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THE COFINALITY OF THE SYMMETRIC GROUP AND THE COFINALITY OF ULTRAPOWERS

BY

HEIKE MILDENBERGER*

Abteilung für Mathematische Logik, Mathematisches Institut
Universität Freiburg, Ernst-Zermelo-Str. 1, 79104 Freiburg im Breisgau, Germany
e-mail: heike.mildenberger@math.uni-freiburg.de

AND

SAHARON SHELAH**

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Edmond Safra Campus, Givat Ram, Jerusalem 9190401, Israel
e-mail: shelah@math.huji.ac.il

ABSTRACT

We prove that $\mathfrak{mcf} < \mathrm{cf}(\mathrm{Sym}(\omega)))$ and $\mathfrak{mcf} > \mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{b}$ are both consistent relative to ZFC. This answers a question by Banakh, Repovš and Zdomskyy and a question from [MS11].

1. Introduction

We compare the cardinal $\mathfrak{mcf}$, the minimal cofinality of the ultrapower $(\omega, <)$ by a non-principal ultrafilter on ω, and the cofinality of the symmetric group on ω, $\mathrm{cf}(\mathrm{Sym}(\omega))$. These two cardinal invariants are closely related: Both are

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cofinalities and hence regular. In ZFC, both cardinals have value in the interval $[\mathfrak{g}, \mathfrak{d}]$, namely Blass and Mildenberger [BM99] showed $\mathfrak{mcf} \geq \mathfrak{g}$, Brendle and Losada [BL03] showed $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \mathfrak{g}$, and Simon Thomas [Tho95] showed $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{d}$. In their relations to $\mathfrak{b}$ the two cardinals behave differently: Obviously $\mathfrak{b} \leq \mathfrak{mcf}$, whereas Sharp and Thomas [ST95, Theorem 1.6] showed that $\mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{b}$ is consistent relative to ZFC. Before our research, in all investigated forcing extensions we have had $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$ and in the forcing extensions in which both $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \mathfrak{b}$ and $\mathfrak{mcf} \geq \mathfrak{b}$, the two cardinal characteristics $\mathrm{cf}(\mathrm{Sym}(\omega))$ and $\mathfrak{mcf}$ coincide. The inequality $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$ is partially due to a mathematical reason: Banakh, Repovš and Zdomskyy showed [BRZ11, Theorem 1.3]: If D is not nearly coherent to a Q-point then

$$\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathrm{cf}((\omega, <)^\omega / D).$$

In particular, if there is no Q-point then

$$\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}.$$

Here we show that indeed an extra assumption is necessary. Our first forcing shows the relative consistency of $\aleph_1 = \mathfrak{b} = \mathfrak{mcf} < \aleph_2 = \mathrm{cf}(\mathrm{Sym}(\omega))$.

In our second forcing we show how to separate the two cardinals in the second direction above $\mathfrak{b}$: $\aleph_1 = \mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{mcf}$ is consistent. We use versions of the oracle-c.c. in the $\aleph_1$-$\aleph_2$-scenario.

There are some known forcings establishing the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$: Three interesting forcings for $\aleph_1 = \mathfrak{b} < \mathfrak{mcf}$ are given in [SS93, SS94]. Since $\mathfrak{b} \leq \mathfrak{u}$ [PS87] and since NCF is equivalent to $\mathfrak{u} < \mathfrak{mcf}$ [Mil01] the NCF-models show the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$. In [MS11] we showed that also $\mathfrak{b}^+ < \mathfrak{mcf}$ is possible. In the second forcing extension of that work we arranged $\mathfrak{b}^+ < \mathfrak{mcf} = \mathrm{cf}(\mathrm{Sym}(\omega))$. In the other forcing extensions for $\mathfrak{b} < \mathfrak{mcf}$ the value of $\mathrm{cf}(\mathrm{Sym}(\omega))$ has not yet been computed or is possibly not determined by the forcing or by NCF.

We recall the definitions: We denote by ${}^\omega\omega$ the set of functions from ω to ω. For $f, g \in {}^\omega\omega$ we write $f \leq^* g$ and say g **eventually dominates** f if

$$(\exists n)(\forall k \geq n)(f(k) \leq g(k)).$$

A set $B \subseteq {}^\omega\omega$ is called **unbounded** if there is no g that dominates all members of B. The **bounding number** $\mathfrak{b}$ is the minimal cardinality of an unbounded set.

Definition 1.1: Let D be a non-principal ultrafilter over ω. By ultrapower we mean the usual modeltheoretic ultrapower: The structure $(\omega,<)^\omega/D$ is defined on the domain $\{[f]_D : f \in {}^\omega\omega\}$ where

$$[f]_D = \{g \in {}^\omega\omega : \{n : f(n) = g(n)\} \in D\}.$$

The order relation is $[f]_D \leq_D [g]_D$ iff $\{n : f(n) \leq g(n)\} \in D$. We write $\mathrm{cf}((\omega,<)^\omega/D)$ for the minimal size of a set that is cofinal in $\leq_D$. The **minimal cofinality of an ultrapower of ω, $\mathfrak{mcf}$,** is defined as the

$$\mathfrak{mcf} = \min\{\mathrm{cf}((\omega,<)^\omega/D) : D \text{ non-principal ultrafilter over } \omega\}.$$

We define the relation $\leq_D$ also on the space ${}^\omega\omega$ by letting $f \leq_D g$ iff $\{n : f(n) \leq g(n)\} \in D$.

Definition 1.2: The group of permutations of ω is denoted by $\mathrm{Sym}(\omega)$. If $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} G_i$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\langle G_i : i < \kappa\rangle$ is strictly increasing, and each G_i is a proper subgroup of $\mathrm{Sym}(\omega)$, we call $\langle G_i : i < \kappa\rangle$ an **increasing decomposition**. We call the minimal κ such that an increasing decomposition of length κ exists the **cofinality of the symmetric group**, and denote it $\mathrm{cf}(\mathrm{Sym}(\omega))$.

Definition 1.3: A subset $\mathcal{G}$ of $[\omega]^\omega$ is called **groupwise dense** if

(1) $(\forall X \in \mathcal{G})(\forall Y \subseteq^* X)(Y \text{ infinite } \to Y \in \mathcal{G})$, and
(2) for every partition of ω into finite intervals $\Pi = \{[\pi_i, \pi_{i+1}) : i \in \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathcal{G}$.

The **groupwise density number, $\mathfrak{g}$**, is the smallest number of groupwise dense families with empty intersection.

An ultrafilter U over ω is called a **Q-point**, if given any strictly increasing function $f\colon \omega \to \omega$ there is an $X \in U$ such that $\forall n$, $X \cap [f(n), f(n+1))$ has just one element. The existence of a Q-point is independent of ZFC; see, e.g., [Can90] for existence and [Mil80] for non-existence. An ultrafilter D is **nearly coherent to an ultrafilter** U if there is a finite-to-one function $f\colon \omega \to \omega$ such that $f(D) = f(U)$. Here

$$f(D) = \{E : f^{-1}[E] \in D\}.$$

Throughout we write $g[X]$ for the set $\{g(x) : x \in X\}$ and $g^{-1}[Y] = \{x : g(x) \in Y\}$. The principle NCF says that any two non-principal ultrafilters over ω are nearly coherent. Its consistency is established in [BS87, BS89, Bla89]. **A base** for an ultrafilter is a subset $\mathcal{B}$ of $\mathscr{U}$ such that $(\forall Y \in \mathscr{U})(\exists X \in \mathcal{B})(X \subseteq Y)$. The character of an ultrafilter is the smallest size of a base. The **ultrafilter characteristic** $\mathfrak{u}$ is the smallest character of a non-principal ultrafilter.

In forcing the **stronger** condition is the larger one. For a forcing order $\mathbb{P}$ and a formula φ, we say $\mathbb{P}$ forces φ if the weakest condition in $\mathbb{P}$ forces φ.

2. Con($\mathfrak{b} = \mathrm{cf}(\omega^\omega/D) < \mathrm{cf}(\mathrm{Sym}(\omega))$)

In this section we prove

THEOREM 2.1: *The constellation $\aleph_1 = \mathfrak{b} = \mathfrak{mcf} < \mathrm{cf}(\mathrm{Sym}(\omega))$ is consistent relative to* ZFC.

We essentially use oracle c.c. [She98, Ch. 4], but in addition to the oracle sequence we construct a sequence $\langle \underset{\sim}{D}_\alpha : \alpha < \omega_1 \rangle$ which approximates a name $\underset{\sim}{D}$ for an ultrafilter. We construct a notion of forcing $\mathbb{P}$ such that for a $\mathbb{P}$-generic filter $\mathbf{G}$, $\underset{\sim}{D}[\mathbf{G}]$ will be an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$. The construction of $\mathbb{P}$ is done via an approximation forcing AP, so that $\mathbb{P} = AP * \underset{\sim}{\mathbb{Q}}$, where $\underset{\sim}{\mathbb{Q}}$ is an AP-name for the AP-generic object.

We recall some oracle technique of [She98, Chapter IV]. Let S be a stationary subset of ω_1. We fix S throughout this section. A set $\mathscr{D} \subseteq \mathcal{P}(S)$ is called a **filter over** S if $\emptyset \notin \mathscr{D}$, $S \in \mathscr{D}$, $\mathscr{D}$ is closed under finite intersections and closed under supersets. A filter $\mathscr{D}$ over S is called **normal** if it contains all sets of the form $[\alpha, \omega_1) \cap S$, $\alpha < \omega_1$, and is closed under diagonal intersections. We recall, given a sequence $\langle D_\delta : \delta \in S \rangle$, that its diagonal intersection is the following set

$$\triangle_{\delta \in S} D_\delta = \left\{ \gamma \in S : \gamma \in \bigcap_{\delta \in \gamma \cap S} D_\delta \right\}.$$

For a filter $\mathscr{D}$ over ω_1 and $X, Y \subseteq \omega_1$ we let $X = Y \bmod \mathscr{D}$ if

$$(X \cap Y) \cup ((\omega_1 \smallsetminus X) \cap (\omega_1 \smallsetminus Y)) \in \mathscr{D},$$

and $X \subseteq Y \mod \mathscr{D}$ if $X \smallsetminus Y = \emptyset \mod \mathscr{D}$.

We recall the notion of a $\diamondsuit_S^-$-sequence. A sequence $\bar{P} = \langle P_\delta : \delta \in S \rangle$ is called a $\diamondsuit_S^-$-sequence if $P_\delta \subseteq \mathcal{P}(\delta)$ is countable and for any $X \subseteq \aleph_1$

$$\{\delta \in S : X \cap \delta \in P_\delta\} \text{ is a stationary subset of } S.$$

It is well known that $\diamondsuit_S^-$ and $\diamondsuit_S$ are equivalent (see [Kun80, Ch. III]).

We fix a sufficiently large regular cardinal χ, indeed $\chi \geq (2^{\aleph_2})^+$ suffices. We fix a well-order $<_\chi$ on $H(\chi)$.

Definition 2.2: We assume that $S \subseteq \omega_1$ is stationary and $\diamondsuit_S$.

(1) (See [She98, IV, Def. 1.1]) An S**-oracle** is a sequence $\bar{M} = \langle M_\delta : \delta \in S\rangle$ such that:
   (a) M_δ is countable and transitive and $\delta + 1 \subseteq M_\delta$.
   (b) $i_\delta \colon (M_\delta, \in, (<_\chi)^{M_\delta}) \hookrightarrow_{\mathrm{elem}} (H(\chi), \in, <_\chi)$ is elementary.
   (c) $M_\delta \models \delta$ is countable.
   (d) For $\delta < \varepsilon \in S$, $M_\delta \subseteq M_\varepsilon$.
   (e) For any $A \subseteq \omega_1$ the set $\{\delta \in S : A \cap \delta \in M_\delta\}$ is stationary in ω_1.

(2) Let M be a countable elementary submodel of $H(\chi)$. A real $\eta \in \omega^\omega$ is called a **Cohen real over** M iff for any $D \in M$ that is dense in $\mathbb{C} = \{p : (\exists n)(p\colon n \to \omega)\}$ (ordered by end-extension) there is an n such that $\eta \restriction n \in D$. Equivalently, for any meagre set $F \subseteq \omega^\omega$ that is coded in M, e.g., by a sequence of nowhere dense trees, we have $\eta \notin F$.

(3) We say that $\langle \bar{M}, \bar{N}, \bar{\eta}\rangle$ is an S**-oracle triple** if
   (a) $\bar{M} = \langle M_\delta : \delta \in S\rangle$ is an S-oracle,
   (b) $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$,
   (c) for $\delta \in S$, η_δ is Cohen over M_δ,
   (d) $\bar{N} = \langle N_\delta : \delta \in S\rangle$,
   (e) $N_\delta = M_\delta[\eta_\delta]$.

(3) Let $\bar{M}$ be an S-oracle sequence. For $A \subseteq H(\omega_1)$, we let

$$I_{\bar{M}}(A) = \{\alpha \in S : A \cap \alpha \in M_\alpha\}$$

and

$$\mathscr{D}_{\bar{M}} = \{X \subseteq \omega_1 : (\exists A \subseteq \omega_1)(X \supseteq I_{\bar{M}}(A))\}.$$

From now on until the end of the section let $S \subseteq \omega_1$ be stationary and assume $\diamondsuit_S$. For L-structures $\mathcal{A}$, $\mathcal{M}$, we write $\mathcal{A} \prec \mathcal{M}$ if $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$. Since for L-structures $\mathcal{A}, \mathcal{B}, \mathcal{M}$ with $\mathcal{A}, \mathcal{B} \prec \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{B}$ also $\mathcal{A} \prec \mathcal{B}$ holds, we have that the structures on any oracle sequence are $\prec$-increasing.

If $f\colon A \to B$ is a function and $C \subseteq A$, then we write $f[C]$ for $\{f(c) : c \in C\}$. We recall the following important properties of $\mathscr{D}_{\bar{M}}$.

LEMMA 2.3 ([She98, IV, Claim 1.4]): *The set* $\{I_{\bar{M}}(A) : A \subseteq \omega_1\}$ *is closed under finite intersections. The filter* $\mathscr{D}_{\bar{M}}$ *contains every end segment of* ω_1*, is normal, and contains any club subset of* S*, and for every* $A \subseteq H(\aleph_1)$*,* $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$*.*

Proof. We prove only the very last statement; the others are proved in [She98, IV, Claim 1.4]. By $\diamondsuit_S$, $|H(\omega_1)| = \omega_1$. Let $f\colon H(\omega_1) \to \omega_1$ be the $<_\chi$-least bijection. Let

$$C = \{\delta \in \omega_1 : \delta \text{ limit and } (\forall \alpha < \delta)(f[M_\alpha] \subseteq \delta)\}.$$

The set $\mathrm{acc}(C)$ of accumulation points of C is club in ω_1. Now we consider $A \subseteq H(\omega_1)$. By definition, $I_{\bar{M}}(f[A]) \in \mathscr{D}_{\bar{M}}$. For any $\delta \in S \cap \mathrm{acc}(C)$ such that $f[A] \cap \delta \in M_\delta$ we have

$$\begin{aligned} M_\delta \ni (i_\delta^{-1}(f^{-1}))[(f[A \cap \delta])] &= \bigcup_{\alpha<\delta} (f^{-1} \restriction f[M_\alpha])[(f[A] \cap \alpha)] \\ &= \bigcup_{\alpha<\delta} A \cap \alpha = A \cap \delta. \end{aligned}$$

Thus we have $I_{\bar{M}}(A) \supseteq I_{\bar{M}}(f[A]) \cap \mathrm{acc}(C)$. By [Jec03, Lemma 14.4], for any club C' in ω_1, any normal filter over S contains the set $S \cap C'$. Since $\mathrm{acc}(C)$ is a club and $\mathscr{D}_{\bar{M}}$ is a normal filter, $\mathrm{acc}(C) \in \mathscr{D}_{\bar{M}}$ and thus $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. ∎

We recall when a notion of forcing $\mathbb{P}$ has the $\bar{M}$-c.c.

Definition 2.4 ([She98, Ch. IV, Def. 1.5]): Let $\bar{M}$ be an S-oracle sequence and let $\mathbb{P}$ be a notion of forcing. We define when $\mathbb{P}$ satisfies the $\bar{M}$-c.c. by cases:

(a) If $|\mathbb{P}| \leq \aleph_0$, always.

(b) If $|\mathbb{P}| = \aleph_1$ and if for every injective $\pi\colon \mathbb{P} \to \omega_1$ the set

$$\begin{aligned} \{\delta \in S :& (\forall A \in M_\delta \cap \mathcal{P}(\delta)) \\ & (((\pi^{-1})[A] \text{ is predense in } (\pi^{-1})[\delta]) \to ((\pi^{-1})[A] \text{ is predense in } \mathbb{P}))\} \end{aligned}$$

is an element of $\mathscr{D}_{\bar{M}}$.

(c) $\mathbb{P}'' \subseteq_{\mathrm{ic}} \mathbb{P}$ means that $\mathbb{P}''$ is an incompatibility preserving suborder of $\mathbb{P}$, i.e., for any $p, q \in \mathbb{P}''$, $p \leq_{\mathbb{P}''} q$ iff $p \leq_{\mathbb{P}} q$ and $p \perp_{\mathbb{P}''} q$ iff $p \perp_{\mathbb{P}} q$.

(d) If $|\mathbb{P}| > \aleph_1$,and for every $\mathbb{P}^\dagger \subseteq \mathbb{P}$ if $|\mathbb{P}^\dagger| \leq \aleph_1$, then there are $\mathbb{P}''$ such that $|\mathbb{P}''| = \aleph_1$ and $\mathbb{P}^\dagger \subseteq \mathbb{P}'' \subseteq_{\mathrm{ic}} \mathbb{P}$ and $\pi\colon \mathbb{P}'' \to \omega_1$ as in (b).

Oracle sequences are not continuous. The requirement $\delta \in M_\delta$ precludes continuity.

LEMMA 2.5: *Assume S is stationary and $\diamondsuit_S$.*

(1) *There is an oracle triple.*
(2) *Let $\langle \bar{M}, \bar{N}, \bar{\eta} \rangle$ be an oracle triple. Then*

$$I := \{\delta \in S : \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\} \in M_\delta\} \in \mathscr{D}_{\bar{M}}.$$

(3) *If $\langle \bar{M}, \bar{N}, \bar{\eta} \rangle$ is an S-oracle triple then $\langle N_\varepsilon : \varepsilon \in I \rangle$ is an I-oracle, with the exception that $(N_\varepsilon, \in)$ is not necessarily an elementary substructure of $H(\chi)$.*[1]

Proof. (1) Let $\langle P_\delta : \delta \in S \rangle$ be a $\diamondsuit_S^-$-sequence. Again we fix the $<_\chi$-least bijection $f\colon H(\omega_1) \to \omega_1$. We choose M_δ, i_δ by induction on δ. Suppose that M_γ, i_γ, $\gamma < \delta$, have been chosen. Let $M'_\delta \prec (H(\chi), \in, <_\chi)$ be a countable elementary substructure with $\langle M_\gamma, i_\gamma : \gamma < \delta \rangle, \delta, P_\delta \in M'_\delta$. Then $\delta + 1 \subseteq M'_\delta$. We let M_δ be the Mostowski collapse of M'_δ. The Mostowski collapse maps P_δ to itself. Moreover, since P_δ is countable, $P_\delta \subseteq M_\delta$, and hence $X \cap \delta \in P_\delta$ implies $X \cap \delta \in M_\delta$. By now, we have taken care of Definition 2.2.(2) (a). For being definite, we let the Cohen forcing $\mathbb{C}$ be the set of finite partial functions from ω to 2, ordered by extension. By the Rasiowa–Sikorski theorem (e.g., [Jec03, Lemma 14.4]) there is a Cohen-generic filter G_δ over M_δ. Then the function $\eta_\delta = \bigcup\{p : p \in G_\delta\} \in {}^\omega 2$ is a Cohen real over M_δ. We let $M_\delta[G_\delta] = N_\delta$.

(2) The set $A = \{(\varepsilon, \eta_\varepsilon) : \varepsilon \in S\} \subseteq H(\omega_1)$. We fix a club C such for $\delta \in C$,

$$f[\{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\}] \subseteq \delta.$$

By Lemma 2.3 we have $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. By normality $C \cap I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. By the choice of C,

$$C \cap I_{\bar{M}}(A) \subseteq \{\delta : \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\} \in M_\delta\}$$

and thus the latter is in $\mathscr{D}_{\bar{M}}$.

(3) Since $\mathscr{D}_{\bar{M}}$ is a normal filter, by [Jec03, Lemma 811], its elements are stationary sets. Hence I is stationary. For $\delta < \varepsilon$, $\delta \in S$, $\varepsilon \in I$, we have $N_\delta \subseteq M_\varepsilon \subseteq N_\varepsilon$. Hence $\langle N_\varepsilon : \varepsilon \in I \rangle$ is increasing. ∎

From now until the end of the section we fix an S-oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$. Note that for $\delta \in I$, $(\forall \alpha < \delta)(M_\alpha[\eta_\alpha] \in M_\delta)$.

[1] In Theorem 2.8 below we will rework the proof of the omitting types theorem for the particular types that shall be omitted and see that the requirement that $(N_\varepsilon, \in)$ fulfil sufficiently much of ZFC and be transitive suffices for our application.

Oracle triples allow for the application of the "Omitting Types Theorem":

LEMMA 2.6 (The Omitting Types Theorem, see [She98, Ch. IV, Lemma 2.1]): *Assume* $\diamondsuit_S$. *Suppose the* $\psi_i(x)$, $i < \omega_1$, *are* Π^1_2 *formulas on reals with a real parameter possibly. Suppose further that there is no solution to* $\bigwedge_{i<\omega_1} \psi_i(x)$ *in* $\mathbf{V}$ *and even if we add a Cohen real to* $\mathbf{V}$ *there will be none. Then there is an* S*-oracle* $\bar{M}'$ *such that for any forcing* $\mathbb{P}$,

$$\text{if } \mathbb{P} \text{ has the } \bar{M}'\text{-c.c, then in } \mathbf{V}^{\mathbb{P}} \text{ there is no solution to } \bigwedge_i \psi_i(x).$$

We let $\psi(x, \eta_i)$ say the following:

$$\begin{aligned} &x = (y, h) \wedge y \in {}^{\omega}2 \text{ and } h \in {}^{\omega}\omega \text{ is increasing and} \\ &(\forall^{\infty} n)(\eta_i \restriction [h(n), h(n+1)) \neq y \restriction [h(n), h(n+1))). \end{aligned} \tag{2.1}$$

By [BJ95, Theorem Ch. 2], any meagre subset of 2^{ω} has a superset of the form

$$M_{(h,y)} = \{z \in {}^{\omega}2 : (\forall^{\infty} n) z \restriction [h(n), h(n+1)) \neq y \restriction [h(n), h(n+1))\}$$

for some strictly increasing function h and some $y \in {}^{\omega}2$. The formula $\psi(x, \eta_i)$ says that η_i is in the meagre set $M_{(h,y)}$. So the type Ψ to be omitted is

$$\bigwedge_{i \in I} \psi(x, \eta_i). \tag{2.2}$$

Actually, we will have a strong form of omission: There is a set Y in a normal filter such that for each $i \in Y$, $x = (y, h) \in M_i[\mathbb{P}]$,

$$(\exists^{\infty} n) \eta_i \restriction [h(n), h(n+1)) = \eta_i \restriction [h(n), h(n+1)).$$

Since $\mathbb{P} \in M_0$ and $\mathbb{P} \subseteq \bigcup\{M_i : i < \omega\}$, thus $\{\eta_i : i \in Y\}$ is not meagre in $\mathbf{V}^{\mathbb{P}}$.

We check that premise of the omitting types theorem is fulfilled in a very local form.

LEMMA 2.7: *Let* M *be a countable transitive model that can be elementarily embedded into* $H(\chi)$, *and let* $\eta \in \mathbf{V}$ *be a Cohen real over* M. *Then there is no* $p \in \mathbb{C}$ *such that* p *forces in Cohen forcing over* $\mathbf{V}$ *that* η *is not Cohen over* $M[\mathbb{C}]$.

Proof. If $\eta \in \mathbf{V}$ is Cohen over M and c is Cohen over $\mathbf{V}$ then c is also Cohen over $M[\eta]$. So $M[\eta][c]$ is an iterated Cohen extension and (η, c) is M-generic for $\mathbb{C} * \mathbb{C}$. Since $\mathbb{C} \times \mathbb{C}$ densely embeds into $\mathbb{C} * \mathbb{C}$, the order of the two Cohen reals does not matter. So c is forced to be Cohen over $M[\eta]$. ∎

By Lemma 2.7, the omitting types theorem shows that there is an oracle $\bar{N}$ for the preservation of η_i' s Coheness over M_i. We review the proof of the omitting types theorem for the preservation of Coheness in order to show that $N_i = M[\eta_i]$ is a strong enough oracle.[2]

Theorem 2.8: *Let $\bar{M}$, $\bar{N}$, S, I be as in Definition 2.2 and Lemma 2.5(2). For each $\mathbb{P}^\dagger$ with the $\bar{N}$-c.c. there is a set $Y \in \mathscr{D}_{\bar{N}}$ such that for any $i \in Y$, η_i is Cohen over $M_i[\mathbb{P}^\dagger]$.*

Proof. We work with the type given in (2.2). We assume $\mathbb{P}^\dagger = \omega_1$. Then by the oracle-c.c.

$$Y' = \{\delta \in S : (\forall A \in N_\delta \cap \mathcal{P}(\delta))(((A \text{ is predense in } (\delta) \rightarrow ((A \text{ is predense in } \mathbb{P}))\}$$

is an element of $\mathscr{D}_{\bar{N}}$.

Let τ be a $\mathbb{P}^\dagger$-name for a real. Since $\mathbb{P}^\dagger = \omega_1$ has the c.c.c. we can assume that $\tau \in H(\omega_1)$. Let $p \in \mathbb{P}^\dagger$. Let Y be the set of $\delta \in Y'$ such that

(a) $\tau \in M_\delta$,
(b) $\tau = \tau^{(N_\delta, \delta)}$,
(c) $\mathbb{P}^\dagger \cap \delta \subseteq_{ic} \mathbb{P}^\dagger$.

Then $Y \in \mathscr{D}_{\bar{N}}$. Let G be $\mathbb{P}^\dagger$-generic over $\mathbf{V}$ and $\delta \in Y$. Then $G \cap \delta$ is $\mathbb{P}^\dagger \cap \delta$-generic over N_δ. Since $\mathbb{P}^\dagger \cap \delta$ is equivalent to Cohen forcing, by Lemma 2.7,

$$N_\delta[G \cap \delta] \models \neg\psi(\underset{\sim}{\tau}[G \cap \delta], \eta_\delta).$$

Since $\mathbb{P}^\dagger \cap \delta \subseteq_{ic} \mathbb{P}^\dagger$, we have $\underset{\sim}{\tau}[G \cap \delta] = \underset{\sim}{\tau}[G]$. By absoluteness,

$$N_\delta[G] \models \neg\psi(\underset{\sim}{\tau}[G], \eta_\delta). \qquad \blacksquare$$

For building up a name for an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$ we introduce some notions for handling names.

Definition 2.9: Let $\mathbb{P}$ be a c.c.c. forcing of size at most $\aleph_1$.

(1) A **canonical $\mathbb{P}$-name** for a subset of ω is a name of the form

$$\tau = \{\langle \check{n}, p \rangle : p \in A_n \rangle\},$$

where the $A_n \subseteq \mathbb{P}$ are countable maximal antichains.

[2] The sequence of the N_i is not an oracle literally, since its entries are not necessarily elementary subsets of $H(\theta)$. However, they are transitive models of a sufficiently large fragment of ZFC. Theorem 2.8 shows that this is sufficient for our specific types. Henceforth we will also call $\bar{N}$ an oracle sequence.

(2) A **canonical** $\mathbb{P}$**-name** for a subset of $\mathcal{P}(\omega)$ is a name of the form

$$\underset{\sim}{K} = \{\langle \tau, q\rangle : q \in A_\tau, \tau \in X\},$$

where X is a set of canonical $\mathbb{P}$-names τ for subsets of ω, for maps π as in (3), and for each $\tau \in X$, the set A_τ is a countable antichain in $\mathbb{P}$.

(3) Let $\pi\colon \mathbb{P} \to \omega_1$ be injective. We let $\pi[\mathbb{P}] = \mathbb{P}'$ and define a partial order (or a quasi order) on $\mathbb{P}'$ such that π is an isomorphism from $(\mathbb{P}, <_{\mathbb{P}})$ to $(\mathbb{P}', <_{\mathbb{P}'})$. Then we lift π to a map $\bar{\pi}\colon \mathbf{V}^{\mathbb{P}} \to \mathbf{V}^{\mathbb{P}'}$-names by letting

$$\bar{\pi}(\tau) = \{\langle \bar{\pi}(\sigma), \pi(p)\rangle : \langle \sigma, p\rangle \in \tau\}.$$

For canonical names τ, $\underset{\sim}{K}$ as above, $\bar{\pi}(\tau) \in H(\omega_1)$, $\bar{\pi}(\underset{\sim}{K}) \subseteq H(\omega_1)$. Thus according to Lemma 2.3, $I_{\bar{M}}(\bar{\pi}(\underset{\sim}{K})) \in \mathscr{D}_{\bar{M}}$. The names $\bar{\pi}(\underset{\sim}{K})$ and $\bar{\pi}(\tau)$ are canonical.

Definition 2.10: Let $\bar{M}$ be an S-oracle sequence and $\mathbb{P}' \subseteq \omega_1$.

(1) We let τ be a canonical $\mathbb{P}'$-name of a subset of ω. We let for $\delta \in \omega_1$,

$$\tau^{(M_\delta,\delta)} = \begin{cases} \tau; & \text{if } \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name, and } \tau \in M_\delta, \\ \text{undefined;} & \text{otherwise.} \end{cases}$$

(2) For a canonical $\mathbb{P}'$-name $\underset{\sim}{K} = \{(\tau, q) : q \in A_\tau, \tau \in X\}$ for a subset of $\mathcal{P}(\omega)$ and $\delta < \omega_1$ we define the M_δ-part as follows:

$$\underset{\sim}{K}^{(M_\delta,\delta)} = \{(\tau, q) : (\tau, q) \in \underset{\sim}{K}, q \in \mathbb{P}' \cap \delta, \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name}, \tau \in M_\delta, A_\tau \subseteq \mathbb{P}' \cap \delta, A_\tau \in M_\delta\}.$$

Note that for a canonical $\mathbb{P}'$-name we have $\underset{\sim}{K}^{(M_\delta,\delta)} \subseteq M_\delta$, however, in general $\underset{\sim}{K}^{(M_\delta,\delta)}$ is not an element of M_δ. By Lemma 2.3 we have though

$$\{\delta \in S : \langle (\varepsilon, \underset{\sim}{K}^{(M_\varepsilon,\varepsilon)}) : \varepsilon < \delta\rangle \in M_\delta\} \in \mathscr{D}_{\bar{M}}.$$

Now we are ready to define the set K^1 of pairs that serve as conditions in the first iterand of our final two-step forcing. The order on K^1 will be defined in Definition 2.18.

Definition 2.11:

(1) For an S-oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$ as above we let K^1 be the set of all $(\mathbb{P}, \underset{\sim}{D})$ with the following properties:
   (a) $\mathbb{P}$ is a c.c.c. forcing with a nonstationary domain $\mathbb{P} \subseteq \omega_1$.
   (b) $\underset{\sim}{D}$ is a canonical $\mathbb{P}$-name of a non-principal ultrafilter over ω.

(c) $Y(\mathbb{P},\underset{\sim}{D}) \in \mathscr{D}_{\bar{N}}$, where $Y(\mathbb{P},\underset{\sim}{D})$ is the set of $\delta \in S$ such that items (α) to (ε) hold:
 - (α) $\mathbb{P} \cap \delta \in M_\delta$.
 - (β) If $E \subseteq \mathbb{P} \cap \delta$ and $E \in N_\delta$ and E is predense in $\mathbb{P} \cap \delta$ then E is predense in $\mathbb{P}$ (so we have that $\mathbb{P}$ has the $\bar{N}$-oracle-c.c.).
 - (γ) $\underset{\sim}{D}^{(M_\delta,\delta)} \in M_\delta$ and $M_\delta \models$ "$\underset{\sim}{D}^{(M_\delta,\delta)}$ is a canonical $\mathbb{P} \cap \delta$-name of an ultrafilter over ω".
 - (δ) $N_\delta \models (\mathbb{P} \cap \delta \Vdash$ "η_δ is Cohen-generic over $M_\delta[\underset{\sim}{\mathbf{G}}_{\mathbb{P}\cap\delta}]$").
 - (ε) $\underset{\sim}{D}^{(N_\delta,\delta)} \in N_\delta$ is a canonical $\mathbb{P}\cap\delta$-name of an ultrafilter over ω such that

$$\mathbb{P} \cap \delta \Vdash (\forall f \in M_\delta[\underset{\sim}{\mathbf{G}}_{\mathbb{P}\cap\delta}] \cap {}^\omega\omega)(f \leq_{\underset{\sim}{D}^{(N_\delta,\delta)}} \eta_\delta).$$

(2) For an oracle triple $(\bar{M},\bar{N},\bar{\eta})$ we let K^2 be the set of $(\mathbb{P},\underset{\sim}{D}) \in H(\aleph_2)$ such that there are a non-stationary $\mathbb{P}' \subseteq \omega_1$ and a bijective $\pi\colon \mathbb{P}' \to \mathbb{P}$ and $(\mathbb{P}',\underset{\sim}{D}') \in K^1$, π is an isomorphism from $\mathbb{P}'$ onto $\mathbb{P}$ with $\bar{\pi}(\underset{\sim}{D}') = \underset{\sim}{D}$.

Remark 2.12: Since we do not add new types that have to be omitted in the course of the iteration, one fixed oracle $\bar{N} \in \mathbf{V}$ is sufficient.

We recall the successor step and the direct limit step for oracle-c.c.

Lemma 2.13 (Lemma [She98, IV 3.2]): *If $\mathbb{P}$ has the $\bar{M}$-c.c. and $\mathbb{P}$ forces that $\mathbb{Q}$ has the $\langle M_\delta[\mathbb{P}] : \delta \in S\rangle$-c.c., then $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ has the $\bar{M}$-c.c.*

Lemma 2.14 (Lemma [She98, IV 3.10]): *If $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \beta\rangle$ is a finite support iteration such that has the $\bar{M}$-c.c. and for $\alpha < \beta$ the forcing $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha$ has the $\langle M_\delta[\mathbb{P}_\alpha] : \delta \in S\rangle$-c.c., then $\mathbb{P}_\beta$ has the $\bar{M}$-c.c.*

If $\pi\colon \mathbb{P}' \to \mathbb{P}$ is an isomorphism between forcing orders, we use it also for its natural extension that maps $\mathbb{P}$-names to $\mathbb{P}'$-names.

Lemma 2.15: *Let $(\bar{M},\bar{N},\bar{\eta})$ be an S-oracle triple and let K^1 be as in Definition 2.11(1). Assume*

(1) $(\mathbb{P},\underset{\sim}{D}) \in H(\aleph_2)$, *$\mathbb{P}$ is a forcing notion, $\mathbb{P} \in H(\omega_2)$ and $\underset{\sim}{D} \in H(\omega_2)$ is a canonical $\mathbb{P}$-name of an ultrafilter over ω.*

(2) *$\mathbb{P}'_\ell$ is a notion of forcing whose domain is a non-stationary subset of ω_1 for $\ell = 1,2$.*

(3) *π_ℓ is an isomorphism from $\mathbb{P}'_\ell$ onto $\mathbb{P}$ for $\ell = 1,2$,*

(4) *$\underset{\sim}{D}'_\ell$ is a $\mathbb{P}'_\ell$-name of a subset of $\mathcal{P}(\omega)$ such that π_ℓ maps $\underset{\sim}{D}'_\ell$ onto $\underset{\sim}{D}$.*

Then $(\mathbb{P}'_1,\underset{\sim}{D}'_1) \in K^1$ iff $(\mathbb{P}'_2,\underset{\sim}{D}'_2) \in K^1$.

Proof. The map $\pi = \pi_2^{-1} \circ \pi_1$ is an isomorphism from $\mathbb{P}'_1$ onto $\mathbb{P}'_2$, and its lifting $\bar{\pi}$ maps $\underset{\sim}{D}'_1$ to $\underset{\sim}{D}'_2$. According to Lemma 2.3,

$$Z = \{\delta \in S : \pi \restriction \delta \text{ is a one-to-one mapping from } \mathbb{P}'_1 \cap \delta \text{ to } \mathbb{P}'_2 \cap \delta \text{ and } \pi \restriction \delta \in M_\delta\}$$

belongs to $\mathscr{D}_{\bar{M}}$. If $\delta \in Z$ then $\delta \in Y(\mathbb{P}'_1, \underset{\sim}{D}'_1)$ iff $\delta \in Y(\mathbb{P}'_2, \underset{\sim}{D}'_2)$, since the defining properties of the sets $Y(\mathbb{P}'_\ell, \underset{\sim}{D}'_\ell)$ are preserved by isomorphisms of forcing orders. ∎

This shows that in Definition 2.11(2) the following is true: If the demand holds for some pair $(\mathbb{P}', \pi)$ then it holds for every such pair. The primed partial orders in Lemma 2.15 shall ensure that the domain is a non-stationary subset of ω_1. Canonical $\mathbb{P}'$-names for reals and for filters over ω are actual subsets of $H(\omega_1)$. According to Lemma 2.15, their properties are invariant under bijections of ω_1. Since any property of the forcing is named modulo $\mathscr{D}_{\bar{N}}$ the particular choice of the injections does not matter. For the actual construction of forcing posets it is convenient to use non-stationary domains for the $\mathbb{P}' \in K^1$, since non-stationarity is preserved by countable unions and by diagonal unions.

The property in Definition 2.11(1)(c)(ε) ensures that $\underset{\sim}{D}$ will be forced to be an ultrafilter such that the weakest condition in the two-step forcing forces $\mathrm{cf}(\omega^\omega/\underset{\sim}{D}) = \aleph_1$, as witnessed by $\langle \eta_\delta : \delta \in S \rangle$. Technically it is more convenient to prove property (δ) by induction and then derive property (ε) from property (δ), though property (ε) is more directly related to $\mathrm{cf}(\omega^\omega/\underset{\sim}{D}) = \aleph_1$. In the case of an $\leq^*$-increasing sequence $\langle \eta_\delta : \delta < S \rangle$ unboundedness is preserved in limits of finite support iterations if each initial segment preserves it [BJ95, Ch. 6, §4]. So it might be possible to prove by induction property (ε) and the negation of (δ). We have not investigated this issue.

Concerning the preservation of (δ), we will frequently use [BJ95, Chapter 6 Section 4]:

Lemma 2.16: *Let $\mathbb{P}_n \lessdot \mathbb{P}_{n+1}$ for $n \in \omega$ and let $\mathbb{P}$ be the direct limit of $\langle \mathbb{P}_n : n \in \omega \rangle$. If $\mathbb{P}_n \Vdash$ "η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}_n}]$" for all n, then $\mathbb{P} \Vdash$ "η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}}]$."*

Let $\mathrm{unif}(\mathcal{M})$ denote the smallest cardinality of a non-meagre set. The following proposition gives the additional information that $\mathrm{unif}(\mathcal{M}) = \aleph_1$ in our forcing extensions, as witnessed by $\{\eta_\delta : \delta \in S\}$.

PROPOSITION 2.17: *If* $(\mathbb{P}, \underset{\sim}{D}) \in K^2$ *then* $\mathbb{P}$ *forces that* $\{\eta_\delta : \delta \in S\}$ *is a non-meagre subset of* ${}^\omega 2$.

Proof. Let $p \in \mathbb{P}$ force that $\{\eta_\delta : \delta \in S\}$ is meagre. Let τ be a name for a meagre F_σ-set. By the c.c.c., there is a $\delta \in Y(\mathbb{P}, \underset{\sim}{D})$ such that $\tau, p \in M_\delta$, $p \in \mathbb{P} \cap \delta$, τ is a $\mathbb{P} \cap \delta$-name, and $p \Vdash \{\eta_\varepsilon : \varepsilon \in S\} \subseteq \tau$. Then $p \Vdash_{\mathbb{P}} \eta_\delta \in \tau$. Since $\delta \in Y(\mathbb{P}, \underset{\sim}{D})$, clause (β) in the definition of $Y(\mathbb{P}, \underset{\sim}{D})$ yields also $p \Vdash_{\mathbb{P}\cap\delta} \eta_\delta \in \tau$. This is a contradiction to Definition 2.11(1)(c)(δ) of the definition of $Y(\mathbb{P}, \underset{\sim}{D})$. ∎

Proposition 2.17 has a sort of an inverse direction for the class of Suslin forcings. A forcing $\mathbb{Q} \subseteq \omega^\omega$ is called Suslin if $\mathbb{Q}$ is an analytic subset of ω^ω and the relations $\leq_{\mathbb{Q}}$ and $\perp_{\mathbb{Q}}$ are analytic sets in $\omega^\omega \times \omega^\omega$. For Suslin proper forcings, not making the ground model meagre is equivalent to preserving the genericity of a Cohen real over any countable model [Gol93, 6.21, 6.22], and then all non-meagre sets in the ground model stay non-meagre.

Now we introduce the approximation forcing $(AP, <_{AP})$:

Definition 2.18: We let K^2 be as above.

(A) Let $\mathbf{p} = (\mathbb{P}_{\mathbf{p}}, \underset{\sim}{D}_{\mathbf{p}}), \mathbf{q} = (\mathbb{P}_{\mathbf{q}}, \underset{\sim}{D}_{\mathbf{q}}) \in K^2$. We define $\mathbf{p} \leq_{AP} \mathbf{q}$, that is, $\mathbf{q}$ is stronger than $\mathbf{p}$, if
  (a) $\mathbb{P}_{\mathbf{p}} \lessdot \mathbb{P}_{\mathbf{q}}$,
  (b) $\Vdash_{\mathbb{P}_{\mathbf{q}}} \underset{\sim}{D}_{\mathbf{p}} \subseteq \underset{\sim}{D}_{\mathbf{q}}$.

(B) For $i = 1, 2$, we let forcing order of approximations be $AP^i = (K^i, \leq_{AP})$. We let $AP = AP^2$.

The following is the parallel of the basic claim on oracle c.c. forcing, [She98, Ch. IV, Claim 3.2]. The forcing $\mathbb{P}_i$ does not mean iteration up to stage i. The variable i, ranging over $\omega + 1$ or $\omega_1 + 1$ or ω_2, is just an index for $\mathbb{P}_i$ being a component of $(\mathbb{P}_i, \underset{\sim}{D}_i) \in K^2$. $\mathbb{P}_i$ is an $\bar{N}$-oracle c.c. forcing and $|\mathbb{P}_i| \leq \aleph_1$.

LEMMA 2.19:

(A) *The structure* $(K^2, \leq_{AP})$ *is a partial order of cardinality* $|H(\aleph_2)|$.

(B) $K^2 \neq \emptyset$.

(C) *If* $\mathbf{p}_n = (\mathbb{P}_n, \underset{\sim}{D}_n) \in K^2$ *for* $n \in \omega$ *and* $\mathbf{p}_n \leq_{AP} \mathbf{p}_{n+1}$, *then the set has an upper bound* $\mathbf{p}_\omega = (\mathbb{P}_\omega, \underset{\sim}{D}_\omega)$ *with* $\mathbb{P}_\omega = \bigcup\{\mathbb{P}_n : n \in \omega\}$.

(D) $(K^2, \leq_{AP})$ *is* $(\omega_1 + 1)$*-strategically closed, that is, for every* $\mathbf{p} \in AP$ *the protagonist has a winning strategy in the following game* $\eth(\mathbf{p})$*: A play lasts* $\omega_1 + 1$ *moves. During the play the player COM, the protagonist,* *chooses for each* $i \leq \omega_1$, $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i) \in K^2$, *and INC, the antagonist, chooses* $\mathbf{q}_i \in K^2$ *such that*

(a) $\mathbf{p}_i \leq_{AP} \mathbf{q}_i$,

(b) $(\forall j < i)(\mathbf{q}_j \leq_{AP} \mathbf{p}_i)$,

(c) $\mathbf{p}_0 = \mathbf{p}$.

The protagonist COM wins the game if they can always move. The hard case is the choice of $\mathbf{p}_{\omega_1}$.

Proof. (A) and (B) are obvious.

(C) Let $\mathbf{p}_n = (\mathbb{P}_n, \underset{\sim}{D}_n)$ and let $\langle \mathbf{p}_n : n \in \omega \rangle$ be $\leq_{AP}$-increasing. We choose $(\mathbb{P}_n, \pi_n, \mathbb{P}'_n, \underset{\sim}{D}'_n)$ by induction on n with the following properties:

(1) $\mathbb{P}'_n \subseteq \omega_1$ is not stationary,

(2) $\pi_n \colon \mathbb{P}'_n \to \mathbb{P}_n$ is an isomorphism of partial orders,

(3) $(\bar{\pi})^{-1}(\underset{\sim}{D}_n) = \underset{\sim}{D}'_n$,

(4) $\pi_n \subseteq \pi_{n+1}$,

(5) $(\mathbb{P}'_n, \underset{\sim}{D}'_n) \in K^1$.

Then we let $\mathbb{P}'_\omega = \bigcup_{n \in \omega} \mathbb{P}'_n$, and the latter is not stationary. Moreover, we let $\pi_\omega = \bigcup_{n \in \omega} \pi_n$.

We fix for $n \in \omega$ a reduction $r_{\mathbb{P}'_\omega, \mathbb{P}'_n} : \mathbb{P}'_\omega \to \mathbb{P}'_n$ and we set

$$C = \{\delta \in S : \delta \text{ limit of } S \text{ and } (\forall n) r_{\mathbb{P}'_\omega, \mathbb{P}'_n}[\mathbb{P}'_\omega \cap \delta] \subseteq \delta\}.$$

Of course C is club in ω_1. We let

$$Y = \bigcap_{k \in \omega} Y(\mathbb{P}'_k, \underset{\sim}{D}'_k) \cap C. \tag{2.3}$$

By [She98, Ch. IV, Claim 3.2], the poset $\mathbb{P}'_\omega$ has the $\bar{N}$-oracle c.c., i.e., $\mathbb{P}'_\omega$ satisfies clause (c)(β) of Definition 2.11. By Lemma 2.16 the set Y is also a witness to clause (c)(δ) for $\mathbb{P}'_\omega \in K^1$.

We show that there is $\underset{\sim}{D}'_\omega$ such that $(\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega)$ is an upper bound of $\langle \mathbf{p}'_n : n < \omega \rangle$ in $\leq_{AP}$. To this end we define an $\mathbb{P}'_\omega$-name $\underset{\sim}{D}'_\omega$ for an ultrafilter such that $\mathbf{p}_\omega = (\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega) \in K^1$ and $Y \subseteq Y(\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega)$. We let

$$\mathbb{P}'_\omega \Vdash \underset{\sim}{E}' = \bigcup_{k \in \omega} \underset{\sim}{D}'_k.$$

Since $\mathbb{P}'_k$ is a complete suborder of $\mathbb{P}'_\omega$ the $\underset{\sim}{D}'_k$ are names for filters and $0_{\mathbb{P}'_{k+1}} \Vdash \underset{\sim}{D}'_k \subseteq \underset{\sim}{D}'_{k+1}$ the weakest element of $\mathbb{P}'_\omega$ forces that $\underset{\sim}{E}'$ is a $\mathbb{P}'_\omega$-name for a filter.

We write $\mathrm{next}(Y,\varepsilon)$ for the next element in Y after ε, i.e.,

$$\mathrm{next}(Y,\varepsilon) = \min\{\delta > \varepsilon : \delta \in Y\}.$$

By induction on $\delta \in Y$, we will define a canonical $\mathbb{P}'_\omega \cap \delta$-name $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ such that

$$\mathbb{P}'_\omega \cap \delta \Vdash \text{“}\underset{\sim}{D}'_\omega(\delta) \supseteq \bigcup\{\underset{\sim}{D}'_\omega(\gamma) : \gamma \in Y \cap \delta\}$$
$$\text{and } \underset{\sim}{D}'_\omega(\delta) \text{ is an ultrafilter in } M_\delta,\text{”}$$

and

$$\mathbb{P}'_\omega \cap \mathrm{next}(Y,\delta) \Vdash \text{“}(\forall f \in M_\delta[\mathbb{P}'_\omega])(\eta_\delta \geq_{\underset{\sim}{D}'_\omega(\mathrm{next}(Y,\delta))} f)$$
$$\text{and } \underset{\sim}{D}'_\omega(\mathrm{next}(Y,\delta)) \cap \mathcal{P}(\omega)^{N_\varepsilon} \text{ is an ultrafilter in } N_\varepsilon.\text{”}$$

The restriction of names, mapping each name X to a name $X^{(M_\delta,\delta)}$, was defined in Definition 2.10(2). We will often write X^{M_δ} instead of $X^{(M_\delta,\delta)}$. For $k \in \omega$ we let

$$Y_k = \{\delta \in Y : \underset{\sim}{D}'_k(\delta) = {\underset{\sim}{D}'_k}^{M_\delta}\}.$$

Then $Y_k \in \mathscr{D}_{\bar{N}}$ and thus also their intersection $Y' = \bigcap_{k\in\omega} Y_k$ is in $\mathscr{D}_{\bar{N}}$. For simplicity, we write just Y for Y'.

Assume that $\langle \underset{\sim}{D}'_\omega(\gamma) : \gamma \in Y \cap \delta\rangle$ has been defined. By the induction hypothesis on $(\mathbf{p}'_k, \pi_k)$, the $\mathbb{P}'_k$-names for ultrafilters $\underset{\sim}{D}'_k$ are defined and increasing in k.

We first consider the limit steps in the induction. Let $\delta \in Y$ be a limit of Y. First case: $\langle \underset{\sim}{D}'_\omega(\gamma) : \gamma < Y \cap \delta\rangle \notin M_\delta$. Then we let

$$1_{\mathbb{P}\cap\delta} \Vdash \underset{\sim}{D}'_\omega(\delta) = \bigcup\{\underset{\sim}{D}'_\omega(\gamma) : \gamma \in Y \cap \delta\}.$$

Second case: $\langle \underset{\sim}{D}'_\omega(\gamma) : \gamma \in Y \cap \delta\rangle \in M_\delta$. We first show

$$1 \Vdash_{\mathbb{P}'_\omega\cap\delta} \underset{\sim}{E}'(\delta) := {\underset{\sim}{E}'}^{M_\delta} \cup \bigcup\{\underset{\sim}{D}'_\omega(\gamma) : \gamma \in Y \cap \delta\} \text{ is a filter base.”}$$

We assume, for a contradiction, that there are a condition $p \in \mathbb{P}'_\omega$, $k \in \omega$, and a $\gamma \in Y \cap \delta$ and there are names X, X', such that p forces that $X \in {\underset{\sim}{D}'_k}^{M_\delta}$ and $X' \in {\underset{\sim}{E}'}^{M_\delta}$, $\gamma \in Y \cap \delta$ such that $X \cap X'$ is empty. Then $p \restriction \mathbb{P}'_k \Vdash X \in \underset{\sim}{\tilde{D}}'_k \restriction \delta$. Let $\mathbf{G}_k$ be $\mathbb{P}'_k$-generic over N_δ with $p \restriction \mathbb{P}'_k \in \mathbf{G}_k$. We let

$$Z[\mathbf{G}_k] = \{n : (\exists \tilde{q} \in \mathbb{P}'_\omega \cap \delta/\mathbf{G}_k)(\tilde{q} \geq p[\mathbf{G}_k] \wedge \tilde{q} \Vdash n \in X'[\mathbf{G}_k] \cap X)\}.$$

Since $\mathbf{p}_k$ is a condition the name $\underset{\sim}{D}'_\omega(\gamma)\restriction\delta$ is an ultrafilter compatible with $\underset{\sim}{D}'_k(\gamma)$. Therefore we have that $p \restriction \mathbb{P}'_k \Vdash_{\mathbb{P}'_k}$ “$Z[\mathbf{G}_k]$ is infinite.” Now we take $n \in \omega$, $\tilde{q}$ as in the definition of $Z[\mathbf{G}_k]$, so that $\tilde{q} \Vdash n \in X \cap X'$. So we have a contradiction. Hence for any $\gamma \in Y \cap \delta$, the weakest condition forces that $\underset{\sim}{E}' \restriction \delta \cup \underset{\sim}{D}'_\omega(\gamma)$ is a filter basis. Since the names $\underset{\sim}{D}'_\omega(\gamma)$ are forced to be increasing with $\gamma \in Y \cap \delta$, also their union, $\underset{\sim}{F}'(\delta)$, is forced to be a filter basis. Now we choose a name $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ for an ultrafilter that extends $\underset{\sim}{F}'(\delta)$.

Now we consider the beginning and the successor steps of the induction. For the beginning, let $\gamma = -1$, $\underset{\sim}{D}'_\omega(-1) = \underset{\sim}{E}'$ and let $\delta = \min(Y)$, and for the successor let δ be the successor of $\gamma \in Y$, i.e., $\delta = \mathrm{next}(Y, \gamma)$. Then $N_\gamma \in M_\delta$. We extend $\underset{\sim}{D}'_\omega(\gamma)$ to $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ so that $\underset{\sim}{D}'_\omega(\delta)$ is a $\mathbb{P}' \cap \delta$-name for an ultrafilter such that

$$1_{\mathbb{P} \cap \delta} \Vdash \underset{\sim}{D}'_\omega(\delta) \supseteq \underset{\sim}{F}(\delta) := (\underset{\sim}{E}' \restriction \delta) \cup \underset{\sim}{D}'_\omega(\gamma)$$
$$\cup \{\{n \in \omega : \eta_\gamma(n) \geq \underset{\sim}{f}(n)\} : \underset{\sim}{f} \in M_\gamma \text{ a } \mathbb{P}'_\omega \cap \delta\text{-name for a function}\}.$$

Since $\gamma \in Y$, we can restrict the considerations to $\mathbb{P}'_\omega \cap \gamma$ names $\underset{\sim}{f}$. Again we show that the weakest condition forces that $\underset{\sim}{F}(\delta)$ has the finite intersection property. Let $q_0 \in \mathbb{P}'_\omega \cap \delta$ be given. Let q_0 force that $\underset{\sim}{A}_1$ be a name of a member of $\underset{\sim}{D}'_k \restriction \delta$ and $q_0 \Vdash \underset{\sim}{A}_2 \in \underset{\sim}{D}'_\omega(\delta)$ and $A_3 = \{n : \eta_\gamma(n) > \underset{\sim}{f}(n)\}$. Now in M_δ we define a $(\mathbb{P}'_k \cap \delta)$-name $\underset{\sim}{A}_{23}$ as follows: if $\mathbf{G}_k \subseteq \mathbb{P}'_{\mathbf{p}_k}$, $q_0 \restriction \mathbb{P}'_k \in G_k$ is $\mathbb{P}'_k$-generic over M_δ we let

$$\underset{\sim}{A}_{23}[\mathbf{G}_k] = \{n : (\exists \hat{q} \in (\mathbb{P}'_\omega \cap \delta)/\mathbf{G}_k)$$
$$(\hat{q} \geq q_0[\mathbf{G}_k] \wedge \hat{q} \Vdash (n \in \underset{\sim}{A}_2[\mathbf{G}_k] \wedge \eta_\gamma(n) \geq \underset{\sim}{f}[\mathbf{G}_{\mathbf{p}_k}](n)))\}.$$

Then $q_0 \restriction \mathbb{P}'_k \Vdash_{\mathbb{P}'_k} \underset{\sim}{A}_1 \cap \underset{\sim}{A}_{23}[\mathbf{G}_k]$ is infinite, since $\mathbb{P}'_k$ is already an approximation and η_γ is Cohen generic also over $M_\gamma[\mathbb{P}'_k]$, and hence $M_\gamma[\mathbb{P}'_k] \models \eta_\gamma \not\leq_{\underset{\sim}{D}'_k} f$. We take $\hat{q} \in (\mathbb{P}'_\omega \cap \delta)/\mathbf{G}_k$ and n as in the definition of $\underset{\sim}{A}_{23}[\mathbf{G}_k]$. Since $q_0 \restriction \mathbb{P}'_k$ is $\mathbb{P}'_k$-generic over M_δ, we may assume that $\hat{q} \restriction \mathbb{P}'_k \geq q_0$ and $\hat{q} \Vdash$ "$n \in \underset{\sim}{A}_1 \cap \underset{\sim}{A}_{23}$." Hence in M_δ there is a name for an ultrafilter $\underset{\sim}{D}'_\omega(\delta)$ containing $\underset{\sim}{F}(\delta)$, and we choose and fix the $<_\chi$-least one and call it $\underset{\sim}{D}'_\omega(\delta)$. Since $N_\gamma \subseteq M_\delta$ and $N_\gamma \in M_\delta$, $\underset{\sim}{D}'_\omega(\delta) \cap \mathcal{P}(\omega)^{N_\gamma}$ is an ultrafilter in N_γ.

Now the induction on $\delta \in Y$ is carried out. We choose a name $\underset{\sim}{D}'_\omega$ such that

$$\mathbb{P}'_\omega \Vdash \underset{\sim}{D}'_\omega = \bigcup \{\underset{\sim}{D}'_\omega(\delta) : \delta \in Y\}.$$

We mirror the construction back to the class K^2: by letting $\underset{\sim}{D}_\omega = \bar{\pi}(\underset{\sim}{D}'_\omega)$.

(D) Let $\mathbf{p} \in K^2$ be given. We write $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i)$, $i < \omega_1$. The strategy of the protagonist is to choose in addition to $\mathbf{p}_i \geq_{AP} \mathbf{q}_j$ for $j < i$, on the side also $\mathbf{p}'_i = (\mathbb{P}'_i, \underset{\sim}{D}'_i) \in K^1$ and $\pi_i \colon \mathbb{P}'_i \to \mathbb{P}_i$ and $\xi_i \in \omega_1$ with the following properties:

(a) $\langle \xi_i : i < \omega_1 \rangle$ is continuously increasing.
(b) $(\mathbb{P}'_i, \underset{\sim}{D}'_i) \in K^1$, $\mathbb{P}'_i \setminus \bigcup\{\mathbb{P}'_j : j < i\} \subseteq [\xi_i + 1, \omega_1)$.
(c) π_i is a isomorphism from $\mathbb{P}'_i$ onto $\mathbb{P}_i$ mapping $\underset{\sim}{D}'_i$ onto $\underset{\sim}{D}_i$.
(d) For $j < i$, $\pi_j \subseteq \pi_i$ (so the $\mathbb{P}'_i$ are $\subseteq$-increasing in ω_1).
(e) For $j < i$, $(\mathbb{P}'_j, \underset{\sim}{D}'_j) \leq_{AP^1} (\mathbb{P}'_i, \underset{\sim}{D}'_i)$ and $(\mathbb{P}_j, \underset{\sim}{D}_j) \leq_{AP} (\mathbb{P}_i, \underset{\sim}{D}_i)$.
(f) If $k < j \leq i$, $p \in \mathbb{P}'_k$ and $q \in \mathbb{P}'_j \cap \xi_i$ and p and q are compatible in $\mathbb{P}'_i$, then they are compatible with a witness in $\mathbb{P}'_i \cap \xi_i$. (Then the proof of [She98, Claim 3.2] for showing that also $\mathbb{P}_i$ has the $\bar{N}$-c.c. works.)
(g) If $i = j + 1 < \omega_1$ is a successor ordinal, then COM chooses $\mathbf{p}_i = \mathbf{q}_j$.
(h) If $i < \omega_1$ is a limit ordinal and if there is $j(*) < i$ such that

$$H = \bigcap\{Y(\mathbb{P}'_j, \underset{\sim}{D}'_j) : j \in [j(*), i)\} \in \mathscr{D}_{\bar{N}},$$

then player COM takes for $\mathbf{p}_i$ the limit of a countable cofinal sequence of $\mathbf{q}_j$'s in the manner described in (C). Thus

$$H \subseteq Y(\mathbb{P}'_i, \underset{\sim}{D}'_i). \tag{2.4}$$

If there is no such $j(*)$, then COM can play just any lower bound of the countable sequence $\mathbf{q}_j$, $j < i$. For a set of $i \in \mathscr{D}_{\bar{N}}$ there is such a $j(*) < i$ with Equation (2.4).

Now if $\mathbf{p}'_i$, $i < \omega_1$, are defined, in the ω_1-limit COM chooses $\mathbb{P}'_{\omega_1}$ as the direct limit. Then Equation (2.4) implies that

$$Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \supseteq \triangle_{i \in \omega_1} Y(\mathbb{P}'_i, \underset{\sim}{D}'_i) \cap \{i : \xi_i = i\},$$

and hence $Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \in \mathscr{D}_{\bar{N}}$. Hence

$$1_{\mathbb{P}'} \Vdash \underset{\sim}{D}'_{\omega_1} = \bigcup_{i<\omega_1} \underset{\sim}{D}'_i \text{ is an ultrafilter extending } \underset{\sim}{D}'_i, i < \omega_1.$$

We mirror the primed objects via $\bigcup_{j<\omega_1} \pi_j$ back to K^2 and thus we get a forcing $\mathbb{P}_{\omega_1} = \bigcup\{\mathbb{P}_i : i < \omega\}$ and a $\mathbb{P}_{\omega_1}$-name $\underset{\sim}{D}_{\omega_1}$ for an ultrafilter over ω. The protagonist COM hence has won the play of the completeness game. ∎

Definition 2.20: Let $\mathbf{G}_{AP}$ be an AP-generic filter. In $\mathbf{V}[\mathbf{G}_{AP}]$ we let

$$\mathbb{Q} = \bigcup\{\mathbb{P} : (\exists \underset{\sim}{D})\ (\mathbb{P}, \underset{\sim}{D}) \in \underset{\sim}{\mathbf{G}}_{AP}\}$$

and let $\underset{\sim}{E}$ be a $\mathbb{Q}$-name such that

$$\mathbb{Q} \Vdash \underset{\sim}{E} = \bigcup\{\underset{\sim}{D} : (\exists \mathbb{P})\ (\mathbb{P}, \underset{\sim}{D}) \in \underset{\sim}{\mathbf{G}}_{AP}\}.$$

We let $\underset{\sim}{\mathbb{Q}}$ be an AP-name for $\mathbb{Q}$ and we use the symbol $\underset{\sim}{E}$ also for an AP-name for $\underset{\sim}{E}$.

LEMMA 2.21:

(a) $\Vdash_{AP} \underset{\sim}{\mathbb{Q}}$ *is a c.c.c. forcing of cardinality* $\aleph_2$,

(b) $\Vdash_{AP} \underset{\sim}{E}$ *is a* $\underset{\sim}{\mathbb{Q}}$*-name of a non-principal ultrafilter and* $\mathfrak{b} = \aleph_1$,

(c) *if* $(\mathbb{P}, \underset{\sim}{D}) \in AP$, *then* $(\mathbb{P}, \underset{\sim}{D}) \Vdash_{AP} (\underset{\sim}{\mathbb{Q}} \Vdash \langle \eta_\delta : \delta \in S\rangle$ *is a* $\leq_{\underset{\sim}{E}}$*-increasing sequence and cofinal in* $\omega^\omega / \underset{\sim}{E})$.

Proof. For (a), see [She98, Ch. IV, Claim 1.6]. Now we prove (b). By the c.c.c. and the construction with direct limits, for every $AP * \underset{\sim}{\mathbb{Q}}$-name τ for a real there are a pair $\mathbf{p} = (\mathbb{P}, \underset{\sim}{D}) \in AP$ and a condition $p \in \mathbb{P}$, and a $\mathbb{P}$-name real τ' for such that $(\mathbf{p}, p) \Vdash_{AP * \underset{\sim}{\mathbb{Q}}} \tau' = \tau$.

(c) We work with the approximation forcing AP^1. Suppose for a contradiction that $((\mathbb{P}, \underset{\sim}{D}), p) \Vdash_{AP^1 * \underset{\sim}{\mathbb{Q}}} (\exists f \in {}^\omega\omega)(f \geq_{\underset{\sim}{E}} \langle \eta_\delta : \delta \in S\rangle)$. Then there is $((\mathbb{P}', \underset{\sim}{D}'), p') \geq_{AP^1} ((\mathbb{P}, \underset{\sim}{D}), p)$ and there is a canonical $\mathbb{P}'$-name $\underset{\sim}{h}$ such that

$$((\mathbb{P}', \underset{\sim}{D}'), p') \Vdash_{AP^1 * \underset{\sim}{\mathbb{Q}}} \underset{\sim}{h} \geq_{\underset{\sim}{E}} \langle \eta_\delta : \delta \in S\rangle. \tag{2.5}$$

Since $\underset{\sim}{h}$ is a name of a real in the c.c.c. forcing $\mathbb{P}'$, there are some $\delta_0 < \omega_1$, $\underset{\sim}{h}' \in M_{\delta_0}$ such that $\underset{\sim}{h}'$ is a $\mathbb{P}' \cap \delta_0$-name such that $((\mathbb{P}', \underset{\sim}{D}'), p') \Vdash_{AP^1 * \underset{\sim}{\mathbb{Q}}} \underset{\sim}{h} = \underset{\sim}{h}'$. We fix such a δ_0, $\underset{\sim}{h}'$. Since $(\mathbb{P}', \underset{\sim}{D}') \in K^1$, by Lemma 2.8 there is $\delta \geq \delta_0$ such that

$$N_\delta \models (\forall h \in M_\delta[G_{\mathbb{P}' \cap \delta}])(h \not\geq_{\underset{\sim}{D}'[G_{\mathbb{P}' \cap \delta}]} \eta_\delta).$$

We take a condition $q \in \mathbb{P}' \cap \delta$, $q \geq_{\mathbb{P}'} p'$, forcing $\forall h \in M_\delta[G_{\mathbb{P}'}] h \not\geq_{\underset{\sim}{D}'} \eta_\delta$. Thus $((\mathbb{P}', \underset{\sim}{D}'), q') \geq ((\mathbb{P}', \underset{\sim}{D}'), p')$ and this is a contradiction to Equation (2.5). ∎

Now we show that the union of the generic filter of the approximation forcing, i.e., the $\underset{\sim}{\mathbb{Q}}$ as given in Lemma 2.21, fulfils $\Vdash_{AP * \underset{\sim}{\mathbb{Q}}} \mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$. The conditions of the form $((\mathbb{P}_*, \underset{\sim}{D}_*), p)$ with $p \in \mathbb{P}_*$ are dense in $AP * \underset{\sim}{\mathbb{Q}}$.

A forcing destroying a given increasing cofinal chain of subgroups $\langle G_i : i < \omega_1\rangle$ of $\mathrm{Sym}(\omega)$ is written down in [MS11]. Such a forcing adds one particular real, a new permutation g that for cofinally many $i < \omega_1$ there is $f_i \in G_{i+1} \smallsetminus G_i$ such that $g \circ f_i \circ g^{-1} \in G_i$. Thus in the extension we have $g \in \mathrm{Sym}(\omega) \smallsetminus \bigcup\{G_i : i < \omega_1\}$ and the sequence $\langle G_i : i < \omega_1\rangle$ is not cofinal any more.

In the rest of this section we construct a variant of such a forcing that adds such a conjugator and at the same time has the $\bar{N}$-oracle c.c. We first show that we can work with convenient supports of permutations.

LEMMA 2.22: *Suppose that a chain of subgroups $\langle G_i : i < \omega_1 \rangle$ is an increasing chain of subgroups of* $\mathrm{Sym}(\omega)$ *such that all permutations that move only finitely many elements are elements of* G_0. *Suppose that* $U \subseteq \omega_1$ *is uncountable and there are*

$$\langle \zeta_i^1, \zeta_i^2, f_i^1, f_i^2 : i \in U \rangle \text{ and } g$$

with the following properties:

(1) *for* $i < j \in U$, $i \leq \zeta_i^1 < \zeta_i^2 < j$,
(2) *for* $i \in U$, $f_i^1 \in G_{\zeta_i^1}$ *and* $f_i^2 \in G_{\zeta_i^2} \setminus G_{\zeta_i^1}$, *and*
(3) *for* $i \in U$, $(\forall^\infty n)((g \circ f_i^1)(n) = (f_i^2 \circ g)(n))$.

Then $g \in \mathrm{Sym}(\omega) \setminus \bigcup\{G_i : i \in \omega_1\}$.

Proof. If $g \in G_{\zeta_i^1}$ for some $i \in U$, then by (3) also $f_i^2 \in G_{\zeta_i^1}$, contradiction. ∎

For carrying this out we use some notions describing permutation groups.

Definition 2.23: Let $f\colon \omega \to \omega$. $\mathrm{supp}(f) = \{n : f(n) \neq n\}$.

Observation 2.24: If $f \in \mathrm{Sym}(\omega)$, then $f[\mathrm{supp}(f)] = \mathrm{supp}(f)$.

For $f \in \mathrm{Sym}(\omega)$, we say f has order 2 if $f \circ f$ is the identity.

For arguing with given supports, we use

LEMMA 2.25 ([MS11, Lemma 3.3]): *If* $\langle G_i : i < \omega_1 \rangle$ *is an increasing sequence of proper subgroups of* $\mathrm{Sym}(\omega)$ *with union* $\mathrm{Sym}(\omega)$, *and* G_0 *contains all permutations with finite support, then for any* $W \in [\omega]^{\aleph_0}$ *the sequence*

$$\langle G_i \cap \{f \in \mathrm{Sym}(\omega) : \mathrm{supp}(f) \subseteq W \wedge f \text{ is of order } 2\} : i < \omega_1 \rangle$$

is not eventually constant.

Now we return to forcing.

LEMMA 2.26: $\Vdash_{AP * \underset{\sim}{\mathbb{Q}}}$ "$\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$".

Proof. Assume towards a contradiction:

$\oplus_1$ $((\mathbb{P}_*, \underset{\sim}{D}_*), p_*) \Vdash_{AP * \underset{\sim}{\mathbb{Q}}}$ "$\langle \underset{\sim}{G}_i : i < \omega_1 \rangle$ is an increasing sequence of proper subgroups of $\mathrm{Sym}(\omega)$ with union $\mathrm{Sym}(\omega)$, and $\underset{\sim}{G}_0$ contains all permutations with finite support".

$\oplus_2$ By Lemma 2.25, $\oplus_1$ implies: $((\mathbb{P}_*, \underset{\sim}{D}_*), p_*) \Vdash_{AP * \underset{\sim}{\mathbb{Q}}}$ "if $W \in [\omega]^{\aleph_0}$ then $\langle \underset{\sim}{G}_i \cap \{f \in \mathrm{Sym}(\omega) : \mathrm{supp}(f) \subseteq W \wedge f$ is of order $2\} : i < \omega_1 \rangle$ is not eventually constant".

$\oplus_3$ We let $\langle m_\eta : \eta \in {}^{\omega>}\omega\rangle$ be a sequence of natural numbers without repetitions. For $\eta \in {}^\omega\omega$ we let $W(\eta) = \{m_{\eta\restriction n} : n \in \omega\}$. Then for $\eta \neq \eta'$ and $k = \min\{n : \eta(n) \neq \eta'(n)\}$ we have

$$W(\eta) \cap W(\eta') = \{m_{\eta\restriction n} : n < k\}.$$

By induction on $i < \omega_1$ we choose $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i) \in AP$, π_i, $\mathbf{p}'_i \in AP^1$, $\xi_i \in \omega_1$, and $(\mathbf{p}_i, \pi_i, \mathbf{p}'_i, \xi_i, \underset{\sim}{\zeta}^1_i, \underset{\sim}{\zeta}^2_i, \underset{\sim}{f}^1_i, \underset{\sim}{f}^2_i, \mathbb{R}'_i, Y(\mathbb{P}'_i, \underset{\sim}{D}'_i))$ such that:

(a) $\mathbf{p}_0 = \mathbf{p}_*$, $Y(\mathbb{P}'_0, \underset{\sim}{D}'_0) = Y(\mathbb{P}_*, \underset{\sim}{D}_*)$.

(b) $\mathbf{p}_i = ((\mathbb{P}_i, \underset{\sim}{D}_i), p_*) \in AP * \underset{\sim}{\mathbb{Q}}$ and $j < i \to \mathbf{p}_j \leq_{AP} \mathbf{p}_i$.

(c) $\mathbf{p}'_i = ((\mathbb{P}'_i, \underset{\sim}{D}'_i), p_*) \in AP^1 * \underset{\sim}{\mathbb{Q}}$ satisfies

(α) $\mathbb{P}'_0 \cap \{\xi_i : i < \omega_1\} = \emptyset$, the set of members of

$$\mathbb{P}'_i \setminus \bigcup\{\mathbb{P}'_j : j < i\} \subseteq [\xi_i + 1, \omega_1),$$

hence $\mathbb{P}'_i \cap \xi_i = \mathbb{P}'_j \cap \xi_i$ for any $j \geq i$,

(β) $\pi_i \colon \mathbb{P}'_i \to \omega_1$ is a one-to-one function mapping $\mathbb{P}'_i$ onto $\mathbb{P}_i$ and mapping $\underset{\sim}{D}'_i$ onto $\underset{\sim}{D}_i$,

(γ) if $j < i$, then $\pi_j \subseteq \pi_i$,

(δ) $\langle \xi_i : i < \omega_1\rangle$ has the properties (a) to (d) of the proof of Lemma 2.19 (D) with respect to the sequence $\langle \mathbf{p}'_i, \pi_i : i < \omega_1\rangle$,

(ε) the set $Y(\mathbb{P}'_i, \underset{\sim}{D}'_i)$ witnesses that $(\mathbb{P}'_i, \underset{\sim}{D}'_1) \in K^1$ as in Definition 2.11(1)(c).

(d) At double successor steps of limit ordinals we add a new Cohen real: If $i = \omega j + 1$ then $\mathbb{P}'_{i+1} = \mathbb{P}'_i * ({}^{\omega>}\omega, \triangleleft)$, we let $\underset{\sim}{\nu}_i$ be a name for a $({}^{\omega>}\omega, \triangleleft)$-generic real. So $\underset{\sim}{\nu}_i$ is a Cohen real over $\mathbf{V}^{\mathbb{P}'_{\omega\cdot j}}$. Since $\mathbf{V}^{\mathbb{P}'_i}$ is unbounded in $\mathbf{V}^{\mathbb{P}'_{i+1}}$ by Lemma 2.7, there is a $\mathbb{P}_{i+1}$-name for an ultrafilter $\underset{\sim}{D}_{i+1}$. The set

$$Y(\mathbb{P}_{i+1}, \underset{\sim}{D}_{i+1}) = Y(\mathbb{P}_i, \underset{\sim}{D}_i) \cap [i+1, \omega) \in \mathscr{D}_{\bar{N}}$$

witnesses that $(\mathbb{P}'_{i+1}, \underset{\sim}{D}'_{i+1}) \in K^1$.

(e) Also, if $i = \omega j + 1$ then we choose $\underset{\sim}{D}'_{i+1}$ such that

$$(\mathbb{P}'_{i+1}, \underset{\sim}{D}'_{i+1}) \geq_{AP} (\mathbb{P}'_i, \underset{\sim}{D}'_i) \quad \text{and} \quad \langle \underset{\sim}{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_j} : \ell < \omega_1\rangle$$

and even $\langle \underset{\sim}{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_i} : \ell < \omega_1\rangle$ is a $\mathbb{P}'_i$-name.

(f) Also at double successors to limit ordinals we fix witnessing functions with the new Cohen ν_i as information in their support, i.e., if $i = \omega \cdot j + 1$ then

(α) for $\ell = 1, 2$, $\mathbf{p}'_{i+1}$ forces that $i < \underset{\sim}{\zeta}^1_i < \underset{\sim}{\zeta}^2_i$,

(β) and for $\ell = 1, 2$, $\mathbf{p}'_{i+1}$ forces that $\underset{\sim}{f}^2_i \in G_{\underset{\sim}{\zeta}^2_i} \setminus G_{\underset{\sim}{\zeta}^1_i}$, $\underset{\sim}{f}^1_i \in G_{\underset{\sim}{\zeta}^1_i}$ is a $\mathbb{P}'_{i+1}$-name of a member of $\mathrm{Sym}(\omega)$ of order 2 such that

$$\mathbb{P}'_{i+1} \Vdash \mathrm{supp}(f^\ell_i) \subseteq \underset{\sim}{w}^\ell_i = W(\langle \ell \rangle ^\frown \underset{\sim}{\nu}_i).$$

Here $\langle \ell \rangle ^\frown \nu$ is the concatenation of the singleton $\langle \ell \rangle$ and ν, i.e., $(\langle \ell \rangle ^\frown \nu)(k) = \ell$ if $k = 0$, and $= \nu(k-1)$ else. Recall that for $\eta \in {}^\omega\omega$, $W(\eta)$ has been defined in $\oplus_3$.

By Lemma 2.25, the desired names for countable ordinals $\underset{\sim}{\zeta}^1_i$, $\underset{\sim}{\zeta}^2_i$ and names $\underset{\sim}{f}^1_i$, $\underset{\sim}{f}^2_i$ exist. The triple $\mathbf{p}'_i \in AP * \underset{\sim}{\mathbb{Q}}$ stays unchanged.

(g) At limit steps $i < \omega_1$, we let $(\mathbb{P}'_i, \underset{\sim}{D}'_i)$ be a lower bound of $(\mathbb{P}_j, \underset{\sim}{D}_j)$, $j < i$, as in Lemma 2.19(C). We let $Y(\mathbb{P}'_i, \underset{\sim}{D}'_i) \in \mathscr{D}_{\bar{N}}$ be a witness to $(\mathbb{P}'_i \underset{\sim}{D}'_i) \in K^1$.

(h) Now finally we explain the order $\mathbb{P}_{i+1}$ for countable limit ordinals i. We let

$$H = \bigcap \{Y(\mathbb{P}'_\varepsilon, \underset{\sim}{D}'_\varepsilon) : \varepsilon < i\}.$$

Then $H \in \mathscr{D}_{\bar{N}}$. We let Y_i, ξ_i be as follows:

$$\begin{aligned} Y_i = \{\delta \in H : (\forall j < i)(\xi_j < \delta) \wedge (\forall j_1 \in i) \\ ((\underset{\sim}{\zeta}^1_{j_1}, \underset{\sim}{\zeta}^2_{j_1}, \underset{\sim}{f}^1_{j_1}, \underset{\sim}{f}^2_{j_1}) \in M_\delta \wedge N_{j_1} \in M_\delta \\ \wedge\, \underset{\sim}{\zeta}^1_{j_1}, \underset{\sim}{\zeta}^2_{j_1}, \underset{\sim}{f}^1_{j_1}, \underset{\sim}{f}^2_{j_1} \text{ are } \mathbb{P}'_i \cap \delta\text{-names})\}, \\ \xi_i = \min(Y_i). \end{aligned} \tag{2.6}$$

Then $Y_i \in \mathscr{D}_{\bar{N}}$. Since any element of $\mathscr{D}_{\bar{N}}$ is unbounded in ω_1, the ordinal ξ_i is well-defined. We define $\underset{\sim}{\mathbb{R}}'_i \in M_{\xi_i}$: $\underset{\sim}{\mathbb{R}}'_i$ is a $\mathbb{P}'_i \cap \xi_i$-name of a c.c.c. forcing notion. Recall that w^1_ε, w^2_ε, $\varepsilon < \xi_i$, ε successor ordinal, are defined in $\oplus_3$(f)(β). The key fact to the $\bar{N}$-c.c. is that these names are so faintly related to the Cohen reals $\langle \eta_\delta : \delta \in S \rangle$. The following is forced by $\mathbb{P}'_i \cap \xi_i$: A member of $\underset{\sim}{\mathbb{R}}'_i$ has the form (u, g) such that:

(α) $u \subseteq \{\omega \cdot j + 1 : \omega \cdot j + 1 \in \xi_i\}$ is finite, g a finite partial permutation of order two, $\mathrm{dom}(g) \subseteq \bigcup_{\varepsilon \in u} w^2_\varepsilon$, such that $\varepsilon \in u$ implies $\mathrm{range}(g) \subseteq w^1_\varepsilon$.

(β) Recall that for $\eta \in {}^{\omega>}\omega$, m_η has been defined in $\oplus_3$. The sets $\mathrm{dom}(g)$ and $\mathrm{range}(g)$ are sufficiently large in the following sense:

- if $\delta \neq \varepsilon \in u$ then we fix n, such that $\nu_\delta \restriction n \neq \nu_\varepsilon \restriction n$ and then require that for $k = 1, 2$ the set

$$\{m_{\langle k \rangle ^\frown \nu_\delta \restriction \ell} : \ell < n\} \subseteq \mathrm{dom}(g) \cap \mathrm{range}(g),$$

- $\forall \varepsilon \in \mathrm{dom}(p)$, if ε is Cohen coordinate (as in $\oplus_3$(d)) and $p(\varepsilon) \in 2^n$, $\ell \leq n$, $k = 1, 2$, then

$$m_{\langle k \rangle ^\frown p(\varepsilon) \restriction \ell} \in \mathrm{dom}(g) \cap \mathrm{range}(g).$$

(γ) If $\varepsilon \in u$ then $\mathrm{dom}(g) \cap w^2_\varepsilon$ is closed under f^1_ε and $\mathrm{range}(g) \cap w^1_\varepsilon$ is closed under f^2_ε.

(δ) For $(u_1, g_1), (u_2, g_2) \in \mathbb{R}'_i$ we let $(u_1, g_1) \leq (u_2, g_2)$ iff
 (i) $u_1 \subseteq u_2$,
 (ii) $g_1 \subseteq g_2$,
 (iii) $(\forall \varepsilon \in u_1)(\forall n \in w^2_\varepsilon \cap (\mathrm{dom}(g_2) \setminus \mathrm{dom}(g_1))(g_2(n) \in w^1_\varepsilon \wedge f^2_\varepsilon(g_2(n)) = g_2(f^1_\varepsilon(n)))$.

We let $\mathbb{P}'_{i+1} = \mathbb{P}'_i * \mathbb{R}'_i$.

Since $\mathbb{R}'_i$ is countable, $\mathbb{P}'_{i+1}$ has the $\bar{N}$-c.c., and again by Lemma 2.7 we find $\underset{\sim}{D}'_{i+1}$ such that $(\mathbb{P}'_{i+1}, \underset{\sim}{D}'_{i+1}) \in K^1$ with witness

$$Y(\mathbb{P}_{i+1}, \underset{\sim}{D}_{i+1}) = Y_i \cap [\xi_i, \omega_1).$$

$\oplus_4$ Once the induction is performed, we define $\mathbf{p}_{\omega_1} = (\mathbb{P}_{\omega_1}, \underset{\sim}{D}_{\omega_1})$ and $\mathbf{p}'_{\omega_1} \in K^1$ and $\pi = \bigcup_{i<\omega_1} \pi_i$ which maps $\mathbf{p}'_{\omega_1}$ onto $\mathbf{p}_{\omega_1}$ as follows:
 (a) $\mathbb{P}'_{\omega_1} = \bigcup\{(\mathbb{P}'_i \cap \xi_i) * \mathbb{R}'_i : i < \omega_1, i \text{ limit}\}$.
 (b) $\mathbb{P}'_{\omega_1} \Vdash \underset{\sim}{D}'_{\omega_1} = \bigcup\{\underset{\sim}{D}'_i : i < \omega_1, i \text{ limit}\}$.
 (c) $\pi = \bigcup_{i<\omega_1} \pi_i$ is an isomorphism from $\mathbb{P}'_{\omega_1}$ onto $\mathbb{P}_{\omega_1}$ mapping $\underset{\sim}{D}'_{\omega_1}$ to $\underset{\sim}{D}_{\omega_1}$.
 (d) $\bigwedge_{i<\omega_1} \mathbf{p}_i \leq \mathbf{p}_{\omega_1} \in K^2$, $\bigwedge_{i<\omega_1} \mathbf{p}'_i \leq \mathbf{p}'_{\omega_1} \in K^1$.

We show that $\mathbf{p}'_{\omega_1} \in K^1$. We let $Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1})$ be the diagonal intersection of the $Y(\mathbb{P}'_i, \underset{\sim}{D}'_i)$ intersected with the set of i such that for any $j < i$, $\xi_j < i$. Since $\mathscr{D}_{\bar{N}}$ is a normal filter, $Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \in \mathscr{D}_{\bar{N}}$. We show that this set witnesses Definition 2.11(1)(c). To this end, we prove the following claim.

Claim: *Suppose that $i \in Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1})$. The forcing $\mathbb{P}'_i \cap \xi_i$ forces the following: If $i_1 < i$, $i_1 \in Y(\mathbb{P}'_i, \underset{\sim}{D}'_i)$, then $\underset{\sim}{\mathbb{R}}'_{i_1} \subseteq_{ic} \underset{\sim}{\mathbb{R}}'_i$ and if $D_0 \in N_{i_1}$ is a predense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \underset{\sim}{\mathbb{R}}'_{i_1}$ then D_0 is predense in $\mathbb{P}'_i \cap \xi_i * \underset{\sim}{\mathbb{R}}'_i$.*

We prove this claim: $\mathbb{P}'_i \cap \xi_i \Vdash \underset{\sim}{\mathbb{R}}'_{i_1} \subseteq_{ic} \underset{\sim}{\mathbb{R}}'_i$ follows from the definition of the orders $\mathbb{R}'_j$.

Assume that $D_0 \in N_{i_1}$ is an open dense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \mathbb{R}_{i_1}$, and $p = (p \restriction \xi_{i_1}, p(\xi_{i_1})) \in (\mathbb{P}'_i \cap i * \mathbb{R}'_i)$. We have to find a condition in $q \in D_0$ that is compatible with p. Assume that $p \cap \xi_{i_1} \Vdash_{\mathbb{P}'_{\xi_{i_1}}} p(i_1) = (u, g)$ and u, g are pinned down in $\mathbf{V}$, not names. After possibly strengthening p and g we can assume that g is so strong that it fulfils:

$$\begin{aligned}\mathrm{dom}(g) \supseteq \{m_{p(\beta)\restriction k} :\ & \beta \in \mathrm{supp}(p), \beta \text{ successor ordinal},\\ & \beta \in u, k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * ({}^{\omega>}\omega, \lhd)\};\\ \mathrm{range}(g) \supseteq \{(f^1_\beta)(m_{p(\beta)}) :\ & \beta \in \mathrm{supp}(p), \beta \text{ successor ordinal},\\ & \beta \in u,\ k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * ({}^{\omega>}\omega, \lhd)\}.\end{aligned}$$

After possibly further strengthening p we can assume that $p \restriction \xi_{i_1}$ determines ζ^j_β for $j = 1, 2$ and determines $\underset{\sim}{f}^2_\beta$ restricted to the set on the right-hand side of the first eqution, and determines $\underset{\sim}{f}^1_\beta$ on the right-hand side of the second equation for any $\beta \in u$. We assume the analogous strength of p' for all triples $(p', (u', g'))$ appearing later in the proof. We assume that $\mathrm{dom}(g) \in \omega$ and that $\mathrm{dom}(g)$ is larger than any $W^2_\varepsilon \cap W^2_\zeta$ for $\varepsilon \neq \zeta \in u$ and that $\mathrm{range}(g)$ is a superset of $W^1_\varepsilon \cap W^1_\zeta$ for $\varepsilon \neq \zeta \in u$.

Now we choose $p_0 = (p \restriction \xi_{i_1}, (u \cap \xi_{i_1}, g)) \in M_{\xi_{i_1}}$. We choose $q_0 = (q_0 \restriction \xi_{i_1}, (u_{q_0}, g_{q_0})) \geq p_0$, $q_0 \in D \cap \xi_{i_1} \cap M_{\xi_{i_1}}$. Then q_0 does not determine more of the Cohen real ν_ε for $\varepsilon \in u_{q_0}$ than p_0 does. Then we take $q_1 \geq q_0$ such that

$$q_1 = (q_0 \restriction \xi_{i_1} \cup \{(\varepsilon, q_1(\varepsilon)) : \varepsilon \in u_{q_0} \setminus \xi_{i_1}\}, (u_{q_0}, g_{q_0}))$$

where for each $\varepsilon \in u \setminus \xi_{i_1}$,

$$\begin{aligned}q_1(\varepsilon) \Vdash & W(0^\frown \underset{\sim}{\nu}_\varepsilon) \cap (\mathrm{dom}(g_{q_0}) \setminus \mathrm{dom}(g)) = \emptyset\\ & \wedge W(1^\frown \underset{\sim}{\nu}_\varepsilon) \cap (\mathrm{range}(g_{q_0}) \setminus \mathrm{range}(g)) = \emptyset.\end{aligned}$$

This special point (not in [She98, Ch. IV], [She06]) is that the $\underset{\sim}{\nu}_i$, i sucessor of a countable limit ordinal, η_δ, $\delta \in S$, are just Cohen reals: Defining relevant generic objects that have a Cohen real as domain allows us to carry on the oracle-c.c. and thus to preserve the Cohenness of the η_δ. This main trick is also used in the next section. Now q_1 is compatible with p.

Thus $Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \in \mathscr{D}_{\bar{N}}$ is a witness for the oracle-c.c. of $\mathbb{P}'_{\omega_1}$, as required in Definition 2.11(1)(c)(β). The other properties in Definition 2.11(1)(c) follow now for $i \in Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1})$ by the inductive definition of the $\mathbb{P}'_i$.

This finishes the construction of a stronger member in AP-forcing.

$\oplus_5$ Let

$$\underset{\sim}{g} = \bigcup\{g : \exists p \exists u (p, (u, g)) \in \mathbf{G}_{\underset{\sim}{\mathbb{P}'_{\omega_1}}}\},$$
$$\underset{\sim}{U} = \bigcup\{u : \exists p \exists g (p, (u, g)) \in \mathbf{G}_{\underset{\sim}{\mathbb{P}'_{\omega_1}}}\}.$$

We show

$$((\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}), p_*) \Vdash_{AP * \underset{\sim}{\mathbb{Q}}} |\underset{\sim}{U} = \aleph_1| \wedge \text{“}\underset{\sim}{g} \notin \bigcup\{\underset{\sim}{G}_i : i < \omega_1\}\text{”}.$$

Proof. We fix a generic filter $\mathbf{G}_{\mathbb{P}'_{\omega_1}}$. By the construction of $\mathbb{P}'_{\omega_1}$ we have

$$(\forall i < j \in S \cap C)(\underset{\sim}{f}^\ell_i \in M_j \wedge \underset{\sim}{f}^\ell_i \text{ is a } \mathbb{P}'_{\omega_1} \cap j\text{-name}).$$

The forcing $\mathbb{P}'_{\omega_1}$ adds a $g\colon \bigcup_{\varepsilon \in U} w^2_\varepsilon \to \bigcup_{\varepsilon \in U} w^1_\varepsilon$ that conjugates for $i \in U$, $f^1_i \in G_{\zeta^1_i}$ and $f^2_i \in G_{\zeta^2_i} \smallsetminus G_{\zeta^1_i}$. If $i \in U$ then

$$\mathrm{dom}(f^\ell_i) = w^\ell_i = W_{\langle \ell \rangle \frown \nu_i}$$

and g conjugates f^1_i and f^2_i up to a finite mistake, by $\oplus_3$ item (i)(δ)(iii). So, for each $i \in U$, $g \circ f^1_i \circ g^{-1} = f^2_i$ up to finitely many arguments. But g is in some subgroup G_j. So for $\zeta^1_i > i > j$, $i \in X$, $f^2_i \in G_{\zeta^1_i}$, contradiction. ∎

End of proof of Theorem 2.1. We assume that $S \subseteq \omega_1$ is stationary and $\mathbf{V} \models \diamondsuit^-_S$. We extend $\mathbf{V}$ with the forcing poset $AP * \underset{\sim}{\mathbb{Q}}$. By Lemma 2.21, $\mathfrak{mcf} = \aleph_1$ in the extension, and by Lemma 2.26, $\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$. ∎

3. On $\mathrm{Con}(\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{mcf})$

Now we show that $\aleph_1 = \mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \aleph_2 = \mathfrak{mcf}$ is consistent relative to ZFC. In [MST06] we established that it is consistent relative to ZFC that $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$. Brendle and Losada showed that $\mathfrak{g} \le \mathrm{cf}(\mathrm{Sym}(\omega))$ in ZFC; see [BL03]. So the following theorem gives another consistency proof for $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$.

THEOREM 3.1: *It is consistent relative to* ZFC *that* $\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \aleph_2 = \mathfrak{mcf}$.

For the proof we will again work with oracle c.c.-forcing. Let $D \subseteq [\omega]^\omega$ be a filter over ω. Then we write D^+ for the D-positive sets, i.e., $X \in D^+$ iff $X \cap Y$ is infinite for any $Y \in D$.

Lemma 3.2: *Let $\kappa \geq \aleph_2$ be a cardinal in $\mathbf{V}$. The $(\mathrm{A})_\kappa$ implies $(\mathrm{B})_\kappa$.*

- $(\mathrm{A})_\kappa$ *For every filter $D \subseteq [\omega]^\omega$ over ω such that $\mathcal{P}(\omega)/D$ has the c.c.c. (that is: for every A_i, $i < \omega_1$, such that $A_i \in D^+$ there are $i \neq j$ such that $A_i \cap A_j \in D^+$), for every regular $\kappa_* < \kappa$, for every sequence $\langle f_i : i < \kappa_* \rangle$ of functions $f_i \in {}^\omega\omega$ there is $g \in {}^\omega\omega$ such that for unboundedly many $i < \kappa_*$, $\neg g \leq_D f_i$.*
- $(\mathrm{B})_\kappa$ *After forcing with a c.c.c. ${}^\omega\omega$-bounding forcing $\mathbb{Q}$, in the extension $\mathbf{V}^\mathbb{Q}$ for every non-principal ultrafilter D on ω, $\mathrm{cf}({}^\omega\omega/D) \geq \kappa$, and $\mathfrak{b}^\mathbf{V} = \mathfrak{b}^{\mathbf{V}^\mathbb{Q}}$.*

Proof. Assume $(A)_\kappa$ and that $q_0 \in \mathbb{Q}$ forces "$\underset{\sim}{D}$ is an ultrafilter over ω and $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa_* \rangle$ is increasing modulo $\underset{\sim}{D}$ and $\kappa_* < \kappa$". So κ_* is regular and uncountable in $\mathbf{V}^\mathbb{Q}$ and hence regular and uncountable in $\mathbf{V}$. We shall show that there is $q_* \geq q_0$,

$$(\boxdot) \qquad q_* \Vdash \exists f \in ({}^\omega\omega) \bigwedge_{\alpha < \kappa_*} \underset{\sim}{f}_\alpha <_{\underset{\sim}{D}} f,$$

and thus we will have established $(B)_\kappa$.

Since $\mathbb{Q}$ is ${}^\omega\omega$-bounding and c.c.c., we can take $g_\alpha \in \mathbf{V}$ for $\alpha \in \kappa_*$ such that $q_0 \Vdash_\mathbb{Q}$ "$\underset{\sim}{f}_\alpha \leq^* g_\alpha$".

We let

$$E = \{A \in \mathcal{P}(\omega)^\mathbf{V} : (\exists q \in \mathbb{Q})(q \geq q_0 \wedge q \Vdash \check{A} \in \underset{\sim}{D})\}$$

and we let

$$D' = \{A \in \mathcal{P}(\omega)^\mathbf{V} : q_0 \Vdash \check{A} \in \underset{\sim}{D}\}.$$

Then we have $E, D' \in \mathbf{V}$ and the following holds:

1. D' is a filter over ω.
2. $E \subseteq (D')^+$. Let $A \in E$, say $q \Vdash A \in \underset{\sim}{D}$, $q \geq q_0$ and let $B \in D'$. Then $q \Vdash A \in \underset{\sim}{D} \wedge B \in \underset{\sim}{D}$, so $q \Vdash$ "$A \cap B$ is infinite." Since $A, B \in \mathbf{V}$, $A \cap B$ is infinite. Since this holds for every $B \in D'$, item (2) is proved.
3. $(D')^+ \subseteq E$. Suppose that $X \notin E$. Then $\forall q \in \mathbb{Q}$, $q \geq q_0$ implies that $q \nVdash X \in \underset{\sim}{D}$, so $q_0 \Vdash X \notin \underset{\sim}{D}$. Since $\underset{\sim}{D}$ is a name of an ultrafilter $q_0 \Vdash X^c \in \underset{\sim}{D}$. So $X^c \in D'$ and $X \notin (D')^+$.
4. So together: $(D')^+ = E$.

(5) q_0 forces that D' is a c.c.c. filter. Proof: Let $q_0 \Vdash_{\mathbb{Q}} A_\alpha \in (D')^+ = E$ for $\alpha \in \omega_1$, via $q_\alpha \geq q_0$. Since $\mathbb{Q}$ is c.c.c. there are $\alpha \neq \beta$ such that $q_\alpha \not\perp q_\beta$. Then there is $r \in \mathbb{Q}$, $r \Vdash A_\alpha \in \underset{\sim}{D}, A_\beta \in \underset{\sim}{D}$, and hence $r \Vdash A_\alpha \cap A_\beta \in \underset{\sim}{D}$ since $\underset{\sim}{D}$ is forced to be a filter. So $A_\alpha \cap A_\beta \in D'^+$.

Let g be as in the condition $(A)_\kappa$, applied to D' and $\langle g_\alpha : \alpha < \kappa\rangle$, so for some cofinal set $u \subseteq \kappa_*$ we have for $\alpha \in u \subseteq \kappa_*$, $\neg g \leq_{D'} g_\alpha$. Hence for $\alpha \in u$, $q_0 \not\Vdash \{n : g(n) \leq g_\alpha(n)\} \in \underset{\sim}{D}$ and there is

$$\tilde{q}_\alpha \geq q_0, \quad \tilde{q}_\alpha \Vdash \{n : g(n) \leq g_\alpha(n)\} \notin \underset{\sim}{D}.$$

Thus $\tilde{q}_\alpha \Vdash \{n : g(n) > g_\alpha(n)\} \in \underset{\sim}{D}$ and the choice of g_α implies

$$\tilde{q}_\alpha \Vdash \{n : g(n) > \underset{\sim}{f}_\alpha(n)\} \in \underset{\sim}{D}.$$

Since $\mathbb{Q}$ has the c.c.c., we have $\mathrm{cf}(\kappa_*) > \omega$. Therefore κ_*-many of the $\tilde{q}_\alpha$ are in the generic filter. So for any $\mathbb{Q}$-generic filter G with $q_0 \in G$ we have $\underset{\sim}{f}_\alpha[G] \leq_{\underset{\sim}{D}[G]} g$ for cofinally many $\alpha \in u$. Hence a condition $q_* \geq q_0$ forces this. Since the sequence $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa_*\rangle$ is $\leq_{\underset{\sim}{D}}$-increasing, we get $q_* \Vdash$ "$(\forall \alpha < \kappa_*)(\underset{\sim}{f}_\alpha \leq_{\underset{\sim}{D}} g)$." Thus Equation (⊡) and the first statement of $(\mathrm{B})_\kappa$ are proved.

Since the forcing $\mathbb{Q}$ is ${}^\omega\omega$-bounding, we have $\mathfrak{b}^{\mathbf{V}} = \mathfrak{b}^{\mathbf{V}^{\mathbb{Q}}}$. ∎

An example for such a $\mathbb{Q}$ is the forcing adding $\aleph_1$ random reals, in a countable support iteration or with the measure algebra over 2^{ω_1}. From now on, we let $\mathbb{Q}$ be one of these forcing for adding $\aleph_1$ random reals. In the extension $\mathbf{V}^{\mathbb{Q}}$ of Lemma 3.2 we have $\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_1$ by [ST95, Theorem 1.6]. So if we succeed to establish the condition $(\mathrm{A})_\kappa$ of the lemma together with $\mathfrak{b} = \aleph_1$ for some $\kappa \geq \aleph_2$, Theorem 3.1 will be proved. We fix a stationary $S \subseteq \omega_1$ and take $\kappa = \aleph_2$ and we work again with oracle-c.c. forcings in order to establish the consistency of $(A)_{\aleph_2}$ and $\mathfrak{b} = \aleph_1$.

Lemma 3.3: *We assume that in $\mathbf{V}$, the set S is stationary in ω_1 and the two diamond principles $\diamondsuit_S$ and $\diamondsuit_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}}$ hold. Then there is an oracle c.c. forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ we have $(A)_{\aleph_2}$ of the previous lemma, and $\mathfrak{b} = \omega_1$.*

Proof. We fix in $\mathbf{V}$ a $\leq^*$-increasing sequence $\langle g_\delta : \delta < \omega_1\rangle$ that is $\leq^*$-unbounded. We fix an oracle $\bar{M} = \langle M_\varepsilon : \varepsilon \in S\rangle$ such that the $\bar{M}$-c.c. ensures that the type $\bigwedge_{\delta<\omega_1} x \geq^* g_\delta$ is omitted. Indeed, $\langle g_\delta : \delta \in \omega_1\rangle \in M_0' \prec H(\chi)$ and M_0 being the Mostowski collapse of M_0' suffices for this. In addition we fix a $\diamondsuit_{\{\alpha<\aleph_2:\mathrm{cf}(\alpha)=\aleph_1\}}$-sequence $\langle T_\alpha : \alpha \in \omega_2, \mathrm{cf}(\alpha) = \aleph_1\rangle \in M_0'$.

In the following α, α' will range over ω_2, $i, j, \varepsilon, \zeta, \xi$ over ω_1, and the letters β, γ, δ will denote particular functions with values in ω_2, ω_1, ω_1. We fix a bijection $b\colon 2^{<\omega} \to \omega$, a bijection $c\colon 2^\omega \cap \mathbf{V} \to \omega_1$ and another bijection $b_2\colon \aleph_2 \to (\mathcal{P}(H(\omega_1)))^2$. By $\diamondsuit_S$ and $\diamondsuit_{\{\alpha<\aleph_2 : \mathrm{cf}(\alpha)=\aleph_1\}}$ such bijections exist.

A finite support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \omega_2, \alpha \le \omega_2 \rangle$ is constructed by induction on $\alpha \le \omega_2$ with the following properties:

(1) $|\mathbb{P}_\alpha| \le \aleph_1$ for $\alpha < \omega_2$,
(2) $\mathbb{P}_\alpha$ has the $\bar{M}$-c.c.

For an odd stage $\alpha \in \omega_2$ we force via $\mathbb{Q}_\alpha = \mathbb{C}$, and we conceive Cohen forcing $\mathbb{C}$ in the form

$$\{p : p \text{ is a partial function from } 2^{<\omega} \text{ to } 2, |p| < \omega\}$$

and fix for $\eta \in 2^\omega \cap \mathbf{V}$ sets

$$A_{\alpha,\eta} = \{b((p(\eta \restriction 0), \dots, p(\eta \restriction n-1))) : n \in \omega, p \in G\} \subseteq \omega$$

in the extension by $\mathbb{C}$, where b is the bijection from above. Note that for $\eta \neq \eta'$, $A_{\alpha,\eta} \cap A_{\alpha,\eta'}$ is finite. We write $A'_{\alpha,\varepsilon} = A_{\alpha,c^{-1}(\varepsilon)}$. Then $|\mathbb{P}_{\alpha+1}| \le \aleph_1$.

For even $\alpha < \omega_2$ we define $\mathbb{Q}_\alpha$ as follows: If $\mathrm{cf}(\alpha) < \omega_1$, we let $\mathbb{Q}_\alpha$ be the trivial forcing, i.e., $\mathbb{Q}_\alpha = \{0\}$. Now let $\alpha > 0$. We assume that $\mathbb{P}_\alpha \subseteq \omega_1$. Then every canonical $\mathbb{P}_\alpha$-name $(\underset{\sim}{D}, \langle \underset{\sim}{f}_i : i < \omega_1 \rangle)$ for a subset of $\mathcal{P}(\omega)$ and an ω_1-sequence of reals is a subset of $H(\omega_1)$. We say that $T \subseteq \alpha$ codes the canonical name $(\underset{\sim}{D}, \langle \underset{\sim}{f}_i : i < \omega_1 \rangle)$ if $b_2[T] = (\underset{\sim}{D}, \langle \underset{\sim}{f}_i : i < \omega_1 \rangle)$.

If $\mathrm{cf}(\alpha) = \omega_1$ and T_α is a canonical $\mathbb{P}_\alpha$-name of a pair $(\underset{\sim}{D}, \langle \underset{\sim}{f}_{\alpha,i} : i < \omega_1 \rangle)$ such that

$$\mathbb{P}_\alpha \Vdash \text{“}\underset{\sim}{D} \text{ contains the cofinite sets and } \mathcal{P}(\omega)/\underset{\sim}{D} \text{ is c.c.c.”}$$

then we first fix in the ground model an increasing sequence $\langle \beta(\alpha, i) : i < \omega_1 \rangle$ that converges to α such that each $\beta(\alpha, i)$ is an odd member of ω_2.

Next we define by induction on $i < \omega$ countable ordinals as follows:

$$\begin{aligned} \gamma(\alpha, 0) &= \min\{\varepsilon < \omega_1 : \underset{\sim}{f}_{\alpha,0} \in \mathbf{V}^{\mathbb{P}_{\beta(\alpha,\varepsilon)}}\}, \\ \gamma(\alpha, i) &= \min\{\varepsilon < \omega_1 : \underset{\sim}{f}_{\alpha,i} \in \mathbf{V}^{\mathbb{P}_{\beta(\alpha,\varepsilon)}} \wedge (\forall j < i)(\varepsilon > \gamma(\alpha, j))\}. \end{aligned} \tag{3.1}$$

Later it will be important that the $\gamma(\alpha, i)$, $i < \omega_1$, are pairwise different.

Then for each $i < \omega_1$ we choose with the maximum principle a name $\delta(\alpha, i) \in \omega_1$ such that

$$\mathbb{P}_\alpha \Vdash (\omega \setminus A_{\beta(\alpha,\gamma(\alpha,i)),\delta(\alpha,i)}) \in \underset{\sim}{D}. \tag{3.2}$$

We do not write the tildes under the names of the δ. For the existence of such $\delta(\alpha, i)$ we use the following claim.

Claim: *For any $i < \omega_1$ there are coboundedly many ε such that*

$$\mathbb{P}_\alpha \Vdash (\omega \setminus A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon}) \in \underset{\sim}{D}.$$

Proof. Assume for a contradiction that $i < \omega_1$ is a counterexample to the claim. Then there are unboundedly many $\varepsilon \in \omega_1$ such that there is $p_\varepsilon \in \mathbb{P}_\alpha$ such that $p_\varepsilon \Vdash A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon)} \in \underset{\sim}{D}^+$. Since $\mathbb{P}_\alpha$ has the c.c.c. there is a $\mathbb{P}_\alpha$-generic G that contains $\aleph_1$ many p_ε as above. Call this uncountable set of ε's X. However, for $\varepsilon \neq \varepsilon' \in X$,

$$\mathbb{P}_\alpha \Vdash A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon} \cap A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon'}$$

is finite. This contradicts the fact that $\mathbb{P}_\alpha \Vdash \mathcal{P}(\omega)/\underset{\sim}{D}$ is c.c.c., and thus the claim is proved. ∎

We use only one $\delta(\alpha, i)$ and its value in ω_1 is not important. However, for the $\gamma(\alpha, i)$, the pairwise inequality $\beta(\alpha, \gamma(\alpha, i)) \neq \beta(\alpha, \gamma(\alpha, j))$ for $i \neq j$ is important, so that there are no conflicts between the various instances of condition (6) below.

Once the sequence $\langle \gamma(\alpha, i), \delta(\alpha, i) : i < \omega_1 \rangle$ is chosen, we define in $\mathbf{V}^{\mathbb{P}_\alpha}$ the forcing $\mathbb{Q}_\alpha$ as follows: $p \in \mathbb{Q}_\alpha$ iff

(1) $p = (u_p, h_p)$,
(2) $u_p \subseteq \omega_1$ is finite,
(3) $h_p \in {}^{\omega>}\omega$.

$\mathbb{Q}_\alpha \models p \leq q$ if

(4) $u_p \subseteq u_q$ and
(5) $h_p \trianglelefteq h_q$ and
(6) if $\xi \in u_p$ and

$$m \in (\omega \setminus A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}) \cap (\mathrm{dom}(h_q) \setminus \mathrm{dom}(h_p))$$

then $f_{\alpha,\xi}(m) < h_q(m)$.

We show by induction on $\alpha \leq \omega_2$ that $\mathbb{P}_\alpha$ has the $\bar{M}$-c.c. and $|\mathbb{P}_\alpha| \leq \aleph_1$ for $\alpha < \omega_1$. Since we take direct limits, the limit steps are covered by [She98, Ch. IV, 3.2]. The start of the induction is trivial. Now we look at the successor steps $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$.

Odd α: $\mathbb{Q}_\alpha$ is the Cohen forcing. Any countable forcing has the $\bar{M}[\mathbb{P}_\alpha]$-c.c. Putting this together with the induction hypothesis, $\mathbb{P}_{\alpha+1}$ has the $\bar{M}$-c.c.

Even α: Since $\mathbb{P}_\alpha$ has the c.c.c., there is a set of representatives of $\mathbb{P}_\alpha$-names of members of $\mathbb{Q}_\alpha$ of size at most $\aleph_1$. Hence we can assume that $|\mathbb{P}_{\alpha+1}| \leq \aleph_1$. To simplify notation, we assume that $\mathbb{P}_\alpha \subseteq \omega_1$ and we assume

$$\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha \cap \varepsilon = \{(u,p) \in \mathbb{Q}_\alpha : u \subseteq \varepsilon\}.$$

We fix a witness $Y(\mathbb{P}_\alpha) \in \mathscr{D}_{\bar{M}}$ for the $\bar{M}$-c.c. of $\mathbb{P}_\alpha$, i.e., for every $\varepsilon \in Y(\mathbb{P}_\alpha)$ for every $I \in M_\varepsilon$ that is a dense subset of $\mathbb{P}_\alpha \cap \varepsilon$, I is dense in $\mathbb{P}_\alpha$.

We intersect $Y(\mathbb{P}_\alpha)$ with the club $C \subseteq \omega_1$ of countable limit ordinals that are closed under the functions $\gamma(\alpha,\cdot)$ and $\delta(\alpha,\cdot)$ that are defined as in Equations (3.1), (3.2). Since $\mathbb{P}_\alpha$ is c.c.c., such a club can be found in the ground model although $\delta(\alpha,\cdot)$ is a name.

Next we prove that $Y(\mathbb{P}_\alpha) \cap C$ witnesses that $\mathbb{P}_{\alpha+1}$ has the $\bar{M}$-c.c. Let $\varepsilon \in Y(\mathbb{P}_\alpha) \cap C$, $D \in M_\varepsilon$ be an open and dense subset of $(\mathbb{P}_\alpha \cap \varepsilon) * (\mathbb{Q} \cap \varepsilon)$. Let $p \in \mathbb{P}_{\alpha+1}$. We have to show that there is $q \in D$ that is compatible with p.

We write $p = (p \restriction \alpha, (u_{p(\alpha)}, h_{p(\alpha)}))$ and we assume that $p \restriction \alpha$ determines the finite sets $u_{p(\alpha)}$ and $h_{p(\alpha)}$ so that they are elements of $[\omega_1]^{<\omega}$ and ${}^{\omega>}\omega$ and that it also determines $\gamma(\alpha,\xi)$ and $\delta(\alpha,\xi)$ for any $\xi \in u_{p(\alpha)}$.

The search for q proceeds in four steps:

First step: We apply the induction hypothesis. We let $D' = D \cap \mathbb{P}_\alpha$; $D' \in M_\varepsilon$ is dense and open in $\mathbb{P}_\alpha \cap \varepsilon$. Since $\mathbb{P}_\alpha$ has the $\bar{M}$-c.c. and $\varepsilon \in Y(\mathbb{P}_\alpha)$ there is $q' \in D' \cap M_\varepsilon$ that is compatible with $p \restriction \alpha$. We fix a witness $r' \in \mathbb{P}_\alpha$ for compatibility.

Second step: We choose $(h', u_{p(\alpha)}) \geq p(\alpha)$ to take a record of r' on its finitely many Cohen coordinates by taking $n \in \omega$ so large such that

$$\begin{aligned}(3.3) \qquad &(\forall m)(\forall \xi \in u_{p(\alpha)})(\forall \beta = \beta(\alpha,\gamma(\alpha,\xi)) \in \operatorname{supp}(r'))\\ &\qquad\qquad ((r' \Vdash (m \notin A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)})) \rightarrow m < n).\end{aligned}$$

Such an n exists since r' pins down only a finite part of the name $A_{\beta(\alpha,\gamma(\beta,\xi)),\delta(\alpha,\xi)}$ for any $\xi \in u_{p(\alpha)}$ with $\beta(\alpha,\gamma(\alpha,\xi)) \in \operatorname{dom}(r')$. Now we let $\operatorname{dom}(h') = n$ and on $n \setminus \operatorname{dom}(h_{p(\alpha)})$ we fix some $h'(k) \geq f_{\alpha,\xi}(k)$ for all $\xi \in u_{p(\alpha)}$. We let $q' = (h', u_{p(\alpha)})$.

Third step: We go again into $D \cap M_\varepsilon$. With the maximum principle we choose $q(\alpha) \in M_\varepsilon$ such that $q' \Vdash \underset{\sim}{q(\alpha)} \geq_{\mathbb{Q}_\alpha} (u_{p(\alpha)} \cap \varepsilon, h') \wedge \underset{\sim}{q(\alpha)} \in D_\alpha[\mathbb{P}_\alpha]$ and let $q = (q', \underset{\sim}{q(\alpha)})$. Then $q = (q', \underset{\sim}{q(\alpha)}) \in M_\varepsilon \cap D$.

Fourth step: We show that p and q are compatible. For any $\xi \in u_{p(\alpha)} \setminus \varepsilon$ we choose $q_1(\beta(\alpha, \gamma(\alpha, \xi))) \geq q'(\beta(\alpha, \gamma(\alpha, \xi)))$ such that

$$q_1(\beta(\alpha,\gamma(\alpha,\xi))) \Vdash_{\mathbb{Q}_{\beta(\alpha,\gamma(\alpha,\xi))}} (\forall n \in \mathrm{dom}(h_{\underset{\sim}{q(\alpha)}} \setminus \mathrm{dom}(h')))(n \in A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}). \tag{3.4}$$

We let

$$r = (q' \cup \{(\beta(\alpha,\gamma(\alpha,\xi)), q_1(\beta(\alpha,\gamma(\alpha,\xi)))) : \xi \in u_{\underset{\sim}{p(\alpha)}} \setminus \varepsilon\}, (u_{\underset{\sim}{p(\alpha)}} \cup u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}})).$$

The condition r is well defined, since for any $\xi \in u_{p(\alpha} \setminus \varepsilon$, the condition $q_1(\beta(\alpha, \gamma(\alpha, \xi)) \in \mathbb{P}_\alpha$ can be chosen to be compatible with $q'(\beta(\alpha, \gamma(\alpha, \xi))$, by the choice of n as in Equation (3.3).

We show that $r \geq p, q$. First $r \restriction \alpha \geq p \restriction \alpha, q'$ and $q' = q \restriction \alpha$. We show that

$$r \restriction \alpha \Vdash_{\mathbb{P}_\alpha} (u_{\underset{\sim}{p(\alpha)}} \cup u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}}) \geq_{\mathbb{Q}_\alpha} (u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}}), (u_{\underset{\sim}{p(\alpha)}}, h').$$

The first is trivial. For the latter, let $\xi \in u_{\underset{\sim}{p(\alpha)}}$. First case: $\xi \in M_\delta$. We chose (after Equation (3.3)) the function $h_{q(\alpha)}(k)$ such that it dominates $f_{\alpha,\xi}(k)$ on any coordinate k not in $\mathrm{dom}(h_{p(\alpha)})$ such that $r' \Vdash k \notin A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}$. Thus $r \restriction \alpha$ forces the relevant instances of clause (6) of $r(\alpha) \geq p(\alpha)$.

Second case: $\xi \in u_{p(\alpha)} \setminus \varepsilon$. Since clause (6) speaks only about $m \in \omega \setminus A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}$, Equation (3.4) implies $r \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{r(\alpha)} \geq \underset{\sim}{q(\alpha)}$. ∎

Remark: We work with the assumption $\diamondsuit_{\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}}$. Alternatively, we could force as in the previous section by approximations of size $\aleph_1$ in a first step and thereafter force with the generic filter of the first forcing. The diamond $\diamondsuit_{\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}}$ hands down at stage α a possible $\mathbb{P}_\alpha$-name for objects D, $\langle g_i : i < \aleph_1 \rangle$ as in property $(\mathrm{A})_{\aleph_2}$ of Lemma 3.2 and thus allows to construct a finite support iteration up to stage ω_2 instead of using an approximation forcing in a first forcing step. So the partial order $\mathbb{P}$ of the sketched alternative construction corresponds in the actually performed forcing $AP * \mathbb{Q}$ to the generic $\underset{\sim}{\mathbb{Q}}$ of the approximation forcing AP.

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