e_j e_k e_k' + e_k' e_k e_j e_i \\ &= e_i e_j e_k e_k' + \underline{\sigma}(e_i e_j e_k e_k') \in \mathrm{Alt}(\mathcal{C}, \underline{\sigma}). \end{aligned}$$

A similar argument shows that $e'_i e'_j$ and $e_i e'_j \in \mathrm{Alt}(\mathcal{C}, \underline{\sigma})$. Consider now

$$\begin{aligned} e_i e'_i + e_j e'_j &= e_i e'_i(e_k e'_k + e'_k e_k) + e_j e'_j(e_k e'_k + e'_k e_k) \\ &= e_i e'_i e_k e'_k + e'_k e_k e_i e'_i + e_j e'_j e_k e'_k + e'_k e_k e_j e'_j. \end{aligned}$$

Using $e_i e'_i + e'_i e_i = 1 = e_j e'_j + e'_j e_j$, we get that

$$\begin{aligned} e_i e'_i + e_j e'_j &= e_i e'_i e_k e'_k + e'_k e_k(1 + e'_i e_i) + e_j e'_j e_k e'_k + e'_k e_k(1 + e'_j e_j) \\ &= e_i e'_i e_k e'_k + e'_k e_k e'_i e_i + e_j e'_j e_k e'_k + e'_k e_k e'_j e_j \\ &= (e_i e'_i e_k e'_k + e_j e'_j e_k e'_k) + \underline{\sigma}(e_i e'_i e_k e'_k + e_j e'_j e_k e'_k). \end{aligned}$$

In particular, $e_i e'_i + e_{i+1} e'_{i+1} \in \mathrm{Alt}(\mathcal{C}, \underline{\sigma})$ for all $1 \leqslant i \leqslant m-1$, and this finishes the proof. ∎

Since the reduced trace is a nonzero linear form, a central simple algebra A always contains an element λ such that $\mathrm{Trd}_A(\lambda) = 1$. Hence, using the previous proposition, we get

Definition 3.3: Let (A, σ, f) be an algebra with quadratic pair, of degree $2m \geqslant 8$ with m even, and assume further $m \equiv 0 \mod 4$ if the characteristic of F is different from 2. Given $\lambda \in A$ with $\mathrm{Trd}_A(\lambda) = 1$, the semi-trace

$$\begin{array}{rcl} \underline{f} : \mathrm{Sym}(\mathcal{C}, \underline{\sigma}) & \to & F \\ s & \mapsto & \mathrm{Trd}_{\mathcal{C}}(c(\lambda) s) \end{array}$$

does not depend on λ. It is called the canonical semi-trace on $(\mathcal{C}, \underline{\sigma})$. We refer to the pair $(\underline{\sigma}, \underline{f})$ as the canonical quadratic pair on $\mathcal{C} = \mathcal{C}(A, \sigma, f)$.

Remark 3.4: Assume the characteristic of F does not divide the degree of A, and let (A, σ) be an algebra with orthogonal involution. Then $\frac{1}{\deg(A)} \in A$ has reduced trace 1, and its image in $\mathcal{C}(A, \sigma)$ is

$$c\Big(\frac{1}{\deg(A)}\Big) = f\Big(\frac{1}{\deg(A)}\Big) = \frac{1}{2}$$

by [17, (5.6)&(8.7)]. So $\underline{f}$ is half the reduced trace of $\mathcal{C}$, as prescribed in this case.

The next proposition provides some evidence that the quadratic pair we have just defined is part of the structure of the Clifford algebra. Let

$$\theta : (A, \sigma, f) \to (B, \tau, g)$$

be an isomorphism of algebras with quadratic pairs. It follows from Definition [17, (8.7)] that θ induces an isomorphism $\mathcal{C}(\theta) : \mathcal{C}(A, \sigma, f) \xrightarrow{\sim} \mathcal{C}(B, \tau, g)$, satisfying

$$\mathcal{C}(\theta)(c_A(a)) = c_B(\theta(a))$$

for all $a \in A$, where c_A (respectively c_B) denotes the canonical map $c_A : A \to \mathcal{C}(A, \sigma, f)$ (respectively $c_B : B \to \mathcal{C}(B, \tau, g)$). Moreover, one may easily check that $\mathcal{C}(\theta)$ preserves the canonical involutions. We claim it is an isomorphism of algebras with quadratic pairs, that is

Proposition 3.5: *Every isomorphism of algebras with quadratic pair*

$$\theta : (A, \sigma, f) \to (B, \tau, g)$$

induces an isomorphism of algebras with quadratic pair

$$\mathcal{C}(\theta) : (\mathcal{C}(A, \sigma, f), \underline{\sigma}, \underline{f}) \xrightarrow{\sim} (\mathcal{C}(B, \tau, g), \underline{\tau}, \underline{g}).$$

Proof. The proposition follows from Proposition 3.1 by functoriality of the Clifford algebra construction. More precisely, for any $\lambda \in A$ with $\mathrm{Trd}_A(\lambda) = 1$, we have

$$\mathrm{Trd}_B(\theta(\lambda)) = \mathrm{Trd}_A(\lambda) = 1.$$

Hence $\underline{f}$ is induced by $c_A(\lambda)$ and $\underline{g}$ is induced by

$$c_B(\theta(\lambda)) = \mathcal{C}(\theta)(c_A(\lambda)).$$

It follows that $\mathcal{C}(\theta)$ preserves the semi-traces as required. ∎

3.2. Explicit description in the split case. Let (V, q) be a nonsingular quadratic space of dimension $2m$, with polar form b_q. We assume that m is even, and further that $m \equiv 0 \mod 4$ if the characteristic of F is different from 2, so that the canonical involution $\underline{\sigma_q}$ of $\mathcal{C}_0(V, q)$ is of orthogonal type in characteristic different from 2, and of symplectic type otherwise. Since q is nonsingular, we may find a pair of vectors (e, e') such that $b_q(e, e') = 1$. Let $u = ee'$ be the corresponding element in $\mathcal{C}_0(V, q)$. We have

$$u + \underline{\sigma_q}(u) = ee' + e'e = b_q(e, e') = 1.$$

Therefore, this element u defines a semi-trace on $(\mathcal{C}_0(V, q), \underline{\sigma_q})$, which we denote by $f_{ee'}$. If V has dimension $2m \geqslant 8$ we claim $f_{ee'}$ coincides with the canonical semi-trace of $(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q})$ under the canonical identification provided in [17, (8.8)]. More precisely, we have

PROPOSITION 3.6: *Let (V, q) be a nonsingular quadratic space of dimension $2m \geqslant 8$, with m even, and assume further that $m \equiv 0 \mod 4$ if the characteristic of F is different from 2. Given a pair of vectors (e, e') such that $b_q(e, e') = 1$, the standard identification $\varphi_q : V \otimes V \to \operatorname{End}_F(V)$ induces an isomorphism of algebras with quadratic pairs*

$$(\mathcal{C}_0(V, q), \underline{\sigma_q}, f_{ee'}) \simeq (\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q}).$$

Proof. In view of [17, (8.8)], it only remains to identify the semi-traces. Denote by P the plane generated by e and e' in V; since q restricts to a nonsingular form on P, by [11, (7.22)], we have $V = P \perp P^{\perp}$. Recall from [17, (5.10)] that $\varphi_q(e \otimes e')$ maps $x \in V$ to $e\, b_q(e', x)$. Hence it vanishes on e' and $P^{\perp}$, and maps e to itself. Therefore, $\varphi_q(e \otimes e') \in \operatorname{End}_F(V)$ has trace 1. By Definition 3.3, the canonical semi-trace of $(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q})$ is determined by the element $c(\varphi_q(e \otimes e'))$; the corresponding element in $\mathcal{C}_0(q)$ is $u = ee'$, and this proves the proposition. ■

Remark 3.7: Under the assumptions of Proposition 3.6, the semi-trace $f_{ee'}$ on $(\mathcal{C}_0(V, q), \underline{\sigma_q})$ does not depend on the choice of the pair (e, e'). This can be directly checked as follows if F has characteristic 2. Consider two pairs of vectors (e, e') and (g, g') with $b_q(e, e') = 1 = b_q(g, g')$, and let $u = ee'$ and $v = gg'$ be the corresponding elements in $\mathcal{C}_0(q)$. Let P and Q be the planes in V respectively generated by (e, e') and (g, g'). The polar form b_q is nondegenerate on both planes, and also on the orthogonal $P^{\perp}$ of the plane P, which has dimension $2m - 2$, see [11, Prop 1.6]. Besides, $P^{\perp} \cap Q^{\perp}$ is a subspace of $P^{\perp}$ of dimension at least $2m - 4$. Since $m \geqslant 4$, $P^{\perp} \cap Q^{\perp}$ has dimension strictly larger than half the dimension of $P^{\perp}$. Therefore, b_q cannot be identically 0 on $P^{\perp} \cap Q^{\perp}$. Hence, there exists a third plane R over which b_q is nondegenerate, and which is orthogonal to P and Q. Let (h, h') be two vectors of R with $b_q(h, h') = 1$, and let $w = hh' \in \mathcal{C}_0(V, q)$. We have

$$\underline{\sigma_q}(u) = u + 1, \quad \underline{\sigma_q}(v) = v + 1 \quad \text{and} \quad \underline{\sigma_q}(w) = w + 1.$$

Moreover, w commutes with u and v. It follows that

$$u = v + (u + v)w + \underline{\sigma_q}((u + v)w).$$

Hence u and v differ by an alternating element, so they define the same semi-trace by [17, (5.7)].

Example 3.8: Assume F has characteristic 2 and (V, q) is a nonsingular quadratic space of dimension $2m \geqslant 8$. Pick an explicit presentation of the quadratic form

$$q = [a_1, b_1] \perp \cdots \perp [a_m, b_m].$$

We use the same notations as in Example 2.1, where the algebra with involution $(\mathcal{C}_0(q), \underline{\sigma_q})$ is described. We claim

$$(3) \qquad \begin{aligned}(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q}) &\simeq (Q_1 \otimes \cdots \otimes Q_{m-1}, \bar{\ } \otimes \cdots \otimes \bar{\ }, f_\otimes) \otimes_F K,\\ &\simeq (Q_{1K} \otimes \cdots \otimes Q_{m-1K}, \bar{\ } \otimes \cdots \otimes \bar{\ }, f_\otimes),\end{aligned}$$

where $Q_i = [a_i b_i, a_i a_m)$, $\bar{\ }$ stands for the canonical involution, K is the quadratic étale extension of F generated by $\mathrm{disc}(q) \in F/\wp(F)$, and $f_\otimes$ is the canonical semi-trace associated to a tensor product as in Proposition 2.3. Indeed, Proposition 3.6 shows that the canonical semi-trace $\underline{f_q}$ corresponds to $f_{e_1 e_1'}$ on $\mathcal{C}_0(V, q)$. Besides, as explained in Example 2.1, $u_1 = e_1 e_1' \in \mathcal{C}_0(V, q)$ corresponds to

$$u_1 \otimes 1 \in Q_{1K} \otimes \Big(\bigotimes_{i=2}^{m-1} Q_{iK}\Big).$$

Therefore, $f_{e_1 e_1'}$ on $(\mathcal{C}_0(V, q), \underline{\sigma_q})$ is the semi-trace induced by f_{u_1} on $(Q_{1K}, \bar{\ })$ as in Section 2.4, which coincides with $f_\otimes$ by Remark 2.4.

Remark 3.9: It follows from Example 3.8 that $(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q})$ is a totally decomposable algebra with quadratic pair, and even more, that it has a totally decomposable descent to F. This extends a well-known result in characteristic different from 2. Indeed, consider a nondegenerate even dimensional quadratic space (V, q) over a field F of characteristic different from 2, with orthogonal basis $(e_1, e_2, \ldots, e_{2m})$. A direct computation shows that the elements

$$\begin{cases} i_1 = e_1 e_2 \\ j_1 = e_1 e_3 \end{cases}, \begin{cases} i_2 = e_1 e_2 e_3 e_4 \\ j_2 = e_1 e_2 e_3 e_5 \end{cases}, \ldots, \begin{cases} i_{m-1} = e_1 \ldots e_{2m-3} e_{2m-2} \\ j_{m-1} = e_1 \ldots e_{2m-3} e_{2m-1} \end{cases}$$

generate pairwise commuting F-quaternion algebras in $\mathcal{C}_0(q)$ that are stable under the canonical involution. Hence, $\mathcal{C}_0(q)$ is isomorphic to the tensor product of those quaternion algebras, extended from F to the center $K = F[e_1 \ldots e_{2m}]$.

Example 3.10: Let π be a 3-fold Pfister form over F. We claim that

$$(4) \qquad (\mathcal{C}(\mathrm{Ad}_\pi), \underline{\sigma_\pi}, \underline{f_\pi}) \simeq \mathrm{Ad}_\pi \times \mathrm{Ad}_\pi.$$

A conceptual argument is given below, see Proposition 4.1, which extends [17, (35.1)] to arbitrary characteristic. Over a base field of characteristic 2, this can also be directly checked as follows. Let

$$\pi = \langle\langle a, b, c]].$$

Using the isometry $x[1, y] \simeq [x, x^{-1}y]$ for $x \in F^\times$ and $y \in F$, we obtain that

$$\pi \simeq [a, a^{-1}c] \perp [b, b^{-1}c] \perp [ab, (ab)^{-1}c] \perp [1, c].$$

Hence, by Example 3.8, we have

$$(\mathcal{C}(\mathrm{Ad}_\pi), \underline{\sigma_\pi}, \underline{f_\pi}) = ([c, a) \otimes [c, b) \otimes [c, ab), \bar{\ } \otimes \bar{\ } \otimes \bar{\ }, f_\otimes) \otimes_F (F \times F).$$

On the other hand,

$$\mathrm{Ad}_\pi \simeq \mathrm{Ad}_{\langle\langle a\rangle\rangle} \otimes \mathrm{Ad}_{\langle\langle b, c]]}.$$

Since $\langle\langle b, c]]$ is the norm form of the quaternion algebra $[c, b)$, using [5, (5.5)] and [10, (2.9)] we get

$$\mathrm{Ad}_\pi \simeq ([0, a), \tau) \otimes ([c, b), \bar{\ }) \otimes ([c, b), \bar{\ }, f),$$

where for the quaternion basis $(1, u, v, w)$ of $[0, a)$ the orthogonal involution τ is characterised by $\tau(u) = u$ and $\tau(v) = v$ and f is any semi-trace on $([c, b), \bar{\ })$. Finally, for a particular choice of the the semi-trace f, the isomorphism from [6, (5.5)] gives us

$$([0, a), \tau) \otimes ([c, b), \bar{\ }, f) \simeq ([c, a) \otimes [c, ab), \bar{\ } \otimes \bar{\ }, f_\otimes).$$

This shows (4) by Remark 2.4.

We also prove the following extension of [17, (8.5)].

Proposition 3.11: *Let q be a nonsingular quadratic form over F of even dimension $2m \geqslant 8$ with m even, and $m \equiv 0 \mod 4$ if F has characteristic not 2. If q is isotropic then $(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q})$ is hyperbolic.*

Proof. For the case where F is of characteristic different from 2, see [17, (8.5)]. We now assume that F is of characteristic 2. As q is isotropic we have

$$q \simeq \mathbb{H} \perp q' \simeq [1, 0] \perp q'$$

for some nonsingular quadratic form q' over F. Hence we may assume $a_1 = 1$ and $b_1 = 0$ in Example 3.8, and we get

$$(\mathcal{C}(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q}) \simeq ([0, c), \bar{\ }, f) \otimes (B, \tau) \otimes_F K$$

for some $c \in F^\times$, some arbitrary choice of a semi-trace f on $([0,c), \bar{\ })$, and some F-algebra with symplectic involution (B,τ) (see Remark 2.5). Since $[0,c)$ is a split algebra, we may choose f so that $([0,c), \bar{\ }, f)$ is the adjoint of a hyperbolic plane, and it follows that the Clifford algebra $(\mathcal{C}_0(\mathrm{Ad}_q), \underline{\sigma_q}, \underline{f_q})$ is hyperbolic. ∎

4. Triality

The purpose of this section is to describe the action of the alternating group A_3 on PGO_8^+ and the induced action on the corresponding Galois cohomology set $H^1(F, \mathrm{PGO}_8^+)$, which can be described in terms of some triples of degree 8 algebras with quadratic pairs, called trialitarian triples. Similar computations were recently made by Alsaody and Gille [2, §4], where they work over a more general base (a unital commutative ring), and consider triples of isometries, while we consider triples of similitudes. Our approach follows [17, §35], with the additional ingredient that the Clifford algebra is endowed with a canonical quadratic pair rather than just an involution.

Recall that a similitude of the algebra with quadratic pair (A,σ,f) is an invertible element $g \in A^\times$ such that

$$\sigma(g)g = \mu(g) \in F^\times$$

and $f(gsg^{-1}) = f(s)$ for all symmetric elements $s \in \mathrm{Sym}(A,\sigma)$. If $A = \mathrm{End}_F(V)$ and (σ, f) is the adjoint of a nonsingular quadratic form q, similitudes of (A,σ,f) coincide with similitudes of the quadratic space (V,q) in the usual sense. Assume in addition that A has even degree $n=2m$. The similitude g with multiplier $\mu(g)$ is called proper if it satisfies the following condition [17, (12.24)(12.32)]:

$$\begin{cases} \text{If } F \text{ has characteristic} \neq 2, & \mathrm{Nrd}_A(g) = \mu(g)^m, \\ \text{If } F \text{ has characteristic } 2, & f(g^{-1}\ell g - \ell) = 0, \end{cases} \tag{5}$$

where $\ell \in A$ is an element defining the semi-trace f. We let

$$\mathrm{PGO}^+(q) = \mathrm{GO}^+(q)/F^\times \quad \text{and} \quad \mathrm{PGO}^+(A,\sigma,f) = \mathrm{GO}^+(A,\sigma,f)/F^\times,$$

where $\mathrm{GO}^+(q)$ (respectively $\mathrm{GO}^+(A,\sigma,f)$) is the group of proper similitudes of q (respectively of (A,σ,f)).

4.1. An action of A_3 on $\mathrm{PGO}^+(n)$. Let $\mathcal{O}$ be a Cayley algebra, and denote by $\star$ its para-Cayley product, defined by $x \star y = \bar{x}\bar{y}$, see [17, §34.A]. The algebra $(\mathcal{O}, \star, n)$ is a symmetric composition algebra, where n is the norm form of $\mathcal{O}$. In particular, the norm form is multiplicative, that is $n(x \star y) = n(x)n(y)$ for all $x, y \in \mathcal{O}$. Moreover, we have

$$x \star (y \star x) = n(x)y = (x \star y) \star x \quad \text{for all } x, y \in \mathcal{O}; \tag{6}$$

see [17, (34.1)]. The key result to define the trialitarian action is the following :

Proposition 4.1: *Let t be a proper similitude of $(\mathcal{O}, n)$ with multiplier $\mu(t)$. There exist proper similitudes (t^+, t^-) of $(\mathcal{O}, n)$ such that for all $x, y \in \mathcal{O}$,*

(a) $t^+(x \star y) = \mu(t^+)\, t(x) \star t^-(y)$;
(b) $t(x \star y) = \mu(t)\, t^-(x) \star t^+(y)$;
(c) $t^-(x \star y) = \mu(t^-)\, t^+(x) \star t(y)$.

The pair (t^+, t^-) is uniquely defined up to a factor (λ^{-1}, λ) for some $\lambda \in F^\times$ and the multipliers satisfy $\mu(t^+)\mu(t)\mu(t^-) = 1$.

From this, we derive an action of the alternating group $A_3 \simeq \mathbb{Z}/3$ on the group of projective similitudes $\mathrm{PGO}^+(n) = \mathrm{GO}^+(n)/F^\times$ as follows. Given a proper similitude t of $(\mathcal{O}, n)$ we denote by $[t]$ its class in $\mathrm{PGO}^+(n)$. It is clear from the relations above that $[t^{++}] = [t^-]$ and $[t^{+++}] = [t]$. Moreover, since t^+ and t^- are unique up to a factor (λ^{-1}, λ) for some $\lambda \in F^\times$, their classes $[t^+]$ and $[t^-]$ are uniquely defined. Hence, we get

Corollary 4.2: *The map*

$$\theta^+ : \mathrm{PGO}^+(n) \to \mathrm{PGO}^+(n)$$
$$[t] \mapsto [t^+]$$

defines an action of A_3 on $\mathrm{PGO}^+(n)$.

Remark 4.3: In [2], Alsaody and Gille give an explicit description of the spin group $\mathrm{Spin}(n)$ with its trialitarian action (see [2, Lem. 3.3 and Thm. 3.9]). Their description is in terms of so-called related triples, which correspond to triples as in our Proposition 4.1, except that t, t^+ and t^- are isometries rather than similitudes. It is clear from their work that the action described in this section is the induced trialitarian action on $\mathrm{PGO}^+(n)$. The explicit description we provide could also be deduced from their results by fppf descent.

The remainder of this section outlines the proof of Proposition 4.1. The argument is mostly borrowed from [17, §34], except for Lemma 4.5 which adds the canonical quadratic pair to the picture.

For all $x \in \mathcal{O}$, we denote by r_x and ℓ_x the endomorphisms of $\mathcal{O}$ defined by $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$ for all $y \in \mathcal{O}$. We first prove

Lemma 4.4: *Let t be a proper similitude of $(\mathcal{O}, n)$. The F-linear map*

$$\psi_t : \mathcal{O} \to M_2(\mathrm{End}_F(\mathcal{O})) \simeq \mathrm{End}_F(\mathcal{O} \oplus \mathcal{O})$$

defined by

$$\psi_t(x) = \begin{pmatrix} 0 & \ell_{t(x)} \\ \mu(t)^{-1} r_{t(x)} & 0 \end{pmatrix}$$

induces isomorphisms of F-algebras

$$\mathcal{C}(n) \xrightarrow{\sim} \mathrm{End}_F(\mathcal{O} \oplus \mathcal{O}) \quad \textit{and}$$
$$\mathcal{C}_0(n) \xrightarrow{\sim} \mathrm{End}_F(\mathcal{O}) \times \mathrm{End}_F(\mathcal{O}),$$

which we denote by Ψ_t.

Proof. A direct computation shows that for all $x, y \in \mathcal{O}$, we have

$$\psi_t(x)\psi_t(y) = \mu(t)^{-1} \begin{pmatrix} \ell_{t(x)} \circ r_{t(y)} & 0 \\ 0 & r_{t(x)} \circ \ell_{t(y)} \end{pmatrix}. \tag{7}$$

In view of (6), it follows that $\psi_t(x)^2 = \mu(t)^{-1} n(t(x)) = n(x)$ for all $x \in \mathcal{O}$. By the universal property of the Clifford algebra, we get a non-trivial algebra morphism $\Psi_t : \mathcal{C}(n) \to \mathrm{End}_F(\mathcal{O} \oplus \mathcal{O})$, which is an isomorphism since both algebras are simple and of the same dimension. By (7), this isomorphism sends $\mathcal{C}_0(n)$ to the direct product $\mathrm{End}_F(\mathcal{O}) \times \mathrm{End}_F(\mathcal{O})$, which embeds diagonally in $\mathrm{End}_F(\mathcal{O} \oplus \mathcal{O})$. ∎

Assume now that $t = \mathrm{Id}$ and consider the corresponding isomorphisms, denoted by Ψ_1. Since the identity map is an isometry, hence has multiplier 1, it satisfies

$$\Psi_1(xy) = \psi_1(x)\psi_1(y) = \begin{pmatrix} \ell_x \circ r_y & 0 \\ 0 & r_x \circ \ell_y \end{pmatrix}. \tag{8}$$

The next lemma is a refined version of [17, (35.1)] (see also [2, Prop. 3.10]):

Lemma 4.5: *The restriction of the isomorphism* Ψ_1 *to the even Clifford algebra induces an isomorphim of algebras with quadratic pairs*

$$\Psi_1 : (\mathcal{C}_0(n), \underline{\sigma_n}, \underline{f_n}) \to \mathrm{Ad}_n \times \mathrm{Ad}_n,$$

where $(\underline{\sigma_n}, \underline{f_n})$ *stands for the canonical quadratic pair on* $\mathcal{C}_0(n)$*.*

Remark 4.6: Since all 3-fold Pfister forms are norm forms of some Cayley algebras, this lemma gives a new proof of Example 3.10.

Proof. By Lemma 4.4, Ψ_1 is an isomorphism of algebras, and one may check it preserves the involution as in [2, Prop. 3.10]. It only remains to prove that it is compatible with the semi-traces. Therefore, we may assume that the characteristic of F is 2. Pick a decomposition of the quadratic form n and a basis $(e_i, e_i')_{1\leqslant i\leqslant 4}$ of $\mathcal{O}$ as in Example 2.1. By Proposition 3.6, the element $e_1e_1' \in \mathcal{C}_0(n)$ determines the canonical semi-trace $\underline{f_n}$ on $\mathcal{C}_0(n)$. Under the isomorphism Ψ_1, it corresponds to the semi-trace determined by the element

$$\Psi_1(e_1e_1') = \psi_1(e_1)\psi_1(e_1') = \begin{pmatrix} \ell_{e_1}\circ r_{e_1'} & 0 \\ 0 & r_{e_1}\circ \ell_{e_1'} \end{pmatrix}$$

by (7). Hence, we have to prove that the elements $\ell_{e_1}\circ r_{e_1'}$ and $r_{e_1}\circ\ell_{e_1'}\in\mathrm{End}_F(\mathcal{O})$ determine the semi-trace f_n associated to the norm form n. By [17, (5.11)], this means we have to check that for all $v \in \mathcal{O}$,

$$\mathrm{Trd}_{\mathrm{End}_F(\mathcal{O})}((\ell_{e_1}\circ r_{e_1'}\circ\varphi_n)(v\otimes v)) = n(v),$$

and similarly for the endomorphisms $(r_{e_1}\circ\ell_{e_1'}\circ\varphi_n)(v\otimes v)$, where φ_n is the standard identification $\mathcal{O}\otimes\mathcal{O}\simeq\mathrm{End}_F(\mathcal{O})$ defined in [17, (5.2)]. Since the restriction of a semi-trace to the space of symmetrized elements is determined, it is enough to prove this equality when v is one of the basis elements $(e_i, e_i')_{1\leqslant i\leqslant 4}$; see the proof of [17, (5.11)]. For all $x \in \mathcal{O}$, we have

$$(\ell_{e_1}\circ r_{e_1'}\circ\varphi_n)(e_i\otimes e_i)(x) = (e_1\star(e_i\star e_1'))\, b_n(e_i, x).$$

Since all basis elements different from e_i' are orthogonal to e_i, this endomorphism maps all elements of the basis to 0, except for e_i'. Hence its trace is the coordinate of $e_1\star(e_i\star e_1')$ on e_i', that is $b_n(e_1\star(e_i\star e_1'), e_i)$. By [17, §34], we get

$$\begin{aligned} b_n(e_1\star(e_i\star e_1'), e_i) &= b_n(e_i, e_1\star(e_i\star e_1')) = b_n(e_i\star e_1, e_i\star e_1') \\ &= n(e_i)b_n(e_1, e_1') = n(e_i), \end{aligned}$$

as required. A similar computation shows the equality also holds for $e'_i \otimes e'_i$, and for the endomorphism $(r_{e_1} \circ \ell_{e'_1} \circ \varphi_n)(v \otimes v)$ instead of $(\ell_{e_1} \circ r_{e'_1} \circ \varphi_n)(v \otimes v)$, so the lemma is proved. ∎

With this in hand, we may now prove Proposition 4.1 as follows. Given a proper similitude t of $(\mathcal{O}, n)$, consider the isomorphisms Ψ_1 and Ψ_t. Identifying the algebras $\mathrm{End}_F(\mathcal{O} \oplus \mathcal{O})$ and $M_2(\mathrm{End}_F(\mathcal{O}))$, by the Skolem-Noether theorem, there exists an invertible element $S \in M_2(\mathrm{End}_F(\mathcal{O}))$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(n) & & \\ {\scriptstyle\Psi_1}\downarrow & \searrow^{\Psi_t} & \\ M_2(\mathrm{End}_F(\mathcal{O})) & \xrightarrow[\mathrm{Int}(S)]{} & M_2(\mathrm{End}_F(\mathcal{O})). \end{array}$$

Restriction to the even part of all three algebras shows that $\mathrm{Int}(S)$ preserves $\mathrm{End}_F(\mathcal{O}) \times \mathrm{End}_F(\mathcal{O}) \subset M_2(\mathrm{End}_F(\mathcal{O}))$, so that

$$S = \begin{pmatrix} s_0 & 0 \\ 0 & s_2 \end{pmatrix} \quad \text{for some } s_0, s_2 \in \mathrm{End}_F(\mathcal{O}).$$

Recall that t also induces an isomorphism $\mathcal{C}_0(t) : \mathcal{C}_0(n) \xrightarrow{\sim} \mathcal{C}_0(n)$ which preserves the canonical quadratic pair by Proposition 3.5. We claim that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C}_0(n) & \xrightarrow{\mathcal{C}_0(t)} & \mathcal{C}_0(n) \\ {\scriptstyle\Psi_1}\downarrow & \searrow^{\Psi_t} & \downarrow{\scriptstyle\Psi_1} \\ \mathrm{End}_F(\mathcal{O}) \times \mathrm{End}_F(\mathcal{O}) & \xrightarrow[\mathrm{Int}(S)]{} & \mathrm{End}_F(\mathcal{O}) \times \mathrm{End}_F(\mathcal{O}). \end{array}$$

Indeed, the lower triangle is obtained from the previous commutative diagram by restriction to the even part. Since $\mathcal{C}_0(t)(xy) = \mu(t)^{-1} t(x) t(y)$ for all $x, y \in \mathcal{O}$ (see [17, (13.1)]), we have by (7) and (8)

$$\Psi_1(\mathcal{C}_0(t)(xy)) = \mu(t)^{-1} \begin{pmatrix} \ell_{t(x)} \circ r_{t(y)} & 0 \\ 0 & r_{t(x)} \circ \ell_{t(y)} \end{pmatrix} = \Psi_t(xy).$$

Therefore, the upper triangle also commutes. In view of Lemma 4.5 and Proposition 3.5, the automorphism $\mathrm{Int}(S)$ preserves the quadratic pair $\mathrm{ad}_n \times \mathrm{ad}_n$, so that s_0 and s_1 are similitudes of $(\mathcal{O}, n)$.

Finally, since $\Psi_t = \mathrm{Int}(S) \circ \Psi_1$, we have for all $x \in \mathcal{O}$, $\psi_t(x) = S\psi_1(x)S^{-1}$. Hence, we get $\mu(t)^{-1} r_{t(x)} = s_2 r_x s_0^{-1}$ and $\ell_t(x) = s_0 \ell_x s_2^{-1}$, so that for all $y \in \mathcal{O}$,

$$\mu(t)^{-1} s_0(y) \star t(x) = s_2(y \star x) \quad \text{and} \quad t(x) \star s_2(y) = s_0(x \star y).$$

Applying the norm n to the second equality, we get $\mu(t)\mu(s_2) = \mu(s_0)$. Hence, the similitudes $t^+ = \mu(s_0)^{-1} s_0$ and $t^- = s_2$ have $\mu(t)\mu(t^+)\mu(t^-) = 1$ and satisfy equations (a) and (c) in Proposition 4.1. Equation (b) follows from (a) and (c), as explained in [17, p. 484]. Since

$$\Psi_1 \circ \mathcal{C}_0(t) \circ \Psi_1^{-1} = \mathrm{Int}\begin{pmatrix} t^+ & 0 \\ 0 & t^- \end{pmatrix}, \tag{9}$$

the pair (t^+, t^-) is unique up to a pair of scalars. The condition on the multipliers guarantees it actually is unique up to (λ^{-1}, λ), for some $\lambda \in F^\times$. It only remains to prove that t^+ and t^- are proper; as explained in [17, §35.B], if one of them was improper, it would satisfy a relation similar to [17, (35.4)(5)(6)] instead of relations (b) and (c) above. This concludes the proof.

Remark 4.7: It follows from the proof that, given a proper similitude $t \in \mathrm{PGO}_8^+$, we have $\theta^+([t]) = [t^+]$ and $\theta^-([t]) = [t^-]$, where $[t^+]$ and $[t^-]$ are characterized by equation (9) above, and Ψ_1 is as in Lemma 4.5.

4.2. Trialitarian triples. A trialitarian triple over F is an ordered triple of degree 8 central simple algebras with quadratic pairs over F,

$$((A, \sigma_A, f_A); (B, \sigma_B, f_B); (C, \sigma_C, f_C)),$$

such that there exists an isomorphism

$$\alpha_A : (\mathcal{C}(A, \sigma_A, f_A), \underline{\sigma_A}, \underline{f_A}) \to (B, \sigma_B, f_B) \times (C, \sigma_C, f_C).$$

Two such triples, denoted by (A, B, C) and (A', B', C') for short, are called isomorphic if there exists isomorphisms of algebras with quadratic pairs

$$\begin{aligned} \phi_A &: (A, \sigma_A, f_A) \to (A', \sigma_{A'}, f_{A'}), \\ \phi_B &: (B, \sigma_B, f_B) \to (B', \sigma_{B'}, f_{B'}), \\ \text{and} \quad \phi_C &: (C, \sigma_C, f_C) \to (C', \sigma_{C'}, f_{C'}), \end{aligned}$$

and α_A and $\alpha_{A'}$ as above such that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{C}(A,\sigma_A,f_A),\underline{\sigma_A},\underline{f_A}) & \xrightarrow{\alpha_A} & (B,\sigma_B,f_B)\times(C,\sigma_C,f_C) \\ {\scriptstyle C(\phi_A)}\downarrow & & \downarrow{\scriptstyle \phi_B\times\phi_C} \\ (\mathcal{C}(A',\sigma_{A'},f_{A'}),\underline{\sigma_{A'}},\underline{f_{A'}}) & \xrightarrow{\alpha_{A'}} & (B',\sigma_{B'},f_{B'})\times(C',\sigma_{C'},f_{C'}). \end{array}$$

Remark 4.8: If (A,B,C) is a trialitarian triple, it follows from the definition that $\mathcal{C}(A,\sigma_A,f_A)$ has center $F\times F$, hence the quadratic pair (σ_A,f_A) has trivial discriminant (see [17, (7.7) & (8.28)]).

Example 4.9: Assume F is of characteristic 2. Let q be an 8-dimensional quadratic form with trivial Arf invariant, and let $(A,\sigma_A,f_A)=\mathrm{Ad}_q$. Pick a presentation

$$q=[a_1,b_1]\perp[a_2,b_2]\perp[a_3,b_3]\perp[a_4,b_4].$$

By Example 3.8, (A,B,B) is a trialitarian triple, where B stands for

$$(B,\sigma_B,f_B)=([a_1b_1,a_1a_4)\otimes[a_2b_2,a_2a_4)\otimes[a_3b_3,a_3a_4),{}^-\otimes{}^-\otimes{}^-,f_\otimes).$$

Hence if the algebra A is split in a trialitarian triple (A,B,C), then B and C are isomorphic. The converse also holds, as we now explain:

LEMMA 4.10: *Let (A,B,C) be a trialitarian triple. The following assertions are equivalent:*

(1) *The algebra A is split.*
(2) *The triples (A,B,C) and (A,C,B) are isomorphic.*
(3) *The algebras with quadratic pairs (B,σ_B,f_B) and (C,σ_C,f_C) are isomorphic.*
(4) *The algebras B and C are Brauer equivalent.*

Proof. Assume A is split, and consider an isometry of determinant -1 of the underlying quadratic space. It induces an automorphism ϕ_A of (A,σ_A,f_A) such that $\mathcal{C}(\phi_A)$ acts non trivially on $F\times F$. Therefore, if $\varepsilon: B\times C\to C\times B$ denotes the switch map, defined by $\varepsilon(x,y)=(y,x)$, then $\varepsilon\circ\alpha_A\circ\mathcal{C}(\phi_A)\circ\alpha_A^{-1}$ is an isomorphism $B\times C\to C\times B$ which acts trivially on $F\times F$. Hence, it is equal to (ϕ_B,ϕ_C) for some isomorphisms of algebras with quadratic pairs $\phi_B: B\to C$ and $\phi_C: C\to B$. This shows that $(\phi_A,\phi_B,\phi_C,\alpha_A,\varepsilon\circ\alpha_A)$ defines an isomorphism of triples between (A,B,C) and (A,C,B). Assertion (3) follows from (2) by definition, and it clearly implies (4). Finally, since A has degree 8, and (σ_A, f_A) has trivial discriminant, by the so-called fundamental relations given in [17, (9.13) and (9.14)], we have $[A] = [B \otimes C]$ and B and C have exponent 2, therefore (4) implies (1). ∎

With these definitions in hand, the usual techniques of Galois descent yield a canonical bijection

$$\boxed{\begin{array}{c} F\text{-isomorphism classes} \\ \text{of trialitarian triples} \end{array}} \longleftrightarrow H^1(F, PGO_8^+). \tag{10}$$

This follows from [17, §29] as follows. Let n_0 be the 8-dimensional hyperbolic quadratic form, so that $\mathrm{PGO}_8^+ = \mathrm{PGO}^+(\mathrm{Ad}_{n_0})$. According to [17, §29.F], there is a canonical bijection between $H^1(F, \mathrm{PGO}_8^+)$ and isomorphism classes of quadruples $(A, \sigma_A, f_A, \varepsilon_A)$, where (A, σ_A, f_A) is a degree 8 algebra with quadratic pair, and $\varepsilon_A : Z_A \to F \times F$ is a fixed isomorphism from the center of the Clifford algebra of (A, σ_A, f_A) and $F \times F$, which is the center of $\mathcal{C}_0(n_0)$. To such a quadruple, we may associate a trialitarian triple (A, B, C), where B and C are defined by

$$\begin{cases} B = \mathcal{C}(A, \sigma_A, f_A)\, e, \\ C = \mathcal{C}(A, \sigma_A, f_A)(1 - e), \\ \text{and } e = \varepsilon_A^{-1}((1, 0)) \in Z_A \subset \mathcal{C}(A, \sigma_A, f_A). \end{cases}$$

Since A has degree 8, the canonical involution $\underline{\sigma_A}$ acts trivially on Z_A. Hence the canonical pair $(\underline{\sigma_A}, \underline{f_A})$ induces quadratic pairs (σ_B, f_B) and (σ_C, f_C) on each component; see Section 3. Moreover, one may check that isomorphic quadruples lead to isomorphic trialitarian triples.

Conversely, given a trialitarian triple (A, B, C) pick an isomorphism α_A between $\mathcal{C}(A)$ and $B \times C$, and define the ε_A to be the restriction of α_A to the center Z_A of the Clifford algebra of A. We claim that the isomorphism class of the quadruple $(A, \sigma_A, f_A, \varepsilon_A)$ does not depend on the choice of α_A. If all such isomorphisms have the same restriction to the centre, this is clear. Assume now that there exists $\alpha_A^{(1)}$ and $\alpha_A^{(2)}$ having different restrictions. Then the composition $\alpha_A^{(2)} \circ (\alpha_A^{(1)})^{-1}$ is an isomorphism of the algebra with quadratic pair $B \times C$ whose restriction to the center $F \times F$ is the non trivial automorphism. Hence, B and C are isomorphic, and A is split by Lemma 4.10. In this case, the algebra with quadratic pair (A, σ_A, f_A) admits improper similitudes, and it follows that the quadruples $(A, \sigma_A, f_A, \varepsilon_A^{(1)})$ and $(A, \sigma_A, f_A, \varepsilon_A^{(2)})$ are isomorphic.

4.3. Action of A_3 on trialitarian triples. The main result of this section is the following, which extends [17, (42.3)] to characteristic 2:

Theorem 4.11: *The action of A_3 on PGO_8^+ induces an action on trialitarian triples, which is given by permutations. In particular, if*

$$(\mathcal{C}(A, \sigma_A, f_A), \underline{\sigma_A}, \underline{f_A}) \simeq (B, \sigma_B, f_B) \times (C, \sigma_C, f_C),$$

then we also have

$$(\mathcal{C}(B, \sigma_B, f_B), \underline{\sigma_B}, \underline{f_B}) \simeq (C, \sigma_C, f_C) \times (A, \sigma_A, f_A)$$

and

$$(\mathcal{C}(C, \sigma_C, f_C), \underline{\sigma_C}, \underline{f_C}) \simeq (A, \sigma_A, f_A) \times (B, \sigma_B, f_B).$$

Proof. Let (A, B, C) be a trialitarian triple, and fix an isomorphism

$$\alpha_A : (\mathcal{C}(A, \sigma_A, f_A), \underline{\sigma_A}, \underline{f_A}) \to (B, \sigma_B, f_B) \times (C, \sigma_C, f_C).$$

As above, we let n_0 be the 8-dimensional hyperbolic form. Recall that Ψ_1 defined as in Lemma 4.5 is an isomorphism

$$\Psi_1 : (\mathcal{C}_0(n_0), \underline{\sigma_{n_0}}, \underline{f_{n_0}}) \to \mathrm{Ad}_{n_0} \times \mathrm{Ad}_{n_0},$$

so that $(\mathrm{Ad}_{n_0}, \mathrm{Ad}_{n_0}, \mathrm{Ad}_{n_0})$ also is a trialitarian triple. After scalar extension to a separable closure F_s of the base field F, both triples are isomorphic. More precisely, consider an arbitrary isomorphism

$$\phi_A : (\mathrm{Ad}_{n_0})_{F_s} \to (A, \sigma_A, f_A)_{F_s}.$$

Composing ϕ_A with an improper similitude of n_0 if necessary, we may assume that the composition $\alpha_A \circ \mathcal{C}_0(\phi_A) \circ \Psi_1^{-1}$ acts trivially on $F \times F$, so that it is given by (ϕ_B, ϕ_C) for some isomorphisms of algebras with quadratic pairs

$$\phi_B : (\mathrm{Ad}_{n_0})_{F_s} \to (B, \sigma_B, f_B)_{F_s} \quad \text{and} \quad \phi_C : (\mathrm{Ad}_{n_0})_{F_s} \to (C, \sigma_C, f_C)_{F_s}.$$

Hence we get an isomorphism of trialitarian triples, that is a commutative diagram

$$\begin{array}{ccc} (\mathcal{C}_0(n_0), \underline{\sigma_{n_0}}, \underline{f_{n_0}})_{F_s} & \xrightarrow{\Psi_1} & (\mathrm{Ad}_{n_0})_{F_s} \times (\mathrm{Ad}_{n_0})_{F_s} \\ \downarrow{\scriptstyle \mathcal{C}(\phi_A)} & & \downarrow{\scriptstyle \phi_B \times \phi_C} \\ (\mathcal{C}(A, \sigma_A, f_A), \underline{\sigma_A}, \underline{f_A}) & \xrightarrow{\alpha_A} & (B, \sigma_B, f_B) \times (C, \sigma_C, f_C). \end{array}$$

Identifying the automorphism group of $(\mathrm{Ad}_{n_0})_{F_s}$ with $\mathrm{PGO}_8^+(F_s)$, we get by Galois descent that the map

$$a : \Gamma_F \to \mathrm{PGO}_8^+(F_s),\ \gamma \mapsto \phi_A^{-1} \circ {}^\gamma\phi_A$$

is a 1-cocycle whose cohomology class determines the triple (A, B, C). Finally, from the commutative diagram above, we have

$$\begin{aligned}\Psi_1 \circ \mathcal{C}_0(\phi_A^{-1} \circ {}^\gamma\phi_A) \circ \Psi_1^{-1} =&(\Psi_1\mathcal{C}(\phi_A^{-1})\alpha_A^{-1}) \circ (\alpha_A\mathcal{C}({}^\gamma\phi_A)\Psi_1^{-1})\\ =&(\phi_B^{-1} \circ {}^\gamma\phi_B, \phi_C^{-1} \circ {}^\gamma\phi_C).\end{aligned}$$

In view of the description of the trialitarian action in Section 4.1, see also Remark 4.7, we get that $\theta^+(a)$ and $\theta^-(a)$ coincide with the cohomology classes of the cocycles $\gamma \mapsto \phi_B^{-1}\circ{}^\gamma\phi_B$ and $\gamma \mapsto \phi_C^{-1}\circ{}^\gamma\phi_C$, respectively. Hence, $\theta^+(A, B, C)$ and $\theta^-(A, B, C)$ are trialitarian triples having respectively B and C as a first slot. Finally, we have $(\theta^+)^2 = \theta^-$ and $\theta^-\theta^+ = \mathrm{Id}$. Applying these formulas to the triple (A, B, C) we get that the second and the third slots in $\theta^+(A, B, C)$ respectively are the first slots in $\theta^-(A, B, C)$ and in (A, B, C), that is

$$\theta^+(A, B, C) = (B, C, A).$$

The same kind of argument shows $\theta^-(A, B, C) = (C, A, B)$, and this finishes the proof. ∎

5. Applications of triality

Theorem 4.11 above shows that the Clifford algebra, viewed as an algebra with quadratic pair, actually is a complete invariant for degree 8 algebras with quadratic pair with trivial discriminant. As a first application of our main result, we now characterize totally decomposable algebras with quadratic pair in degree 8; see Theorem 5.1. The proof uses Lemma 4.10, which describes all triples including a split algebra. Using direct sums of algebras with quadratic pairs, we then provide examples of triples, in which all three slots decompose as a sum of two degree 4 totally decomposable algebras with quadratic pair. Finally, we prove that all trialitarian triples that include two algebras of index at most 2 are of this shape.

5.1. Totally decomposable quadratic pairs. Using the trialitarian action described in the previous section, we may characterize totally decomposable degree 8 algebras with quadratic pair as follows:

THEOREM 5.1: *Let (A,σ,f) be an F-algebra of degree 8 with quadratic pair. Then (A,σ,f) is totally decomposable if and only if it has trivial discriminant and its Clifford algebra has a split factor.*

Proof. If the base field F has characteristic different from 2, the result follows immediately from [17, (42.11)] by uniqueness of the semi-trace. Let us now consider a base field F of characteristic 2. Assume first that (A,σ,f) has trivial discriminant, and its Clifford algebra has a split factor. This means (A,σ,f) is part of a trialitarian triple (A,B,C) with B or C split. By Theorem 4.11, it is also part of a triple whose first slot is split, and in view of Lemma 4.10, we get a quadratic form q such that

$$(\mathcal{C}(\mathrm{Ad}_q),\underline{\sigma_q},\underline{f_q})\simeq(A,\sigma,f)\times(A,\sigma,f).$$

By the explicit computation given in Example 3.8, it follows that (A,σ,f) is totally decomposable.

Assume conversely that (A,σ,f) is totally decomposable, and pick a decomposition

$$(A,\sigma,f)=([a_1,b_1)\otimes[a_2,b_2)\otimes[a_3,b_3),{}^-\otimes{}^-\otimes{}^-,f_\otimes).$$

Let $q=[b_1,a_1b_1^{-1}]\perp[b_2,a_2b_2^{-1}]\perp[b_3,a_3b_3^{-1}]\perp[1,a_1+a_2+a_3]$. Then q has trivial discriminant, and applying again Example 3.8, we get

$$(\mathcal{C}(\mathrm{Ad}_q),\underline{\sigma_q},\underline{f_q})\simeq(A,\sigma,f)\times(A,\sigma,f).$$

Hence by Theorem 4.11, we have

$$(\mathcal{C}(A,\sigma,f),\underline{\sigma},\underline{f})\simeq(A,\sigma,f)\times\mathrm{Ad}_q\,.$$

This proves (A,σ,f) has trivial discriminant and its Clifford algebra has a split component. ∎

5.2. EXAMPLES OF TRIALITARIAN TRIPLES. Given a trialitarian triple (A,B,C), it follows from [13, Thm. 1.5] that either all three involutions σ_A, σ_B and σ_C are isotropic, or all three are anisotropic. The triple is called isotropic or anisotropic accordingly. In this section, we provide explicit examples of trialitarian triples, and we prove all triples that include at least two algebras of Schur index at most 2 are of this shape, as well as all isotropic triples. We use the following definition, which was first introduced for algebras with involution by Dejaiffe [8], and later extended to quadratic pairs in [12, p. 379] (see also [7, Def. 1.4] and [14, Prop. 7.4.2]).

Definition 5.2: The algebra with quadratic pair (A, σ, f) is called an orthogonal sum of (A_1, σ_1, f_1) and (A_2, σ_2, f_2), and we write

$$(A, \sigma, f) \in (A_1, \sigma_1, f_1) \boxplus (A_2, \sigma_2, f_2),$$

if there are symmetric orthogonal idempotents e_1 and e_2 in the algebra A such that for $i \in \{1, 2\}$,

$$(e_iAe_i, \sigma_{|e_iAe_i}) \simeq (A_i, \sigma_i),$$

so that we may identify A_i with a subset of A, and

$$f(s_i) = f_i(s_i) \quad \text{for all } s_i \in \mathrm{Sym}(A_i, \sigma_i).$$

Note that the identification of A_i with its image in A is compatible with the reduced trace. More precisely, we have

$$\mathrm{Trd}_A(a_i) = \mathrm{Trd}_{A_i}(a_i) \quad \text{for all } a_i \in A_i \simeq e_iAe_i \subset A;$$

see the matrix description of the orthogonal sum given in [8, §2]. Moreover, the direct product $(A_1, \sigma_1, f_1) \times (A_2, \sigma_2, f_2)$ embeds in (A, σ, f), meaning there is an embedding of the direct product of algebras with involution, and the restriction of f to the image of $\mathrm{Sym}(A_i, \sigma_i)$ coincides with f_i for $i \in \{1, 2\}$.

Example 5.3: Let (V_1, q_1) and (V_2, q_2) be two nonsingular quadratic spaces over F. For all $\mu \in F^\times$, we have

$$\mathrm{Ad}_{q_1 \perp \langle\mu\rangle q_2} \in \mathrm{Ad}_{q_1} \boxplus \mathrm{Ad}_{q_2}.$$

This follows directly from the description of Ad_q given in [17, §5.B], and the definition above, taking e_1 and e_2 in

$$A = \mathrm{End}_F(V_1 \oplus V_2)$$

to be the orthogonal projections on V_1 and V_2 respectively.

As this example shows, an orthogonal sum (A, σ, f) is not uniquely determined by its summands (A_1, σ_1, f_1) and (A_2, σ_2, f_2) and $(A_1, \sigma_1, f_1) \boxplus (A_2, \sigma_2, f_2)$ should be considered as a set.

With this in hand, we may produce examples of trialitarian triples as follows. Let Q_1, Q_2, Q_3 and Q_4 be quaternion algebras such that $\bigotimes_{i=1}^4 Q_i$ is split. For all i and j with $i \neq j$, we denote by f_{ij} the semi-trace $f_\otimes$ on $\mathrm{Sym}(Q_i \otimes Q_j, \bar{\ } \otimes \bar{\ })$ associated to the tensor product decomposition $(Q_i, \bar{\ }) \otimes (Q_j, \bar{\ })$ as in Notation 2.6. We have the following

PROPOSITION 5.4: *Let (A,σ,f) be an F-algebra with quadratic pair such that*

$$(A,\sigma,f)\in(Q_1\otimes Q_2,{}^-\otimes{}^-,f_{12})\boxplus(Q_3\otimes Q_4,{}^-\otimes{}^-,f_{34}).$$

Then (σ,f) has trivial discriminant, and the Clifford algebra $\mathcal{C}(A,\sigma,f)$, with its canonical quadratic pair, is a direct product of

$$\begin{aligned}\mathcal{C}^+(A,\sigma,f)&\in(Q_1\otimes Q_3,{}^-\otimes{}^-,f_{13})\boxplus(Q_2\otimes Q_4,{}^-\otimes{}^-,f_{24}),\\ \text{and}\quad \mathcal{C}^-(A,\sigma,f)&\in(Q_1\otimes Q_4,{}^-\otimes{}^-,f_{14})\boxplus(Q_2\otimes Q_3,{}^-\otimes{}^-,f_{23}).\end{aligned}$$

Proof. In characteristic different from 2, the algebra with involution version of this result is stated and proved in [24, Prop. 6.6], and the proposition follows immediately by uniqueness of the semi-trace in this case. So we may assume F has characteristic 2. The same argument as in characteristic different from 2 applies, and we get a description of $(\mathcal{C}(A,\sigma,f),\underline{\sigma})$ as an algebra with involution. By Definition 5.2, it only remains to check that the canonical semi-trace $\underline{f}$ acts as f_{ij} on each subset

$$\operatorname{Sym}(Q_i\otimes Q_j,{}^-\otimes{}^-)\subset\operatorname{Sym}(\mathcal{C}(A,\sigma,f),\underline{\sigma}).$$

For $i\in\{1,2\}$, let u_i in Q_i be a quaternion such that $\bar{u}_i+u_i=1$. Identifying $Q_1\otimes Q_2$ to a subset of A as above, we have

$$\operatorname{Trd}_A(u_1\otimes u_2)=\operatorname{Trd}_{Q_1\otimes Q_2}(u_1\otimes u_2)=\operatorname{Trd}_{Q_1}(u_1)\operatorname{Trd}_{Q_2}(u_2)=1.$$

Therefore, the canonical semi-trace $\underline{f}$ on $\mathcal{C}(A,\sigma,f)$ is determined by the element $c(u_1\otimes u_2)$ in $\mathcal{C}(A,\sigma,f)$ by Definition 3.3.

Recall from [17, (15.12)] that the Clifford algebra of $(Q_i\otimes Q_j,{}^-\otimes{}^-,f_{ij})$, with its canonical involution, is the direct product $(Q_i,{}^-)\times(Q_j,{}^-)$. The same argument as in the proof of [8, Prop. 3.5] shows that the embedding of the direct product $(Q_1\otimes Q_2,{}^-\otimes{}^-,f_{12})\times(Q_3\otimes Q_4,{}^-\otimes{}^-,f_{34})$ in (A,σ,f) induces an embedding of the tensor product of the corresponding Clifford algebras

$$((Q_1,{}^-)\times(Q_2,{}^-))\otimes((Q_3,{}^-)\times(Q_4,{}^-))\hookrightarrow(\mathcal{C}(A,\sigma,f),\underline{\sigma}).$$

It follows that $c(u_1\otimes u_2)$ is $c_{12}(u_1\otimes u_2)\otimes(1,1)$, where c_{12} is the canonical map from $(Q_1\otimes Q_2,{}^-\otimes{}^-,f_{12})$ to its Clifford algebra $(Q_1,{}^-)\times(Q_2,{}^-)$. This map is described in [17, (8.19)], and we get

$$c(u_1\otimes u_2)=(u_1,u_2)\otimes(1,1)=(u_1\otimes 1,u_2\otimes 1,u_1\otimes 1,u_2\otimes 1),$$

in $(Q_1 \otimes Q_3) \times (Q_2 \otimes Q_4) \times (Q_1 \otimes Q_4) \times (Q_2 \otimes Q_3)$. Therefore, the canonical semi-trace acts on $\mathrm{Sym}(Q_i \otimes Q_k, {}^{-} \otimes {}^{-})$ for $i \in \{1,2\}$ and $k \in \{3,4\}$ by

$$\underline{f}(x) = \mathrm{Trd}_{Q_i \otimes Q_k}((u_i \otimes 1)x).$$

In particular, it vanishes on $\mathrm{Sym}(Q_i, {}^{-}) \otimes \mathrm{Sym}(Q_k, {}^{-})$ and coincides with f_{ik} by [17, (5.20)]; see also Section 2.4. This finishes the proof. ∎

Remark 5.5: (1) If one of the quaternion algebras, say Q_4, is split, then the algebras with quadratic pair $(Q_i \otimes Q_4, {}^{-} \otimes {}^{-}, f_{i4})$ are hyperbolic for $i \in \{1,2,3\}$; see [17, (15.14)]. Hence, we get an isotropic trialitarian triple in which the algebras with quadratic pairs are respectively Witt equivalent to

$$(Q_1 \otimes Q_2, {}^{-} \otimes {}^{-}, f_{12}), \quad (Q_1 \otimes Q_3, {}^{-} \otimes {}^{-}, f_{13}) \text{ and } (Q_2 \otimes Q_3, {}^{-} \otimes {}^{-}, f_{23}),$$

where Q_1 Q_2 and Q_3 are quaternion algebras with $Q_1 \otimes Q_2 \otimes Q_3$ split. In particular, all three algebras in the triple have index at most 2.

(2) The argument in the proof of [24, Prop. 6.12] extends to this setting and it follows that all trialitarian triples which are isotropic are either as in (1) or with a split component. In the second case, they coincide up to permutation with

$$(\mathrm{Ad}_{q \perp \mathbb{H}}, M_2(D), M_2(D)),$$

where q is an Albert form, D is the corresponding biquaternion algebra, which may have index $1, 2$ or 4 depending on q, and $M_2(D)$ is endowed with a hyperbolic quadratic pair; see Proposition 3.11.

The purpose of the remaining part of this section is to prove the following theorem, which provides a description of all trialitarian triples including at least two algebras of Schur index at most 2.

Theorem 5.6: *Let (A,B,C) be a trialitarian triple over F such that at least two of the algebras A, B and C have Schur index at most 2. Then there exist F-quaternion algebras Q_1, Q_2, Q_3 and Q_4, with $Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4$ split, such that*

$$\begin{aligned} (A, \sigma_A, f_A) &\in (Q_1 \otimes Q_2, {}^{-} \otimes {}^{-}, f_{12}) \boxplus (Q_3 \otimes Q_4, {}^{-} \otimes {}^{-}, f_{34}), \\ (B, \sigma_B, f_B) &\in (Q_1 \otimes Q_3, {}^{-} \otimes {}^{-}, f_{13}) \boxplus (Q_2 \otimes Q_4, {}^{-} \otimes {}^{-}, f_{24}), \\ \text{and} \quad (C, \sigma_C, f_C) &\in (Q_1 \otimes Q_4, {}^{-} \otimes {}^{-}, f_{14}) \boxplus (Q_2 \otimes Q_3, {}^{-} \otimes {}^{-}, f_{23}), \end{aligned}$$

where f_{ij} is the semi-trace $f_\otimes$ on $\mathrm{Sym}(Q_i \otimes Q_j, {}^{-} \otimes {}^{-})$ associated to the tensor product decomposition $(Q_i, {}^{-}) \otimes (Q_j, {}^{-})$ as in Notation 2.6.

In characteristic 2, the proof of the theorem uses the so-called generalized quadratic forms, as defined by Tits in [26]. We first recall the definition and a few well-known facts. Let D be a central simple F-algebra with involution θ of the first kind. A generalized quadratic form over (D,θ) is a pair (V,q) where V is a finitely generated right projective D-module and q is a map $q : V \to D/\operatorname{Symd}(D,\theta)$ subject to the following conditions:

(a) $q(xd) = \theta(d)q(x)d$ for all $x \in V$ and $d \in D$.
(b) There exists a hermitian form h defined on V and with values in (D,θ) such that for all $x, y \in V$ we have

$$q(x+y) - q(x) - q(y) = h(x,y) + \operatorname{Symd}(D,\theta).$$

In this case the hermitian form (V,h) is uniquely determined (see [16, (5.2)]) and we call it the polar form of (V,q). Note that it follows from (b) that

$$h(x,x) \in \operatorname{Symd}(A,\theta), \quad \text{for all } x \in V,$$

hence the polar form of any quadratic form over (D,θ) is alternating. We call (V,q) nonsingular if its polar form is nondegenerate. We say that (V,q) represents an element $d \in D$ if $q(x) = d + \operatorname{Symd}(D,\theta)$ for some $x \in V \setminus \{0\}$. We call (V,q) isotropic if it represents 0 and anisotropic otherwise. For a field extension K/F we write

$$(D,\theta)_K = (D \otimes_F K, \theta \otimes \operatorname{Id}), \quad V_K = V \otimes_F K,$$

and by q_K we mean the unique generalized quadratic form

$$q : V_K \to D_K/\operatorname{Symd}(D_K,\theta_K)$$

such that

$$q_K(x \otimes k) = q(x)k^2 + \operatorname{Symd}(D_K,\theta_K)$$

for all $x \in V$and $k \in K$.

Assume now that D is a division algebra over F. For $a_1,\dots,a_n \in D$, we denote by $\langle a_1,\dots,a_n\rangle$ the quadratic form (D^n,q) over (D,θ) where q is defined by

$$q(x_1,\dots,x_n) = \sum_{i=1}^{n} \theta(x_i)a_ix_i + \operatorname{Symd}(D,\theta)$$

for all $(x_1,\dots,x_n) \in D^n$. We call such a form a diagonal form. We call a quadratic form diagonalisable if it is isometric to a diagonal form.

The proof of Theorem 5.6 uses the following:

LEMMA 5.7: *Let $(Q, \overline{})$ be an F-quaternion division algebra endowed with its canonical involution, and let (V, q) over $(Q, \overline{})$ be a skew-hermitian space in characteristic different from 2 and a generalized quadratic space in characteristic 2. Let K be a separable quadratic extension of F contained in Q and pick $u \in Q$, a pure quaternion in characteristic different from 2 and a quaternion with trace 1 in characteristic 2, such that $K \simeq F[u]$. If the form q_K is isotropic, then q represents $u\lambda$ for some $\lambda \in F^\times$.*

Proof. The result is clear if q is isotropic, since (V, q) contains a hyperbolic plane, which does represent u; in characteristic 2, this follows from the fact that all 2-dimensional isotropic generalized quadratic forms are isometric, therefore q contains a subform isometric to $\langle u, -u\rangle$; see [16, (5.6.1)]. We may therefore assume that q is anisotropic.

Let us write $K = F[\alpha]$ where α satisfies $\alpha^2 = a = u^2$ in characteristic different from 2 and $\alpha^2 + \alpha = a = u^2 + u$ in characteristic 2. Assume q_K is isotropic; there exist $x, y \in V$ such that

$$q_K(x \otimes 1 + y \otimes \alpha) = 0.$$

We claim that $q(x + yu) = u\lambda$ for some $\lambda \in F^\times$.

In order to prove this, assume first that F has characteristic 2, so that q is a generalized quadratic form with polar form h. Since $\mathrm{Symd}(Q, \overline{}) = F$, the forms q and q_K respectively have values in Q/F and Q_K/K. We have

$$\begin{aligned} q_K(x \otimes 1 + y \otimes \alpha) =& q_K(x \otimes 1) + q_K(y \otimes \alpha) + h_K(x \otimes 1, y \otimes \alpha) \\ =& q(x) \otimes 1 + q(y) \otimes \alpha^2 + h(x, y) \otimes \alpha \\ =& (q(x) + q(y)a) \otimes 1 + (h(x, y) + q(y))) \otimes \alpha \\ =& 0 \mod K. \end{aligned}$$

Therefore, since $Q_K/K \simeq Q/F \otimes 1 \oplus Q/F \otimes \alpha$, we get that $q(x) + q(y)a \in F$ and $h(x, y) + q(y) \in F$. Hence, we have

$$\begin{aligned} q(x + yu) =& q(x) + q(yu) + h(x, yu) = q(x) + (u + 1)q(y)u + h(x, y)u \\ =& (q(x) + uq(y)u) + (h(x, y) + q(y))u \\ =& (q(y)a + uq(y)u) + (h(x, y) + q(y))u \in Q/F. \end{aligned}$$

Take any $\xi \in Q$ such that $q(y) = \xi \mod F$. The quaternion $\xi a + u\xi u$ commutes with u, hence belongs to $F[u] \subset Q$. Therefore, $q(x + yu) \in F[u]/F$, and since q is anisotropic, it is equal to $u\lambda \mod F$ for some $\lambda \in F^\times$.

Assume now F has characteristic different from 2, so that q is a skew-hermitian form over $(Q, \bar{\ })$, which we denote by h in the computations below to avoid confusion. A similar computation as above shows

$$h_K(x \otimes 1 + y \otimes \alpha, x \otimes 1 + y \otimes \alpha) \\ = (h(x,x) + h(y,y)a) \otimes 1 + (h(x,y) + h(y,x)) \otimes \alpha.$$

Hence, $h(x,x) = -ah(y,y)$. Moreover, $h(x,y) = -h(y,x)$, and since h is skew-hermitian, it follows that $h(y,x) \in F$. From this, we get

$$h(x+yu, x+yu) = (h(x,x) - uh(y,y)u) + 2h(x,y)u \\ = -(ah(y,y) + uh(y,y)u) + 2h(x,y)u.$$

Again, since $ah(y,y) + uh(y,y)u$ commutes with u and $h(x,y) \in F$, this proves that $h(x+yu, x+yu)) \in F[u]$. The form h is anisotropic and skew-symmetric, so we get $h(x+yu, x+yu) = u\lambda$ for some $\lambda \in F^\times$ as required. ∎

With this in hand, we may now prove Theorem 5.6.

Proof. Let (A, σ, f) be an F-algebra with quadratic pair. Assume A has degree 8, (σ, f) has trivial discriminant and two of A, $\mathcal{C}^+(A, \sigma, f)$ and $C^-(A, \sigma, f)$ have index at most 2. By triality, we may assume that A and at least one component of the Clifford algebra of (A, σ, f) have index at most 2. By Proposition 5.4, it is enough to prove there exist quaternion algebras Q_1, Q_2, Q_3 and Q_4 such that

$$(A, \sigma, f) \in (Q_1 \otimes Q_2, \bar{\ } \otimes \bar{\ }, f_{12}) \boxplus (Q_3 \otimes Q_4, \bar{\ } \otimes \bar{\ }, f_{34}). \tag{11}$$

Let us first consider the split case, so that $(A, \sigma, f) \simeq \mathrm{Ad}_q$ for some nonsingular quadratic form q over F with trivial discriminant. By assumption, we have in addition $\mathrm{ind}(\mathcal{C}_0(q)) \leqslant 2$. Therefore by Proposition 2.2 there exist a 4-dimensional symmetric bilinear form B and a 2-dimensional nonsingular quadratic form ϕ over F such that $q \simeq B \otimes \phi$. In particular, we may write

$$q \simeq B_1 \otimes \phi \perp B_2 \otimes \phi$$

for some 2-dimensional symmetric bilinear forms B_1 and B_2. Since these summands are similar to 2-fold Pfister forms, taking the adjoint quadratic pair gives the result, by Remark 2.5 and Example 5.3.

Assume now $\mathrm{ind}(A) = 2$. Let Q be an F-quaternion division algebra such that A is Brauer equivalent to Q.

If F has characteristic different from 2, there exists a skew-hermitian form q over $(Q, \bar{\ })$ such that $\sigma = \mathrm{ad}_q$; since the semi-trace is unique in this case, the quadratic pair (σ, f) is determined by q, so we say it is adjoint to q and we write

$$(\sigma, f) \simeq \mathrm{ad}_q .$$

Let u be a pure quaternion represented by q. We have $q = \langle u \rangle \perp q'$ for some rank 3 skew-hermitian form q' over $(Q, \bar{\ })$, so that

$$(A, \sigma, f) \in \mathrm{Ad}_{\langle u \rangle} \boxplus \mathrm{Ad}_{q'} .$$

A similar decomposition may be obtained if F has characteristic 2. Pick a generalized quadratic space (V, q) over $(Q, \bar{\ })$ such that $(A, \sigma, f) = \mathrm{Ad}_q$, see [12, Thm 1.5]. The form q is nonsingular, hence by [9, (6.3) and (7.5)] it is diagonalisable, with entries that are non-symmetric elements of $(Q, \bar{\ })$. Hence, $q = \langle u_1 \rangle \perp q'$ for some quaternion u_1 with nonzero trace and some nonsingular generalized quadratic form q'. Let $u = u_1 \,\mathrm{Trd}_Q(u_1)^{-1}$; u is a quaternion with reduced trace 1, and it satisfies

$$\mathrm{Ad}_{\langle u_1 \rangle} \simeq \mathrm{Ad}_{\langle u \rangle} .$$

Therefore, we get

$$(A, \sigma, f) \in \mathrm{Ad}_{\langle u \rangle} \boxplus \mathrm{Ad}_{q'} .$$

Finally, let $K = F[u]$. In both cases, K is a quadratic separable field extension of F that splits Q, and the quadratic pair $(\mathrm{Ad}_{\langle u \rangle})_K$ has trivial discriminant. Therefore, the hypothesis on (A, σ, f) guarantees that the remaining part $(\mathrm{Ad}_{q'})_K$ is adjoint to a quadratic form over K which is 6-dimensional, has trivial discriminant, and has Clifford invariant of index 2. By [17, (16.5)], it is isotropic. Hence, Lemma 5.7 shows that $q' = \langle u\lambda \rangle \perp q''$ for some nonsingular q'', which is a skew-hermitian form or a generalized quadratic form over $(Q, \bar{\ })$, depending on the characteristic of F. In both cases, we get

$$(A, \sigma, f) \in \mathrm{Ad}_{q_1} \boxplus \mathrm{Ad}_{q_2},$$

where both q_1 and q_2 have trivial discriminant. Hence both Ad_{q_1} and Ad_{q_2} are decomposable by [17, (15.12)], and this finishes the proof. ∎

6. Appendix: Canonical semi-trace on the full Clifford algebra of a quadratic form

In this appendix, we will show how one can construct a canonical semi-trace on the full Clifford algebra of a quadratic form. This semi-trace will be closely related to the semi-trace constructed in Section 3. If the field is of characteristic different from 2, then the full Clifford algebra has a unique semi-trace if and only if the canonical involution is orthogonal. Therefore, throughout this section, we assume that F is a field of characteristic 2.

We first give a construction of the full Clifford algebra of a nonsingular quadratic form. For this, we use the following presentation of quaternion algebras. When the characteristic of F is 2, a quaternion algebra may be defined as F-algebra generated by two elements r, s subject to $r^2, s^2 \in F$ and $rs + sr = 1$. If $s^2 \neq 0$, then $(1, sr, s, sr^2)$ is a quaternion basis of this algebra, in the sense of §2.1, and otherwise the algebra is split (see [17, p. 25]).

Example 6.1: Let q be a nonsingular quadratic form over F with polar form b. Pick a decomposition

$$q \simeq [a_1, b_1] \perp \cdots \perp [a_m, b_m],$$

and let $(e_i, e_i')_{1\leqslant i\leqslant m}$ be a basis of the underlying vector space V such that for all i with $1 \leqslant i \leqslant m$, we have

$$q(e_i) = a_i, \quad q(e_i') = b_i, \quad b(e_i, e_i') = 1$$

and all other pairs of basis vectors are orthogonal. We may assume $a_i \neq 0$ for all i.

The full Clifford algebra of q is generated by the elements $\{e_i, e_i'\}_{1\leqslant i\leqslant m}$, subject to the following relations for all $i \in \{1, \ldots, m\}$:

$$e_i^2 = a_i, \quad {e_i'}^2 = b_i, \quad e_ie_i' + e_i'e_i = 1.$$

In addition, any pair of elements in the basis other than (e_i, e_i') commute. By definition, the elements e_i and e_i' are fixed under the canonical involution $\underline{\sigma_q}$ on $\mathcal{C}(q)$. Therefore, the pairs $(e_i, e_i')_{1\leqslant i\leqslant m}$ each generate pairwise commuting $\underline{\sigma_q}$-stable F-quaternion subalgebras of $\mathcal{C}(q)$, respectively isomorphic to $[a_ib_i, a_i)$, and $\underline{\sigma_q}$ restricts to the canonical involution on each of these quaternion subalgebras. In particular, the canonical involution on $\mathcal{C}(q)$ is always symplectic.

PROPOSITION 6.2: *Let (V,q) be a nonsingular quadratic space of even dimension $2m \geqslant 6$. Given a pair $(e,e') \in V^2$ with $b_q(e,e')=1$, the map*

$$f : \operatorname{Sym}(\mathcal{C}(q), \underline{\sigma_q}) \to F, \qquad x \mapsto \operatorname{Trd}_{\mathcal{C}(q)}(ee'x),$$

is a semi-trace and does not depend on the choice of (e,e').

We will refer to this semi-trace as the canonical semi-trace on the full Clifford algebra, and use the same notation, $\underline{f_q}$, as for the canonical semi-trace on the even Clifford algebra.

Remark 6.3: (1) If m is even, we may define $\underline{f_q}$ as the semi-trace on the full Clifford algebra $\mathcal{C}(q)$ induced by the canonical semi-trace on $\mathcal{C}_0(q)$, in the sense of §2.4. Note though that restricting the canonical semi-trace of $\mathcal{C}(q)$, viewed as a map, to the even part $\operatorname{Sym}(\mathcal{C}_0(q), \underline{\sigma_q})$ does not produce a semi-trace, since the values are in F, while the center of $\mathcal{C}_0(q)$ is a quadratic étale extension of F.

(2) Since the involution $\underline{\sigma_q}$ on $\mathcal{C}(q)$ always is symplectic, m need not be even here, so this remark is not enough to prove the proposition.

Proof. Let $(e,e') \in V^2$ be two vectors such that $b_q(e,e')=1$. One computes that the element $u = ee' \in \mathcal{C}(q)$ satisfies $u + \underline{\sigma_q}(u) = 1$, hence it determines a semi-trace $f_{ee'}$ on $\mathcal{C}(q)$.

If $m \geqslant 4$, the same computations as in Remark 3.7 show that $f_{ee'}$ does not depend on the choice of (e,e'). If $m=3$ we may argue as follows. There exists a base $\varepsilon = (e,e',e_2,e_2',e_3,e_3')$ such that $b_q(e_2,e_2') = b_q(e_3,e_3') = 1$ and all other pairs of basis vectors are orthogonal. The computation at the end of Remark 3.7 shows that $f_{ee'} = f_{e_2e_2'} = f_{e_3e_3'}$; this semi-trace, which is canonically associated to the basis ε, is denoted by f_ε in the sequel. Consider now another pair of vectors (g,g') with $b_q(g,g')=1$ which is part of another base $\xi = (g,g',g_2,g_2',g_3,g_3')$ with $b_q(g_2,g_2') = b_q(g_3,g_3') = 1$ and all others pairs of basis vectors are orthogonal. By Revoy's Proposition [25, Prop. 3], there is a chain of symplectic bases $\eta_1 = \varepsilon, \eta_2, \dots, \eta_r = \xi$ of (V,q) such that for all i, $1 \leqslant i \leqslant r-1$, the bases η_i and η_{i+1} have a common symplectic pair. Hence, again by the computations of Remark 3.7, the semi-traces f_{η_i} and $f_{\eta_{i+1}}$ coincide, and this concludes the proof. ∎

Remark 6.4: Using Example 6.1 and the same arguments as in the proof of Example 3.8, we see that if $q = [a_1, b_1] \perp \cdots \perp [a_m, b_m]$, with $m \geqslant 3$, then

$$(\mathcal{C}(q), \underline{\sigma_q}, \underline{f_q}) \simeq ([a_1b_1, a_1) \otimes \cdots \otimes [a_mb_m, a_m), ^- \otimes \cdots \otimes ^-, f_\otimes).$$

In particular, two decompositions of the algebra with involution $(\mathcal{C}(q), \underline{\sigma_q})$ arising from two different presentations of q give rise to the same canonical semi-trace. Compare with Remark 2.5.

Corollary 6.5: *Let q be a nonsingular quadratic form over F. If q is isotropic then $(\mathcal{C}(q), \underline{\sigma_q}, \underline{f_q})$ is hyperbolic.*

Proof. This follows from Remark 6.4 using a similar argument as that in Proposition 3.11. ∎

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