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EXTERIOR SQUARE GAMMA FACTORS FOR CUSPIDAL REPRESENTATIONS OF GL_n: FINITE FIELD ANALOGS AND LEVEL-ZERO REPRESENTATIONS

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ABSTRACT

We follow Jacquet–Shalika [7], Matringe [12] and Cogdell–Matringe [3] to define exterior square gamma factors for irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. These exterior square gamma factors are expressed in terms of Bessel functions associated to the cuspidal representations. We also relate our exterior square gamma factors over finite fields to those over local fields through level-zero representations.

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1. Introduction

Let F be a p-adic local field of characteristic zero with residue field $\mathfrak{f}$. Fix a non-trivial additive character ψ of F. In their work [7], Jacquet and Shalika define important integrals which we call local Jacquet–Shalika integrals, see [7, Sections 7, 9.3]. These Jacquet–Shalika integrals enable them to introduce integral representations for the exterior square L-function $L(s,\pi,\wedge^2)$ of a (generic) representation π of $\mathrm{GL}_n(F)$. Later, Matringe [12] and Cogdell–Matringe [3] prove local functional equations for these local exterior square L-functions, in which local factors $\gamma(s,\pi,\wedge^2,\psi)$ and $\epsilon(s,\pi,\wedge^2,\psi)$ play an important role. These local factors are related via the following equation:

$$\gamma(s,\pi,\wedge^2,\psi) = \frac{\epsilon(s,\pi,\wedge^2,\psi)L(1-s,\widetilde{\pi},\wedge^2)}{L(s,\pi,\wedge^2)}. \tag{1.1}$$

If π is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$, then it is generic, and the local factors $\gamma(s,\pi,\wedge^2,\psi)$ and $\epsilon(s,\pi,\wedge^2,\psi)$ are defined. By type theory of Bushnell and Kutzko [2], π is constructed from some maximal simple type (J,λ); see [2, Section 6] for details. Partially, λ comes from some irreducible cuspidal representation π_0 of $\mathrm{GL}_m(\mathfrak{e})$, where $\mathfrak{e}$ is some finite extension of $\mathfrak{f}$. In this paper, we are interested in defining the exterior square gamma factor $\gamma(\pi_0,\wedge^2,\psi_0)$ for some non-trivial additive character ψ_0 of $\mathfrak{e}$, and we relate π to π_0 via their exterior square gamma factors, in the case where π is of level-zero. To be concrete, the main result of this paper is that if π is a level-zero representation constructed from π_0, a cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$ which does not admit a Shalika vector, then $\gamma(s,\pi,\wedge^2,\psi) = \gamma(\pi_0,\wedge^2,\psi_0)$.

In Section 2, we work with a general finite field $\mathbb{F}$ and a non-trivial additive character ψ. For a generic representation π of $\mathrm{GL}_n(\mathbb{F})$ where $n = 2m$ or $n = 2m+1$, we follow [7] to define the Jacquet–Shalika integral $J_{\pi,\psi}(W,\phi)$ and its dual $\tilde{J}_{\pi,\psi}(W,\phi)$, where W is a Whittaker function of π and ϕ is a complex valued function on $\mathbb{F}^m$. Using ideas from [12, 3], we define the exterior square gamma factor $\gamma(\pi,\wedge^2,\psi)$ of π in

Theorem: *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ that does not admit a Shalika vector. Then there exists a non-zero constant $\gamma(\pi,\wedge^2,\psi)$ such that*

$$\tilde{J}_{\pi,\psi}(W,\phi) = \gamma(\pi,\wedge^2,\psi)\cdot J_{\pi,\psi}(W,\phi),$$

for any Whittaker function $W \in \mathcal{W}(\pi,\psi)$ and any complex valued function ϕ on $\mathbb{F}^m$. Here $n = 2m$ or $n = 2m+1$.

This theorem is presented as Theorem 2.15 in Section 2. Its proof is separated into the even and odd cases, since the Jacquet–Shalika integrals differ in these cases. The proof relies heavily on some multiplicity one theorems, whose details can be found in Appendix A. If π admits a Shalika vector, we need to borrow results from the local field case in order to define the exterior square gamma factor, which is discussed in Theorem 3.13. By choosing some suitable W and ϕ, we are able to express $\gamma(\pi, \wedge^2, \psi)$ as a summation of the Bessel function $\mathcal{B}_{\pi,\psi}$ over some tori of $\mathrm{GL}_n(\mathbb{F})$, see Theorem 2.26 and Theorem 2.27. The ability to write local factors closely related to the exterior square gamma factor in terms of integrals of partial Bessel functions over tori as in [4, Proposition 4.6], is one of the key ingredients that Cogdell, Shahidi and Tsai use in order to prove the stability of exterior square gamma factors for local fields.

In Section 3, we assume that π is a level-zero representation of $\mathrm{GL}_n(F)$, where F is a p-adic local field of characteristic zero with residue field $\mathfrak{f}$. Let ψ be a non-trivial additive character of F, which descends to a non-trivial additive character ψ_0 of $\mathfrak{f}$. By type theory of Bushnell and Kutzko, π is constructed from a maximal simple type $(\mathrm{GL}_n(\mathfrak{o}), \pi_0)$, where $\mathfrak{o}$ is the ring of integers of F and π_0 is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$. Our main result is in Theorem 3.14, which states that if π_0 does not admit a Shalika vector, then

$$\gamma(s, \pi, \wedge^2, \psi) = \gamma(\pi_0, \wedge^2, \psi_0).$$

$\gamma(\pi_0, \wedge^2, \psi_0)$ is a non-zero constant, thus the above equation implies that $\gamma(s, \pi, \wedge^2, \psi)$ is a non-zero constant independent of s. The case where π_0 admits a Shalika vector is also treated in Theorem 3.15. We are also able to specify the local exterior square factors $L(s, \pi, \wedge^2)$ and $\epsilon(s, \pi, \wedge^2, \psi)$ explicitly along with the equalities of the exterior square gamma factors.

This paper grows out of part of a thesis project of the first author, and the master's thesis [16] of the second author.

2. The Jacquet–Shalika integral over a finite field

In this section, we define analogs of the local Jacquet–Shalika integrals [7, Section 7], [3, Section 3.2] over a finite field. We then prove that they satisfy a functional equation, which defines an important invariant, the exterior square gamma factor. This functional equation is valid only under some assumption on the relevant representation, which we restate in several equivalent ways. We then express the exterior square gamma factor in terms of the Bessel function of Gelfand [5].

2.1. Preliminaries and notations.

2.1.1. *Notations.* Let $\mathbb{F}$ be a finite field and denote $q = |\mathbb{F}|$. Fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. For every positive integer m, we denote by $\mathbb{F}_m$ the unique field extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$ of degree m. Let $\psi : \mathbb{F} \to \mathbb{C}^*$ be a non-trivial additive character. For a non-negative integer m, we denote $\mathcal{S}(\mathbb{F}^m) = \{f : \mathbb{F}^m \to \mathbb{C}\}$, the space of complex valued functions on $\mathbb{F}^m$. If $\phi \in \mathcal{S}(\mathbb{F}^m)$, we define its Fourier transform with respect to ψ by the formula

$$\mathcal{F}_\psi \phi(y) = q^{-\frac{m}{2}} \sum_{x \in \mathbb{F}^m} f(x) \psi(\langle x, y \rangle),$$

where $\langle x, y \rangle$ is the standard bilinear form on $\mathbb{F}^m$. We have the following Fourier inversion formulas:

$$\mathcal{F}_\psi \mathcal{F}_\psi \phi(x) = \phi(-x), \quad \mathcal{F}_{\psi^{-1}} \mathcal{F}_\psi \phi(x) = \phi(x).$$

For an irreducible generic representation π of $\mathrm{GL}_n(\mathbb{F})$, we denote by $\mathcal{W}(\pi, \psi)$ the Whittaker model of π with respect to the character ψ. For $W \in \mathcal{W}(\pi, \psi)$, we define $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$ by $\widetilde{W}(g) = W(w_n g^\iota)$, where

$$g^\iota = {}^t g^{-1} \quad \text{and} \quad w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let $n_1, \dots, n_r \geq 1$ with $n_1 + \cdots + n_r = n$ and $g_i \in \mathrm{GL}_{n_i}(\mathbb{F})$ for every i. We denote

$$\mathrm{antidiag}(g_1, \dots, g_r) = \begin{pmatrix} & & & g_1 \\ & & g_2 & \\ & \iddots & & \\ g_r & & & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}).$$

2.1.2. *The Bessel function.* Let (π, V_π) be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. By [5, Propositions 4.2, 4.3] there exists a unique element $\mathcal{B}_{\pi,\psi} \in \mathcal{W}(\pi, \psi)$ satisfying $\mathcal{B}_{\pi,\psi}(I_n) = 1$ and $\mathcal{B}_{\pi,\psi}(u_1 g u_2) = \psi(u_1)\psi(u_2)\mathcal{B}_{\pi,\psi}(g)$, for every $g \in \mathrm{GL}_n(\mathbb{F})$ and $u_1, u_2 \in N_n$, where N_n is the upper triangular unipotent subgroup of $\mathrm{GL}_n(\mathbb{F})$.

Theorem 2.1 ([5, Proposition 4.9]): *Let $w, d \in \mathrm{GL}_n(\mathbb{F})$, where w is a permutation matrix and d is a diagonal matrix. Suppose that $\mathcal{B}_{\pi,\psi}(dw) \neq 0$. Then $dw = \mathrm{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$, where $n_1 + \cdots + n_r = n$ and $\lambda_1, \dots, \lambda_r \in \mathbb{F}^*$.*

2.2. The Jacquet–Shalika integral. In this section, we define the Jacquet–Shalika integral analog over the finite field $\mathbb{F}$.

2.2.1. *The even case.* Let (π, V_π) be an irreducible generic representation of $\mathrm{GL}_{2m}(\mathbb{F})$. For $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}(\mathbb{F}^m)$ we define the Jacquet–Shalika integral as

$$J_{\pi,\psi}(W,\phi) = \frac{1}{[G:N][M:\mathcal{B}]} \times \sum_{g\in N\backslash G}\sum_{X\in\mathcal{B}\backslash M} W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}X)\phi(\varepsilon g).$$

Here $G = \mathrm{GL}_m(\mathbb{F})$, $N \le G$ is the upper triangular unipotent subgroup, $M = M_m(\mathbb{F})$, $\mathcal{B} \le M$ is the upper triangular matrix subspace, $\varepsilon = \varepsilon_m = \begin{pmatrix} 0 & \cdots & 0 & 1\end{pmatrix}$; σ_{2m} is the column permutation matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & | & m+1 & m+2 & \cdots & 2m \\ 1 & 3 & 5 & \cdots & 2m-1 & | & 2 & 4 & \cdots & 2m\end{pmatrix}.$$

We also define the dual Jacquet–Shalika integral as

$$\tilde{J}_{\pi,\psi}(W,\phi) = J_{\widetilde{\pi},\psi^{-1}}\left(\widetilde{\pi}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\widetilde{W}, \mathcal{F}_\psi\phi\right).$$

It is immediate from the definition that

Proposition 2.2 (Double-duality):

$$\tilde{J}_{\widetilde{\pi},\psi^{-1}}\left(\widetilde{\pi}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\widetilde{W}, \mathcal{F}_\psi\phi\right) = J_{\pi,\psi}(W,\phi).$$

One can show the following

Proposition 2.3:

$$\tilde{J}_{\pi,\psi}(W,\phi) = \frac{1}{[G:N][M:\mathcal{B}]} \times \sum_{g\in N\backslash G}\sum_{X\in\mathcal{B}\backslash M} W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}X)\mathcal{F}_\psi\phi(\varepsilon_1 g^\iota),$$

where $\varepsilon_1 = \begin{pmatrix} 1 & 0 & \cdots & 0\end{pmatrix}$.

The following proposition is a simple exercise that follows from Theorem 2.1. It will also follow as a special case of Lemma 2.28.

Proposition 2.4: *Let* $W = [G:N][M:\mathcal{B}]\pi(\sigma_{2m}^{-1})\mathcal{B}_{\pi,\psi}$, *and let*

$$\phi(x) = \delta_\varepsilon(x) = \begin{cases} 1 & x = \varepsilon, \\ 0 & \textit{else}. \end{cases}$$

Then

$$J_{\pi,\psi}(W,\phi) = 1.$$

Definition 2.5: The **Shalika subgroup** $S_{2m} \leq \mathrm{GL}_{2m}(\mathbb{F})$ is defined as

$$S_{2m} = \left\{ \begin{pmatrix} g & X \\ & g \end{pmatrix} \mid g \in \mathrm{GL}_m(\mathbb{F}),\ X \in M_m(\mathbb{F}) \right\}.$$

S_{2m} acts on $\mathcal{S}(\mathbb{F}^m)$ by

$$\left(\rho \begin{pmatrix} g & X \\ & g \end{pmatrix} \phi \right)(y) = \phi(yg).$$

We define a character $\Psi : S_{2m} \to \mathbb{C}^*$ by

$$\Psi \begin{pmatrix} g & X \\ & g \end{pmatrix} = \psi(\mathrm{tr}(Xg^{-1})).$$

One easily checks

Proposition 2.6 (Equivariance property): *Let* $s \in S_{2m}$ *and* $B \in \{J_{\pi,\psi}, \tilde{J}_{\pi,\psi}\}$*; we have*

$$B(\pi(s)W, \rho(s)\phi) = \Psi(s)B(W,\phi).$$

Definition 2.7: A vector $0 \neq v \in V_\pi$ is called a **Shalika vector**, if for every $s \in S_{2m}$ we have

$$\pi(s)v = \Psi(s)v.$$

Remark 2.8: Suppose that π admits a Shalika vector v. Then for every $a \in \mathbb{F}^*$ we have $aI_{2m} \in S_{2m}$ and $\omega_\pi(a)v = \pi(aI_{2m})v = v$, and therefore π has trivial central character.

2.2.2. *The odd case.* Let (π, V_π) be an irreducible generic representation of $\mathrm{GL}_{2m+1}(\mathbb{F})$. For $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}(\mathbb{F}^m)$ we define the Jacquet–Shalika integral as

$$\begin{aligned} &J_{\pi,\psi}(W,\phi)\\ &=\frac{1}{[G:N][M:\mathcal{B}]|M_{1\times m}(\mathbb{F})|}\\ &\quad\times\sum_{g\in N\backslash G}\sum_{X\in\mathcal{B}\backslash M}\sum_{Z\in M_{1\times m}(\mathbb{F})} W\left(\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1\end{pmatrix}\right)\\ &\qquad\times\psi(-\mathrm{tr}X)\phi(Z). \end{aligned}$$

Here the notations are the same as in the even case, except for σ_{2m+1} which is the column permutation matrix corresponding to the permutation

$$\sigma=\begin{pmatrix} 1 & 2 & 3 & \cdots & m & | & m+1 & m+2 & \cdots & 2m & | & 2m+1\\ 1 & 3 & 5 & \cdots & 2m-1 & | & 2 & 4 & \cdots & 2m & | & 2m+1\end{pmatrix}.$$

We also define the dual Jacquet–Shalika integral as

$$\tilde{J}_{\pi,\psi}(W,\phi)=J_{\widetilde{\pi},\psi^{-1}}\left(\widetilde{\pi}\begin{pmatrix} & I_m & \\ I_m & & \\ & & 1\end{pmatrix}\widetilde{W},\mathcal{F}_\psi\phi\right).$$

Again, we get immediately from the definition the following analog of Proposition 2.2:

Proposition 2.9 (Double-duality):

$$\tilde{J}_{\widetilde{\pi},\psi^{-1}}\left(\widetilde{\pi}\begin{pmatrix} & I_m & \\ I_m & & \\ & & 1\end{pmatrix}\widetilde{W},\mathcal{F}_\psi\phi\right)=J_{\pi,\psi}(W,\phi).$$

One can show the following two propositions which are similar to Propositions 2.3 and 2.4, respectively.

Proposition 2.10:

$$\tilde{J}_{\pi,\psi}(W,\phi) = \frac{1}{[G:N][M:\mathcal{B}]|M_{1\times m}(\mathbb{F})|} \times \sum_{g\in N\backslash G}\sum_{X\in\mathcal{B}\backslash M}\sum_{Z\in M_{1\times m}(\mathbb{F})} W\left(\begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix}\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1\end{pmatrix}\right) \times \psi(-\mathrm{tr}X)\mathcal{F}_\psi\phi(Z).$$

Proposition 2.11: *Let* $W = [G:N][M:\mathcal{B}]|M_{1\times m}(\mathbb{F})|\pi(\sigma_{2m+1}^{-1})\mathcal{B}_{\pi,\psi}$*, and* $\phi=\delta_0$ *be the indicator function of* $0\in\mathbb{F}^m$*. Then*

$$J_{\pi,\psi}(W,\phi)=1.$$

Definition 2.12: The **Shalika subgroup** $S_{2m+1}\le \mathrm{GL}_{2m+1}(\mathbb{F})$ is defined as

$$S_{2m+1} = \left\{ \begin{pmatrix} g & X & Y \\ & g & \\ & Z & 1\end{pmatrix} \mid \begin{array}{l} g\in\mathrm{GL}_m(\mathbb{F}),\ X\in M_m(\mathbb{F}), \\ Y\in M_{m\times 1}(\mathbb{F}),\ Z\in M_{1\times m}(\mathbb{F})\end{array} \right\}.$$

Let $P_{2m+1}\le \mathrm{GL}_{2m+1}(\mathbb{F})$ be the Mirabolic subgroup, i.e., the subgroup of matrices having $\varepsilon_{2m+1} = \begin{pmatrix} 0 & \cdots & 0 & 1\end{pmatrix}$ as their last row. We define a character $\Psi : S_{2m+1}\cap P_{2m+1}\to\mathbb{C}^*$ by

$$\Psi\begin{pmatrix} g & X & Y \\ & g & \\ & & 1\end{pmatrix} = \psi(\mathrm{tr}(Xg^{-1})).$$

The Shalika subgroup acts on $\mathcal{S}(\mathbb{F}^m)$ by the following relations [3, Proposition 3.1]:

- $\rho\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\phi(x) = \phi(xg).$
- $\rho\begin{pmatrix} I_m & z_0 & \\ & I_m & \\ & & 1\end{pmatrix}\phi(x) = \psi(-\mathrm{tr}z_0)\phi(x).$

- $\rho \begin{pmatrix} I_m & & y_0 \\ & I_m & \\ & & 1 \end{pmatrix} \phi(x) = \psi(\langle x, y_0 \rangle)\phi(x).$
- $\rho \begin{pmatrix} I_m & & \\ & I_m & \\ & x_0 & 1 \end{pmatrix} \phi(x) = \phi(x + x_0).$

As in [3, Proposition 3.1], one has

Proposition 2.13: *The action of S_{2m+1} on $\mathcal{S}(\mathbb{F}^m)$ is equivalent to*

$$\mathrm{Ind}_{P_{2m+1} \cap S_{2m+1}}^{S_{2m+1}}(\Psi^{-1}),$$

given by mapping $f \in \mathrm{Ind}_{P_{2m+1} \cap S_{2m+1}}^{S_{2m+1}}(\Psi^{-1})$ to $\phi \in \mathcal{S}(\mathbb{F}^m)$, defined as

$$\phi(x) = f\left(\begin{smallmatrix} I_m & & \\ & I_m & \\ & x & 1 \end{smallmatrix}\right).$$

As in [3, Lemma 3.2, 3.3], we have the following analog of Proposition 2.6:

Proposition 2.14 (Equivariance property): *Let $B \in \{J_{\pi,\psi}, \tilde{J}_{\pi,\psi}\}$. Then for every $s \in S_{2m+1}$,*

$$B(\pi(s)W, \rho(s)\phi) = B(W, \phi).$$

Since in the odd case, the character Ψ is defined only on the subgroup $S_{2m+1} \cap P_{2m+1}$, we don't have a definition for a Shalika vector in this case.

2.3. The functional equation. In this section, we prove the functional equation satisfied by the Jacquet–Shalika integral. This allows us to define the exterior square gamma factor of an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$.

Theorem 2.15 (The functional equation): *Let $n = 2m$ or $n = 2m + 1$, and let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$. If n is even, suppose that π does not admit a Shalika vector. Then there exists a non-zero constant $\gamma(\pi, \wedge^2, \psi) \in \mathbb{C}^*$, such that for every $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(\mathbb{F}^m)$, we have*

$$\tilde{J}_{\pi,\psi}(W, \phi) = \gamma(\pi, \wedge^2, \psi) \cdot J_{\pi,\psi}(W, \phi).$$

The proof of this theorem is based on the proofs for the local field case of the local Jacquet–Shalika integrals [12, Section 4], [3, Section 3.3]. The Shalika subgroup S_n plays an important role in the proof. We treat the even case and the odd case separately. In both cases, the main idea is to show that the space of S_n equivariant bilinear forms

$$B : \mathcal{W}(\pi, \psi) \times \mathcal{S}(\mathbb{F}^m) \to \mathbb{C}$$

is at most one-dimensional. Since $J_{\pi,\psi}$ and $\tilde{J}_{\pi,\psi}$ define non-zero elements of this space (see Propositions 2.4 and 2.6 for the even case, and Propositions 2.11 and 2.14 for the odd case), the theorem follows. Since the proofs of the even case and odd case differ by only a little, we will only stress the proof of the even case.

Definition 2.16: We call $\gamma(\pi, \wedge^2, \psi)$ in the above theorem the **exterior square gamma factor** of π with respect to the character ψ.

Remark 2.17: By double duality (Proposition 2.2 and Proposition 2.9), we have that

$$\gamma(\pi, \wedge^2, \psi) \cdot \gamma(\widetilde{\pi}, \wedge^2, \psi^{-1}) = 1.$$

Substituting in the functional equation the functions from Proposition 2.4 and Proposition 2.11 in the even case and the odd case respectively, and using the fact that $\mathcal{B}_{\widetilde{\pi},\psi^{-1}} = \overline{\mathcal{B}_{\pi,\psi}}$, we get that $\overline{\gamma(\pi, \wedge^2, \psi)} = \gamma(\widetilde{\pi}, \wedge^2, \psi^{-1})$, and therefore $|\gamma(\pi, \wedge^2, \psi)| = 1$.

The proof of the functional equation relies on the following lemma.

LEMMA 2.18: *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S_n \cap P_n}(\pi, \Psi) \leq 1.$$

Here $P_n \leq \mathrm{GL}_n(\mathbb{F})$ is the mirabolic subgroup.

Proof. We only prove the lemma in the case $n = 2m$. The case $n = 2m + 1$ is similar. We define a homomorphism

$$\Lambda : \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi) \to \mathrm{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1)$$

by

$$\Lambda(L)(v) = \frac{1}{|\mathrm{GL}_m(\mathbb{F})|} \sum_{g \in \mathrm{GL}_m(\mathbb{F})} L\left(\pi\begin{pmatrix} g & \\ & I_m \end{pmatrix} v\right),$$

where $M_{m,m} \le \mathrm{GL}_{2m}(\mathbb{F})$ is the Levi subgroup corresponding to the partition (m,m). We claim that Λ is injective: suppose that $L \neq 0$ and that $v \in V_\pi$ satisfies $L(v) \neq 0$. We define, for a function $\Phi : M_m(\mathbb{F}) \to \mathbb{C}$, a vector $v_\Phi \in V_\pi$ by

$$v_\Phi = \frac{1}{\sqrt{|M_m(\mathbb{F})|}} \sum_{X \in M_m(\mathbb{F})} \Phi(X)\pi \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} v.$$

Then

$$\Lambda(L)v_\Phi = \frac{1}{|\mathrm{GL}_m(\mathbb{F})|} \sum_{g \in \mathrm{GL}_m(\mathbb{F})} \mathcal{F}_\psi \Phi(g) L \left(\pi \begin{pmatrix} g & \\ & I_m \end{pmatrix} v \right).$$

Choosing Φ such that $\mathcal{F}_\psi \Phi = \delta_{I_m}$, we get that $\Lambda(L)v_\Phi = \frac{1}{|\mathrm{GL}_m(\mathbb{F})|} v \neq 0$, and therefore $\Lambda(L) \neq 0$. Thus, Λ is injective. The lemma then follows from the multiplicity one theorem below. ∎

THEOREM 2.19: *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1) \le 1.$$

Here $M_{m,m} \le \mathrm{GL}_{2m}(\mathbb{F})$ is the Levi subgroup corresponding to the partition (m,m).

We have a similar multiplicity one theorem for $n = 2m+1$, but with $M_{m,m}$ replaced by a conjugation L_{2m+1} of $M_{m+1,m}$. The proof of Theorem 2.19 will be given in the appendix, as it is a detour from the main line of the paper. Theorem 2.19 is restated and proved in Theorem A.1, while the one for the odd case is Theorem A.2.

Proof of Theorem 2.15, $n = 2m$. The idea is to show that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(\mathbb{F}^m), \Psi) \le 1.$$

Since $J_{\pi,\psi}$ and $\tilde{J}_{\pi,\psi}$ define non-zero elements of $\mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(\mathbb{F}^m), \Psi)$, it follows that such a constant exists.

We first claim that the restriction map

$$\mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(\mathbb{F}^m), \Psi) \to \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(\mathbb{F}^m \setminus \{0\}), \Psi)$$

is injective, where $\mathcal{S}(\mathbb{F}^m \setminus \{0\})$ is realized as a subspace of $\mathcal{S}(\mathbb{F}^m)$ by the set of elements of $\mathcal{S}(\mathbb{F}^m)$, vanishing at zero.

Suppose that this restriction map is not injective. Then there exists

$$0 \neq B : V_\pi \times \mathcal{S}(\mathbb{F}^m) \to \mathbb{C}$$

satisfying

$$B(\pi(s)v, \rho(s)\phi) = \Psi(s)B(v, \phi)$$

for every $s \in S_{2m}$, such that $b \in \widetilde{V_\pi}$ defined by $b(v) = B(v, \delta_0)$ is not the zero functional, where δ_0 is the indicator function of 0 in $\mathbb{F}^m$. Let $(\cdot, \cdot)$ be an inner product on V_π with respect to which π is unitary, and let $0 \neq v_0 \in V_\pi$, such that $b(v) = (v, v_0)$. Then since $\rho(s)\delta_0 = \delta_0$ for every $s \in S_{2m}$, it follows from the equivariance property of B that

$$b(\pi(s)v) = B(\pi(s)v, \delta_0) = B(\pi(s)v, \rho(s)\delta_0) = \Psi(s)B(v, \delta_0) = \Psi(s)b(v).$$

Thus, for any $s \in S_{2m}$ and $v \in V_\pi$, we have

$$\begin{aligned}(v, \pi(s)v_0) &= (\pi(s^{-1})v, v_0) = b(\pi(s^{-1})v) \\ &= \Psi(s^{-1})b(v) = \overline{\Psi(s)}(v, v_0) = (v, \Psi(s)v_0).\end{aligned}$$

This shows that v_0 is a Shalika vector, which is a contradiction.

Next we write a sequence of isomorphisms from which it follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(\mathbb{F}^m \setminus \{0\}), \Psi) \leq 1.$$

Since $(S_{2m} \cap P_{2m})\backslash S_{2m} \cong \mathbb{F}^m \setminus \{0\}$ by the map $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \mapsto \varepsilon g$, we have that

$$\mathcal{S}(\mathbb{F}^m \setminus \{0\}) \cong \mathrm{Ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)$$

and we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{S_{2m}}(\pi \otimes \mathrm{Ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1), \Psi) &\cong \mathrm{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \widetilde{\mathrm{Ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)}) \\ &\cong \mathrm{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \mathrm{Ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(\widetilde{1})).\end{aligned}$$

Thanks to Frobenius reciprocity, we get that

$$\begin{aligned}\mathrm{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \mathrm{Ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)) &\cong \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\Psi^{-1} \otimes \pi, 1) \\ &\cong \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi).\end{aligned}$$

By Lemma 2.18, the last space has dimension ≤ 1. ∎

Remark 2.20: As seen in the proof, the proof fails if π admits a Shalika vector. In this case, a modified functional equation is valid. This is discussed in Theorem 3.13.

2.3.1. *Equivalent conditions for admitting a Shalika vector.* Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F})$. In this section we state equivalent conditions for π to admit a Shalika vector.

Proposition 2.21: *The representation π admits a Shalika vector if and only if there exists $W \in \mathcal{W}(\pi, \psi)$ such that $J_{\pi,\psi}(W, 1) \neq 0$, where 1 denotes the constant function valued 1 on $\mathbb{F}^m$.*

Proof. Suppose that there exists $W \in \mathcal{W}(\pi, \psi)$ such that $J_{\pi,\psi}(W, 1) \neq 0$. Define $W_0 \in \mathcal{W}(\pi, \psi)$ by

$$W_0(h) = \frac{1}{[G : N]} \frac{1}{[M : \mathcal{B}]} \sum_{g \in N \backslash G} \sum_{X \in \mathcal{B} \backslash M} W\left(h \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(-\mathrm{tr} X).$$

Then $W_0 \neq 0$ as $W_0(\sigma_{2m}) = J_{\pi,\psi}(W, 1)$. By changing variables, we can show that

$$W_0(hs) = \Psi(s) W_0(h),$$

for every $s \in S_{2m}$, thus W_0 is a Shalika vector.

For the other direction, assume π admits a non-zero Shalika vector $W_0 \in \mathcal{W}(\pi, \psi)$. Choose an inner product $(\cdot, \cdot)$ on $\mathcal{W}(\pi, \psi)$, with respect to which π is unitary. Then W_0 defines a non-zero element $T_{W_0} \in \mathrm{Hom}_{S_{2m}}(\pi, \Psi)$ by

$$T_{W_0}(W') = (W', W_0).$$

We have that

$$\mathrm{Hom}_{S_{2m}}(\pi, \Psi) \subseteq \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi).$$

By Lemma 2.18, we have in this case that

$$0 \neq \mathrm{Hom}_{S_{2m}}(\pi, \Psi) = \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi).$$

We define $W \in \mathcal{W}(\pi, \psi)$ by

$$W(h) = \sum_{g \in N \backslash P} \sum_{X \in \mathcal{B} \backslash M} \mathcal{B}_{\pi,\psi}\left(h \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1}\right) \psi(-\mathrm{tr} X),$$

where $P = P_m(\mathbb{F}) = \{g \in \mathrm{GL}_m(\mathbb{F}) \mid \varepsilon_m g = \varepsilon_m\}$ is the mirabolic subgroup. We can show that for $s \in S_{2m} \cap P_{2m}$,

$$W(hs) = \Psi(s) W(h).$$

Therefore, W defines an element $T_W \in \mathrm{Hom}_{S_{2m}\cap P_{2m}}(\pi, \Psi)$ by

$$T_W(W') = (W', W).$$

Thus $T_W \in \mathrm{Hom}_{S_{2m}}(\pi, \Psi)$. This implies that W is a Shalika vector. We can also show that $J_{\pi,\psi}(W, 1) = W(\sigma_{2m})$. By Proposition 2.4, we have that $W(\sigma_{2m}) = 1$, hence $J_{\pi,\psi}(W, 1) = W(\sigma_{2m}) \neq 0$. ∎

Corollary 2.22: *If π admits a Shalika vector, then the functional equation in Theorem 2.15 does not hold.*

Proof. Let $W \in \mathcal{W}(\pi, \psi)$, such that $J_{\pi,\psi}(W, 1) = 1$. Then since $\mathcal{F}_\psi 1 = q^{\frac{m}{2}} \cdot \delta_0$, we have that $\tilde{J}_{\pi,\psi}(W, 1) = 0$, and therefore no such non-zero constant exists. ∎

Suppose that π is associated with the regular character $\theta : \mathbb{F}^*_{2m} \to \mathbb{C}^*$ [6]. We now recall the work of Prasad [14] in order to classify when π admits a Shalika vector, in terms of θ.

Let $N_{m,m}$ be the unipotent radical of $\mathrm{GL}_{2m}(\mathbb{F})$ corresponding to the partition (m, m). Denote by $V_{\pi_{N_{m,m},\Psi}}$ the twisted Jacquet submodule of $N_{m,m}$ with respect to the character Ψ:

$$V_{\pi_{N_{m,m},\Psi}} = \left\{ v \in V_\pi \mid \forall X \in M_m(\mathbb{F}),\ \pi \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} v = \psi(\mathrm{tr} X) v \right\}.$$

Then $V_{\pi_{N_{m,m},\Psi}}$ is a subspace invariant under the action $\pi_{N_{m,m},\Psi}$ of $\mathrm{GL}_m(\mathbb{F})$, defined by $\pi_{N_{m,m},\Psi}(g)v = \pi\begin{pmatrix} g & \\ & g \end{pmatrix} v$, where $g \in \mathrm{GL}_m(\mathbb{F})$, $v \in V_{\pi_{N_{m,m},\Psi}}$. Prasad shows

Theorem 2.23 ([14, Theorem 1]): $\pi_{N_{m,m},\Psi} \cong \mathrm{Ind}_{\mathbb{F}^*_m}^{\mathrm{GL}_m(\mathbb{F})}(\theta \restriction_{\mathbb{F}^*_m})$.

From the definition of a Shalika vector and the definition of the representation $\pi_{N_{m,m},\Psi}$, we see that π admits a Shalika vector if and only if

$$0 \neq \mathrm{Hom}_{\mathrm{GL}_m(\mathbb{F})}(1, \pi_{N_{m,m},\Psi}) \cong \mathrm{Hom}_{\mathbb{F}^*_m}(1, \theta \restriction_{\mathbb{F}^*_m}).$$

The isomorphism above comes from Frobenius reciprocity. The latter space is non-zero if and only if $\theta \restriction_{\mathbb{F}^*_m} = 1$, and then it is one-dimensional. Therefore we get the following

Corollary 2.24: *π admits a Shalika vector if and only if $\theta \restriction_{\mathbb{F}^*_m} = 1$, and in this case, the space of Shalika vectors is*

We conclude this section with a theorem.

Theorem 2.25 (Equivalent conditions for admitting a Shalika vector): *Suppose π is an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F})$ associated with the regular character*

$$\theta : \mathbb{F}_{2m}^* \to \mathbb{C}^*.$$

The following are equivalent.

1. (1) *π admits a Shalika vector.*
2. (2) *There exists $W \in \mathcal{W}(\pi, \psi)$, such that $J_{\pi,\psi}(W, 1) \neq 0$.*
3. (3) *$\theta \restriction_{\mathbb{F}_m^*} = 1$.*

Moreover, in these cases, the space of Shalika vectors is one-dimensional.

2.4. An expression for the exterior square gamma factor. Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$. If n is even, suppose that π does not admit a Shalika vector.

In this section we express the exterior square gamma factor of π in terms of the Bessel function. One may use Proposition 2.4 and Proposition 2.11 in order to get such an expression, but the Jacquet–Shalika integral sums over too many elements, some of which are not in the support of the Bessel function. We find a more accurate expression which involves only elements of the form of Theorem 2.1.

Our main results of this section are the following theorems:

Theorem 2.26: *If $n = 2m$, the exterior square gamma factor is given by the formula*

$$\begin{aligned}\gamma(\pi, \wedge^2, \psi) &= q^{-\frac{m}{2} + 2\binom{m}{2}} \sum_{\substack{m_1,\dots,m_r \geq 1 \\ m_1 + \dots + m_r = m \\ \lambda_1,\dots,\lambda_r \in \mathbb{F}^*}} q^{-\sum_{i=1}^r 2\binom{m_i}{2}} \\ &\qquad \times \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r})^{-1}) \psi(\lambda_r \delta_{m_r,1}),\end{aligned}$$

where

$$\delta_{m_r,1} = \begin{cases} 1 & m_r = 1, \\ 0 & m_r \neq 1. \end{cases}$$

Theorem 2.27: *If $n = 2m+1$, the exterior square gamma factor is given by the formula*

$$\gamma(\pi, \wedge^2, \psi) = q^{\frac{m}{2}+2\binom{m}{2}} \sum_{\substack{m_1,\dots,m_r \geq 1 \\ m_1+\dots+m_r=m \\ \lambda_1,\dots,\lambda_r \in \mathbb{F}^*}} q^{-\sum_{i=1}^r 2\binom{m_i}{2}} \times \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r}, 1 I_1)^{-1}).$$

The following lemma indicates which representatives for g and X in the Jacquet–Shalika integral contribute to the sum. It is a key to the proofs of Theorem 2.26 and Theorem 2.27. We will only give the proof of Theorem 2.26, as the proof of Theorem 2.27 is quite similar.

Lemma 2.28: *Let $g \in \mathrm{GL}_m(\mathbb{F})$ and $X \in \mathcal{N}_m^-(\mathbb{F})$ a lower triangular nilpotent matrix (i.e., a lower triangular matrix with zeros on its diagonal). Suppose that $g = wdu$, where $u \in N_m(\mathbb{F})$, w is a permutation matrix and d is a diagonal matrix. Denote by τ the permutation defined by the columns of w. Write $X = (x_{ij})$. Suppose that*

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} \in N_{2m}\, \mathrm{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) N_{2m}, \tag{2.1}$$

where $\lambda_1, \dots, \lambda_r \in \mathbb{F}^*$, and $n_1 + \dots + n_r = 2m$. Then:

1. *$wd = \mathrm{antidiag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})$, where $n_i = 2m_i$, for every $1 \leq i \leq r$ (and therefore $m_1 + \dots + m_r = m$).*
2. *$x_{ij} = 0$ for every (i,j) satisfying $j < i$ and $\tau^{-1}(j) < \tau^{-1}(i)$.*

Furthermore, in this case,

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \sigma_{2m}^{-1} = \mathrm{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r}) \cdot v,$$

where $v \in N_{2m}(\mathbb{F})$ is an upper triangular unipotent matrix with zeros right above its diagonal.

Proof. Since $\sigma_{2m} \left(\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right) \sigma_{2m}^{-1} \in N_{2m}$, we have by Equation (2.1) that

$$\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} wd & \\ & wd \end{pmatrix} \sigma_{2m}^{-1} \in N_{2m}\, \mathrm{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) N_{2m}.$$

Let

$$w' = \sigma_{2m} \begin{pmatrix} w & \\ & w \end{pmatrix} \sigma_{2m}^{-1}.$$

Then w' is a column permutation matrix of the permutation τ', where $\tau'(2j) = 2\tau(j)$, and $\tau'(2j-1) = 2\tau(j) - 1$, for every $1 \leq j \leq m$, therefore

$$w' = \begin{pmatrix} e_{2\tau(1)-1} & e_{2\tau(1)} & \dots & e_{2\tau(m)-1} & e_{2\tau(m)} \end{pmatrix},$$

where e_i is the i-th standard column vector in $\mathbb{F}^{2m}$.

Let $d = \mathrm{diag}(a_1, \dots, a_m)$ and $d' = \sigma_{2m}\left(\begin{smallmatrix} d & \\ & d \end{smallmatrix}\right)\sigma_{2m}^{-1}$. Then

$$d' = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_m, a_m).$$

Let

$$U_X = \sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \sigma_{2m}^{-1}.$$

Since $X = (x_{ij})$ is a lower triangular nilpotent matrix, U_X is a lower triangular unipotent matrix, and its columns are given by $\mathrm{column}_{2j-1}(U_X) = e_{2j-1}$ and

$$\mathrm{column}_{2j}(U_X) = e_{2j} + \sum_{i=j+1}^{m} x_{ij} e_{2i-1}.$$

Let

$$Z = \sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} wd & \\ & wd \end{pmatrix} \sigma_{2m}^{-1} = U_X w' d'.$$

We get that $\mathrm{column}_{2j-1}(Z) = a_j e_{2\tau(j)-1}$ and

$$\mathrm{column}_{2j}(Z) = a_j e_{2\tau(j)} + \sum_{i=\tau(j)+1}^{m} a_j x_{i\tau(j)} e_{2i-1}.$$

We claim that Z has the form

$$Z = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda_1 I_{2m_1} \\ 0 & 0 & \dots & \lambda_2 I_{2m_2} & * \\ 0 & 0 & \iddots & * & * \\ 0 & \lambda_{r-1} I_{2m_{r-1}} & \cdots & * & * \\ \lambda_r I_{2m_r} & * & \cdots & * & * \end{pmatrix},$$

where $2m_i = n_i$, for every $1 \leq i \leq r$.

The proof of the claim is by applying row and column reduction carefully, in order to obtain the diagonal and permutation matrices that are involved in the Bruhat decomposition of Z. Every $h \in \mathrm{GL}_n(\mathbb{F})$ has a Bruhat decomposition $h = u_1 w_h d_h u_2$, where $u_1, u_2 \in N_n(\mathbb{F})$, w_h is a permutation matrix and d_h is a diagonal matrix. Such w_h and d_h are unique. Throughout the text, we refer to the uniqueness of w_h and d_h as the **uniqueness of the Bruhat decomposition**. We will first show that Z is of the form

$$Z = \begin{pmatrix} 0 & * \\ \lambda_r I_{2m_r} & * \end{pmatrix}.$$

By the above description of the columns of Z, we have that the first column of Z is $a_1 \cdot e_{2\tau(1)-1}$. Let $2m - 2\tau(1) + 2 = 2m_r$. We show by induction that for every $1 \le l \le 2m_r$,

$$\mathrm{column}_l(Z) = \lambda_r e_{2m-2m_r+l}.$$

For $l = 1$, we have $\mathrm{column}_1(Z) = a_1 e_{2m-2m_r+1}$. By assumption

$$Z \in N_{2m} \operatorname{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) N_{2m}$$

and therefore by the uniqueness of the Bruhat decomposition of Z, we must have $a_1 = \lambda_r$. Assume that the claim is true for all columns before l, that is to say, $\mathrm{column}_i(Z) = \lambda_r e_{2m-2m_r+i}$ for $1 \le i < l$. Since

$$\mathrm{column}_{l-1}(Z) = \lambda_r e_{2m-2m_r+l-1},$$

we expect $\mathrm{column}_l(Z) = \lambda_r e_{2m-2m_r+l}$.

If $l = 2j-1$ is odd, then since $\mathrm{column}_{2j-1}(Z) = a_j e_{2\tau(j)-1}$, in order for Z to have the required Bruhat decomposition $Z = u_1 \operatorname{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) u_2$ for some $u_1, u_2 \in N_{2m}(\mathbb{F})$, we must have that

$$a_j = \lambda_r \quad \text{and} \quad 2\tau(j) - 1 = 2m - 2m_r + 2j - 1,$$

i.e., $\tau(j) = m - m_r + j$.

If $l = 2j$ is even, then we have that

$$\mathrm{column}_{2j}(Z) = \lambda_r e_{2m-2m_r+2j} + \sum_{i=m-m_r+j+1}^{m} \lambda_r x_{i,m-m_r+j} e_{2i-1}.$$

If any of the $x_{i,m-m_r+j}$ is non-zero, then by applying row reduction (by multiplying from the left by an upper triangular unipotent matrix that annihilates all the elements above the lowest element of $\mathrm{column}_{2j}(Z)$), we get that the Bruhat decomposition of Z is not of the required form. Therefore we get that $\mathrm{column}_{2j}(Z) = \lambda_r e_{2m-2m_r+2j}$, as required.

We have shown that for every $1 \leq l \leq 2m_r$, $\mathrm{column}_l(Z) = \lambda_r e_{2m-2m_r+l}$. By the uniqueness of the Bruhat decomposition of Z, we must have $n_r = 2m_r$.

We now have that

$$Z = \begin{pmatrix} 0 & Z' \\ \lambda_r I_{2m_r} & A \end{pmatrix},$$

where $Z' \in \mathrm{GL}_{2m-2m_r}(\mathbb{F})$ and $A \in M_{2m_r \times (2m-2m_r)}(\mathbb{F})$. We have that

$$Z \cdot \begin{pmatrix} I_{2m_r} & -\lambda_r^{-1} A \\ 0 & I_{2m-2m_r} \end{pmatrix} = \begin{pmatrix} 0 & Z' \\ \lambda_r I_{2m_r} & 0 \end{pmatrix}.$$

By uniqueness of the Bruhat decomposition of Z, we must have that

$$Z' \in N_{2m-2m_r} \operatorname{antidiag}(\lambda_1 I_{n_1}, \dots, \lambda_{n_{r-1}} I_{n_{r-1}}) N_{2m-2m_r}.$$

Since Z' inherits an analogous column description from Z but is of smaller size than that of Z, we get by induction on the size of the matrix, i.e., by repeating the above steps applied to Z, that

$$Z' = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda_1 I_{2m_1} \\ 0 & 0 & \dots & \lambda_2 I_{2m_2} & * \\ 0 & 0 & \iddots & * & * \\ 0 & \lambda_{r-2} I_{2m_{r-2}} & \cdots & * & * \\ \lambda_{r-1} I_{2m_{r-1}} & * & \cdots & * & * \end{pmatrix},$$

and that $n_i = 2m_i$ for every $1 \leq i \leq r-1$, and therefore Z has the desired form.

By the above claim, we get that the Bruhat decomposition of Z is $Z = w_Z d_Z u_Z$, where $w_Z d_Z = \operatorname{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r})$ (w_Z is a permutation matrix, d_Z is a diagonal matrix, and u_Z is an upper triangular unipotent matrix). We conclude from $\sigma_{2m} \left(\begin{smallmatrix} wd & \\ & wd \end{smallmatrix} \right) \sigma_{2m}^{-1} = w_Z d_Z$ that

$$wd = \operatorname{antidiag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r}),$$

which is the first part of the lemma. We also conclude from the induction process of the above claim that $x_{\tau(i)\tau(j)} = 0$ for every $i > j$ with $\tau'(2i-1) > \tau'(2j)$, which is equivalent to $x_{ij} = 0$ for every $i > j$ with $\tau^{-1}(i) > \tau^{-1}(j)$. This finishes the second part of the lemma.

Finally, we write $Z = w'd'(d'^{-1} w'^{-1} U_X w' d')$. We claim that $w'^{-1} U_X w'$ is an upper triangular matrix with zeros right above its diagonal: the only non-zero non-diagonal components of U_X are located in the positions of the form $(2i-1, 2j)$ with values x_{ij} for $j < i$, and these move in the conjugation $w'^{-1} U_X w'$

to $(\tau'^{-1}(2i-1),\tau'^{-1}(2j)) = (2\tau^{-1}(i)-1, 2\tau^{-1}(j))$. If $2\tau^{-1}(j) < 2\tau^{-1}(i)-1$, then we get that $x_{ij}=0$, and therefore $d'^{-1}w'^{-1}U_X w'd'$ is an upper unipotent matrix with zeros right above its diagonal. Since

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1} = Z\sigma_{2m}\begin{pmatrix} u & \\ & u \end{pmatrix}\sigma_{2m}^{-1},$$

and since $\sigma_{2m}\left(\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right)\sigma_{2m}^{-1}$ is an upper triangular unipotent matrix with zeros right above its diagonal, it follows that

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1} = \operatorname{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r}) \cdot v,$$

for some upper triangular unipotent matrix $v \in N_{2m}(\mathbb{F})$ with zeros right above its diagonal. ∎

We need one more lemma, regarding the number of elements involved in the Jacquet–Shalika integral, such that g has a given Bruhat decomposition. As we'll see in the proofs of Theorem 2.26 and Theorem 2.27, in order for a coset $g \in N\backslash G$ to contribute to the Jacquet–Shalika integral, we must have $g \in N \operatorname{antidiag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})N$. Let $wd = \operatorname{antidiag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})$. Given $g \in NwdN$ and $X \in \mathcal{N}^-$ as in Lemma 2.28, we'll see in the following proofs that the summand of the Jacquet–Shalika integral on a special choice of functions, depends only on wd. To evaluate the Jacquet–Shalika integral, we should count the number of cosets in the set $\{Nwdu \mid u \in N\} \subseteq N\backslash G$ and the number of options for a matrix $X \in \mathcal{N}^-$ satisfying the condition of Lemma 2.28.

Lemma 2.29: *Let* $g = wd = \operatorname{antidiag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})$, *where*

$$w = \operatorname{antidiag}(I_{m_1}, \dots, I_{m_r})$$

is a permutation matrix, $d = \operatorname{diag}(\lambda_r I_{m_r}, \dots, \lambda_1 I_{m_1})$ *is a diagonal matrix,* $m_1 + \dots + m_r = m$, *and* $\lambda_1, \dots, \lambda_r \in \mathbb{F}^*$.

(1) *Consider the right action of the upper triangular unipotent subgroup* $N = N_m$ *on* $N\backslash G$, *where* $G = \mathrm{GL}_m(\mathbb{F})$. *Then the orbit of* Ng *is of size* $q^{\binom{m}{2} - \sum_{i=1}^r \binom{m_i}{2}}$.

(2) *Let* τ *be the permutation corresponding to columns of* w. *Then the set*

$$\{X \in \mathcal{N}_m^- \mid x_{ij} = 0, \forall 1 \le j < i \le m, \text{ s.t. } \tau^{-1}(j) < \tau^{-1}(i)\}$$

is of cardinality $q^{\binom{m}{2} - \sum_{i=1}^r \binom{m_i}{2}}$.

Proof. By the orbit-stabilizer theorem, we have that the orbit of Nwd is of size $[N : \mathrm{stab}_N(Nwd)]$. We have that

$$\begin{aligned}\mathrm{stab}_N(Nwd) &= N \cap w^{-1}Nw \\ &= \{(u_{ij}) \in N \mid u_{ij} = 0, \forall i < j \text{ s.t. } \tau^{-1}(i) > \tau^{-1}(j)\}.\end{aligned}$$

Therefore

$$\log_q |\mathrm{stab}_N(Nwd)| = |\{(i,j) \mid i < j \text{ and } \tau^{-1}(i) < \tau^{-1}(j)\}| = \sum_{i=1}^{r} \binom{m_i}{2},$$

and the first part is proved. The second part follows from the fact that

$$|\{(i,j) \mid j < i \text{ and } \tau^{-1}(j) < \tau^{-1}(i)\}| = \sum_{i=1}^{r} \binom{m_i}{2}. \qquad \blacksquare$$

We are now ready to prove Theorem 2.26.

Proof of Theorem 2.26. We have $\mathcal{F}_\psi \delta_{-\varepsilon_1}(x) = q^{-\frac{m}{2}} \psi(\langle -x, \varepsilon_1 \rangle) = q^{-\frac{m}{2}} \psi(-x_1)$ and therefore by the Fourier inversion formula, if $\phi(x) = \psi(-x_1)$, then

$$\mathcal{F}_\psi \phi = q^{\frac{m}{2}} \delta_{\varepsilon_1}.$$

We compute $\tilde{J}_{\pi,\psi}(W, \phi)$, for $W = [G:N][M:\mathcal{B}]\pi(\sigma_{2m}^{-1})\mathcal{B}_{\pi,\psi}$ and $\phi(x) = \psi(-x_1)$. We have from Proposition 2.3 that

$$\begin{aligned}&q^{-\frac{m}{2}} \tilde{J}_{\pi,\psi}(W, \phi) \\ &= \sum_{X \in \mathcal{B}\backslash M} \sum_{g \in N\backslash G} \mathcal{B}_{\pi,\psi}\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1}\right)\psi(-\mathrm{tr}X)\delta_{\varepsilon_1}(\varepsilon_1 g^\iota).\end{aligned}$$

Notice that $\varepsilon_1 g^\iota = \varepsilon_1$ if and only if g has $e_1 = {}^t\varepsilon_1$ as its first column. It follows now (similarly to the proof of Proposition 2.4) that $\tilde{J}_{\pi,\psi}(W, \phi) = q^{\frac{m}{2}}$, and therefore

$$\begin{aligned}&q^{\frac{m}{2}} \cdot \gamma(\pi, \wedge^2, \psi)^{-1} = J_{\pi,\psi}(W, \phi) \\ &= \sum_{g \in N\backslash G} \sum_{X \in \mathcal{B}\backslash M} \mathcal{B}_{\pi,\psi}\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\sigma_{2m}^{-1}\right)\psi(-\mathrm{tr}X)\psi(-g_{m1}).\end{aligned} \tag{2.2}$$

Since the Jacquet–Shalika integral sums over cosets of the form $g \in N\backslash G$ (and is constant on these), it follows from the Bruhat decomposition that it suffices to consider elements of the form $g = wdu$, where w is a permutation matrix, d is a diagonal matrix and u is an upper triangular unipotent matrix. By Lemma 2.28, we only need to consider w, d such that $wd = \mathrm{antidiag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r})$.

We get from Lemma 2.28 that

$$\mathcal{B}_{\pi,\psi}\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} wdu & \\ & wdu \end{pmatrix}\sigma_{2m}^{-1}\right)$$
$$= \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r})).$$

Implicitly, we see that it does not depend on X and u. By Lemma 2.29, given such wd, we have $q^{\binom{m}{2}-\sum_{i=1}^{r}\binom{m_i}{2}}$ options for $u \in N$, and the same number of options for $X \in \mathcal{N}^-$.

Therefore we get the following formula:

$$J_{\pi,\psi}(W,\phi) = \sum_{\substack{m_1,\dots,m_r \geq 1 \\ m_1+\dots+m_r=m \\ \lambda_1,\dots,\lambda_r \in \mathbb{F}^*}} q^{2(\binom{m}{2}-\sum_{i=1}^{r}\binom{m_i}{2})}.$$
$$\cdot \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(\lambda_1 I_{2m_1}, \dots, \lambda_r I_{2m_r})) \cdot \psi(-\lambda_r \cdot \delta_{m_r,1}).$$

The theorem now follows from Equation (2.2), [13, Proposition 2.15], and Remark 2.17. ∎

Remark 2.30: Let

$$S_0 = \sum_{\substack{m_0 > 1 \\ m_1,\dots,m_{r-1} \geq 1 \\ m_0+\dots+m_{r-1}=m \\ \lambda_0,\dots,\lambda_{r-1} \in \mathbb{F}^*}} q^{-2\sum_{i=0}^{r-1}\binom{m_i}{2}} \cdot \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(\lambda_0 I_{2m_0}, \dots, \lambda_{r-1} I_{2m_{r-1}})).$$

Then for every $a \in F^*$, $S_0 = \omega_\pi(a) \cdot S_0$, and therefore if π has a non-trivial central character, then $S_0 = 0$.

Also let

$$S_1 = \sum_{\substack{m_1,\dots,m_{r-1} \geq 1 \\ m_1+\dots+m_{r-1}=m-1 \\ \lambda_1,\dots,\lambda_{r-1} \in \mathbb{F}^*}} q^{-2\sum_{i=1}^{r-1}\binom{m_i}{2}} \cdot \mathcal{B}_{\pi,\psi}(\mathrm{antidiag}(I_2, \lambda_1 I_{2m_1}, \dots, \lambda_{r-1} I_{2m_{r-1}})).$$

Then by a change of variables, we have that

$$\gamma(\pi, \wedge^2, \psi) = q^{-\frac{m}{2}+2\binom{m}{2}}\left(S_0 + S_1 \cdot \sum_{a \in \mathbb{F}^*} \omega_\pi(a^{-1})\psi(a)\right).$$

3. The relation with the Jacquet–Shalika integral over a local field

In this section we relate our theory of the Jacquet–Shalika integral over a finite field, to the theory of the Jacquet–Shalika integral over a local non-archimedean field, via level-zero representations.

3.1. Preliminaries and notations.

3.1.1. *Notations.* Let F be a local non-archimedean field. Denote by $\mathfrak{o}$ the ring of integers of F, by $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$. Let ϖ be a uniformizer of F, a generator of $\mathfrak{p}$. Denote by $\mathfrak{f} = \mathfrak{o}/\mathfrak{p}$ the residue field of F and $q = |\mathfrak{f}|$. We use the standard normalization for the absolute value, so that $|\varpi| = \frac{1}{q}$.

Denote by $\nu : \mathfrak{o} \to \mathfrak{f}$ the quotient map. We continue to denote by ν the maps induced by ν from $\mathfrak{o}^m \to \mathfrak{f}^m$, $M_m(\mathfrak{o}) \to M_m(\mathfrak{f})$, $\mathrm{GL}_m(\mathfrak{o}) \to \mathrm{GL}_m(\mathfrak{f})$, etc. Let $\psi : F \to \mathbb{C}^*$ be a non-trivial additive character, with conductor $\mathfrak{p}$, i.e., ψ is trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$. Then ψ defines a non-trivial additive character $\psi_0 : \mathfrak{f} \to \mathbb{C}^*$ by

$$\psi_0 \circ \nu = \psi \restriction_{\mathfrak{o}} .$$

We denote by $\mathcal{S}(F^m)$ the space of Schwartz functions $f : F^m \to \mathbb{C}$—locally constant and compactly supported functions. We choose the standard normalizations for the Haar measures on F and F^*, i.e., $\int_{\mathfrak{o}} dx = 1$ and $\int_{\mathfrak{o}^*} d^\times x = 1$.

For $f \in \mathcal{S}(F^m)$, we denote its Fourier transform

$$\mathcal{F}_\psi f(y) = q^{\frac{m}{2}} \int_{F^m} f(x) \cdot \psi(\langle x, y \rangle) dx,$$

where $\langle x, y \rangle$ is the standard bilinear form on F^m. The Fourier inversion formulas for this normalization are given by

$$\mathcal{F}_{\psi^{-1}} \mathcal{F}_\psi f(x) = f(x) \quad \text{and} \quad \mathcal{F}_\psi \mathcal{F}_\psi f(x) = f(-x).$$

If (π, V_π) is an irreducible generic representation of $\mathrm{GL}_n(F)$, we denote by $\mathcal{W}(\pi, \psi)$, as in Section 2.1, its Whittaker model with respect to the character ψ. We also denote for an element $W \in \mathcal{W}(\pi, \psi)$, an element $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$, defined by $\widetilde{W}(g) = W(w_n g^\iota)$, where

$$g^\iota = {}^t g^{-1} \quad \text{and} \quad w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

3.1.2. *The local Jacquet–Shalika integral.* We briefly review the theory of the local Jacquet–Shalika integral.

Let (π, V_π) be a generic irreducible representation of $\mathrm{GL}_n(F)$. We first give the formal formulas for the Jacquet–Shalika integral and its dual. These should initially be treated as formal expressions. Theorem 3.1 and the discussion afterwards explain how to interpret these integrals for general $s \in \mathbb{C}$.

Suppose $n = 2m$. The Jacquet–Shalika integral of π with respect to the character ψ is defined as follows: for every $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$,

$$J_{\pi,\psi}(s, W, \phi) = \int_{N\backslash G} \int_{\mathcal{B}\backslash M} W\left(\sigma_{2m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \times \psi(-\mathrm{tr} X) dX \cdot |\det g|^s \phi(\varepsilon g) d^\times g,$$

where the notations are the same as in Section 2.2, this time defined over F.

We define the dual Jacquet–Shalika integral as

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = J_{\widetilde{\pi},\psi^{-1}}\left(1 - s, \widetilde{\pi}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix} \widetilde{W}, \mathcal{F}_\psi \phi\right).$$

Now suppose $n = 2m+1$. In this case, the Jacquet–Shalika integral of π with respect to the character ψ is defined as

$$\begin{aligned} &J_{\pi,\psi}(s, W, \phi) \\ &= \int_{N\backslash G} \int_{\mathcal{B}\backslash M} \int_{M_{1\times m}(F)} W\left(\sigma_{2m+1} \begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_m & & \\ & I_m & \\ & Z & 1 \end{pmatrix}\right) \\ &\quad \times \psi(-\mathrm{tr} X) \phi(Z) |\det g|^{s-1} dZ\, dX\, d^\times g. \end{aligned}$$

In this case, we define the dual Jacquet–Shalika integral as

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = J_{\widetilde{\pi},\psi^{-1}}\left(1 - s, \widetilde{\pi}\begin{pmatrix} & I_m & \\ I_m & & \\ & & 1 \end{pmatrix} \widetilde{W}, \mathcal{F}_\psi \phi\right).$$

These Jacquet–Shalika integrals converge in suitable half planes. From [7], we have the following theorems (for $n = 2m$ or $n = 2m + 1$) regarding their convergence:

Theorem 3.1 ([7, Section 7, Proposition 1; Section 9, Proposition 3]): *There exists* $r_{\pi,\wedge^2} \in \mathbb{R}$*, such that for every* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > r_{\pi,\wedge^2}$*, the integral* $J_{\pi,\psi}(s, W, \phi)$ *converges, for every* $W \in \mathcal{W}(\pi, \psi)$ *and* $\phi \in \mathcal{S}(F^m)$*.*

Correspondingly, the dual Jacquet–Shalika integrals $\tilde{J}_{\pi,\psi}(s, W, \phi)$ converge in a left half plane ($\mathrm{Re}(s) < 1 - r_{\widetilde{\pi},\wedge^2}$).

By [9, Proposition 2.3], [3, Lemma 3.1], for fixed W and ϕ, $J_{\pi,\psi}(s, W, \phi)$ converges (in its domain of convergence) to an element of $\mathbb{C}(q^{-s})$ (a rational function in q^{-s}) and therefore has a meromorphic continuation to the entire complex plane, which we continue to denote $J_{\pi,\psi}(s, W, \phi)$. Similarly, we continue to denote the meromorphic continuation of $\tilde{J}_{\pi,\psi}(s, W, \phi)$ with the same notation. Moreover, the set

$$I = \mathrm{span}_{\mathbb{C}}\{J_{\pi,\psi}(s, W, \phi) \mid W \in \mathcal{W}(\pi, \psi), \phi \in \mathcal{S}(F^m)\}$$

is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$. By a non-vanishing theorem of D. Belt (see [1, Theorem 2.2]), there exists a unique polynomial $p \in \mathbb{C}[Z]$, such that $p(0) = 1$ and $I = \frac{1}{p(q^{-s})}\mathbb{C}[q^{-s}, q^s]$. We define the exterior square L-function $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})}$ as in [10, Definition 3.4].

Similar to Proposition 2.3 and Proposition 2.10, we can express $\tilde{J}_{\pi,\psi}(s, W, \phi)$ as in the following

Proposition 3.2: (1) *For* $n = 2m$*,*

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \int_{N\backslash G} \int_{\mathcal{B}\backslash M} W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \times \psi(-\mathrm{tr}X)dX \cdot |\det g|^{s-1} \cdot \mathcal{F}_\psi\phi(\varepsilon_1 g^\iota)d^\times g.$$

(2) *For* $n = 2m + 1$*,*

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \int_{N\backslash G} \int_{\mathcal{B}\backslash M} \int_{M_{1\times m}(F)} W\left(\begin{pmatrix} & 1 \\ I_{2m} & \end{pmatrix}\sigma_{2m+1}\begin{pmatrix} I_m & X & \\ & I_m & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_m & & -{}^tZ \\ & I_m & \\ & & 1 \end{pmatrix}\right) \times \psi(-\mathrm{tr}X)\mathcal{F}_\psi\phi(Z)|\det g|^s dZ\, dX\, d^\times g.$$

We now introduce the important local exterior square factors $\gamma(s,\pi,\wedge^2,\psi)$ and $\epsilon(s,\pi,\wedge^2,\psi)$, relating the Jacquet–Shalika integral $J_{\pi,\psi}(s,W,\phi)$ to its dual $\tilde{J}_{\pi,\psi}(s,W,\phi)$.

Theorem 3.3 ([12, Theorem 4.1], [3, Theorem 3.1]): *There exists an element $\gamma(s,\pi,\wedge^2,\psi)\in\mathbb{C}(q^{-s})$, such that for every $W\in\mathcal{W}(\pi,\psi)$ and $\phi\in\mathcal{S}(F^m)$,*

$$\tilde{J}_{\pi,\psi}(s,W,\phi)=\gamma(s,\pi,\wedge^2,\psi)\cdot J_{\pi,\psi}(s,W,\phi).$$

Furthermore

$$\gamma(s,\pi,\wedge^2,\psi)=\epsilon(s,\pi,\wedge^2,\psi)\cdot\frac{L(1-s,\widetilde{\pi},\wedge^2)}{L(s,\pi,\wedge^2)},$$

where $\epsilon(s,\pi,\wedge^2,\psi)$ is an invertible element of $\mathbb{C}[q^{-s},q^s]$.

In the case where π is a supercuspidal representation, the following result regrading $L(s,\pi,\wedge^2)$ is known

Theorem 3.4 ([8, Theorem 3.6]):

(1) *If $n=2m+1$, then $L(s,\pi,\wedge^2)=1$.*

(2) *If $n=2m$, then*

$$L(s,\pi,\wedge^2)=\frac{1}{p(q^{-s})},$$

where $p(Z)\in\mathbb{C}[Z]$ is a polynomial dividing $1-\omega_\pi(\varpi)Z^m$, satisfying $p(0)=1$.

Actually a more precise version of this theorem is known, in which $L(s,\pi,\wedge^2)$ is expressed in terms of twisted Shalika functionals in the even case. This is discussed in Section 3.4.1; see Theorem 3.19 for instance.

3.1.3. *Level-zero supercuspidal representations.* Let (π_0,V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$. Let $\chi:F^*\to\mathbb{C}^*$ be a multiplicative character, such that $\chi\restriction_{\mathfrak{o}^*}=\omega_{\pi_0}\circ\nu\restriction_{\mathfrak{o}^*}$. Let $(\pi_0\cdot\chi,V_{\pi_0})$ be the representation of the group $F^*\cdot\mathrm{GL}_n(\mathfrak{o})$, defined by the formula

$$\pi_0\cdot\chi(zk)=\chi(z)\pi_0(\nu(k)),$$

where $z\in F^*$ and $k\in\mathrm{GL}_n(\mathfrak{o})$.

Theorem 3.5 ([15, Theorem 6.2]): *Let*

$$\pi=\mathrm{ind}_{F^*\cdot\mathrm{GL}_n(\mathfrak{o})}^{\mathrm{GL}_n(F)}(\pi_0\cdot\chi).$$

Then π is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$, with central character χ.

In fact, this theorem is a special case of the construction of supercuspidal representations using type theory, due to Bushnell and Kutzko. See [2, Chapter 6] for details. We say that π is a level-zero supercuspidal representation of $\mathrm{GL}_n(F)$ constructed from the representation π_0, with respect to the central character χ, or simply a level-zero supercuspidal representation constructed from the representation π_0 (as we can recover χ via the central character ω_π). Throughout the paper, we also use the term a **level-zero representation** to refer to the term a **level-zero supercuspidal representation**.

3.1.4. *Lifts.* In order to relate Jacquet–Shalika integrals of cuspidal representations of $\mathrm{GL}_n(\mathfrak{f})$ to the local Jacquet–Shalika integrals of their corresponding level-zero representations, we need to be able to lift Schwartz functions and Whittaker functions to corresponding functions defined over the local field. We describe here briefly the process of doing so, leaving the details to the reader.

Lifts of Schwartz functions. Denote for a function $\phi_0 \in \mathcal{S}(\mathfrak{f}^m)$ a lift $\mathcal{L}(\phi_0) \in \mathcal{S}(F^m)$ defined by

$$\mathcal{L}(\phi_0)(x) = \begin{cases} \phi_0(\nu(x)) & x \in \mathfrak{o}^m, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify the following relation between the Fourier transforms and the lifts.

Proposition 3.6: *Let $\phi_0 \in \mathcal{S}(\mathfrak{f}^m)$. Then*

$$\mathcal{F}_\psi \mathcal{L}(\phi_0) = \mathcal{L}(\mathcal{F}_{\psi_0}\phi_0).$$

Lifts of Whittaker functions. Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$. Let π be a level-zero supercuspidal constructed from π_0. Let $0 \neq T_0 \in \mathrm{Hom}_{N_n(\mathfrak{f})}(\pi_0, \psi_0)$ be a Whittaker functional of π_0. The following proposition explains how to lift T_0 to a Whittaker functional of π.

Proposition 3.7 ([16, Theorem 4.3]): *Denote by $T : V_\pi \to \mathbb{C}$ the functional*

$$\langle T, f\rangle = \int_{N_n(\mathfrak{o})\backslash N_n(F)} \langle T_0, f(u)\rangle \psi^{-1}(u) d^\times u,$$

where $f \in V_\pi$ (recall from Theorem 3.5 that $f : \mathrm{GL}_n(F) \to V_{\pi_0}$). Then this integral converges and $0 \neq T \in \mathrm{Hom}_{N_n(F)}(\pi, \psi)$ is a Whittaker functional.

Proof. Let $f \in V_\pi$. We have that f is supported on a set of the form

$$F^* \cdot \mathrm{GL}_n(\mathfrak{o}) \cdot K_f,$$

where K_f is a compact subset of $\mathrm{GL}_n(F)$. Suppose that $u \in N_n(F) \cap \mathrm{supp}(f)$. Then $u = zkk'$, where $z \in F^*$, $k \in \mathrm{GL}_n(\mathfrak{o})$ and $k' \in K_f$. Taking determinants, we get from $z = u(k')^{-1}k^{-1}$ that $|z|^n = |\det k'|^{-1}$. Since K_f is a compact set, we have that there exist $c_f, C_f > 0$, such that $c_f \le |\det k'|^{-1} \le C_f$ for all $k' \in K_f$. Therefore, we have for z as above, $c_f^{\frac{1}{n}} \le |z| \le C_f^{\frac{1}{n}}$. This implies that

$$N_n(F) \cap \mathrm{supp}(f) \subseteq \{z \in F^* \mid c_f^{\frac{1}{n}} \le |z| \le C_f^{\frac{1}{n}}\} \cdot \mathrm{GL}_n(\mathfrak{o}) \cdot K_f.$$

The right-hand side is a compact subset of $\mathrm{GL}_n(F)$ as a product of compact subsets, and the left hand side is a closed subset, and hence compact. We proved that the integral defining T is over a compact domain, and therefore converges.

The functional T is not zero: let $v_0 \in V_{\pi_0}$. We consider the function $f_{v_0} : \mathrm{GL}_n(F) \to V_{\pi_0}$ defined by

$$f_{v_0}(g) = \begin{cases} \omega_\pi(z)\pi_0(\nu(k))v_0 & g = zk,\, z \in F^*,\, k \in \mathrm{GL}_n(\mathfrak{o}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_{v_0} \in V_\pi$ with $\mathrm{supp} f_{v_0} = F^* \cdot \mathrm{GL}_n(\mathfrak{o})$. Since $\mathrm{supp} f_{v_0} \cap N_n(F) = N_n(\mathfrak{o})$, we have that

$$\langle T, f_{v_0} \rangle = \int_{N_n(\mathfrak{o}) \backslash N_n(\mathfrak{o})} \langle T_0, f_{v_0}(u) \rangle \psi^{-1}(u) d^\times u = \langle T_0, f_{v_0}(I_n) \rangle = \langle T_0, v_0 \rangle.$$

Choosing v_0 such that $\langle T_0, v_0 \rangle \ne 0$, we get that $\langle T, f_{v_0} \rangle \ne 0$, and therefore $T \ne 0$, as required.

The functional T is a Whittaker functional: let $f \in V_\pi$, $u_0 \in N_n(F)$ and $g \in \mathrm{GL}_n(F)$. Then $(\pi(u_0)f)(g) = f(gu_0)$. Therefore

$$\langle T, \pi(u_0)f \rangle = \int_{N_n(\mathfrak{o}) \backslash N_n(F)} \langle T_0, f(uu_0) \rangle \psi^{-1}(u) d^\times u.$$

Substituting $u' = uu_0$, we get

$$\langle T, \pi(u_0)f \rangle = \int_{N_n(\mathfrak{o}) \backslash N_n(F)} \langle T_0, f(u') \rangle \psi^{-1}(u' u_0^{-1}) d^\times u = \psi(u_0) \langle T, f \rangle,$$

as required. ∎

Using the lifted Whittaker functional T, we are now able to define a lift of a Whittaker function.

Definition 3.8: Let $W_0 \in \mathcal{W}(\pi_0, \psi_0)$. Let $v_{W_0} \in V_{\pi_0}$ be the unique vector such that

$$W_0(g) = \langle T_0, \pi_0(g) v_{W_0} \rangle,$$

for every $g \in \mathrm{GL}_n(\mathfrak{f})$. Let $f_{W_0} \in V_\pi$ be defined as

$$f_{W_0}(g) = \begin{cases} \omega_\pi(z)\pi_0(\nu(k))v_{W_0}, & g = zk,\ z \in F^*,\ k \in \mathrm{GL}_n(\mathfrak{o}). \\ 0, & \text{otherwise.} \end{cases}$$

We define $\mathcal{L}(W_0) \in \mathcal{W}(\pi, \psi)$ by

$$\mathcal{L}(W_0)(g) = \langle T, \pi(g) f_{W_0} \rangle$$

for $g \in \mathrm{GL}_n(F)$.

We have that $\mathcal{L}(W_0)$ is given by a simple formula:

Proposition 3.9 ([16, Proposition 4.4]): *$\mathcal{L}(W_0)$ is supported on*

$$N_n(F) \cdot F^* \cdot \mathrm{GL}_n(\mathfrak{o})$$

and

$$\mathcal{L}(W_0)(u_0 z k) = \psi(u_0) \cdot \omega_\pi(z) \cdot W_0(\nu(k)),$$

for any $u_0 \in N_n(F)$, $z \in F^*$, $k \in \mathrm{GL}_n(\mathfrak{o})$.

Proof. Let $g \in \mathrm{GL}_n(F)$ with $f(g) \neq 0$. Then

$$\mathcal{L}(W_0)(g) = \langle T, \pi(g) f_{W_0} \rangle = \int_{N_n(\mathfrak{o}) \backslash N_n(F)} \langle T_0, f_{W_0}(ug) \rangle \psi^{-1}(u) d^\times u.$$

Since f_{W_0} is supported on $F^* \cdot \mathrm{GL}_n(\mathfrak{o})$, we must have $ug \in F^* \cdot \mathrm{GL}_n(\mathfrak{o})$ for some $u \in N_n(F)$, i.e., $g \in N_n(F) \cdot F^* \cdot \mathrm{GL}_n(\mathfrak{o})$.

Write $g = u_0 z k$ for $u_0 \in N_n(F)$, $z \in F^*$, $k \in \mathrm{GL}_n(\mathfrak{o})$. Then

$$\mathcal{L}(W_0)(u_0 z k) = \langle T, \pi(u_0 z k) f_{W_0} \rangle = \psi(u_0) \omega_\pi(z) \langle T, \pi(k) f_{W_0} \rangle,$$

where we used the fact that T is a Whittaker functional and that $\pi(z) = \omega_\pi(z) \mathrm{id}_{V_\pi}$. Finally, write

$$\langle T, \pi(k) f_{W_0} \rangle = \int_{N_n(\mathfrak{o}) \backslash N_n(F)} \langle T_0, f_{W_0}(uk) \rangle \psi^{-1}(u) d^\times u.$$

This integral is supported on $uk \in F^* \cdot \mathrm{GL}_n(\mathfrak{o})$, i.e., $u \in F^* \cdot \mathrm{GL}_n(\mathfrak{o})$. We have that $u \in (F^* \cdot \mathrm{GL}_n(\mathfrak{o})) \cap N_n(F) = N_n(\mathfrak{o})$, and therefore we get

$$\begin{aligned}\langle T, \pi(k) f_{W_0} \rangle &= \int_{N_n(\mathfrak{o}) \backslash N_n(\mathfrak{o})} \langle T_0, f_{W_0}(uk) \rangle \psi^{-1}(u) d^\times u \\ &= \langle T_0, f_{W_0}(k) \rangle \\ &= \langle T_0, \pi_0(\nu(k)) v_{W_0} \rangle = W_0(\nu(k)).\end{aligned}$$

Therefore we have $\mathcal{L}(W_0)(u_0 z k) = \psi(u_0)\omega_\pi(z) W_0(\nu(k))$, as required. ∎

3.2. A relation between the Jacquet–Shalika integrals. In this section, we find a relation between local Jacquet–Shalika integrals of level-zero representations and their corresponding residue field Jacquet–Shalika integrals. Our main result is the following:

Theorem 3.10: *Let $n = 2m$ or $n = 2m+1$. Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$. If n is even, suppose that π_0 does not admit a Shalika vector. Let π be a level-zero supercuspidal (irreducible) representation constructed from π_0. Then for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$, $\phi_0 \in \mathcal{S}(\mathfrak{f}^m)$, $s \in \mathbb{C}$,*

$$\begin{aligned}J_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) &= J_{\pi_0,\psi_0}(W_0, \phi_0), \\ \tilde{J}_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) &= \tilde{J}_{\pi_0,\psi_0}(W_0, \phi_0).\end{aligned}$$

Theorem 3.10 implies that if π_0 does not admit a Shalika vector, the Jacquet–Shalika integrals of its corresponding level-zero representation for fixed lifted functions result in constant elements of $\mathbb{C}(q^{-s})$, i.e., they are independent of s. In the case where n is even, we have a modified theorem that also handles the case in which the representation π_0 admits a Shalika vector:

Theorem 3.11: *Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathfrak{f})$. Then for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$, $\phi_0 \in \mathcal{S}(\mathfrak{f}^m)$, $s \in \mathbb{C}$,*

$$\begin{aligned}&J_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) \\ &\quad = J_{\pi_0,\psi_0}(W_0, \phi_0) + q^{-ms}\omega_\pi(\varpi)\phi_0(0) L(ms, \omega_\pi) J_{\pi_0,\psi_0}(W_0, 1), \\ &\tilde{J}_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) \\ &\quad = \tilde{J}_{\pi_0,\psi_0}(W_0, \phi_0) + q^{-m(1-s)}\omega_\pi^{-1}(\varpi)\mathcal{F}_{\psi_0}\phi_0(0) L(m(1-s), \omega_\pi^{-1}) J_{\pi_0,\psi_0}(W_0, 1).\end{aligned}$$

By Theorem 2.25, we have that Theorem 3.11 implies Theorem 3.10 in the even case. We only need to prove the odd case in Theorem 3.10 once we prove Theorem 3.11. But the proof of the odd case of Theorem 3.10 uses similar techniques and ideas from that of Theorem 3.11, so we will give only the proof of Theorem 3.11, and leave the proof of Theorem 3.10 out. We will need the following lemma in the proof.

Lemma 3.12: *Suppose that* $\sigma_{2m}\left(\begin{smallmatrix} I_m & X \\ & I_m \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right) = \lambda \cdot u \cdot k$, *where*

$$a = \mathrm{diag}(a_1, \ldots, a_m) \in A_m$$

is an invertible diagonal matrix, $X \in \mathcal{N}_m^-(F)$ *is a lower triangular nilpotent matrix,* $\lambda \in F^*$, $u \in N_{2m}$ *and* $k \in K_{2m} = \mathrm{GL}_{2m}(\mathfrak{o})$. *Then*

(1) $|a_1| = \cdots = |a_m| = |\lambda|$,
(2) $X \in M_m(\mathfrak{o})$.

Proof. Denote $Z = a^{-1}Xa$, and

$$u_Z = \sigma_{2m} \begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix} \sigma_{2m}^{-1}.$$

Also denote

$$b = \sigma_{2m} \begin{pmatrix} a & \\ & a \end{pmatrix} \sigma_{2m}^{-1} = \mathrm{diag}(a_1, a_1, a_2, a_2, \ldots, a_m, a_m).$$

Then we have that $bu_Z\sigma_{2m} = \lambda uk$. Writing $u_Z = n_Z t_Z k_Z$ as in [7, Section 5, Proposition 4], we get that

$$\lambda^{-1} b t_Z = (b n_Z^{-1} b^{-1} u) \cdot (k \sigma_{2m}^{-1} k_Z^{-1}).$$

Since

$$A_{2m} \cap (N_{2m} \cdot K_{2m}) = A_{2m} \cap K_{2m} = (\mathfrak{o}^*)^{2m},$$

we get that $\lambda^{-1}bt_Z$ is a diagonal matrix having units on its diagonal. Writing $t_Z = \mathrm{diag}(t_1, \ldots, t_{2m})$, we have that $|t_{2i-1}| = |t_{2i}| = \frac{|\lambda|}{|a_i|}$, for every $1 \le i \le m$. By [7, Section 5, Proposition 4], we get that $|t_i| = 1$ for every $1 \le i \le 2m$, and therefore $|a_i| = |\lambda|$, for every $1 \le i \le m$. Finally, by [7, Section 5, Proposition 5], we get that $Z \in M_m(\mathfrak{o})$. Since $\frac{|a_i|}{|a_j|} = \frac{|\lambda|}{|\lambda|} = 1$, for every $1 \le i, j \le m$, we get that $X \in M_m(\mathfrak{o})$, as required. ∎

Throughout the proof of Theorem 3.11, we will use the following symbols: $W = \mathcal{L}(W_0)$, $\phi = \mathcal{L}(\phi_0)$; $A = A_m$ is the diagonal subgroup of G; $K = \mathrm{GL}_m(\mathfrak{o})$; $\mathcal{N}^- = \mathcal{N}_m^-(F) \leq M_m(F)$ is the subspace consisting of lower triangular nilpotent matrices. Let $X \in \mathcal{N}^-$. Also let $g = ak$, where $a = \mathrm{diag}(a_1, \dots, a_m) \in A$, $k \in K$. Then by the Iwasawa decomposition of $\mathrm{GL}_m(F)$, we have

$$d^\times g = \delta_{B_m}^{-1}(a) \prod_{i=1}^{m} d^\times a_i \cdot d^\times k,$$

where

$$\delta_{B_m}^{-1}(a) = \prod_{1 \leq i < j \leq m} |\frac{a_j}{a_i}|.$$

Proof of Theorem 3.11. We prove only the equation regarding $J_{\pi,\psi}$. The proof of the equation regarding $\tilde{J}_{\pi,\psi}$ is similar. It can also be deduced from the first equation. The proof consists of two parts. In the first part we find the supports of g, X in the Jacquet–Shalika integral. In the second part, we evaluate the integral on the supports.

Suppose that $\sigma_{2m}\left(\begin{smallmatrix} I_m & X \\ & I_m \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)\left(\begin{smallmatrix} k & \\ & k \end{smallmatrix}\right) \in \mathrm{supp}(W) \subseteq F^* N_{2m} K_{2m}$ (see Proposition 3.9). Then by Lemma 3.12, we have that $|a_i| = |a_m|$, for every $1 \leq i \leq m$, and $X \in M_m(\mathfrak{o})$. Therefore, we get that $a_i = u_i \cdot a_m$, where $u_i \in \mathfrak{o}^*$, $d^\times u_i = d^\times a_i$, for every $1 \leq i \leq m-1$. We also get $\delta_{B_m}^{-1}(a) = \prod_{1 \leq i < j \leq m} |\frac{a_j}{a_i}| = 1$. Therefore, the Jacquet Shalika integral is integrated on $X \in \mathcal{N}_m^-(\mathfrak{o})$, $g = ak$, where $k \in K$, $a = a_m \cdot \mathrm{diag}(u_1, \dots, u_{m-1}, 1)$, $u_i \in \mathfrak{o}^*$ for every $1 \leq i \leq m-1$,

$$d^\times g = d^\times a_m \cdot \prod_{i=1}^{m-1} d^\times u_i \cdot d^\times k,$$

and by replacing the variable k with $\mathrm{diag}(u_1, \dots, u_{m-1}, 1)^{-1} \cdot k$, we have that $J_{\pi,\psi}(s, W, \phi)$ is given by

$$\int_{F^*} \int_K \int_{\mathcal{N}_m^-(\mathfrak{o})} W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} k & \\ & k \end{pmatrix}\right) \phi(\varepsilon k a_m)|a_m|^{ms}\omega_\pi(a_m) dX d^\times k d^\times a_m.$$

Since $\phi = \mathcal{L}(\phi_0)$ is a lift of the Schwartz function ϕ_0, we have that for a fixed k, $\phi(\varepsilon k a_m) = 0$ for $|a_m| > 1$ and $\phi(\varepsilon k a_m) = \phi_0(0)$ for $|a_m| < 1$. Therefore

$$\int_{F^*} \phi(\varepsilon k a_m)|a_m|^{ms}\omega_\pi(a_m) d^\times a_m = \int_{\mathfrak{o}^*} \phi(\varepsilon k a_m)\omega_\pi(a_m) d^\times a_m + \phi_0(0) I(s),$$

where

$$I(s)=\sum_{i=1}^{\infty}\int_{\varpi^i\cdot\mathfrak{o}^*}|a_m|^{ms}\omega_\pi(a_m)d^\times a_m=\int_{\mathfrak{o}^*}\omega_\pi(a_m)d^\times a_m\cdot\sum_{i=1}^{\infty}q^{-ims}\omega_\pi(\varpi)^i.$$

Therefore, $I(s)=q^{-ms}\omega_\pi(\varpi)L(ms,\omega_\pi)$ if ω_π is unramified (i.e., $\omega_\pi\restriction_{\mathfrak{o}^*}$ trivial, which happens if and only if ω_{π_0} is trivial), and otherwise $\int_{\mathfrak{o}^*}\omega_\pi(a_m)d^\times a_m=0$, so $I(s)=0$. We are left to evaluate

$$\int_{\mathfrak{o}^*}\int_K\int_{\mathcal{N}_m^-(\mathfrak{o})}W\left(\sigma_{2m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}k & \\ & k\end{pmatrix}\right)\\ \times\omega_\pi(a_m)(\phi(\varepsilon k a_m)+\phi_0(0)I(s))dX\,d^\times k\,d^\times a_m.$$

Since W, ϕ and ω_π are lifts of W_0, ϕ_0 and ω_{π_0} respectively, and the expression is constant on the quotient spaces $\mathrm{GL}_m(\mathfrak{o})/1+\varpi M_m(\mathfrak{o})\cong\mathrm{GL}_m(\mathfrak{f})$, $\mathcal{N}^-(\mathfrak{o})/\mathcal{N}^-(\mathfrak{p})\cong\mathcal{N}^-(\mathfrak{f})$ and $\mathfrak{o}^*/(1+\mathfrak{p})\cong\mathfrak{f}^*$, we get that this sum equals

$$\frac{1}{|\mathfrak{f}^*|}\frac{1}{|\mathrm{GL}_m(\mathfrak{f})|}\frac{1}{|\mathcal{N}^-(\mathfrak{f})|}\sum_{a_m\in\mathfrak{f}^*}\sum_{k\in\mathrm{GL}_m(\mathfrak{f})}\sum_{X\in\mathcal{N}^-(\mathfrak{f})}W_0\left(\sigma_{2m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}k & \\ & k\end{pmatrix}\right)\\ \times\omega_{\pi_0}(a_m)(\phi_0(\varepsilon k a_m)+\phi_0(0)I(s)).$$

By replacing the variable k with $k=k'a_m^{-1}$, we get

$$\frac{1}{|\mathrm{GL}_m(\mathfrak{f})|}\frac{1}{|\mathcal{N}^-(\mathfrak{f})|}\\ \times\sum_{k'\in\mathrm{GL}_m(\mathfrak{f})}\sum_{X\in\mathcal{N}^-(\mathfrak{f})}W_0\left(\sigma_{2m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}k' & \\ & k'\end{pmatrix}\right)(\phi_0(\varepsilon k')+\phi_0(0)I(s)).$$

Since this expression is constant on cosets of $k'\in N_m(\mathfrak{f})\backslash\mathrm{GL}_m(\mathfrak{f})$ and since $\mathcal{B}_m(\mathfrak{f})\backslash M_m(\mathfrak{f})\cong\mathcal{N}_m^-(\mathfrak{f})$, we get that

$$J_{\pi,\psi}(s,W,\phi)=J_{\pi_0,\psi_0}(W_0,\phi_0)+\phi_0(0)I(s)J_{\pi_0,\psi_0}(W_0,1).$$

Finally, we claim that

$$\phi_0(0)I(s)J_{\pi_0,\psi_0}(W_0,1)=\phi_0(0)q^{-ms}\omega_\pi(\varpi)L(ms,\omega_\pi)J_{\pi_0,\psi_0}(W_0,1).$$

If ω_{π_0} is trivial, we already saw that this is true. If ω_{π_0} is non-trivial, π_0 does not admit a Shalika vector, and we get from Theorem 2.25 that $J_{\pi_0,\psi_0}(W,1)=0$, and therefore the result follows. ∎

3.3. The modified functional equation in the even case. As a result of Theorem 3.11, we get the following modified functional equation, a generalization of Theorem 2.15, this time valid for all irreducible cuspidal representations of $\mathrm{GL}_{2m}(\mathbb{F})$, regardless whether they admit a Shalika vector.

Theorem 3.13 (The modified functional equation): *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Let $\psi_0 : \mathbb{F} \to \mathbb{C}^*$ be a non-trivial (additive) character. Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{F})$. Then there exists $\gamma(s, \pi_0, \wedge^2, \psi_0) \in \mathbb{C}(q^{-s})$, such that for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$ and $\phi_0 \in \mathcal{S}(\mathbb{F}^m)$, we have*

$$\begin{aligned}\tilde{J}_{\pi_0,\psi_0}(W_0, \phi_0) &+ q^{-m(1-s)}\mathcal{F}_{\psi_0}\phi_0(0)L(m(1-s), 1)J_{\pi_0,\psi_0}(W_0, 1)\\ &= \gamma(s, \pi_0, \wedge^2, \psi_0) \cdot (J_{\pi_0,\psi_0}(W_0, \phi_0) + q^{-ms}\phi_0(0)L(ms, 1)J_{\pi_0,\psi_0}(W_0, 1)).\end{aligned}$$

Proof. If π_0 does not admit a Shalika vector, then from Theorem 2.25 for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$, we have that $J_{\pi_0,\psi_0}(W_0, 1) = 0$, and therefore we get the same functional equation as in Theorem 2.15.

Suppose that π_0 admits a Shalika vector. Then by Remark 2.8, π_0 has a trivial central character. Choose any local field F with $\mathbb{F}$ as its residue field, and $\psi : F \to \mathbb{C}^*$, an additive character, such that $\psi \restriction_{\mathfrak{o}} = \psi_0 \circ \nu$. Let (π, V_π) be the level-zero supercuspidal representation constructed from π_0, with respect to the trivial central character. The statement now follows from Theorem 3.11 and Theorem 3.3. ∎

3.4. Exterior square gamma factors for level-zero supercuspidal representations. Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathfrak{f})$, and let (π, V_π) be a level-zero representation of $\mathrm{GL}_n(F)$ constructed from π_0.

As a corollary of Theorem 3.10 we obtain the following main theorems of the paper.

Theorem 3.14: *If π_0 does not admit a Shalika vector, then $\gamma(s, \pi, \wedge^2, \psi)$ is an invertible constant (i.e., independent of s), and*

$$\gamma(s, \pi, \wedge^2, \psi) = \gamma(\pi_0, \wedge^2, \psi_0).$$

Furthermore,

$$L(s, \pi, \wedge^2) = 1, \quad \epsilon(s, \pi, \wedge^2, \psi) = \gamma(\pi_0, \wedge^2, \psi_0).$$

Proof. We can choose $W_0 \in \mathcal{W}(\pi_0, \psi_0)$ and $\phi_0 \in \mathcal{S}(\mathfrak{f}^m)$, such that $J_{\pi_0,\psi_0}(W_0, \phi_0)$ is non-zero (for instance take the functions in Proposition 2.4, Proposition 2.11). By Theorem 3.10, we have that

$$J_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = J_{\pi_0,\psi_0}(W_0, \phi_0)$$

and

$$\tilde{J}_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = \tilde{J}_{\pi_0,\psi_0}(W_0, \phi_0).$$

We get from the functional equations in Theorem 2.15 and Theorem 3.3 that

$$\gamma(s, \pi, \wedge^2, \psi) = \frac{\tilde{J}_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0))}{J_{\pi,\psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0))} = \frac{\tilde{J}_{\pi_0,\psi_0}(W_0, \phi_0)}{J_{\pi_0,\psi_0}(W_0, \phi_0)} = \gamma(\pi_0, \wedge^2, \psi_0).$$

This proves the result regarding the gamma factors.

If $n = 2m + 1$, we have from Theorem 3.4 that $L(s, \pi, \wedge^2) = 1$. Suppose $n = 2m$, and denote

$$L(s, \pi, \wedge^2) = \frac{1}{p_1(q^{-s})}, \quad L(s, \widetilde{\pi}, \wedge^2) = \frac{1}{p_2(q^{-s})},$$

where $p_1(Z), p_2(Z) \in \mathbb{C}[Z]$ are polynomials with $p_1(0) = p_2(0) = 1$. Since the gamma factor $\gamma(s, \pi, \wedge^2, \psi)$ is a constant, by Theorem 3.3 we must have that

$$\frac{p_1(q^{-s})}{p_2(q^{-(1-s)})} = \frac{L(1 - s, \widetilde{\pi}, \wedge^2)}{L(s, \pi, \wedge^2)} = c \cdot q^{ks},$$

where $k \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. This implies that $p_1(Z)$ and $p_2(q^{-1}Z^{-1})$ have the same non-zero roots. By Theorem 3.4, we have that $p_1(Z)$ divides $1 - \omega_\pi(\varpi)Z^m$ and p_2 divides $1 - \omega_\pi(\varpi)^{-1}Z^m$, and therefore $p_1(Z)$ and $p_2(q^{-1}Z^{-1})$ can't have mutual roots, as roots r of $p_1(Z)$ satisfy $r^m = \omega_\pi(\varpi)^{-1}$, while roots r' of $p_2(q^{-1}Z^{-1})$ satisfy $r'^m = q^{-m}\omega_\pi(\varpi)^{-1}$. Therefore $p_1(Z), p_2(Z)$ are constants and $p_1(Z) = p_2(Z) = 1$, which implies that $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})} = 1$. The result regarding $\epsilon(s, \pi, \wedge^2, \psi)$ now follows from the equation in Theorem 3.3. ∎

Theorem 3.15: *If $n = 2m$ and π_0 admits a Shalika vector, then*

$$\gamma(s, \pi, \wedge^2, \psi) = \frac{q^{ms}}{q^{\frac{m}{2}}\omega_\pi(\varpi)} \cdot \frac{L(m(1 - s), \omega_\pi^{-1})}{L(ms, \omega_\pi)}.$$

Furthermore, $L(s, \pi, \wedge^2) = L(ms, \omega_\pi)$, and $\epsilon(s, \pi, \wedge^2, \psi) = \frac{q^{ms}}{q^{\frac{m}{2}}\omega_\pi(\varpi)}$.

Also in this case $\gamma(s, \pi, \wedge^2, \psi) = \gamma(s - s_0, \pi_0, \wedge^2, \psi_0)$, where $q^{ms_0} = \omega_\pi(\varpi)$ (see also Theorem 3.13).

Proof. By Remark 2.8, since π_0 admits a Shalika vector, the central character ω_{π_0} is trivial. Thus, the central character ω_π is unramified. Therefore,

$$L(s,\omega_\pi) = \frac{1}{1-\omega_\pi(\varpi)q^{-s}}.$$

By Theorem 2.25, there exists $W_0 \in \mathcal{W}(\pi_0,\psi_0)$, such that $J_{\pi_0,\psi_0}(W_0,1) = 1$. We substitute in Theorem 2.25 such W_0 and $\phi_0 = 1$, $\mathcal{F}_{\psi_0}\phi_0 = q^{\frac{m}{2}}\delta_0$ to get

$$J_{\pi,\psi}(s,\mathcal{L}(W_0),\mathcal{L}(\phi_0)) = 1 + q^{-ms}\omega_\pi(\varpi)L(ms,\omega_\pi) = L(ms,\omega_\pi), \tag{3.1}$$

and

$$\tilde{J}_{\pi,\psi}(s,\mathcal{L}(W_0),\mathcal{L}(\phi_0)) = q^{\frac{m}{2}}q^{-m(1-s)}\omega_\pi(\varpi)^{-1}L(m(1s),\omega_\pi^{-1}).$$

The result regarding $\gamma(s,\pi,\wedge^2,\psi)$ follows as

$$\gamma(s,\pi,\wedge^2,\psi) = \frac{\tilde{J}_{\pi,\psi}(s,\mathcal{L}(W_0),\mathcal{L}(\phi_0))}{J_{\pi,\psi}(s,\mathcal{L}(W_0),\mathcal{L}(\phi_0))}.$$

Regarding $L(s,\pi,\wedge^2)$, denote $L(s,\pi,\wedge^2) = \frac{1}{p(q^{-s})}$, where $p(Z) \in \mathbb{C}[Z]$ is a polynomial with $p(0) = 1$. By Theorem 3.4, we have that $p(Z) \mid 1-\omega_\pi(\varpi)Z^m$. From Equation (3.1), we have that

$$J_{\pi,\psi}(s,\mathcal{L}(W_0),\mathcal{L}(\phi_0)) = L(ms,\omega_\pi) = \frac{1}{1-\omega_\pi(\varpi)q^{-ms}} \in \frac{1}{p(q^{-s})}\mathbb{C}[q^s,q^{-s}],$$

so $1-\omega_\pi(\varpi)Z^m \mid p(Z)$. Therefore we must have $p(Z) = 1-\omega_\pi(\varpi)Z^m$, and the result $L(s,\pi,\wedge^2) = L(ms,\omega_\pi)$ follows. The result regarding $\epsilon(s,\pi,\wedge^2,\psi)$ now follows from the equation in Theorem 3.3. ∎

Theorem 3.14 and Theorem 3.15 establish a connection between a cuspidal representation π_0 and its corresponding level-zero representation π via the local exterior square factors of π. Moreover, these theorems demonstrate a close connection between the existence of Shalika vectors and the existence of poles of the local exterior square L-function.

Corollary 3.16: *Let π be a level-zero representation constructed from an irreducible cuspidal representation π_0. Then π_0 admits a Shalika vector if and only if $L(s,\pi,\wedge^2)$ has a pole.*

Proof. On one hand, if π_0 admits a Shalika vector, then by Theorem 3.15, $L(s,\pi,\wedge^2) = L(ms,\omega_\pi)$ has a pole. On the other hand, if π_0 does not admit a Shalika vector, then by Theorem 3.14, $L(s,\pi,\wedge^2) = 1$ does not have any poles. ∎

Remark 3.17: Once we establish the connection between Shalika vectors of π_0 and Shalika functionals of π in the next section, we can see that Corollary 3.16 is a special case of a more general fact relating existence of Shalika functionals to poles of exterior square L functions. Corollary 4.4 of [9], which is a consequence of [9, Theorem 4.3], states that for an irreducible square integrable representation π, a sufficient condition for π to have a non-zero Shalika functional is that $L(s, \pi, \wedge^2)$ has a pole at $s = 0$. Actually, it is also a necessary condition; see [12, Proposition 6.1] or [8, Propositon 3.12].

3.4.1. *Relation with Shalika functionals.* Shalika vectors in V_{π_0} are closely related to poles of $L(s, \pi, \wedge^2)$, which in turn are closely akin to (twisted) Shalika functionals on $\mathcal{W}(\pi, \psi)$. After we relate Shalika vectors and Shalika periods, we will give another explanation for the computation of $L(s, \pi, \wedge^2)$ in the case where π_0 admits a Shalika vector. We begin with the introduction of twisted Shalika functionals Λ_s on $\mathcal{W}(\pi, \psi)$.

Definition 3.18: Let (π, V_π) be a representation with an unramified central character. For any $s \in \mathbb{C}$ satisfying $q^{ms} = \omega_\pi(\varpi)$, the twisted Shalika period Λ_s on $\mathcal{W}(\pi, \psi)$ is defined to be the following linear functional $\Lambda_s : \mathcal{W}(\pi, \psi) \to \mathbb{C}$,

$$\Lambda_s(W) = \int_{F^* N \backslash G} \int_{\mathcal{B} \backslash M} W\left(\sigma_{2m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(-\mathrm{tr} X) dX |\det g|^s d^\times g.$$

For any $s \in \mathbb{C}$, we set ν^s to be the one-dimensional representation of $\mathrm{GL}_{2m}(F)$ given by

$$\nu^s(g) = |\det g|^s$$

for any $g \in \mathrm{GL}_{2m}(F)$. Applying [9, Lemma 4.2] to the representation $\pi \otimes \nu^{\frac{s}{2}}$, we know that Λ_s converges absolutely. Λ_s is a twisted Shalika functional in the sense that for any $h \in S_{2m}$ and any $W \in \mathcal{W}(\pi, \psi)$, we have

$$\Lambda_s(\pi(h)W) = |\det h|^{-s} \Psi(h) \Lambda_s(W). \tag{3.2}$$

By Equation (3.2), we have

$$\Lambda_s \in \mathrm{Hom}_{S_{2m}(F)}(\pi \otimes \nu^{\frac{s}{2}}, \Psi).$$

By [8], we have

Theorem 3.19 ([8, Theorem 3.6]): *Let (π, V_π) be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$, with an unramified central character. Then*

$$L(s, \pi, \wedge^2) = \prod_\alpha (1 - \alpha q^{-s})^{-1},$$

where the product runs over all $\alpha = q^{s_0}$, such that $\alpha^m = \omega_\pi(\varpi)$ and $\Lambda_{s_0} \neq 0$. Equivalently, the product runs over all $\alpha = q^{s_0}$ such that

$$\mathrm{Hom}_{S_{2m}(F)}(\pi \otimes \nu^{\frac{s_0}{2}}, \Psi) \neq 0.$$

As before, let (π_0, V_{π_0}) be an irreducible representation of $\mathrm{GL}_{2m}(\mathfrak{f})$ and let (π, V_π) be a level-zero representation constructed from π_0. We recall that the central character ω_{π_0} is trivial if and only if the central character ω_π is unramified. Similarly to the proof of Theorem 3.11, one can show

Proposition 3.20: *Suppose ω_{π_0} is trivial and equivalently ω_π is unramified. For any $W_0 \in \mathcal{W}(\pi_0, \psi_0)$ and any $s \in \mathbb{C}$, such that $q^{ms} = \omega_\pi(\varpi)$, we have*

$$\Lambda_s(\mathcal{L}(W_0)) = J_{\pi_0, \psi_0}(W_0, 1).$$

We remark that the right-hand side of the equation in the above proposition is independent of $s \in \mathbb{C}$.

We can now use Theorem 3.19 and Proposition 3.20 to give another explanation why in the case that π_0 admits a Shalika vector, $L(s, \pi, \wedge^2) = L(ms, \omega_\pi)$: Since π_0 admits a Shalika vector, we have by Remark 2.8 that the central character ω_{π_0} is trivial. We also have by Theorem 2.25 and Proposition 3.20 that for any $s \in \mathbb{C}$, with $q^{ms} = \omega_\pi(\varpi)$, $\Lambda_s \neq 0$. Therefore, the condition $\alpha = q^{s_0}$ with $\Lambda_{s_0} \neq 0$ is always valid. Since α in the product in Theorem 3.19 runs over the m-th roots of $\omega_\pi(\varpi)$, we get

$$\begin{aligned} L(s, \pi, \wedge^2) =& \prod_\alpha (1 - \alpha q^{-s})^{-1} \\ =& (1 - \omega_\pi(\varpi) q^{-ms})^{-1} = L(ms, \omega_\pi). \end{aligned}$$

We conclude this section by giving a characterization of the existence of Shalika vectors in terms of the existence of (twisted) Shalika functionals.

Proposition 3.21: *The space $\mathrm{Hom}_{S_{2m}(\mathfrak{f})}(\pi_0, \Psi_0)$ is non-zero if and only if the space $\mathrm{Hom}_{S_{2m}(F)}(\pi \otimes \nu^{s/2}, \Psi)$ is non-zero for some $s \in \mathbb{C}$. Here $\Psi_0 : S_{2m}(\mathfrak{f}) \to \mathbb{C}^*$ is the character on the Shalika subgroup, defined by the character ψ_0 (see Definition 2.5).*

Proof. With a choice of inner product on V_{π_0}, with respect to which π_0 is unitary, we can show that the existence of a Shalika vector is equivalent to $\mathrm{Hom}_{S_{2m}(\mathfrak{f})}(\pi_0, \Psi_0) \neq 0$. By Theorem 3.19, $L(s, \pi, \wedge^2)$ has a pole at $s \in \mathbb{C}$ if and only if $\mathrm{Hom}_{S_{2m}(F)}(\pi \otimes \nu^{s/2}, \Psi) \neq 0$. Therefore, the proposition now follows from Corollary 3.16. ∎

Appendix A. Multiplicity one theorems

In the appendix, we follow [11] in order to prove the following multiplicity one theorems, which are keys to the proofs of the functional equation for the finite field case (Theorem 2.15).

Let $\mathbb{F}$ be a finite field.

THEOREM A.1: *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1) \leq 1,$$

where $M_{m,m}$ is the Levi subgroup corresponding to the partition (m, m), and P_{2m} is the mirabolic subgroup.

THEOREM A.2: *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m+1}(\mathbb{F})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{L_{2m+1} \cap P_{2m+1}}(\pi, 1) \leq 1,$$

where P_{2m+1} is the mirabolic subgroup and L_{2m+1} is the following maximal (non-standard) Levi subgroup of $\mathrm{GL}_{2m+1}(\mathbb{F})$ corresponding to the partition $(m+1, m)$:

$$L_{2m+1} = \left\{ \begin{pmatrix} g_1 & & u \\ & g_2 & \\ v & & \lambda \end{pmatrix} \;\middle|\; \begin{matrix} g_1, g_2 \in \mathrm{GL}_m(\mathbb{F}),\, u \in M_{m\times 1}(\mathbb{F}), \\ v \in M_{1\times m}(\mathbb{F}), \lambda \in \mathbb{F} \end{matrix} \right\} \cap \mathrm{GL}_{2m+1}(\mathbb{F}).$$

The proofs of these theorems require some preparation. For a finite group G and a vector space V over $\mathbb{C}$, denote

$$\mathcal{S}(G, V) = \{f : G \to V\}.$$

Denote by ρ and λ the right and left actions of G on $\mathcal{S}(G, V)$ respectively:

$$(\rho(g_0)f)(g) = f(gg_0), \quad (\lambda(g_0)f)(g) = f({g_0}^{-1}g).$$

Let $\psi : \mathbb{F} \to \mathbb{C}$ be a non-trivial additive character of $\mathbb{F}$.

For every positive integer n, we denote $G_n = \mathrm{GL}_n(\mathbb{F})$. In the following, we use the convention that for $k < n$, G_k is embedded in G_n by mapping $g \mapsto \left(\begin{smallmatrix} g & \\ & I_{n-k} \end{smallmatrix}\right)$. Suppose that $p + q = n$, where $p \geq q \geq 0$. Let $r = p - q$, and let $\sigma_{p,q}$ be the following permutation:

$$\begin{pmatrix} 1 & \cdots & r & \mid & r+1 & r+2 & \cdots & p & \mid & p+1 & p+2 & \cdots & p+q \\ 1 & \cdots & r & \mid & r+1 & r+3 & \cdots & p+q-1 & \mid & r+2 & r+4 & \cdots & p+q \end{pmatrix}.$$

Let $w_{p,q}$ be the column permutation matrix corresponding to $\sigma_{p,q}$.

Let $H_{p,q}^{(n)} = w_{p,q} M_{p,q} w_{p,q}^{-1}$, where $M_{p,q}$ is the Levi subgroup corresponding to the partition (p,q). If $q \geq 1$, denote

$$H_{p,q-1}^{(n)} = w_{p,q} \left\{ \begin{pmatrix} m & \\ & 1 \end{pmatrix} \mid m \in M_{p,q-1} \right\} w_{p,q}^{-1},$$

where $M_{p,q-1}$ is the Levi subgroup of $\mathrm{GL}_{n-1}(\mathbb{F})$ corresponding to the partition $(p, q-1)$. Note that since $\sigma_{p,q}(n) = n$, we have that $H_{p,q-1}^{(n)}$ is a subgroup of G_{n-1}. If $q \geq 1$, also denote

$$H_{p-1,q-1}^{(n)} = \left\{ \begin{pmatrix} h & \\ & I_2 \end{pmatrix} \mid h \in H_{p-1,q-1}^{(n-2)} \right\}.$$

Let

$$U_n = \left\{ \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix} \mid x \in M_{n-1 \times 1}(\mathbb{F}) \right\}$$

be the unipotent radical of G_n, corresponding to the partition $(n-1,1)$. Let $P_n \leq G_n$ be the mirabolic subgroup. We have that $P_n = U_n \rtimes G_{n-1}$. ψ defines a character on U_n by $\psi(u) = \psi(u_{n-1,n})$.

Recall the definition of the following Bernstein–Zelevinsky derivative: for a representation π of P_{n-1}, we define

$$\Phi^+(\pi) = \mathrm{Ind}_{P_{n-1}U_n}^{P_n}(\pi \otimes \psi),$$

where $(\pi \otimes \psi)(pu) = \psi(u) \cdot \pi(p)$, for $p \in P_{n-1}$ and $u \in U_n$. We have the following relation between this functor and irreducible cuspidal representations of G_n:

THEOREM A.3 ([5, Theorem 2.3]): *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$. Then the restriction of π to the mirabolic subgroup P_n is isomorphic to the representation $(\Phi^+)^{n-1}(1)$.*

We start with the following lemmas from [11]. These are purely algebraic statements, which are proved exactly as in [11].

Lemma A.4 ([11, Lemma 3.1]): *Let $p \geq q \geq 1$ with $p+q=n$, and let*

$$S_{p,q}^{(n)} = \{g \in G_{n-1} \mid \psi(gug^{-1}) = 1, \forall u \in U_n \cap H_{p,q}^{(n)}\}.$$

Then $S_{p,q}^{(n)} = P_{n-1} \cdot H_{p,q-1}^{(n)}$.

Lemma A.5 ([11, Lemma 3.2]): *Let $p \geq q \geq 1$ with $p+q=n$, and let*

$$S_{p,q-1}^{(n)} = \{g \in G_{n-2} \mid \psi(gug^{-1}) = 1, \forall u \in U_{n-1} \cap H_{p,q-1}^{(n)}\}.$$

Then $S_{p,q-1}^{(n)} = P_{n-2} \cdot H_{p-1,q-1}^{(n)}$.

The proofs of both multiplicity one theorems rely on the following propositions:

Proposition A.6: *Suppose $p \geq q \geq 1$ with $p+q=n$. Let (σ, V_σ) be a representation of P_{n-1}. Then there exists an embedding*

$$\mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}(\Phi^+(\sigma), 1) \hookrightarrow \mathrm{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}(\sigma, 1).$$

Proposition A.7: *Suppose $p \geq q \geq 1$ with $p+q=n$. Let (σ, V_σ) be a representation of P_{n-2}. Then there exists an embedding*

$$\mathrm{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}(\Phi^+(\sigma), 1) \hookrightarrow \mathrm{Hom}_{P_{n-2} \cap H_{p-1,q-1}^{(n)}}(\sigma, 1).$$

We prove only Proposition A.6. The proof of Proposition A.7 is similar.

Proof of Proposition A.6. Denote

$$W = \Phi^+(\sigma) = \mathrm{Ind}_{P_{n-1}U_n}^{P_n}(\sigma'),$$

where $\sigma' = \sigma \otimes \psi$.

Let $A : \mathcal{S}(P_n, V_\sigma) \to W$ be the projection operator

$$(Af)(p) = \frac{1}{|P_{n-1}U_n|} \sum_{y \in P_{n-1}U_n} \sigma'(y)^{-1} f(yp).$$

Let $L \in \mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}(\Phi^+(\sigma), 1)$. We can define using A and L a distribution $T = L \circ A : \mathcal{S}(P_n, V_\sigma) \to \mathbb{C}$. One easily checks that this distribution satisfies

$$\langle T, \rho(h) f\rangle = \langle T, f\rangle \qquad \forall h \in P_n \cap H_{p,q}^{(n)}, \tag{A.1}$$

$$\langle T, \lambda(y_0) f\rangle = \langle T, \sigma'(y_0)^{-1} f\rangle \qquad \forall y_0 \in P_{n-1}U_n. \tag{A.2}$$

Since A is onto, we have that the map $L \mapsto L \circ A$ is an injection from $\mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}(\Phi^+(\sigma), 1)$ to the space of all distributions satisfying Equation (A.1) and Equation (A.2).

Let $\Psi : P_n \to \mathbb{C}$ be the function defined by $\Psi(ug) = \psi(u)$, where $u \in U_n$ and $g \in G_{n-1}$. It is well defined, since $U_n \cap G_{n-1} = \{I_n\}$. One has

$$\Psi(up) = \psi(u)\Psi(p)$$

for $u \in U_n$ and $p \in P_n$.

Let T be a distribution on $\mathcal{S}(P_n, V_\sigma)$, satisfying Equation (A.1) and Equation (A.2). We denote by $\Psi \cdot T$ the distribution defined by $\langle \Psi \cdot T, f\rangle = \langle T, \Psi \cdot f\rangle$. One easily checks using Equation (A.2) that

$$\langle \lambda(u)(\Psi \cdot T), f\rangle = \langle \Psi \cdot T, f\rangle, \tag{A.3}$$

for every $u \in U_n$. Define for $f : P_n \to V_\sigma$, $f' : G_{n-1} \to V_\sigma$ by

$$f'(g) = \sum_{u \in U_n} f(ug),$$

and define a distribution S on $\mathcal{S}(G_{n-1}, V_\sigma)$ by defining it on the basis by

$$\langle S, \delta_{g_0}\rangle = \langle \Psi \cdot T, \delta_{g_0}\rangle,$$

for every $g_0 \in G_{n-1}$. Then one easily checks using Equation (A.3) that $\langle S, f'\rangle = \langle \Psi \cdot T, f\rangle$, for every $f \in \mathcal{S}(G_{n-1}, V_\sigma)$. It follows that

$$\langle \Psi \cdot T, \rho(u_0)f\rangle = \langle \Psi \cdot T, f\rangle,$$

for every $f \in \mathcal{S}(P_n, V_\sigma)$ and $u_0 \in U_n$.

Suppose that $g_0 \in G_{n-1}$ is in the support of S, i.e., $\langle S, \delta_{g_0}\rangle \neq 0$. Then for every $u_0 \in U_n \cap H^{(n)}_{p,q}$, we have

$$\Psi(g_0)\langle T, \delta_{g_0}\rangle = \langle \Psi \cdot T, \delta_{g_0}\rangle = \langle \Psi \cdot T, \rho(u_0)\delta_{g_0}\rangle = \Psi(g_0 u_0^{-1})\langle T, \delta_{g_0}\rangle.$$

Since $\langle S, \delta_{g_0}\rangle \neq 0$, we get that $\langle T, \delta_{g_0}\rangle \neq 0$, and therefore $\Psi(g_0 u_0^{-1}) = \Psi(g_0)$, which implies that $\psi(g_0 u_0 g_0^{-1}) = 1$. Thus we have that $\mathrm{supp} S \subseteq S^{(n)}_{p,q}$, and by Lemma A.4, $\mathrm{supp} S \subseteq P_{n-1} \cdot H^{(n)}_{p,q-1}$, and hence

$$\mathrm{supp} T \subseteq U_n P_{n-1} H^{(n)}_{p,q-1} = P_{n-1} U_n H^{(n)}_{p,q-1}.$$

Hence the restriction map $T \to T \restriction_{\mathcal{S}(P_{n-1}U_n H^{(n)}_{p,q-1}, V_\sigma)}$, from the space of distributions on $\mathcal{S}(P_n, V_\sigma)$ satisfying Equation (A.1) and Equation (A.2) to the space on distributions of $\mathcal{S}(P_{n-1}U_n H^{(n)}_{p,q-1}, V_\sigma)$ satisfying Equation (A.1) and Equation (A.2), is injective.

Consider the projection

$$B : \mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V_\sigma) \to \mathcal{S}(P_{n-1}U_nH^{(n)}_{p,q-1}, V_\sigma)$$

given by

$$(Bf)(y^{-1}h) = \frac{1}{|P_{n-1} \cap H^{(n)}_{p,q-1}|} \sum_{a \in P_{n-1} \cap H^{(n)}_{p,q-1}} f(ay, ah),$$

for $y \in P_{n-1}U_n$, $h \in H^{(n)}_{p,q-1}$. This is well defined, and B is a projection in the sense that if $f(y,h) = g(y^{-1}h)$ for $g \in \mathcal{S}(P_{n-1}U_nH^{(n)}_{p,q-1}, V_\sigma)$, then $Bf = g$. In particular B is onto.

Consider the linear isomorphism from $\mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V_\sigma)$ to itself, $\phi \mapsto \tilde{\phi}$, given by

$$\tilde{\phi}(y,h) = \sigma'(y)^{-1}\phi(y,h).$$

Let $y_0 \in P_{n-1}U_n$, $h_0 \in H^{(n)}_{p,q-1}$, $\phi \in \mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V_\sigma)$ and denote $\phi_1 = \rho(y_0, h_0)\phi$. One checks that

$$\widetilde{\phi_1} = \sigma'(y_0)(\rho(y_0,h_0)\tilde{\phi})$$

and that

$$B(\widetilde{\phi_1}) = \rho(h_0)\lambda(y_0)\sigma'(y_0)B(\tilde{\phi}),$$

which implies for a distribution T on $\mathcal{S}(P_{n-1}U_nH^{(n)}_{p,q-1}, V_\sigma)$ satisfying Equation (A.1) and Equation (A.2), we have

$$\langle T, B(\tilde{\phi})\rangle = \langle T, B(\widetilde{\phi_1})\rangle.$$

For T as above, we define a distribution D_T on $\mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V_\sigma)$ by

$$\langle D_T, \phi\rangle = \langle T, B(\tilde{\phi})\rangle.$$

The map $T \mapsto D_T$ is injective, as B is surjective and $\phi \mapsto \tilde{\phi}$ is an isomorphism. The above discussion shows that $\rho(y_0,h_0)D_T = D_T$, for every $y_0 \in P_{n-1}U_n$, $h_0 \in H^{(n)}_{p,q-1}$. This means that D_T is determined by the functional $\xi_T : V_\sigma \to \mathbb{C}$, defined by

$$\langle \xi_T, v\rangle = \langle D_T, v \cdot \delta_{(I_n,I_n)}\rangle,$$

and is given by the formula

$$\text{(A.4)} \qquad \langle D_T, \phi\rangle = \sum_{\substack{y \in P_{n-1}U_n \\ h \in H^{(n)}_{p,q-1}}} \langle \xi_T, \phi(y,h)\rangle.$$

We now show that $\xi_T \in \mathrm{Hom}_{P_{n-1} \cap H^{(n)}_{p,q-1}}(\sigma, 1)$. Let $\phi \in \mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1})$, let $b \in P_{n-1} \cap H^{(n)}_{p,q-1}$, and let $\phi_1 = \lambda(b,b)\phi$. Then an easy computation shows that

$$\lambda(b,b)\widetilde{\sigma'(b)^{-1}\phi} = \widetilde{\phi_1}.$$

Since $B(\lambda(b,b)f) = B(f)$, for every $f \in \mathcal{S}(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V_\sigma)$, we get that

$$\langle T, \lambda(b,b)\phi \rangle = \langle T, \sigma'(b)^{-1}\phi \rangle.$$

It follows from Equation (A.4) that

$$\langle T, \lambda(b,b)\phi \rangle = \langle T, \phi \rangle.$$

Therefore $\langle \xi_T, v \rangle = \langle \xi_T, \sigma'(b)v \rangle$, for every $v \in V_\sigma$ and every $b \in P_{n-1} \cap H^{(n)}_{p,q-1}$, as required. ∎

We are now able to prove Theorem A.1 and Theorem A.2.

Proof of Theorem A.1. Since

$$H^{(2m)}_{m,m} = w_{m,m} M_{m,m} w_{m,m}^{-1} \quad \text{and} \quad P_{2m} = w_{m,m} P_{2m} w_{m,m}^{-1},$$

we get that

$$\mathrm{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1) \cong \mathrm{Hom}_{P_{2m} \cap H^{(2m)}_{m,m}}(\pi, 1)$$

by mapping $L \in \mathrm{Hom}_{P_{2m} \cap H^{(2m)}_{m,m}}(\pi, 1)$ to $L\pi(w_{m,m}) \in \mathrm{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1)$. Therefore it suffices to prove that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_{2m} \cap H^{(2m)}_{m,m}}(\pi, 1) \leq 1.$$

Since π is an irreducible cuspidal representation, we have from Theorem A.3 that

$$\pi \restriction_{P_{2m}} \cong (\Phi^+)^{2m-1}(1).$$

Using Proposition A.6 and Proposition A.7 repeatedly, and using the fact that $H^{(2m)}_{m-1,m-1} \cong H^{(2m-2)}_{m-1,m-1}$, one gets an embedding

$$\mathrm{Hom}_{P_{2m} \cap H^{(2m)}_{m,m}}((\Phi^+)^{2m-1}(1), 1) \hookrightarrow \mathrm{Hom}_{P_2 \cap H^{(2)}_{1,0}}(1, 1).$$

The last space is one-dimensional, and therefore we get the required result. ∎

Proof of Theorem A.2. Let τ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & m & | & m+1 & | & m+2 & m+3 & \dots & 2m+1 \\ 1 & 2 & 3 & \dots & m & | & 2m+1 & | & m+1 & m+2 & \dots & 2m \end{pmatrix},$$

and let w_τ be the column permutation matrix corresponding to τ. Then $L_{2m+1} = w_\tau M_{m+1,m} w_\tau^{-1}$, where $M_{m+1,m}$ is the standard Levi subgroup corresponding to the partition $(m+1, m)$. A simple calculation shows that

$$L_{2m+1} \cap P_{2m+1} = w_\tau \left\{ \begin{pmatrix} p & \\ & g \end{pmatrix} \mid p \in P_{m+1}, g \in \mathrm{GL}_m(\mathbb{F}) \right\} w_\tau^{-1}.$$

Similarly, an easy calculation shows that

$$H^{(2m+2)}_{m+1,m} \cap P_{2m+1} = w_{m+1,m+1} \left\{ \begin{pmatrix} p & & \\ & g & \\ & & 1 \end{pmatrix} \mid \begin{array}{l} p \in P_{m+1} \\ g \in \mathrm{GL}_m(\mathbb{F}) \end{array} \right\} w_{m+1,m+1}^{-1}.$$

Therefore $P_{2m+1} \cap H^{(2m+2)}_{m+1,m}$ and $P_{2m+1} \cap L_{2m+1}$ are conjugate as subgroups of $\mathrm{GL}_{2m+1}(\mathbb{F})$ (actually even as subgroups of P_{2m+1}), which implies that

$$\mathrm{Hom}_{P_{2m+1} \cap L_{2m+1}}(\pi, 1) \cong \mathrm{Hom}_{P_{2m+1} \cap H^{(2m+2)}_{m+1,m}}(\pi, 1).$$

Thus it suffices to prove that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_{2m+1} \cap H^{(2m+2)}_{m+1,m}}(\pi, 1) \le 1.$$

As in the previous proof, by Theorem A.3, and by using Proposition A.6 and Proposition A.7 repeatedly, we get an embedding

$$\mathrm{Hom}_{P_{2m+1} \cap H^{(2m+2)}_{m+1,m}}(\pi, 1) \hookrightarrow \mathrm{Hom}_{P_2 \cap H^{(2)}_{1,0}}(1, 1),$$

and the statement follows. ∎

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