LOCALLY ANALYTIC REPRESENTATIONS IN THE ÉTALE COVERINGS OF THE LUBIN–TATE MODULI SPACE

BY

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ABSTRACT

The Lubin–Tate moduli space X_0^{rig} is a p-adic analytic open unit polydisc which parametrizes deformations of a formal group H_0 of finite height defined over an algebraically closed field of characteristic p. It is known that the natural action of the automorphism group $\mathrm{Aut}(H_0)$ on X_0^{rig} gives rise to locally analytic representations on the topological duals of the spaces $H^0(X_0^{\mathrm{rig}}, (\mathcal{M}_0^s)^{\mathrm{rig}})$ of global sections of certain equivariant vector bundles $(\mathcal{M}_0^s)^{\mathrm{rig}}$ over X_0^{rig}. In this article, we show that this result holds in greater generality. On the one hand, we work in the setting of deformations of formal modules over the valuation ring of a finite extension of $\mathbb{Q}_p$. On the other hand, we also treat the case of representations arising from the vector bundles $(\mathcal{M}_m^s)^{\mathrm{rig}}$ over the deformation spaces X_m^{rig} with Drinfeld level-m-structures. Finally, we determine the space of locally finite vectors in $H^0(X_m^{\mathrm{rig}}, (\mathcal{M}_m^s)^{\mathrm{rig}})$. Essentially, all locally finite vectors arise from the global sections of invertible sheaves over the projective space via pullback along the Gross–Hopkins period map.

CONTENTS

1. Introduction

The theory of locally analytic representations provides a suitable framework to study continuous p-adic representations of p-adic reductive groups in the context of the p-adic Langlands program. Introduced by Schneider–Teitelbaum and later developed by Emerton, the notion of a locally analytic representation V of a p-adic Lie group G is (roughly) defined by the property that, for each vector $v \in V$, the orbit map

$$G \longrightarrow V, \quad g \longmapsto g(v),$$

is locally on G given by a convergent power series with coefficients in V (cf. [Eme17], [ST02ii]). Thus the category of locally analytic representations encompasses classical smooth representations, finite-dimensional algebraic representations as well as tensor products of these two, the so-called locally algebraic representations. A fundamental theorem of locally analytic representation theory establishes an anti-equivalence between the category of locally analytic representations on vector spaces of compact type and the category of continuous $D(G)$-modules on nuclear Fréchet spaces via duality functor, where $D(G)$ is the algebra of locally analytic distributions on G (cf. [ST02ii], Corollary 3.4).

First non-trivial examples of locally analytic representations coming from geometry were found by Morita in his investigation of the p-adic upper half plane or **Drinfeld's upper half space** of dimension 1 (cf. [Mor85]). In general, if K is a finite extension of $\mathbb{Q}_p$ then Drinfeld's upper half space Y_0^{rig} of dimension $h-1$ is obtained by deleting all K-rational hyperplanes from the projective space $\mathbb{P}_K^{h-1}$. The natural action of $GL_h(K)$ on the projective space stabilizes Y_0^{rig}. Restricting any $GL_h(K)$-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}_K^{h-1}$ to Y_0^{rig} gives rise to a locally analytic $GL_h(K)$-representation on the strong topological dual of the nuclear Fréchet space $\mathcal{F}(Y_0^{\mathrm{rig}})$ of its global sections (cf. [Orl08], [ST02i]). The upper half space and its étale coverings give rise to Drinfeld's upper half space Y_∞^{rig} **at infinity** which is a moduli space parametrizing certain EL Rapoport–Zink data. The dual space to Y_∞^{rig} is the **Lubin–Tate moduli space X_∞^{rig} at infinity** parametrizing the dual EL Rapoport–Zink data (cf. [SW13], Section 7). Analogous to the general linear group action on the upper half space, there is a natural action of another p-adic Lie group Γ on the Lubin–Tate moduli space X_0^{rig} and its étale covers. While examining this action of Γ, Kohlhaase first showed that, in this case too, one obtains locally

analytic representations on the dual space of the global sections of certain equi-variant vector bundles over the Lubin–Tate moduli space provided $K = \mathbb{Q}_p$ (cf. [Koh14], Theorem 3.5). The aim of this article is twofold, firstly to gener-alize Kohlhaase's result to any finite base extension K of $\mathbb{Q}_p$ and extend it to the finite étale coverings of the Lubin–Tate moduli space, secondly to compute locally finite (algebraic) vectors in the concerned representations in order to understand their structure.

Let us consider the Γ-equivariant vector bundles over X_m^{rig} induced by the s-fold tensor power of the Lie algebra of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$ at

To describe our results in detail, let p be a prime number and K be a finite ex-tension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, uniformizer ϖ and residue class field k. Let us denote by $\breve{K}$ the completion of the maximal unramified extension of K and by $\breve{\mathfrak{o}}$ its ring of integers. Fix a (unique) one-dimensional formal $\mathfrak{o}$-module H_0 over an algebraic closure $\overline{k}$ of k of finite height h. The Lubin–Tate moduli space is a formal scheme X_0 parametrizing deformations of H_0 to complete local $\breve{\mathfrak{o}}$-algebras with residue field $\overline{k}$.[1] Adding level structures to the moduli problem, Drinfeld showed that the formal scheme X_m parametrizing deforma-tions equipped with a level-m-structure is a finite flat covering of X_0, and X_0 is (non-canonically) isomorphic to the formal spectrum $\mathrm{Spf}(\breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]])$ (cf. [Dri74], Proposition 4.2 and Proposition 4.3). Passing to the generic fibres of the formal schemes, one obtains a tower of rigid $\breve{K}$-analytic spaces $(X_m^{\mathrm{rig}})_{m \in \mathbb{N}_0}$ carrying commuting actions of the covering group $GL_h(\mathfrak{o})$ and of the automor-phism group $\Gamma \cong \mathfrak{o}_{B_h}^\times$ of H_0, where $\mathfrak{o}_{B_h}$ is the maximal order of the central K-division algebra B_h of invariant $1/h$. The covering group action on X_m^{rig} fac-tors through the finite group $GL_h(\mathfrak{o}/\varpi^m \mathfrak{o})$ making X_m^{rig} an étale Galois cover of the open unit polydisc X_0^{rig} with Galois group $GL_h(\mathfrak{o}/\varpi^m \mathfrak{o})$. On the other hand, the Γ-action on X_m^{rig} is much more complicated and is the one we are interested in. These group actions are of significance, as they realize the lo-cal Jacquet–Langlands correspondence on the l-adic étale cohomology of the Lubin–Tate tower, as conjectured by Carayol (cf. [Car90]. [Str08ii]).

Let us consider the Γ-equivariant vector bundles over X_m^{rig} induced by the s-fold tensor power of the Lie algebra of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$ at level m for any integer s, and denote by M_m^s the global sections of these vector bundles. The Γ-action on the nuclear $\breve{K}$-Fréchet space M_m^s is semi-linear for its action on $\mathcal{O}_{X_m^{\mathrm{rig}}}(X_m^{\mathrm{rig}}) = M_m^0$. In our first main result, we prove that the strong topological $\breve{K}$-linear dual $(M_m^s)_b'$ of M_m^s is a locally K-analytic representation

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[1] We will mostly refer to the generic fibre X_0^{rig} as the Lubin–Tate moduli space.

of Γ for all $s \in \mathbb{Z}$ and $m \geq 0$ (cf. Theorem 3.3.5 and Theorem 3.4.6). The proof of local analyticity essentially follows the Kohlhaase's approach in [Koh14] and consists of the following two steps:

(1) The Gross–Hopkins' p-adic period map $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}^{h-1}_{\breve{K}}$ constructed in [GH94] can be used to explicitly find out the Γ-action on the fundamental domain D of X_0^{rig}. We first show that the explicitly known Γ-action on the sections M_D^s over D is locally K-analytic by direct computations.

(2) Using the structure theory of the locally $\mathbb{Q}_p$-analytic distribution algebra $D(\Gamma_{\mathbb{Q}_p})$, we then show that the continuous Γ-action on M_m^s extends to a continuous action of $D(\Gamma_{\mathbb{Q}_p})$. Finally, to deduce that this action factors through a continuous action of the locally K-analytic distribution algebra $D(\Gamma)$, we use the step 1 and the étaleness of the covering morphisms.

Our second main result concerns computing the subspace of $(M_m^s)_{\mathrm{lf}}$ of locally finite vectors in the Γ-representations M_m^s. A **locally finite vector** is a vector contained in a finite-dimensional subrepresentation of some open subgroup of Γ. Consider the $\breve{K}$-linear algebraic representation $B_h \otimes_{K_h} \breve{K}$ on which Γ acts by the left multiplication, and let $\breve{K}_m$ denote the m-th Lubin–Tate extension of $\breve{K}$ equipped with a smooth Γ-action via

$$\mathfrak{o}_{B_h}^\times \xrightarrow{\mathrm{Nrd}} \mathfrak{o}^\times \twoheadrightarrow (\mathfrak{o}/\varpi^m \mathfrak{o})^\times \cong \mathrm{Gal}(\breve{K}_m/\breve{K}).$$

We show that there is an isomorphism

$$(M_m^s)_{\mathrm{lf}} \cong \breve{K}_m \otimes_{\breve{K}} \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K}) \cong \breve{K}_m \otimes_{\breve{K}} \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$$

of Γ-representations for all $m \geq 0$ and $s \in \mathbb{Z}$ (cf. Corollary 4.1.11, Theorem 4.2.5, Theorem 4.2.13). Moreover, $(M_m^s)_{\mathrm{lf}}$ is a finite-dimensional semi-simple locally algebraic representation. To prove the above isomorphism, we extensively use the action of the Lie algebra of Γ obtained from the Gross–Hopkins' period map. The other key ingredients of the proof are the generic flatness of the line bundle induced by the Lie algebra of the universal additive extension (cf. [GH94], Section 21), Strauch's result on geometrically connected components of X_m^{rig} (cf. [Str08i]) and Fargues' cellular decomposition of the Lubin–Tate tower (cf. [FGL08], Section I.7).

We expect both our results to hold in much greater generality. The Γ-equivariant vector bundles that we consider arise as the pullbacks of the invertible sheaves $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)$ on the projective space along the Gross–Hopkins' period map (cf. Remark 3.1.14). Given any Γ-equivariant vector bundle $\mathcal{G}$ over the projective space $\mathbb{P}^{h-1}_{\breve{K}}$, we believe that the representations realized on the global sections over X^{rig}_m of its pullback along the period map are dual to locally analytic representations, and the locally algebraic part again comes from the global sections of $\mathcal{G}$. However, we don't have a proof as of now. Another major open question concerning the locally analytic Γ-representations $(M^s_m)'_b$ is whether they are admissible or not in the sense of [ST03], Section 6. The similar representations in the example of Drinfeld's upper half space and its first étale covering are known to be admissible (cf. [PSS19]). However, the presence of the spherical Hecke algebra action on the space of global rigid analytic functions on the Lubin–Tate moduli space raises questions on the admissibility of $(M^s_m)'_b$ (cf. [Koh13], Proposition 3.3 and Remark 3.5). We would also like to mention the work [Lo15] of Chi Yu Lo, showing the analyticity of the action of a certain rigid analytic group associated to Γ on a particular closed polydisc of X^{rig}_0, which will likely be relevant in further investigations of locally analytic representations coming from the Lubin–Tate moduli space.

NOTATION AND CONVENTIONS. $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of non-negative integers respectively. If $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r_0$ is an r-tuple of non-negative integers and $T = (T_1, \ldots, T_r)$ is a family of indeterminates for some $r \in \mathbb{N}$, then we set $|\alpha| := \alpha_1 + \cdots + \alpha_r$, and $T^\alpha := T^{\alpha_1}_1 \cdots T^{\alpha_r}_r$. Unless stated otherwise, all rings are considered to be commutative with identity. A ring extension $A \subseteq B$ will be denoted by $B|A$, and its degree by $[B : A]$ if it is finite and free. Let p be a fixed prime number and let K be a finite field extension of $\mathbb{Q}_p$ with the valuation ring $\mathfrak{o}$. We fix a uniformizer ϖ of K and

let $k := \mathfrak{o}/\varpi\mathfrak{o}$ denote its residue class field of characteristic p and cardinality q. The absolute value $|\cdot|$ of K is assumed to be normalized through $|p| = p^{-1}$. We denote by $\breve{K}$ the completion of the maximal unramified extension of K, and by $\breve{\mathfrak{o}}$ its valuation ring. We denote by σ the Frobenius automorphism of an algebraic closure $\overline{k}$ of k, as well as its unique lift to a ring automorphism of $\breve{\mathfrak{o}}$ and the induced field automorphism of $\breve{K}$. We also fix an algebraic closure $\overline{\breve{K}}$ of $\breve{K}$ and denote its valuation ring by $\overline{\breve{\mathfrak{o}}}$. The absolute value $|\cdot|$ on K extends uniquely to $\breve{K}$, and to $\overline{\breve{K}}$. For a positive integer h, let K_h be the unramified extension of K of degree h, $\mathfrak{o}_h$ be its valuation ring, and B_h be the central K-division algebra of invariant $1/h$. We fix an embedding $K_h \hookrightarrow B_h$ and a uniformizer Π of B_h, satisfying $\Pi^h = \varpi$. Let $\mathrm{Nrd} : B_h \longrightarrow K$ denote the reduced norm of B_h over K. The symbol $\mathbb{P}^{h-1}_{\breve{K}}$ always denotes the $(h-1)$-dimensional rigid analytic projective space over $\breve{K}$.

2. Drinfeld's coverings of the Lubin–Tate moduli space and the group actions

We begin with a quick introduction to the Lubin–Tate deformation problem equipped with Drinfeld's level structures. Then we prove the main result of this section, namely the continuity of the Γ-action on the universal deformation rings.

2.1. DEFORMATIONS OF FORMAL $\mathfrak{o}$-MODULES WITH LEVEL STRUCTURES. Recall from [GH94] that a one-dimensional formal $\mathfrak{o}$-module F over a local $\mathfrak{o}$-algebra A is, after having fixed a formal coordinate, given by a formal power series $F(X, Y) \in A[[X, Y]]$, together with a ring homomorphism $[\,\cdot\,]_F : \mathfrak{o} \longrightarrow \mathrm{End}(F)$ such that $[\lambda]_A(X) \equiv i_A(\lambda)X \pmod{\deg 2}$, where $i_A : \mathfrak{o} \longrightarrow A$ is the structure morphism. Let H_0 be a one-dimensional formal $\mathfrak{o}$-module of finite height h over $\overline{k}$ which is defined over k. According to [Dri74], Proposition 1.6 and 1.7, the formal module H_0 is unique up to isomorphism, and one has

$$(2.1.1) \qquad\qquad \mathrm{End}(H_0) \cong \mathfrak{o}_{B_h}$$

where $\mathfrak{o}_{B_h}$ is the valuation ring of the central K-division algebra B_h of invariant $1/h$. Let $\mathcal{C}$ be the category of commutative unital complete Noetherian local $\breve{\mathfrak{o}}$-algebras $R = (R, \mathfrak{m}_R)$ with residue class field $\overline{k}$. The Lubin–Tate deformation problem considers liftings of H_0 to the objects of $\mathcal{C}$ together with certain additional data defined below.

Definition 2.1.2: Let R be an object of $\mathcal{C}$ and H be a formal $\mathfrak{o}$-module over R, given by a power series $H(X, Y) \in R[[X, Y]]$.

(1) A pair (H, ρ), where $\rho : H_0 \xrightarrow{\sim} H \otimes_R \overline{k}$ is an isomorphism of formal $\mathfrak{o}$-modules over $\overline{k}$, is called **deformation** of H_0 to R.

(2) Denote by $(\mathfrak{m}_R, +_H)$ the abstract $\mathfrak{o}$-module $\mathfrak{m}_R$ in which addition and $\mathfrak{o}$-multiplication are defined as

$$x +_H y := H(x, y) \quad \text{and} \quad ax := [a]_H(x)$$

respectively for all $x, y \in \mathfrak{m}_R$, $a \in \mathfrak{o}$. For a non-negative integer m, a **Drinfeld level-m-structure on H** is a homomorphism

$$\phi : \left(\frac{\varpi^{-m} \mathfrak{o}}{\mathfrak{o}} \right)^h \longrightarrow (\mathfrak{m}_R, +_H)$$

of abstract $\mathfrak{o}$-modules such that $\prod_{\alpha \in (\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h}(X - \phi(\alpha))$ divides $[\varpi^m]_H(X)$ in $R[[X]]$.

(3) We call the triple (H, ρ, ϕ) a deformation of H_0 to R **with level-m-structure** if (H, ρ) is a deformation of H_0 to R and ϕ is a Drinfeld level-m-structure on H.

Two deformations (H, ρ, ϕ) and (H', ρ', ϕ') of H_0 to R with level-m-structures are isomorphic if there is an isomorphism $f : H \xrightarrow{\sim} H'$ of formal $\mathfrak{o}$-modules over R making the following diagrams commutative:
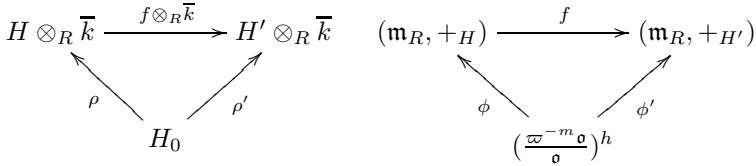
For any integer $m \geq 0$, consider the set valued functor $\mathrm{Def}_m : \mathcal{C} \longrightarrow \mathrm{Set}$, which associates to an object R of $\mathcal{C}$ the set of isomorphism classes of deformations of H_0 to R with level-m-structures. For a morphism $\varphi : R \longrightarrow R'$ in $\mathcal{C}$, $\mathrm{Def}_m(\varphi)$ is defined by sending a class $[(H, \rho, \phi)]$ to the class $[(H \otimes_R R', \rho, \varphi \circ \phi)]$. Notice that

$$\rho : H_0 \xrightarrow{\sim} H \otimes_R \overline{k} \cong (H \otimes_R R') \otimes_{R'} \overline{k}.$$

We denote the triple $(H \otimes_R R', \rho, \varphi \circ \phi)$ by $\varphi_*(H, \rho, \phi)$ for simplicity.

THEOREM 2.1.3 (Lubin–Tate, Drinfeld):

(1) *The functor* Def_m *is representable by a regular local ring* R_m *of dimension* h *for all* $m \geq 0$.

(2) *For any two integers* $0 \leq m \leq m'$, *the natural transformation* $\mathrm{Def}_{m'} \longrightarrow \mathrm{Def}_m$ *of functors defined by sending a class* $[(H, \rho, \phi)]$ *to* $[(H, \rho, \phi|_{(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h})]$ *induces a homomorphism of local rings* $R_m \longrightarrow R_{m'}$ *which is finite and flat.*

(3) *The ring* R_0 *is non-canonically isomorphic to the ring* $\breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ *of formal power series in* $h - 1$ *indeterminates over* $\breve{\mathfrak{o}}$.

Proof. See [Dri74], Proposition 4.2 and 4.3. ∎

Let us denote the universal deformation of H_0 to R_m with level-m-structure by the triple $(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$. Here $\mathbb{H}^{(m)} = \mathbb{H}^{(0)} \otimes_{R_0} R_m$, i.e., the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$ over R_m is given by the base change of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(0)}$ over R_0 under the map $R_0 \longrightarrow R_m$ induced by Part 2, Theorem 2.1.3. We note that, since $R_0 \cong \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ is an integral domain, the flatness of the map $R_0 \longrightarrow R_m$ implies $R_0 \hookrightarrow R_m$ for all $m \geq 0$.

By the universal property, given an object R of $\mathcal{C}$ and a deformation (H, ρ, ϕ) of H_0 to R with level-m-structure, there is a unique $\breve{\mathfrak{o}}$-linear local ring homomorphism $\varphi : R_m \longrightarrow R$ such that

$$\mathrm{Def}_m(\varphi)([(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})]) = [\varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})] = [(H, \rho, \phi)].$$

The unique isomorphism between the deformations $\varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$ and (H, ρ, ϕ) over R will be denoted by $[\varphi] : \varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) \overset{\sim}{\longrightarrow} (H, \rho, \phi)$ (cf. [GH94], Proposition 12.10).

2.2. THE GROUP ACTIONS. For all $m \geq 0$, the functor Def_m admits natural commuting left actions of the groups $\Gamma := \mathrm{Aut}(H_0)$ and $G_0 := GL_h(\mathfrak{o})$ for which the morphisms $\mathrm{Def}_{m'} \longrightarrow \mathrm{Def}_m$ of functors mentioned in Part 2, Theorem 2.1.3 are equivariant. On R-valued points, they are given by

$$[(H, \rho, \phi)] \longmapsto [(H, \rho \circ \gamma^{-1}, \phi)] \quad \text{and} \quad [(H, \rho, \phi)] \longmapsto [(H, \rho, \phi \circ g^{-1})]$$

for $\gamma \in \Gamma, g \in G_0$. Here $g^{-1} \in GL_h(\mathfrak{o})$ acts on the free $(\frac{\mathfrak{o}}{\varpi^m \mathfrak{o}})$-module $(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h$ by considering it as an $\mathfrak{o}$-module via the natural reduction map $\mathfrak{o} \twoheadrightarrow \frac{\mathfrak{o}}{\varpi^m \mathfrak{o}}$. Because of the representability, these actions give rise to commuting left actions of Γ and G_0 on the universal deformation rings R_m. We use the same letters γ

and g to denote the automorphisms of R_m induced by $\gamma \in \Gamma$ and by $g \in G_0$ respectively. It is immediate from the definition that the G_0-action on R_m factors through a quotient by the m-th principal congruence subgroup

$$G_m := 1 + \varpi^m M_h(\mathfrak{o})$$

of G_0. For $m' \geq m \geq 0$, the induced action of $G_m/G_{m'}$ makes $R_{m'}[\frac{1}{\varpi}]$ étale and Galois over $R_m[\frac{1}{\varpi}]$ with Galois group $G_m/G_{m'}$ (cf. Theorem 2.1.2 (ii), [Str08ii]).

The actions of Γ and G_0 on R_m induce semilinear actions of Γ and G_0 on the Lie algebra $\mathrm{Lie}(\mathbb{H}^{(m)})$ of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$. Recall that the Lie algebra $\mathrm{Lie}(\mathbb{H}^{(m)})$ of $\mathbb{H}^{(m)}$ is the tangent space $\mathrm{Hom}_{R_m}((X)/(X)^2, R_m)$ of its coordinate ring $R_m[[X]]$ (equipped with the trivial Lie bracket). We now describe the Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})$; the G_0-action is defined likewise. Given $\gamma \in \Gamma$, extend the ring automorphism γ of R_m to $R_m[[X]]$ by sending X to itself. This induces a homomorphism

$$\gamma_* : \mathrm{Lie}(\mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\gamma_* \mathbb{H}^{(m)})$$

of additive groups. The isomorphism $[\gamma] : \gamma_* \mathbb{H}^{(m)} \xrightarrow{\sim} \mathbb{H}^{(m)}$ also induces a natural R_m-linear map

$$\mathrm{Lie}([\gamma]) : \mathrm{Lie}(\gamma_* \mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)}).$$

We define $\gamma : \mathrm{Lie}(\mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)})$ as the composite of these two maps, i.e., $\gamma := \mathrm{Lie}([\gamma]) \circ \gamma_*$.

Given another element $\gamma' \in \Gamma$, let $\gamma'_*[\gamma] : \gamma'_*(\gamma_* \mathbb{H}^{(m)}) \xrightarrow{\sim} \gamma'_* \mathbb{H}^{(m)}$ be the isomorphism obtained by applying γ' to the coefficients of $[\gamma]$. Then $[\gamma'] \circ \gamma'_*[\gamma]$ is an isomorphism between the formal $\mathfrak{o}$-modules $(\gamma'\gamma)_* \mathbb{H}^{(m)}$ and $\mathbb{H}^{(m)}$ over R_m. Therefore by uniqueness, we have $[\gamma'\gamma] = [\gamma'] \circ \gamma'_*[\gamma]$. One also checks easily that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Lie}(\gamma_* \mathbb{H}^{(m)}) & \xrightarrow{\ \mathrm{Lie}([\gamma])\ } & \mathrm{Lie}(\mathbb{H}^{(m)}) \\
{\scriptstyle \gamma'_*} \downarrow & & \downarrow {\scriptstyle \gamma'_*} \\
\mathrm{Lie}(\gamma'_*(\gamma_* \mathbb{H}^{(m)})) & \xrightarrow[\ \mathrm{Lie}(\gamma'_*[\gamma])\ ]{} & \mathrm{Lie}(\gamma'_* \mathbb{H}^{(m)})
\end{array}
$$

Then it follows that

$$
\begin{aligned}
\mathrm{Lie}([\gamma'\gamma]) \circ (\gamma'\gamma)_* &= \mathrm{Lie}([\gamma']) \circ \mathrm{Lie}(\gamma'_*[\gamma]) \circ \gamma'_* \circ \gamma_* \\
(2.2.1) \qquad &= \mathrm{Lie}([\gamma']) \circ (\gamma'_* \circ \mathrm{Lie}([\gamma]) \circ (\gamma'_*)^{-1}) \circ \gamma'_* \circ \gamma_* \\
&= \mathrm{Lie}([\gamma']) \circ \gamma'_* \circ \mathrm{Lie}([\gamma]) \circ \gamma_*.
\end{aligned}
$$

Thus we obtain an action of Γ (and of G_0) on the additive group $\mathrm{Lie}(\mathbb{H}^{(m)})$ which is semilinear for the action of Γ (and of G_0 respectively) on R_m because γ_* is semilinear. Given a positive integer s, we denote by $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ the s-fold tensor product of $\mathrm{Lie}(\mathbb{H}^{(m)})$ over R_m with itself. This is a free R_m-module of rank 1 with a semi-linear action of Γ defined by

$$\gamma(\delta_1 \otimes \cdots \otimes \delta_s) := \gamma(\delta_1) \otimes \cdots \otimes \gamma(\delta_s).$$

Set

$$\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes 0} := R_m$$

and

$$\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} := \mathrm{Hom}_{R_m}(\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes(-s)}, R_m)$$

if s is a negative integer. In the latter case, a semi-linear action of Γ is defined by

$$\gamma(\varphi)(\delta_1 \otimes \cdots \otimes \delta_{-s}) := \gamma(\varphi(\gamma^{-1}(\delta_1) \otimes \cdots \otimes \gamma^{-1}(\delta_{-s}))).$$

The semi-linear actions of G_0 on the s-fold tensor products are defined similarly. As before, for all $s \in \mathbb{Z}$, the G_0-action on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ factors through G_0/G_m.

Remark 2.2.2: Using that the group actions of Γ and G_0 on R_m commute, one can show that they commute on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ as follows. It suffices to show the commutativity for $s = 1$. Since the G_0-action is defined likewise, we may use (2.2.1) for $\gamma \in \Gamma$ and $g \in G_0$. As a result, we get

$$\begin{aligned}
\mathrm{Lie}([g]) \circ g_* \circ \mathrm{Lie}([\gamma]) \circ \gamma_* &= \mathrm{Lie}([g\gamma]) \circ (g\gamma)_* \\
&= \mathrm{Lie}([\gamma g]) \circ (\gamma g)_* \\
&= \mathrm{Lie}([\gamma]) \circ \gamma_* \circ \mathrm{Lie}([g]) \circ g_*.
\end{aligned}$$

We are primarily interested in the action of Γ, a p-adic Lie gorup, on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$. Before describing the underlying Lie group structure of Γ, we refer the reader to [Sch11], page 38, page 47 and page 89 for the definitions of a locally analytic map, a locally analytic manifold and a locally analytic group, respectively. By (2.1.1), we have $\Gamma \cong \mathfrak{o}_{B_h}^\times$. Recall that the division algebra B_h is a K_h-vector space of dimension h with basis $\{\Pi^i\}_{0 \le i \le h-1}$ whose multiplication is determined by the relations $\Pi^h = \varpi$ and $\Pi\lambda = \lambda^\sigma \Pi$ for all $\lambda \in K_h$ (λ^σ denotes the image of λ under the Frobenius automorphism σ). Thus, any $\gamma \in \Gamma = \mathfrak{o}_{B_h}^\times$

can be uniquely written as

$$\gamma = \sum_{i=0}^{h-1} \lambda_i \Pi^i$$

with $\lambda_0 \in \mathfrak{o}_h^\times$ and $\lambda_1, \ldots, \lambda_{h-1} \in \mathfrak{o}_h$. The map

$$\psi : \Gamma \longrightarrow K_h^h$$

(2.2.3)
$$\sum_{i=0}^{h-1} \lambda_i \Pi^i \longmapsto (\lambda_0, \lambda_1 \ldots \lambda_{h-1})$$

identifies Γ with a compact open subset $\mathfrak{o}_h^\times \times \mathfrak{o}_h^{h-1}$ of K_h^h making it into a compact open locally K_h-analytic submanifold of K_h^h. The composition map

$$\psi(\Gamma) \times \psi(\Gamma) \xrightarrow{\psi^{-1} \times \psi^{-1}} \Gamma \times \Gamma \xrightarrow{\text{multiplication}} \Gamma \xrightarrow{\psi} \psi(\Gamma)$$

from an open subset in K_h^{2h} to K_h^h can be easily seen to be locally K-analytic since each component of this map is a composition of a polynomial and a K-linear Frobenius automorphism σ, both being locally K-analytic. Therefore, Γ is a locally K-analytic group. However, notice that Γ is not a locally K_h-analytic group because $\sigma : \mathfrak{o}_h^\times \longrightarrow \mathfrak{o}_h^\times$ is not locally K_h-analytic unless $h = 1$.

Now, being a compact and a totally disconnected Hausdorff topological group, Γ is a profinite topological group. A basis of neighbourhoods of the identity is given by the normal subgroups

$$\Gamma_i := 1 + \varpi^i \mathfrak{o}_{B_h} = 1 + \varpi^i \text{End}(H_0), \quad i \geq 1$$

of finite index. Let us put $\Gamma_0 := \Gamma$. Our aim is to show that the Γ-action on $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is continuous, i.e., the action map $\Gamma \times \text{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow \text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is continuous for the $\mathfrak{m}_{R_m}$-adic topology on $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$, and for the product of profinite and $\mathfrak{m}_{R_m}$-adic topology on the left-hand side. But, first we need a couple of lemmas. For any two non-negative integers n and m, set

$$\mathbb{H}_n^{(m)} := \mathbb{H}^{(m)} \otimes_{R_m} (R_m / \mathfrak{m}_{R_m}^{n+1}).$$

We have $H_0 \cong \mathbb{H}_0^{(m)}$ via $\rho^{(m)}$ for all $m \geq 0$.

LEMMA 2.2.4: *If n and m are non-negative integers, then the homomorphism of $\mathfrak{o}$-algebras $\text{End}(\mathbb{H}_{n+1}^{(m)}) \longrightarrow \text{End}(\mathbb{H}_n^{(m)})$, induced by reduction modulo $\mathfrak{m}_{R_m}^{n+1}$, is injective.*

Proof. Let $m \geq 0$ be arbitrary. We show by induction on n that the ring homomorphism $i_n : \mathrm{End}(\mathbb{H}_n^{(m)}) \longrightarrow \mathrm{End}(\mathbb{H}_0^{(m)})$, induced by reduction modulo the maximal ideal, is injective for every $n \in \mathbb{N}_0$. The case $n = 0$ is trivial. Let $n \geq 1$ and assume that i_{n-1} is injective. Since H_0 is of height h, we have

$$[\varpi]_{\mathbb{H}_0^{(m)}}(X) \equiv \overline{u}X^{q^h} \mod \deg q^h + 1$$

for some $u \in R_m^\times$. Then $[\varpi]_{\mathbb{H}_0^{(m)}} = i_n([\varpi]_{\mathbb{H}_n^{(m)}})$ implies that

$$[\varpi]_{\mathbb{H}_n^{(m)}}(X) \equiv \overline{\varpi}X + \overline{b_2}X^2 + \cdots + \overline{b_{q^h-1}}X^{q^h-1} + \overline{u}X^{q^h} \mod \deg q^h + 1$$

for some $b_2, \ldots, b_{q^h-1} \in \mathfrak{m}_{R_m}$.

Now let

$$f(X) = \sum_{i=1}^{\infty} \overline{a_i}X^i \in \mathrm{End}(\mathbb{H}_n^{(m)})$$

such that $i_n(f) = 0$, i.e., $a_i \in \mathfrak{m}_{R_m}$ for all $i \geq 1$. We need to show that $a_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $i \geq 1$. However, the induction hypothesis implies that $a_i \in \mathfrak{m}_{R_m}^n$. Thus $[\varpi]_{\mathbb{H}_n^{(m)}} \circ f = 0$. Since $[\varpi]_{\mathbb{H}_n^{(m)}} \circ f = f \circ [\varpi]_{\mathbb{H}_n^{(m)}}$, we get $a_i u^i \in \mathfrak{m}_{R_m}^{n+1}$ by induction i and hence $a_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $i \geq 1$. ∎

The above lemma allows us to consider all the $\mathfrak{o}$-algebras $\mathrm{End}(\mathbb{H}_n^{(m)})$ as subalgebras of $\mathrm{End}(\mathbb{H}_0^{(m)})$.

PROPOSITION 2.2.5: *For all $n \geq 0$, $m \geq 0$, the subalgebra $\mathrm{End}(\mathbb{H}_n^{(m)})$ of $\mathrm{End}(\mathbb{H}_0^{(m)})$ contains $\varpi^n \mathrm{End}(\mathbb{H}_0^{(m)})$.*

Proof. Let $m \geq 0$ be arbitrary. We proceed by induction on n, the case $n = 0$ being trivial. Let $n \geq 1$ and assume the assertion to be true for $n - 1$. Let $\varphi \in \varpi^n \mathrm{End}(\mathbb{H}_0^{(m)})$. By induction hypothesis, we have $\varphi \in \varpi \mathrm{End}(\mathbb{H}_{n-1}^{(m)})$. Now for any $\psi \in \mathrm{End}(\mathbb{H}_{n-1}^{(m)})$, choose a power series $\tilde{\psi} \in (R_m/\mathfrak{m}_{R_m}^{n+1})[[X]]$ with trivial constant term such that $\tilde{\psi} \mod \mathfrak{m}_{R_m}^n = \psi$. The power series

$$\varpi\tilde{\psi} = [\varpi]_{\mathbb{H}_n^{(m)}} \circ \tilde{\psi}$$

is a lift of

$$\varpi\psi = [\varpi]_{\mathbb{H}_{n-1}^{(m)}} \circ \psi.$$

We claim that $\varpi\tilde{\psi} \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and $(\varpi\psi \longmapsto \varpi\tilde{\psi}) : \varpi\mathrm{End}(\mathbb{H}_{n-1}^{(m)}) \longrightarrow \mathrm{End}(\mathbb{H}_n^{(m)})$ is a well-defined injective map. The proposition then follows from the claim.

First, let us see why $\varpi\tilde{\psi}$ defines an endomorphism of $\mathbb{H}_n^{(m)}$. Since $\psi \in \text{End}(\mathbb{H}_{n-1}^{(m)})$, we have

$$0 = \psi(X +_{\mathbb{H}_{n-1}^{(m)}} Y) -_{\mathbb{H}_{n-1}^{(m)}} \psi(X) -_{\mathbb{H}_{n-1}^{(m)}} \psi(Y)$$
$$= (\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y)) \mod \mathfrak{m}_{R_m}^n.$$

Thus all the coefficients of the power series

$$(\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y))$$

lie in $\mathfrak{m}_{R_m}^n/\mathfrak{m}_{R_m}^{n+1}$. Since $\varpi \in \mathfrak{m}_{R_m}$ and $(\mathfrak{m}_{R_m}^n)^k \subseteq \mathfrak{m}_{R_m}^{n+1}$ for all integers $k > 1$, we get $[\varpi]_{\mathbb{H}_n^{(m)}} \circ (\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y)) = 0$. Consequently,

$$\varpi\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) = \varpi\tilde{\psi}(X) +_{\mathbb{H}_n^{(m)}} \varpi\tilde{\psi}(Y).$$

Similarly one shows that

$$0 = [\varpi]_{\mathbb{H}_n^{(m)}} \circ ([a]_{\mathbb{H}_n^{(m)}} \circ \tilde{\psi} -_{\mathbb{H}_n^{(m)}} \tilde{\psi} \circ [a]_{\mathbb{H}_n^{(m)}})$$
$$= [a]_{\mathbb{H}_n^{(m)}} \circ \varpi\tilde{\psi} -_{\mathbb{H}_n^{(m)}} \varpi\tilde{\psi} \circ [a]_{\mathbb{H}_n^{(m)}}$$

for all $a \in \mathfrak{o}$. Therefore $\varpi\tilde{\psi} \in \text{End}(\mathbb{H}_n^{(m)})$.

To see that the above map is well-defined, take another lift $\tilde{\psi}'$ of ψ with trivial constant terms. Then

$$(\tilde{\psi}' -_{\mathbb{H}_n^{(m)}} \tilde{\psi}) \mod \mathfrak{m}_{R_m}^n = \psi -_{\mathbb{H}_{n-1}^{(m)}} \psi = 0.$$

Thus $[\varpi]_{\mathbb{H}_n^{(m)}} \circ (\tilde{\varphi}' -_{\mathbb{H}^{(m)}} \tilde{\varphi}) = 0$ as above. Hence $\varpi\tilde{\psi}' = \varpi\tilde{\psi}$. Finally, the injectivity is clear because $\varpi\tilde{\psi}_1 = \varpi\tilde{\psi}_2$ implies $\varpi\psi_1 = \varpi\psi_2$ after reduction modulo $\mathfrak{m}_{R_m}^n$. ∎

THEOREM 2.2.6: *For all $n \geq 0$, $m \geq 0$, the induced action of Γ_{n+m} on $R_m/\mathfrak{m}_{R_m}^{n+1}$ is trivial. Thus the map $((\gamma, f) \mapsto \gamma(f)) : \Gamma \times R_m \longrightarrow R_m$ is continuous where the left hand side carries the product topology.*

Proof. Let n and m be arbitrary non-negative integers. Let $\gamma \in \Gamma_{n+m}$ and $\text{pr}_n^{(m)} : R_m \longrightarrow R_m/\mathfrak{m}_{R_m}^{n+1}$ denote the natural projection. Consider the level-m-structure

$$\phi_n^{(m)} := \text{pr}_n^{(m)} \circ \phi^{(m)}$$

on $\mathbb{H}_n^{(m)}$ and consider the deformation $(\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)})$ of H_0 to $R_m/\mathfrak{m}_{R_m}^{n+1}$ with this level-m-structure. Let $\gamma_n^{(m)} : R_m \longrightarrow R_m/\mathfrak{m}_{R_m}^{n+1}$ denote the unique

ring homomorphism for which there exists an isomorphism

$$[\gamma_n^{(m)}] : (\gamma_n^{(m)})_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) \xrightarrow{\sim} (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)}).$$

Note that also the ring homomorphism $\mathrm{pr}_n^{(m)} \circ \gamma : R_m \longrightarrow R_m/\mathfrak{m}_m^{n+1}$ admits an isomorphism of deformations

$$(\mathrm{pr}_n^{(m)} \circ \gamma)_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) = (\mathrm{pr}_n^{(m)})_*(\gamma_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}))$$
$$\cong (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)}).$$

Therefore by uniqueness, we have $\mathrm{pr}_n^{(m)} \circ \gamma = \gamma_n^{(m)}$ and $[\gamma_n^{(m)}] = [\gamma] \mod \mathfrak{m}_{R_m}^{n+1}$.

Since the map

$$(\sigma \mapsto \rho^{(m)} \circ \sigma \circ (\rho^{(m)})^{-1})$$

is an isomorphism $\mathrm{End}(H_0) \xrightarrow{\sim} \mathrm{End}(\mathbb{H}_0^{(m)})$ of $\mathfrak{o}$-algebras, Proposition 2.2.5 shows that $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} \in 1 + \varpi^m \mathrm{End}(\mathbb{H}_n^{(m)}) \subseteq \mathrm{Aut}(\mathbb{H}_n^{(m)})$. We claim that $(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \phi_n^{(m)} = \phi_n^{(m)}$. Write

$$\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} = 1 + \varepsilon\varpi^m$$

for some $\varepsilon \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and let $\alpha \in (\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h$ be arbitrary. Then

$$(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1})(\phi_n^{(m)}(\alpha)) = (1 + \varepsilon\varpi^m)(\phi_n^{(m)}(\alpha))$$
$$= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\varpi^m(\phi_n^{(m)}(\alpha)))$$
$$= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\phi_n^{(m)}(\varpi^m\alpha))$$
$$= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\phi_n^{(m)}(0))$$
$$= \phi_n^{(m)}(\alpha).$$

Therefore, the automorphism $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}$ of $\mathbb{H}_n^{(m)}$ defines an isomorphism of deformations

$$(\mathbb{H}_n^{(m)}, \rho^{(m)}, \phi_n^{(m)})$$
$$\cong (\mathbb{H}_n^{(m)}, (\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \rho^{(m)}, (\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \phi_n^{(m)})$$
$$= (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)}).$$

However, $(\mathbb{H}_n^{(m)}, \rho^{(m)}, \phi_n^{(m)}) = (\mathrm{pr}_n^{(m)})_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$. By uniqueness again, we have $\mathrm{pr}_n^{(m)} = \mathrm{pr}_n^{(m)} \circ \gamma = \gamma_n^{(m)}$. This implies that Γ_{n+m} acts trivially on $R_m/\mathfrak{m}_{R_m}^{n+1}$ and $[\gamma] \mod \mathfrak{m}_{R_m}^{n+1} = \rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}$. ∎

The R_m-module $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is complete and Hausdorff for the $\mathfrak{m}_{R_m}$-adic topology because it is free of finite rank. By the semi-linearity of the Γ-action, the R_m-submodules $\mathfrak{m}_{R_m}^n \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ are Γ-stable for any non-negative integer n.

THEOREM 2.2.7: *Let s, n, m be integers with $n \geq 0$ and $m \geq 0$. The induced action of Γ_{2n+m+1} on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is trivial. Thus the map*

$$((\gamma, \delta) \mapsto \gamma(\delta)) : \Gamma \times \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$$

is continuous where the left-hand side carries the product topology.

Proof. If we assume the assertion to be true for $s = 1$, then by the definition of the action, it is easy to see that it holds for all positive s. On the other hand, let $\overline{\varphi} \in \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}$ and $\gamma \in \Gamma_{2n+m+1}$. Then by assumption, $\gamma(\delta) - \delta \in \mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})$. Write

$$\gamma^{-1}(\delta) = \delta + \sum_{i=1}^{r} \alpha_i \eta_i$$

with $\alpha_i \in \mathfrak{m}_{R_m}^{n+1}$ and $\eta_i \in \mathrm{Lie}(\mathbb{H}^{(m)})$. Then

$$
\begin{aligned}
(\varphi - \gamma(\varphi))(\delta) &= \varphi(\delta) - \gamma(\varphi)(\delta) = \varphi(\delta) - \gamma(\varphi(\gamma^{-1}(\delta))) \\
&= \varphi(\delta) - \gamma\left(\varphi\left(\delta + \sum_{i=1}^{r} \alpha_i \eta_i\right)\right) \\
&= \varphi(\delta) - \gamma(\varphi(\delta)) - \sum_{i=1}^{r} \gamma(\alpha_i)\gamma(\varphi(\eta_i)).
\end{aligned}
$$

Since $2n + m + 1 \geq n + m$, by Theorem 2.2.6, we have $\varphi(\delta) - \gamma(\varphi(\delta)) \in \mathfrak{m}_{R_m}^{n+1}$. Also $\gamma(\alpha_i) \in \mathfrak{m}_{R_m}^{n+1}$. Therefore $(\varphi - \gamma(\varphi))(\delta) \in \mathfrak{m}_{R_m}^{n+1}$. If δ_0 is a basis of $\mathrm{Lie}(\mathbb{H}^{(m)})$ over R_m, and $\psi \in \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}$ is defined by

$$\psi(\delta_0) = 1,$$

then

$$\varphi - \gamma(\varphi) = (\varphi - \gamma(\varphi))(\delta_0)\psi \in \mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}.$$

Thus $\overline{\varphi} = \gamma(\overline{\varphi})$. A similar argument like this can be used to show that the assertion is true for all higher negative s. Hence it is sufficient to prove the theorem for $s = 1$.

Let $\gamma \in \Gamma_{2n+m+1}$. By identifying

$$\mathrm{Lie}(\mathbb{H}^{(m)})/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)}) = \mathrm{Lie}(\mathbb{H}_n^{(m)}),$$

Theorem 2.2.6 and its proof show that the map

$$\gamma \quad \mathrm{mod}\ \mathfrak{m}_{R_m}^{n+1} : \mathrm{Lie}(\mathbb{H}_n^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}_n^{(m)})$$

is given by $\mathrm{Lie}(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1})$ where

$$\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} \in 1 + \varpi^{2n+m+1}\mathrm{End}(\mathbb{H}_0^{(m)}) \subseteq 1 + \varpi^{n+m+1}\mathrm{End}(\mathbb{H}_n^{(m)}).$$

Therefore it suffices to show that the natural action of

$$1 + \varpi^{n+m+1}\mathrm{End}(\mathbb{H}_n^{(m)}) \subset \mathrm{End}(\mathbb{H}_n^{(m)})$$

on $\mathrm{Lie}(\mathbb{H}_n^{(m)})$ is trivial. However, if $\varphi \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and $\delta \in \mathrm{Lie}(\mathbb{H}_n^{(m)})$, then

$$\begin{aligned}
(\mathrm{Lie}(1 + \varpi^{n+m+1}\varphi)(\delta))(\overline{X}) &= \delta(\overline{(1 + \varpi^{n+m+1}\varphi)(X)}) \\
&= \delta(\overline{X +_{\mathbb{H}_n^{(m)}} \varpi^{n+m+1}\varphi(X)}) \\
&= \delta(\overline{X + \varpi^{n+m+1}\varphi(X)}) \\
&= \delta(\overline{X})
\end{aligned}$$

because $\varpi^{n+m+1} \in \mathfrak{m}_{R_m}^{n+1}$. ∎

Remark 2.2.8: The Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ gives rise to an action of the group ring $\breve{\mathfrak{o}}[\Gamma]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$. By Theorem 2.2.7, the induced action of $\breve{\mathfrak{o}}[\Gamma]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ factors through $(\breve{\mathfrak{o}}/\varpi^{n+1}\breve{\mathfrak{o}})[\Gamma/\Gamma_{2n+m+1}]$ such that the following diagram with the horizontal action maps and the vertical reduction maps commutes for all n:

$$
\begin{array}{ccc}
\frac{\breve{\mathfrak{o}}}{(\varpi^{n+1})}\left[\frac{\Gamma}{\Gamma_{2n+m+1}}\right] \times \frac{\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}}{\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}} & \longrightarrow & \frac{\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}}{\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}} \\
\downarrow & & \downarrow \\
\frac{\breve{\mathfrak{o}}}{(\varpi^n)}\left[\frac{\Gamma}{\Gamma_{2(n-1)+m+1}}\right] \times \frac{\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}}{\mathfrak{m}_{R_m}^n\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}} & \longrightarrow & \frac{\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}}{\mathfrak{m}_{R_m}^n\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}}
\end{array}
$$

Taking projective limits over n, we obtain an action of the Iwasawa algebra $\breve{\mathfrak{o}}[[\Gamma]]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ that extends the action of Γ.

2.3. Rigidification and the equivariant vector bundles. Berthelot's rigidification functor associates to every locally Noetherian adic formal ŏ-scheme whose reduction is a scheme locally of finite type over $\mathrm{Spec}(\overline{k})$, a rigid $\breve{K}$-analytic space (cf. [Jong95], Section 7). For an affine formal ŏ-scheme $\mathrm{Spf}(A)$, there is a bijection between the closed points of its generic fibre $\mathrm{Spec}(A \otimes_{\breve{o}} \breve{K})$ and the points of the associated rigid analytic space. Let us denote by X_m^{rig} the rigidification of the affine formal ŏ-scheme $X_m = \mathrm{Spf}(R_m)$ under Berthelot's functor, and by $R_m^{\mathrm{rig}} := \mathcal{O}_{X_m^{\mathrm{rig}}}(X_m^{\mathrm{rig}})$ the $\breve{K}$-algebra of the global rigid analytic functions on X_m^{rig}.

By functoriality, X_m^{rig} and R_m^{rig} carry commuting (left) actions of Γ and G_0, thus an action of the product group $\Gamma \times G_0$, and the G_0-action factors through G_0/G_m. For $m' \geq m \geq 0$, let

$$\pi_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$$

denote the morphism of rigid analytic spaces induced by Part 2, Theorem 2.1.3 and by functoriality. It follows from the properties of the rigidification functor that the morphism $\pi_{m',m}$ is a finite étale Galois covering with Galois group $G_m/G_{m'}$ (cf. [Jong95], Section 7). Consequently, the ring extension $R_{m'}^{\mathrm{rig}}|R_m^{\mathrm{rig}}$ is finite Galois with Galois group $G_m/G_{m'}$. We note that all covering morphisms are $(\Gamma \times G_0)$-equivariant.

It follows from the isomorphism $R_0 \cong \breve{o}[[u_1, \ldots, u_{h-1}]]$ that X_0^{rig} is isomorphic to the $(h-1)$-dimensional rigid analytic open unit polydisc over $\breve{K}$, and the isomorphism $R_0 \cong \breve{o}[[u_1, \ldots, u_{h-1}]]$ extends to an isomorphism

$$(2.3.1) \quad R_0^{\mathrm{rig}} \cong \left\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \mid c_\alpha \in \breve{K} \text{ and } \lim_{|\alpha| \to \infty} |c_\alpha| r^{|\alpha|} = 0 \text{ for all } 0 < r < 1 \right\}$$

of $\breve{K}$-algebras. This allows us to view R_0^{rig} as a topological $\breve{K}$-Fréchet algebra whose topology is defined by the family of norms $\| \cdot \|_l$, given by

$$\left\| \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \right\|_l := \sup_{\alpha \in \mathbb{N}_0^{h-1}} \{ |c_\alpha| |\varpi|^{|\alpha|/l} \}$$

for any positive integer l. Let $R_{0,l}^{\mathrm{rig}}$ be the completion of R_0^{rig} with respect to the norm $\| \cdot \|_l$. Then

$$R_{0,l}^{\mathrm{rig}} \cong \left\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \mid c_\alpha \in \breve{K}, \lim_{|\alpha| \to \infty} |c_\alpha| |\varpi|^{|\alpha|/l} = 0 \right\}$$

is the $\breve{K}$-Banach algebra of rigid analytic functions on the affinoid subdomain

$$\mathbb{B}_l := \{x \in X_0^{\mathrm{rig}} \mid |u_i(x)| \leq |\varpi|^{1/l} \text{ for all } 1 \leq i \leq h-1\}$$

of X_0^{rig}. Further,

$$R_0^{\mathrm{rig}} \cong \varprojlim_l R_{0,l}^{\mathrm{rig}}$$

is the topological projective limit of the $\breve{K}$-Banach algebras $R_{0,l}^{\mathrm{rig}}$.

Since R_0 is a local ring, the finite flat R_0-module R_m is free by [Mat87], Theorem 7.10, and has rank $r = |G_0/G_m|$. Let $R_{m,l}^{\mathrm{rig}}$ denote the affinoid $\breve{K}$-algebra of the rigid analytic functions on the affinoid subdomain

$$\mathbb{B}_{m,l} := \pi_{m,0}^{-1}(\mathbb{B}_l)$$

of X_m^{rig}. Then by [Jong95], Lemma 7.2.2, we have

(2.3.2) $R_{m,l}^{\mathrm{rig}} \cong R_m \otimes_{R_0} R_{0,l}^{\mathrm{rig}}$

as $R_m|R_0$ is finite. Let us fix a basis $\{e_1, \ldots, e_r\}$ of R_m over R_0 and view it as an $R_{0,l}^{\mathrm{rig}}$-basis of $R_{m,l}^{\mathrm{rig}} = R_m \otimes_{R_0} R_{0,l}^{\mathrm{rig}}$. The next lemma shows that $R_{m,l}^{\mathrm{rig}}$ is a $\breve{K}$-Banach algebra with respect to the norm

$$\|(f_1 e_1 + \cdots + f_r e_r)\|_l := \max_{1 \leq i \leq r} \{\|f_i\|_l\},$$

where $f_i \in R_{0,l}^{\mathrm{rig}}$ for all i, by showing that it is indeed an algebra norm.

LEMMA 2.3.3: Let $f = f_1 e_1 + \cdots + f_r e_r$, $g = g_1 e_1 + \cdots + g_r e_r \in R_{m,l}^{\mathrm{rig}}$. Then

$$\|fg\|_l \leq \|f\|_l \|g\|_l.$$

Proof. Let $e_i e_j = \sum_{k=1}^r a_{ijk} e_k$ for all $1 \leq i, j \leq r$. Note that

$$a_{ijk} \in R_0 = \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$$

and thus $\|a_{ijk}\|_l \leq 1$ for all $1 \leq i, j, k \leq r$. Also $\|\cdot\|_l$ is multiplicative on $R_{0,l}^{\mathrm{rig}}$. Therefore

$$\|fg\|_l = \max_{1 \leq k \leq r} \left\{ \left\| \sum_{1 \leq i,j \leq r} f_i g_j a_{ijk} \right\|_l \right\}$$

$$\leq \max_{1 \leq k \leq r} \{ \max_{1 \leq i,j \leq r} \|f_i g_j a_{ijk}\|_l \}$$

$$\leq \max_{1 \leq i,j \leq r} \{\|f_i\|_l \|g_j\|_l\} \leq \|f\|_l \|g\|_l. \quad \blacksquare$$

It then follows from [BGR84], (6.1.3), Proposition 2 that the affinoid topology on $R_{m,l}^{\mathrm{rig}}$ coincides with Banach topology given by the aforementioned norm $\|\cdot\|_l$. The natural maps $R_{m,l+1}^{\mathrm{rig}} \longrightarrow R_{m,l}^{\mathrm{rig}}$ induced from $R_{0,l+1}^{\mathrm{rig}} \longrightarrow R_{0,l}^{\mathrm{rig}}$ endow the projective limit

$$R_m^{\mathrm{rig}} \cong \varprojlim_l R_{m,l}^{\mathrm{rig}}$$

with the structure of a $\breve{K}$-Fréchet algebra. Indeed, this projective limit is isomorphic to the $\breve{K}$-algebra of global rigid analytic functions on X_m^{rig} by [Jong95], Lemma 7.2.2. Thus we have

$$(2.3.4) \qquad\qquad R_m^{\mathrm{rig}} \cong R_m \otimes_{R_0} R_0^{\mathrm{rig}}$$

as R_m is a finite free R_0-module, and $R_{m,l}^{\mathrm{rig}}$ can be viewed as the Banach completion of R_m^{rig} with respect to the norm $\|\cdot\|_l$ defined as before by $\|(f_1 e_1 + \cdots + f_r e_r)\|_l := \max_{1 \le i \le r}\{\|f_i\|_l\}$, with $f_i \in R_0^{\mathrm{rig}}$.

The Γ-action on X_0^{rig} stabilizes the affinoid subdomains $\mathbb{B}_l$ for all positive integers l. Indeed, since $\gamma(u_i)$ belongs to the maximal ideal $(\varpi, u_1, \ldots, u_{h-1})$ of R_0, $\|\gamma(u_i)\|_l \le |\varpi|^{1/l}$ for all $1 \le i \le h-1$. This implies that $\|\gamma(f)\|_l \le \|f\|_l$ for all $f \in R_0^{\mathrm{rig}}$. Thus the Γ-action on R_0^{rig} extends to its completion $R_{0,l}^{\mathrm{rig}}$ for all positive integers l. As a consequence, the affinoid subdomains $\mathbb{B}_{m,l}$ of X_m^{rig} are stable under the $(\Gamma \times G_0)$-action for all m and l, and the isomorphism (2.3.2) is $(\Gamma \times G_0)$-equivariant for the diagonal $(\Gamma \times G_0)$-action on the right. Similarly the isomorphism (2.3.4) is also $(\Gamma \times G_0)$-equivariant for the diagonal $(\Gamma \times G_0)$-action on the right.

Remark 2.3.5: By [BGR84], (6.1.3), Theorem 1, the $\breve{K}$-algebra automorphism of an affinoid $\breve{K}$-algebra $R_{m,l}^{\mathrm{rig}}$ corresponding to $(\gamma, g) \in \Gamma \times G_0$ is automatically continuous for its $\breve{K}$-Banach topology defined by the norm $\|\cdot\|_l$. Since the $\breve{K}$-Fréchet topology of R_m^{rig} is given by the family of norms $\|\cdot\|_l$, $l \in \mathbb{N}$, the group $\Gamma \times G_0$ acts on R_m^{rig} by continuous $\breve{K}$-algebra automorphisms for all $m \ge 0$.

Now recall from [GH94], Section 15 that a Γ-equivariant vector bundle $\mathcal{M}$ on the formal scheme X_m is a locally free $\mathcal{O}_{X_m}$-module $\mathcal{M}$ of finite rank equipped with a (left) Γ-action that is compatible with the Γ-action on X_m. Since X_m is formally affine, a Γ-equivariant vector bundle $\mathcal{M}$ on X_m is completely determined by its global sections $\mathcal{M}(X_m)$. Hence, for all $s \in \mathbb{Z}$ and $m \ge 0$, the free R_m-module $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ of rank 1 equipped with a semilinear Γ-action gives

rise to a Γ-equivariant line bundle

$$\mathcal{M}_m^s := \mathcal{L}\mathrm{ie}(\mathbb{H}^{(m)})^{\otimes s}$$

on X_m. Its rigidification $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a locally free $\mathcal{O}_{X_m^{\mathrm{rig}}}$-module of rank 1 by [Jong95], 7.1.11. Let

$$M_m^s := (\mathcal{M}_m^s)^{\mathrm{rig}}(X_m^{\mathrm{rig}})$$

denote its global sections. Because of the fact that X_m is affine, the natural map

(2.3.6) $$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow M_m^s$$

is an isomorphism. By functoriality, Γ acts on $(\mathcal{M}_m^s)^{\mathrm{rig}}$ in such a way that the map (2.3.6) is Γ-equivariant for the diagonal Γ-action on the left and for the Γ-action induced by functoriality on the right. In particular, the Γ-action on M_m^s is semilinear for its action on R_m^{rig}, and thus $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a rigid Γ-equivariant line bundle on X_m^{rig}. In a similar fashion, it can be seen that $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a rigid G_0-equivariant line bundle on X_m^{rig}, and the actions of Γ and G_0 commute (cf. Remark 2.2.2). By functoriality again, the G_0-action on $(\mathcal{M}_m^s)^{\mathrm{rig}}$ factors through the quotient group G_0/G_m.

   For all s, m and l, set

$$M_{m,l}^s := (\mathcal{M}_m^s)^{\mathrm{rig}}(\mathbb{B}_{m,l}).$$

Then $M_{m,l}^s$ is a free $R_{m,l}^{\mathrm{rig}}$-module of rank 1 for which the natural $R_{m,l}^{\mathrm{rig}}$-linear map

(2.3.7) $$R_{m,l}^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow M_{m,l}^s$$

is an isomorphism (cf. [Jong95], 7.1.11), and is $(\Gamma \times G_0)$-equivariant for the diagonal $(\Gamma \times G_0)$-action on the left. Endowing M_m^s and $M_{m,l}^s$ with the natural topologies of finitely generated modules over R_m^{rig} and $R_{m,l}^{\mathrm{rig}}$ respectively, makes them a $\check{K}$-Fréchet space and a $\check{K}$-Banach space respectively. One then has a topological isomorphism

$$M_m^s \cong \varprojlim_{l} M_{m,l}^s$$

for the projective limit topology on the right, and the group $\Gamma \times G_0$ acts on M_m^s by continuous $\check{K}$-vector space automorphisms for all s and m.

3. Locally analytic representations at higher levels

We saw in the previous section that the group $\Gamma = \text{Aut}(H_0)$ acts on the Fréchet space M_m^s of the global sections of the rigid Γ-equivariant line bundle $(\mathcal{M}_m^s)^{\text{rig}}$ over X_m^{rig} by continuous vector space automorphisms. The goal of this section is to show that the strong topological $\breve{K}$-linear dual $(M_m^s)_b'$ of M_m^s with the induced Γ-action is a locally K-analytic representation of Γ for all $s \in \mathbb{Z}$ and levels $m \geq 0$.[2] Recall from [ST02ii], Section 3 that a locally K-analytic representation V of Γ (over $\breve{K}$) is a barrelled Hausdorff locally convex $\breve{K}$-vector space equipped with a Γ-action by continuous linear endomorphisms such that for each $v \in V$, the orbit map $\rho_v = (g \mapsto g(v)) \in C^{an}(\Gamma, V)$, i.e., ρ_v is a V-valued locally K-analytic function on Γ.

As stated in the introduction, to show the local K-analyticity, we are going to make use of the Gross–Hopkins' period morphism Φ and of the fundamental domain D. The next subsection is intended to give a brief overview of these two technical tools.

3.1. THE PERIOD MORPHISM AND THE GROSS–HOPKINS FUNDAMENTAL DOMAIN. Let $\mathbb{E}^{(m)}$ denote the **universal additive extension** of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$. It sits in the exact sequence

$$(3.1.1) \qquad 0 \longrightarrow \text{Lie}((\mathbb{H}^{(m)})') \xrightarrow{\alpha^{(m)}} \text{Lie}(\mathbb{E}^{(m)}) \xrightarrow{\beta^{(m)}} \text{Lie}(\mathbb{H}^{(m)}) \longrightarrow 0$$

where

$$(\mathbb{H}^{(m)})' = \mathbb{G}_a \otimes_{R_m} \text{Hom}_{R_m}(\text{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a), R_m)$$

is the additive formal $\mathfrak{o}$-module of dimension $h - 1$ associated with the free R_m-module dual to the module $\text{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)$ of extensions of $\mathbb{H}^{(m)}$ by $\mathbb{G}_a$ (cf. [GH94], Section 11). The universality induces commuting semi-linear actions of Γ and G_0 on $\text{Lie}((\mathbb{H}^{(m)})')$ and on $\text{Lie}(\mathbb{E}^{(m)})$ for which the maps $\alpha^{(m)}$ and $\beta^{(m)}$ are equivariant, and the G_0-action factors through G_0/G_m. By rigid-ification, the short exact sequence (3.1.1) gives rise to an exact sequence

$$0 \longrightarrow (\mathcal{L}\text{ie}((\mathbb{H}^{(m)})')^{\otimes s})^{\text{rig}} \longrightarrow (\mathcal{L}\text{ie}(\mathbb{E}^{(m)})^{\otimes s})^{\text{rig}} \longrightarrow (\mathcal{M}_m^s)^{\text{rig}} \longrightarrow 0$$

of the corresponding rigid $(\Gamma \times G_0)$-equivariant line bundles on X_m^{rig} for all non-negative s, and for negative s in the opposite direction. Since X_m is an affine

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[2] In the notation $(M_m^s)_b'$, the superscript $'$ indicates the continuous linear dual of the topological vector space M_m^s, while the subscript b stands for bounded and implies that $(M_m^s)_b'$ is equipped with the topology of bounded convergence, i.e., the strong topology.

formal scheme, by taking global sections, we get an exact sequence

$$(3.1.2) \quad 0 \longrightarrow R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}((\mathbb{H}^{(m)})')^{\otimes s} \longrightarrow R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \longrightarrow M_m^s \longrightarrow 0$$

of $\check{K}$-linear $(\Gamma \times G_0)$-representations for $s \geq 0$, and in the opposite direction for $s < 0$.

PROPOSITION 3.1.3: *The Γ-equivariant line bundle $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(0)})$ is generically flat, i.e., there exists a basis $\{c_0, c_1, \ldots, c_{h-1}\}$ of $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ over R_0^{rig} such that the $\check{K}$-subspace of $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ spanned by c_i's is Γ-stable. Let $B_h \otimes_{K_h} \check{K}$ be the h-dimensional $\check{K}$-linear Γ-representation where the action of $\Gamma \cong \mathfrak{o}_{B_h}^{\times}$ is given by left multiplication. Then we have an isomorphism*

$$(3.1.4) \qquad R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \cong R_0^{\mathrm{rig}} \otimes_{\check{K}} (B_h \otimes_{K_h} \check{K})$$

of $R_0^{\mathrm{rig}}[\Gamma]$-modules with Γ acting diagonally on both sides.

Proof. See [GH94], Proposition 21.8 and Proposition 22.4. ∎

COROLLARY 3.1.5: *The $(\Gamma \times G_0)$-equivariant line bundle $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s}$ is generically flat for all $m \geq 0$ and $s \in \mathbb{Z}$.*

Proof. Since $\mathbb{H}^{(m)} = \mathbb{H}^{(0)} \otimes_{R_0} R_m$, it follows from the universality of $\mathbb{E}^{(m)}$ that

$$\mathbb{E}^{(m)} = \mathbb{E}^{(0)} \otimes_{R_0} R_m.$$

The isomorphism $\mathrm{Lie}(\mathbb{E}^{(m)}) \cong R_m \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ of $R_m[\Gamma \times G_0]$-modules gives rise to an isomorphism

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)}) \cong R_m^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$$

of $R_m^{\mathrm{rig}}[\Gamma \times G_0]$-modules. Then using (2.3.4) and (3.1.4), we have an isomorphism

$$(3.1.6) \qquad R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)}) \cong R_m^{\mathrm{rig}} \otimes_{\check{K}} (B_h \otimes_{K_h} \check{K})$$

of $R_m^{\mathrm{rig}}[\Gamma \times G_0]$-modules, where Γ and G_0 act diagonally on both sides. The action of G_0 on $B_h \otimes_{K_h} \check{K}$ by convention is trivial. The corollary follows after taking tensor powers on both sides. ∎

Let v_i denote the images of the basis elements c_i under the map

$$R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \longrightarrow M_0^1.$$

According to [GH94], Proposition 23.2, the global sections $\{v_i\}_{0 \leq i \leq h-1}$ of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ have no common zeros on X_0^{rig}, and are linearly independent over $\check{K}$. If $\mathbb{V}$ denotes the $\check{K}$-subspace of M_0^1 spanned by them, then $\mathbb{V}$ is

Γ-stable, and is isomorphic to $B_h \otimes_{K_h} \breve{K}$ as a Γ-representation. Let $\mathbb{P}(\mathbb{V})$ be the projective space of all hyperplanes in $\mathbb{V}$. Then the map

$$\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}(\mathbb{V})$$
$$x \longmapsto \{v \in \mathbb{V} | v(x) = 0\}$$

is an étale surjective morphism of rigid analytic spaces, if $\mathbb{P}(\mathbb{V})$ is identified with the $(h-1)$-dimensional rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$ (cf. [GH94], Proposition 23.5). The morphism $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$ is called the **period morphism**. In homogeneous projective coordinates, it is given by

$$\Phi(x) = [\varphi_0(x) : \ldots : \varphi_{h-1}(x)]$$

where $\varphi_0, \ldots, \varphi_{h-1} \in R_0^{\mathrm{rig}}$ are certain global rigid analytic functions without any common zero. These functions can be constructed from the logarithm

$$g_0(X) = \sum_{n \geq 0} a_n X^{q^n}$$

of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(0)}$ over R_0 as the limits

$$(3.1.7) \qquad \begin{aligned} \varphi_0 &:= \lim_{n \to \infty} \varpi^n a_{nh} \\ \varphi_i &:= \lim_{n \to \infty} \varpi^{n+1} a_{nh+i}, \quad \text{if } 1 \leq i \leq h-1 \end{aligned}$$

in the Fréchet topology of R_0^{rig} (cf. [GH94], (21.6) and (21.13)).

An important property of the period morphism Φ is that it is Γ-equivariant for the Γ-action on $\mathbb{P}_{\breve{K}}^{h-1}$ by fractional linear transformations via the following injective group homomorphism (cf. [Koh13], Remark 1.4):

$$j : \Gamma \hookrightarrow GL_h(K_h)$$

$$(3.1.8) \qquad \sum_{i=0}^{h-1} \lambda_i \Pi^i \longmapsto \begin{pmatrix} \lambda_0 & \varpi\lambda_1 & \varpi\lambda_2 & \cdots & \cdots & \varpi\lambda_{h-1} \\ \lambda_{h-1}^{\sigma} & \lambda_0^{\sigma} & \lambda_1^{\sigma} & \cdots & \cdots & \lambda_{h-2}^{\sigma} \\ \lambda_{h-2}^{\sigma^2} & \varpi\lambda_{h-1}^{\sigma^2} & \lambda_0^{\sigma^2} & \cdots & \cdots & \lambda_{h-3}^{\sigma^2} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \lambda_1^{\sigma^{h-1}} & \varpi\lambda_2^{\sigma^{h-1}} & \cdots & \cdots & \varpi\lambda_{h-1}^{\sigma^{h-1}} & \lambda_0^{\sigma^{h-1}} \end{pmatrix}$$

The Gross–Hopkins fundamental domain D is the affinoid subdomain of X_0^{rig} defined as follows:

$$(3.1.9) \qquad D := \{x \in X_0^{\mathrm{rig}} \,\big|\, |u_i(x)| \leq |\varpi|^{(1-\frac{i}{h})} \text{ for all } 1 \leq i \leq h-1\}.$$

According to [GH94], Lemma 23.14, the function φ_0 does not have any zeroes on D, hence is a unit in $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Setting $w_i := \frac{\varphi_i}{\varphi_0}$ for $1 \leq i \leq h-1$, [GH94], Lemma 23.14 implies that the affinoid $\breve{K}$-algebra $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is isomorphic to the generalized Tate algebra:

$$(3.1.10) \quad \begin{aligned} &\mathcal{O}_{X_0^{\mathrm{rig}}}(D) \\ &\cong \breve{K}\langle \varpi^{-(1-\frac{1}{h})}w_1, \ldots, \varpi^{-(1-\frac{h-1}{h})}w_{h-1}\rangle \\ &:= \left\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \breve{K}[[w_1, \ldots, w_{h-1}]] \,\Big|\, \lim_{|\alpha|\to\infty} |c_\alpha||\varpi|^{\sum_{i=1}^{h-1} \alpha_i(1-\frac{i}{h})}=0 \right\} \end{aligned}$$

It follows from [FGL08], Remarque I.3.2 that D is stable under the Γ-action on X_0^{rig}. Also, the Γ-equivariant period morphism Φ restricts to an isomorphism $\Phi : D \xrightarrow{\sim} \Phi(D)$ over D (cf. [GH94], Corollary 23.15). As a result, we have an explicit formula for the Γ-action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ similar to the one of Devinatz–Hopkins (cf. [Koh13], Proposition 1.3):

PROPOSITION 3.1.11: *Fix i with $1 \leq i \leq h-1$, and let $\gamma = \sum_{j=0}^{h-1} \lambda_j \Pi^j \in \Gamma$. Then*

$$(3.1.12) \qquad \gamma(w_i) = \frac{\sum_{j=1}^i \lambda_{i-j}^{\sigma^j} w_j + \sum_{j=i+1}^h \varpi \lambda_{h+i-j}^{\sigma^j} w_j}{\lambda_0 + \sum_{j=1}^{h-1} \lambda_{h-j}^{\sigma^j} w_j}.$$

The group Γ acts on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ by continuous $\breve{K}$-algebra endomorphisms extending its action on R_0^{rig}.

Proof. This is straightforward since γ acts on the projective homogeneous coordinates $[\varphi_0 : \ldots : \varphi_{h-1}]$ through right multiplication with the matrix $j(\gamma)$ in (3.1.8). By [BGR84], (6.1.3), Theorem 1, the induced $\breve{K}$-algebra endomorphism γ of the affinoid $\breve{K}$-algebra $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is automatically continuous. ∎

Remark 3.1.13: A **rigidified extension** (E, s) of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ is an extension E of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ together with a section $s : \mathrm{Lie}(\mathbb{H}^{(0)}) \longrightarrow \mathrm{Lie}(E)$. The set $\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a)$ of isomorphism classes of rigidified extensions of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ is a free R_0-module of rank h, and has a basis $\{g_0, g_1, \ldots, g_{h-1}\}$ where $g_0 \in R_0[[X]]$ is the logarithm of $\mathbb{H}^{(0)}$, and $g_i := \frac{\partial g_0}{\partial u_i}$ for $1 \leq i \leq h-1$

(cf. [GH94], Proposition 9.8). Moreover, the R_0-module $\omega(\mathbb{E}^{(0)})$ of invariant differentials on the universal additive extension is isomorphic to $\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a)$ (cf. [GH94], (11.4)). Thus,

$$R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \cong \mathrm{Hom}_{R_0}(\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a), R_0^{\mathrm{rig}}).$$

The functions φ_i in (3.1.7) are precisely $c_i(g_0)$, and the basis dg_0 of $\omega(\mathbb{H}^{(0)})$ is mapped to g_0 under the natural map $\omega(\mathbb{H}^{(0)}) \longrightarrow \omega(\mathbb{E}^{(0)})$. As a result, the global sections v_i and v_j of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ (see paragraph after Corollary 3.1.5) are related by the relation $\varphi_j v_i = \varphi_i v_j$ for all $0 \le i, j \le h-1$. Consequently, we have $\varphi_j^s v_i^s = \varphi_i^s v_j^s$ in M_0^s. Let $U_i \subset X_0^{\mathrm{rig}}$ be the non-vanishing locus of φ_i; then on $U_i \cap U_j$, we get $v_i^s = \frac{\varphi_i^s}{\varphi_j^s} v_j^s$ and $v_j^s = \frac{\varphi_j^s}{\varphi_i^s} v_i^s$. The U_i's cover X_0^{rig} as the functions φ_i's do not vanish simultaneously at any point on X_0^{rig}. This means that

$$(\mathcal{M}_0^s)^{\mathrm{rig}}|_{U_i} \cong \mathcal{O}_{X_0^{\mathrm{rig}}}|_{U_i} \varphi_i^s \quad \text{for all } 0 \le i \le h-1.$$

In particular, for $i = 0$, we have an isomorphism

$$M_D^s := (\mathcal{M}_0^s)^{\mathrm{rig}}(D) \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$$

of $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-modules which is also Γ-equivariant. The Γ-action on M_D^s is semilinear for its action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$.

Remark 3.1.14: The discussion in Remark 3.1.13 shows that the generating global sections v_i's of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ are the pullbacks $\Phi^*(\varphi_i)$ of φ_i's along the period morphism Φ for all $0 \le i \le h-1$. As a consequence, it follows that

$$(\mathcal{M}_0^1)^{\mathrm{rig}} \cong \Phi^* \mathcal{O}_{\mathbb{P}_{\tilde{K}}^{h-1}}(1).$$

By the general properties of the inverse image functor, we then have

$$(\mathcal{M}_0^s)^{\mathrm{rig}} \cong \Phi^* \mathcal{O}_{\mathbb{P}_{\tilde{K}}^{h-1}}(s) \quad \text{for all } s \in \mathbb{Z}.$$

3.2. LOCAL ANALYTICITY OF THE Γ-ACTION ON M_D^s. In this subsection, we show that the orbit map $(\gamma \mapsto \gamma(f\varphi_0^s)) : \Gamma \longrightarrow M_D^s$ explicitly given by Proposition 3.1.11 and Remark 3.1.13 is locally K-analytic for all $f\varphi_0^s \in M_D^s$.

Let $M_h(K_h)$ denote the ring of $h \times h$ matrices with entries from K_h. It carries an induced topology from the identification with $K_h^{h^2}$, which endows it with a structure of a locally analytic K_h-manifold. The subset $GL_h(K_h)$ of invertible

matrices is open and forms a locally K_h-analytic group. Consider the subgroup P of $GL_h(K_h)$ defined as follows:

$$P := \{a = (a_{ij})_{0 \leq i,j \leq h-1} \in GL_h(\mathfrak{o}_h) \,|\, a_{ij}, a_{0k} \in \varpi \mathfrak{o}_h$$

$$\text{for all } 1 \leq i, j, k \leq h-1 \text{ with } i > j\}$$

It is conjugate to a standard Iwahori subgroup of $GL_h(K_h)$. The conditions on the entries of a matrix in P force all of its diagonal entries to lie in $\mathfrak{o}_h^\times$. Since P contains the ball of radius $|\varpi^2|$ around a for any $a \in P$, P is open in $GL_h(K_h)$. Thus, P is a locally K_h-analytic subgroup of $GL_h(K_h)$. The inclusion map $j : \Gamma \hookrightarrow GL_h(K_h)$ mentioned in (3.1.8) has image in P.

LEMMA 3.2.1: *The inclusion map $j : \Gamma \hookrightarrow P$ in (3.1.8) is locally K-analytic.*

Proof. The global chart for P induced from that for $M_h(K_h)$ sends a in P to

$$(a_{00}, a_{01}, \ldots a_{0(h-1)}, a_{10}, a_{11}, \ldots, a_{(h-1)(h-2)}, a_{(h-1)(h-1)})$$

in $K_h^{h^2}$. Recall the global chart ψ for Γ from (2.2.3). Using the global charts for both groups, it is easy to see that the corresponding map from the open subset $\psi(\Gamma)$ in K_h^h to $K_h^{h^2}$ is locally K-analytic since each component of this map is either a linear polynomial or a K-linear Frobenius automorphism σ or a composition of both, all being locally K-analytic. As before, we remark that j is generally not locally K_h-analytic because $\sigma : \mathfrak{o}_h^\times \longrightarrow \mathfrak{o}_h^\times$ is not locally K_h-analytic unless $h = 1$. $\blacksquare$

The algebra $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is a $\breve{K}$-Banach algebra with respect to the multiplicative norm $\|\cdot\|_D$ defined as follows: for $f = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$,

$$\|f\|_D := \sup_{\alpha \in \mathbb{N}_0^{h-1}} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1 - \frac{i}{h})}$$

(cf. [BGR84], Section 6.1.5, Proposition 1 and 2). Let P act on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ by $\breve{K}$-linear ring automorphisms by defining

$$(3.2.2) \qquad a(w_i) := \frac{a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j}$$

for $a \in P$ and for $1 \leq i \leq h-1$. This gives an action of P on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ by continuous $\breve{K}$-linear ring automorphisms which, when restricted to Γ via j,

coincides with the Γ-action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ (cf. Proposition 3.1.11). Indeed, note that

$$a_{00} + \sum_{j=1}^{h-1} a_{j0}w_j = a_{00}\left(1 + \sum_{j=1}^{h-1} a_{00}^{-1}a_{j0}w_j\right) \in (\mathcal{O}_{X_0^{\mathrm{rig}}}(D))^{\times}$$

is a unit of norm 1, and

$$\left\|a_{0i} + \sum_{j=1}^{h-1} a_{ji}w_j\right\|_D = \|w_i\|_D$$

by the strict triangle inequality. Altogether, $\|a(w_i)\|_D = \|w_i\|_D$ which ensures that P acts on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ via

$$\sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \longmapsto \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha a(w_1)^{\alpha_1} \cdots a(w_{h-1})^{\alpha_{h-1}}$$

in a well-defined way.

We now show that the above action is locally K-analytic.

LEMMA 3.2.3:   *The map $\iota : P \longrightarrow P$, $(a_{ij})_{0 \leq i,j \leq h-1} \longmapsto (\iota(a)_{ij})_{0 \leq i,j \leq h-1}$ defined by*

$$\iota(a)_{ij} = \begin{cases} a_{ij}^{-1}, & \text{if } i = j = 0; \\ a_{ij}, & \text{otherwise} \end{cases}$$

is locally K_h-analytic, and thus locally K-analytic.

Proof. This follows from [Sch11], Proposition 13.6, and the fact that $K_h^{\times}$ is a locally K_h-analytic group. The local K-analyticity of ι follows due to restriction of scalars. ∎

PROPOSITION 3.2.4: *The action of P on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is locally K_h-analytic, and thus locally K-analytic, i.e., the orbit maps of the action are locally K-analytic.*

Proof. By Lemma 3.2.3, it is enough to show that, for each $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, the map $\iota(a) \longmapsto a(f)$ from P to $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is locally K_h-analytic. Consider the open neighbourhood U of 0 in $K_h^{h^2}$ defined as follows:

$$U := \{x = (x_1, x_2, \ldots, x_{h^2}) \in \mathfrak{o}_h^{h^2} \mid x_i, x_{qh+r} \in \varpi\mathfrak{o}_h \text{ for all } 2 \leq i \leq h$$
$$\text{and for all } q \geq r \text{ with } q, r > 1\}.$$

Let $T = (T_1, T_2, \ldots, T_{h^2})$, and let $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ denote the set of power series in T with coefficients from $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ which converge on U.

Like $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ is also a $\check{K}$-Banach algebra with respect to the multiplicative norm

$$\left\| \sum_{\alpha \in \mathbb{N}_0^{h^2}} f_\alpha T^\alpha \right\|_U$$

$$:= \sup_{\alpha \in \mathbb{N}_0^{h^2}} \|f_\alpha\|_D |\varpi|^{(\alpha_2 + \alpha_3 + \cdots + \alpha_h + \alpha_{2h+2} + \alpha_{3h+2} + \alpha_{3h+3} + \cdots + \alpha_{(h-1)h+h-1})}$$

(cf. [Sch11], Proposition 5.3). Under the global chart of P in the proof of Lemma 3.2.1, we now show that, for a monomial $w^\alpha \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, the map $\iota(a) \longmapsto a(w^\alpha)$ belongs to $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ for every $\alpha \in \mathbb{N}_0^{h-1}$.

By (3.2.2), we have

$$
\begin{aligned}
a(w^\alpha) &= a(w_1)^{\alpha_1} \ldots a(w_{h-1})^{\alpha_{h-1}} \\
&= \left(\frac{a_{01} + \sum_{j=1}^{h-1} a_{j1} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j} \right)^{\alpha_1} \cdots \left(\frac{a_{0(h-1)} + \sum_{j=1}^{h-1} a_{j(h-1)} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j} \right)^{\alpha_{h-1}} \\
\text{(3.2.5)} \qquad &= \left(\prod_{i=1}^{h-1} \left(a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j \right)^{\alpha_i} \right) (a_{00}^{-1})^{|\alpha|} \left(1 + a_{00}^{-1} \sum_{j=1}^{h-1} a_{j0} w_j \right)^{-|\alpha|} \\
&= \left(\prod_{i=1}^{h-1} \left(a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j \right)^{\alpha_i} \right) (a_{00}^{-1})^{|\alpha|} \left(\sum_{l=0}^{\infty} \left(-a_{00}^{-1} \sum_{j=1}^{h-1} a_{j0} w_j \right)^l \right)^{|\alpha|}.
\end{aligned}
$$

Thus the expression of $a(w^\alpha)$ is a product of $(a_{00}^{-1})^{|\alpha|}$ and two big brackets. The first big bracket in (3.2.5) is a product of polynomials in a_{ij}'s with coefficients from $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, and hence is the evaluation at $\iota(a)$ of an element in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$. Similarly, $(a_{00}^{-1})^{|\alpha|}$ is the evaluation at $\iota(a)$ of the monomial $T_1^{|\alpha|}$ which belongs to $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$. The second big bracket is the $|\alpha|$-th power of a certain geometric series. The l-th term in that series is the evaluation of the polynomial $(-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j)^l$ at $\iota(a)$, and

$$\left\| \left(-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \right)^l \right\|_U = \left(\left\| -T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \right\|_U \right)^l = |\varpi|^{\frac{l}{h}}.$$

Hence, the series $\sum_{l=0}^{\infty} (-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j)^l$ converges in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$, and the map $\iota(a) \longmapsto a(w^\alpha) \in \mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ for every $\alpha \in \mathbb{N}_0^{h-1}$.

Let us calculate the norms $\| \cdot \|_U$ of the above power series corresponding to the terms in the expression (3.2.5) or find an upper bound for them. First, $\|T_1^{|\alpha|}\|_U = 1$. Since

$$\left\| \sum_{l=0}^{\infty} \left(-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \right)^l \right\|_U \le \sup_{l \ge 0} \left\| \left(-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \right)^l \right\|_U = \sup_{l \ge 0} |\varpi|^{\frac{l}{h}} = 1,$$

the power series corresponding to the second big bracket in (3.2.5) has the norm ≤ 1. The first big bracket is obtained by evaluating

$$\prod_{i=1}^{h-1} \left(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \right)^{\alpha_i}$$

at $\iota(a)$, and

$$\left\| \prod_{i=1}^{h-1} \left(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \right)^{\alpha_i} \right\|_U = \prod_{i=1}^{h-1} \left\| \left(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \right) \right\|_U^{\alpha_i}$$

$$= \prod_{i=1}^{h-1} |\varpi|^{\alpha_i \left(1 - \frac{i}{h} \right)}.$$

Therefore, the power series corresponding to the first big bracket has the norm $|\varpi|^{\sum_{i=1}^{h-1} \alpha_i \left(1 - \frac{i}{h} \right)}$. So, for every $\alpha \in \mathbb{N}_0^{h-1}$, the map $\iota(a) \longmapsto a(w^\alpha)$ is given by an element in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ whose norm is bounded above by $|\varpi|^{\sum_{i=1}^{h-1} \alpha_i \left(1 - \frac{i}{h} \right)}$.

Now given $f = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$,

$$a(f) = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha(a(w^\alpha)),$$

and for every $\alpha \in \mathbb{N}_0^{h-1}$, the map $\iota(a) \longmapsto c_\alpha(a(w^\alpha))$ is represented by a power series in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ having norm $\le |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i \left(1 - \frac{i}{h} \right)}$. Since

$$\lim_{|\alpha| \to \infty} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i \left(1 - \frac{i}{h} \right)} = 0,$$

we see that the map $\iota(a) \longmapsto a(f)$ from P to $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is given by a convergent power series in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$. As a is arbitrary, this implies that the action of P on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is locally K_h-analytic, and thus locally K-analytic by the restriction of scalars. ∎

PROPOSITION 3.2.6: *The $\check{K}$-Banach space $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is a locally K-analytic representation of Γ.*

Proof. This follows from Lemma 3.2.1 and Proposition 3.2.4. ■

THEOREM 3.2.7: *Let s be any integer. Then the $\check{K}$-Banach space M_D^s is a locally K-analytic representation of Γ.*

Proof. Due to Remark 3.1.13, we have a Γ-equivariant, $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-linear isomorphism $M_D^s \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D).\varphi_0^s$ of free $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-modules of rank 1. Then M_D^s obtains a structure of a $\check{K}$-Banach space with respect to the norm defined as

$$\|f\varphi_0^s\|_{M_D^s} := \|f\|_D.$$

Since the Γ-action on M_D^s is semilinear for its action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, we have $\gamma(f\varphi_0^s) = \gamma(f)\gamma(\varphi_0^s)$ for all $\gamma \in \Gamma$ and $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Now, as mentioned in the proof of Proposition 3.1.11,

$$\gamma(\varphi_0^s) = \gamma(\varphi_0)^s = (\lambda_0\varphi_0 + \lambda_{h-1}^\sigma\varphi_1 + \cdots + \lambda_1^{\sigma^{h-1}}\varphi_{h-1})^s$$
$$= (\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}}w_{h-1})^s\varphi_0^s.$$

So the orbit map from Γ to M_D^s is given by sending γ to

$$\gamma(f\varphi_0^s) = \gamma(f)(\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}}w_{h-1})^s\varphi_0^s.$$

The map $\gamma \mapsto \gamma(f)$ is locally K-analytic by Proposition 3.2.6, and the map $\gamma \mapsto (\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}}w_{h-1})^s\varphi_0^s$ is also locally K-analytic since it is given by a linear polynomial in the coordinates of γ. Thus, the orbit map, being a product of these two maps, is locally K-analytic. Therefore, M_D^s is a locally analytic Γ-representation for all integers s. ■

3.3. LOCAL ANALYTICITY OF THE Γ-ACTION ON M_0^s. Let $D(\Gamma)$ denote the algebra of $\check{K}$-valued locally K-analytic distributions on Γ (cf. [ST02ii], Section 2). The strong topological duality induces an anti-equivalence between the category of locally K-analytic representations of Γ on the $\check{K}$-vector spaces of compact type and the category of continuous $D(\Gamma)$-modules on the nuclear $\check{K}$-Fréchet spaces (cf. [ST02ii], Corollary 3.4). Using the local K-analyticity of the Γ-action on M_D^s obtained in the previous subsection, we now show that, at level $m = 0$, the induced Γ-action on the vector space $(M_0^s)_b'$ of compact type is locally K-analytic by showing that its strong topological dual M_0^s is a continuous $D(\Gamma)$-module.

The continuity of the Γ-action on the universal deformation ring R_0 (cf. Theorem 2.2.6) leads to a continuous Γ-action on R_0^{rig}. This is implied by the next proposition.

PROPOSITION 3.3.1: *Let n and l be integers with $n \geq 0$ and $l \geq 1$. If $\gamma \in \Gamma_n$, and if $f \in R_0^{\mathrm{rig}}$, then*

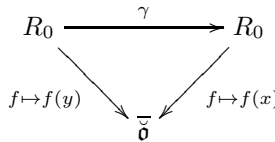$$\|\gamma(f) - f\|_l \leq |\varpi|^{n/l}\|f\|_l.$$

Proof. Note that $R_{0,l}^{\mathrm{rig}}$ is a generalized Tate algebra over $\check{K}$ in the variables $(\varpi^{-1/l}u_i)_{1 \leq i \leq h-1}$. Then by [BGR84], (6.1.5), Proposition 5, we have

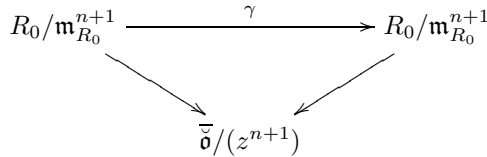$$\|g\|_l = \sup\{|g(x)| | x \in \mathbb{B}_l(\overline{\check{K}})\} \text{for any} g \in R_{0,l}^{\mathrm{rig}},$$

where

$$\mathbb{B}_l(\overline{\check{K}}) = \{x \in (\overline{\check{K}})^{h-1} | |x_i| \leq |\varpi|^{1/l} \text{ for all } 1 \leq i \leq h-1\}.$$

Let us first prove the assertion for $f = u_i$ for some $1 \leq i \leq h-1$. If $x \in \mathbb{B}_l(\overline{\check{K}})$ and $y = (y_j) := (\gamma(u_j)(x))$, then we need to show that $|x_i - y_i| \leq |\varpi|^{(n+1)/l}$ because $\|u_i\|_l = |\varpi|^{1/l}$. Consider the commutating diagram

$$
\begin{array}{ccc}
R_0 & \xrightarrow{\quad\gamma\quad} & R_0 \\
& \searrow{\scriptstyle f \mapsto f(y)} \quad \swarrow{\scriptstyle f \mapsto f(x)} & \\
& \overline{\check{\mathfrak{o}}} &
\end{array}
$$

of homomorphisms of $\check{\mathfrak{o}}$-algebras. Choosing $z \in \overline{\check{\mathfrak{o}}}$ with $|z| = |\varpi|^{1/l}$, we have $x_j \in z\overline{\check{\mathfrak{o}}}$ for all j. Further, $\varpi \in z\overline{\check{\mathfrak{o}}}$ because $l \geq 1$. As a consequence, the right oblique arrow of the above diagram maps $\mathfrak{m}_{R_0} = (\varpi, u_1, \ldots, u_{h-1})$ to $z\overline{\check{\mathfrak{o}}}$. Note that $\gamma(u_j) \in \mathfrak{m}_{R_0}$ so we obtain $y_j \in z\overline{\check{\mathfrak{o}}}$ as well. Therefore, also the left oblique arrow maps $\mathfrak{m}_{R_0}$ to $z\overline{\check{\mathfrak{o}}}$. Now consider the induced diagram

$$
\begin{array}{ccc}
R_0/\mathfrak{m}_{R_0}^{n+1} & \xrightarrow{\quad\gamma\quad} & R_0/\mathfrak{m}_{R_0}^{n+1} \\
& \searrow \qquad \swarrow & \\
& \overline{\check{\mathfrak{o}}}/(z^{n+1}) &
\end{array}
$$

According to Theorem 2.2.6, the upper horizontal arrow is the identity. It follows that $x_i - y_i \in z^{n+1}\overline{\check{\mathfrak{o}}}$, i.e.,

$$|x_i - y_i| \leq |\varpi|^{(n+1)/l}.$$

We now prove the assertion for $f = u^\alpha$ by induction on $|\alpha|$. The case $|\alpha| = 0$ is trivial. Let $|\alpha| > 0$. Choose an index i with $\alpha_i > 0$. Define $\beta_j := \alpha_j$ if $j \neq i$, and $\beta_i := \alpha_i - 1$. Then for $x \in \mathbb{B}_l(\check{\overline{K}})$,

$$
\begin{aligned}
|\gamma(u^\alpha)(x) - u^\alpha(x)| &= |y^\alpha - x^\alpha| = |y_i y^\beta - x_i x^\beta| \\
&\leq \max\{|y_i||y^\beta - x^\beta|, |y_i - x_i||x^\beta|\}.
\end{aligned}
$$

Now $|y_i||y^\beta - x^\beta| \leq |\varpi|^{1/l}\|\gamma(u^\beta) - u^\beta\|_l \leq |\varpi|^{(n+1)/l}\|u^\beta\|_l = |\varpi|^{n/l}\|u^\alpha\|_l$ by the induction hypothesis and $|y_i - x_i||x^\beta| \leq |\varpi|^{(n+1)/l}|\varpi|^{|\beta|/l} = |\varpi|^{n/l}\|u^\alpha\|_l$ as seen above. Thus we obtain $|\gamma(u^\alpha)(x) - u^\alpha(x)| \leq |\varpi|^{n/l}\|u^\alpha\|_l$ for all $x \in \mathbb{B}_l(\check{\overline{K}})$ as required.

Therefore if $f = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \in R_0^{\mathrm{rig}}$, then by continuity of γ, we get

$$
\begin{aligned}
\|\gamma(f) - f\|_l &= \left\| \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha(\gamma(u^\alpha) - u^\alpha) \right\|_l \\
&\leq \sup_{\alpha \in \mathbb{N}_0^{h-1}} |c_\alpha| \|\gamma(u^\alpha) - u^\alpha\|_l \\
&\leq \sup_{\alpha \in \mathbb{N}_0^{h-1}} |c_\alpha| |\varpi|^{n/l}\|u^\alpha\|_l \\
&= |\varpi|^{n/l}\|f\|_l. \qquad \blacksquare
\end{aligned}
$$

We write $\Gamma_{\mathbb{Q}_p}$ for Γ when viewed as a locally $\mathbb{Q}_p$-analytic group, and $\mathfrak{g}_{\mathbb{Q}_p}$ for its Lie algebra $\mathfrak{g}$ when considered as a $\mathbb{Q}_p$-vector space. Let $d := [K : \mathbb{Q}_p]$. Since $\Gamma_{\mathbb{Q}_p}$ is a compact locally $\mathbb{Q}_p$-analytic group of dimension $t := dh^2$, it contains an open subgroup Γ_o which is a uniform pro-p group of rank t (cf. [DDMS03], Theorem 8.32). The subgroups in its lower p-series $P_i(\Gamma_o)$ $(i \geq 1)$ form a basis of open neighbourhoods of the identity in Γ_o and are also uniform pro-p groups of rank t (cf. [DDMS03], Proposition 1.7, Proposition 1.11 (i), Theorem 3.6 (i) and Proposition 4.4). Let n be a positive integer such that $\Gamma_n \subseteq \Gamma_o$. As Γ_n is open in Γ_o, it contains $\Gamma_* := P_i(\Gamma_o)$ for some $i \geq 1$. In what follows, we view Γ_* as a locally $\mathbb{Q}_p$-analytic group.

Let us denote by $\Lambda(\Gamma_*) := \check{\mathfrak{o}}[[\Gamma_*]]$ the Iwasawa algebra of Γ_* over $\check{\mathfrak{o}}$. Set

$$
b_i := \gamma_i - 1 \in \Lambda(\Gamma_*) \quad \text{and} \quad b^\alpha := b_1^{\alpha_1} \cdots b_t^{\alpha_t}
$$

for any $\alpha \in \mathbb{N}_0^t$ where $\{\gamma_1, \ldots, \gamma_t\}$ is a minimal topological generating set of Γ_*. By [DDMS03], Theorem 7.20, any element $\mu \in \Lambda(\Gamma_*)$ admits a unique

expansion of the form

$$\mu = \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \quad \text{with } d_\alpha \in \breve{\mathfrak{o}} \, \forall \alpha \in \mathbb{N}_0^t.$$

For any $l \geq 1$, this allows us to define the $\breve{K}$-norm $\| \cdot \|_l$ on the algebra $\Lambda(\Gamma_*)_{\breve{K}} := \Lambda(\Gamma_*) \otimes_{\breve{\mathfrak{o}}} \breve{K}$ through

$$(3.3.2) \qquad \left\| \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \right\|_l := \sup_{\alpha \in \mathbb{N}_0^t} \{ |d_\alpha| |\varpi|^{|\alpha|/l} \}.$$

By [ST03], Proposition 4.2, the norm $\| \cdot \|_l$ on $\Lambda(\Gamma_*)_{\breve{K}}$ is submultiplicative. As a consequence, the completion

$$\Lambda(\Gamma_*)_{\breve{K},l} = \left\{ \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \mid d_\alpha \in \breve{K}, \lim_{|\alpha| \to \infty} |d_\alpha| |\varpi|^{|\alpha|/l} = 0 \right\}$$

of $\Lambda(\Gamma_*)_{\breve{K}}$ with respect to $\| \cdot \|_l$ is a $\breve{K}$-Banach algebra. The natural inclusions $\Lambda(\Gamma_*)_{\breve{K},l+1} \hookrightarrow \Lambda(\Gamma_*)_{\breve{K},l}$ endow the projective limit

$$D(\Gamma_*) = \varprojlim_l \Lambda(\Gamma_*)_{\breve{K},l}$$

with the structure of a $\breve{K}$-Fréchet algebra. By Amice's theorem, the above projective limit is indeed equal to the algebra of $\breve{K}$-valued locally $\mathbb{Q}_p$-analytic distributions on Γ_* (cf. [ST03], Section 4). By fixing coset representatives $\{\gamma_1' = 1, \gamma_2', \ldots, \gamma_s'\}$ of Γ_* in $\Gamma_{\mathbb{Q}_p}$, the natural topological isomorphism

$$C^{an}(\Gamma_{\mathbb{Q}_p}, \breve{K}) \cong \prod_{i=1}^s C^{an}(\gamma_i' \Gamma_*, \breve{K})$$

of locally convex $\breve{K}$-vector spaces induces a topological isomorphism

$$(3.3.3) \qquad D(\Gamma_{\mathbb{Q}_p}) \cong \bigoplus_{i=1}^s \delta_{\gamma_i'} D(\Gamma_*) \quad (\delta_{\gamma_i'}\text{'s are Dirac distributions})$$

by dualizing (cf. [Fé99], Korollar 2.2.4). This defines a $\breve{K}$-Fréchet algebra structure on $D(\Gamma_{\mathbb{Q}_p})$ given by the family of norms

$$\|\delta_{\gamma_1'} \mu_1 + \cdots + \delta_{\gamma_s'} \mu_s\|_l := \max_{i=1}^s \{\|\mu_i\|_l\} \quad \text{with } l \geq 1$$

(cf. [ST03], Theorem 5.1).

Note that $D(\Gamma_{\mathbb{Q}_p})$ is not the same as the distribution algebra $D(\Gamma)$ of $\breve{K}$-valued locally K-analytic distributions on Γ. In fact, the natural embedding $C^{an}(\Gamma, \breve{K}) \hookrightarrow C^{an}(\Gamma_{\mathbb{Q}_p}, \breve{K})$ induces a map $D(\Gamma_{\mathbb{Q}_p}) \longrightarrow D(\Gamma)$ which is a strict

surjection and a homomorphism of $\breve{K}$-algebras by [Koh07], Lemma 1.3.1. According to [Koh07], Lemma 1.3.2 and Lemma 1.3.3, the kernel I of the surjection $D(\Gamma_{\mathbb{Q}_p}) \twoheadrightarrow D(\Gamma)$ is the closure of the ideal generated by all elements of the form $i(\lambda \mathfrak{x}) - \lambda i(\mathfrak{x})$ with $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$, $\lambda \in K$ and $i : \mathfrak{g}_{\mathbb{Q}_p} \hookrightarrow D(\Gamma_{\mathbb{Q}_p})$ denoting the natural inclusion as explained on page 450 of [ST02ii].

THEOREM 3.3.4: *The action of $\Gamma_{\mathbb{Q}_p}$ on R_0^{rig} extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p})$, which then factors through a continuous action of $D(\Gamma)$ on R_0^{rig}. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(R_0^{\mathrm{rig}})'_b$ of R_0^{rig} is locally K-analytic.*

Proof. First, we show that $R_{0,l}^{\mathrm{rig}}$ is a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$ for all $l \geq 1$. To show this, let us prove by induction on $|\alpha|$ that $\|b^\alpha(f)\|_l \leq \|b^\alpha\|_l \|f\|_l$ for any $f \in R_0^{\mathrm{rig}}$. This is clear if $|\alpha| = 0$. Let $|\alpha| > 0$ and let i be the minimal index such that $\alpha_i > 0$. Define $\beta_j := \alpha_j$ if $j \neq i$, and $\beta_i := \alpha_i - 1$. Since $\Gamma_* \subseteq \Gamma_n$, Proposition 3.3.1 and the induction hypothesis imply

$$\|b^\alpha(f)\|_l = \|((\gamma_i - 1)b^\beta)(f)\|_l = \|(\gamma_i - 1)(b^\beta(f))\|_l \leq |\varpi|^{n/l}\|b^\beta(f)\|_l$$
$$\leq |\varpi|^{1/l}\|b^\beta\|_l\|f\|_l \leq |\varpi|^{1/l}|\varpi|^{|\beta|/l}\|f\|_l$$
$$= |\varpi|^{(|\beta|+1)/l}\|f\|_l = \|b^\alpha\|_l\|f\|_l$$

as required.

By Remark 2.2.8, we then have

$$\|\mu(f)\|_l \leq \|\mu\|_l\|f\|_l$$

for all $\mu \in \Lambda(\Gamma_*)_{\breve{K}}$ and $f \in R_0[\frac{1}{\varpi}] = R_0 \otimes_{\breve{\mathfrak{o}}} \breve{K}$. Hence the map

$$\Lambda(\Gamma_*)_{\breve{K}} \times R_0\left[\frac{1}{\varpi}\right] \longrightarrow R_0\left[\frac{1}{\varpi}\right] \quad ((\mu, f) \mapsto \mu(f))$$

is continuous if $\Lambda(\Gamma_*)_{\breve{K}}$ and $R_0[\frac{1}{\varpi}]$ are endowed with the respective $\| \cdot \|_l$-topologies, and if the left-hand side carries the product topology. Since $R_0[\frac{1}{\varpi}]$ is dense in $R_{0,l}^{\mathrm{rig}}$, we obtain a map $\Lambda(\Gamma_*)_{\breve{K},l} \times R_{0,l}^{\mathrm{rig}} \longrightarrow R_{0,l}^{\mathrm{rig}}$ by passing to completions. By continuity, it gives $R_{0,l}^{\mathrm{rig}}$ the structure of a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$. Taking projective limits over l, we obtain a continuous map $D(\Gamma_*) \times R_0^{\mathrm{rig}} \longrightarrow R_0^{\mathrm{rig}}$ giving R_0^{rig} the structure of a continuous module over $D(\Gamma_*)$. Because of the topological isomorphism (3.3.3), R_0^{rig} becomes a continuous module over $D(\Gamma_{\mathbb{Q}_p})$.

To see that the $D(\Gamma_{\mathbb{Q}_p})$-action on R_0^{rig} factors through a continuous action of $D(\Gamma)$, it suffices to check that $i(\lambda\mathfrak{x})(f) = \lambda i(\mathfrak{x})(f)$ for all $\lambda \in K$, $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p} \subseteq D(\Gamma_{\mathbb{Q}_p})$ and $f \in R_0^{\mathrm{rig}}$. Here we make use of Theorem 3.2.7. Being a locally K-analytic Γ-representation, $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is a $D(\Gamma)$-module and thus carries an action of the Lie algebra $\mathfrak{g}$ (cf. [ST02ii], Proposition 3.2). Thus, $i(\lambda\mathfrak{x})(f) = \lambda i(\mathfrak{x})(f)$ holds for all $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. As the K-linear inclusion $R_0^{\mathrm{rig}} \hookrightarrow \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is continuous, it is $\mathfrak{g}_{\mathbb{Q}_p}$-equivariant. Hence the equality $i(\lambda\mathfrak{x})(f) = \lambda i(\mathfrak{x})(f)$ is also true for all $f \in R_0^{\mathrm{rig}}$.

Now it follows from [Sch02], Proposition 19.9 and the arguments proving the claim on page 98, that the $\check{K}$-Fréchet space R_0^{rig} is nuclear. Therefore, [ST02ii], Corollary 3.4 implies that the locally convex $\check{K}$-vector space $(R_0^{\mathrm{rig}})_b'$ is of compact type and that the action of Γ obtained by dualizing is locally K-analytic.   ∎

The preceding theorem can be generalized as follows. Let Γ_* be a uniform pro-p group contained in Γ_{2n+1} for some positive integer n.

THEOREM 3.3.5: *The action of $\Gamma_{\mathbb{Q}_p}$ on M_0^s extends to a continuous action of the $\check{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p})$, which then factors through a continuous action of $D(\Gamma)$ on M_0^s. Hence the action of Γ on the strong continuous $\check{K}$-linear dual $(M_0^s)_b'$ of M_0^s is locally K-analytic for any $s \in \mathbb{Z}$.*

Proof. Choose a generator δ of the R_0-module $\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$. Then by (2.3.6) and (2.3.7), $M_0^s = R_0^{\mathrm{rig}}\delta$ and $M_{0,l}^s = R_{0,l}^{\mathrm{rig}}\delta$. The topology on $M_{0,l}^s$ is defined by the norm $\|f\delta\|_l := \|f\|_l$. Let $\gamma(\delta) = f_0\delta$; then by Γ-equivariance, we have $\gamma(f\delta) = \gamma(f)\gamma(\delta) = \gamma(f)f_0\delta$ for all $f\delta \in M_0^s$. Hence

$$(3.3.6) \qquad \begin{aligned} \gamma(f\delta) - f\delta &= (\gamma(f)f_0 - f)\delta = (\gamma(f)f_0 - ff_0 + ff_0 - f)\delta \\ &= ((\gamma(f) - f)f_0 + f(f_0 - 1))\delta. \end{aligned}$$

Now if $\gamma \in \Gamma_* \subseteq \Gamma_{2n+1}$ and if $f\delta \in M_0^s$, then

$$\|\gamma(f) - f\|_l \le |\varpi|^{\frac{2n+1}{l}}\|f\|_l$$

by Proposition 3.3.1 and

$$\gamma(\delta) - \delta = (f_0 - 1)\delta \in \mathfrak{m}_{R_0}^{n+1}\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$$

by Theorem 2.2.7, i.e.,

$$f_0 - 1 \in \mathfrak{m}_{R_0}^{n+1}.$$

Since $\|y\|_l \leq |\varpi|^{1/l}$ for any $y \in \mathfrak{m}_{R_0} = (\varpi, u_1, \ldots, u_{h-1})$, $\|f_0 - 1\|_l \leq |\varpi|^{\frac{n+1}{l}}$ and $\|f_0\|_l \leq \max\{\|f_0 - 1\|_l, 1\} = 1$. Thus by the multiplicativity of the norm $\|\cdot\|_l$ on R_0^{rig} and by (3.3.6), we have

$$\|\gamma(f\delta) - f\delta\|_l = \|(\gamma(f) - f)f_0 + f(f_0 - 1)\|_l$$
$$\leq \max\{\|(\gamma(f) - f)\|_l\|f_0\|_l, \|f\|_l\|(f_0 - 1)\|_l\}$$
$$\leq \max\{|\varpi|^{\frac{2n+1}{l}}\|f\|_l, |\varpi|^{\frac{n+1}{l}}\|f\|_l\}$$
$$= |\varpi|^{\frac{n+1}{l}}\|f\|_l = |\varpi|^{\frac{n+1}{l}}\|f\delta\|_l.$$

The rest of the proof now proceeds along the same lines as in the proof of Theorem 3.3.4. ∎

3.4. LOCAL ANALYTICITY OF THE Γ-ACTION ON M_m^s WITH $m > 0$. In this subsection, as the title indicates, we extend the theorems of the previous subsection to higher levels $m > 0$. The following observation together with the continuity of the Γ-action on R_m and on R_0^{rig} (cf. Theorem 2.2.6 and Proposition 3.3.1 respectively) allows us to show the continuity of the Γ-action on R_m^{rig} for $m > 0$.

LEMMA 3.4.1: *For every* $m \geq 0$, *there exists a positive integer* k_m *such that* $\mathfrak{m}_{R_m}^n \subseteq \mathfrak{m}_{R_0} R_m$ *for all* $n \geq k_m$.

Proof. Since R_m is a finite free module over R_0, $R_m/\mathfrak{m}_{R_0} R_m$ is a finite-dimensional vector space over $R_0/\mathfrak{m}_{R_0} = \overline{k}$. Moreover, $R_m/\mathfrak{m}_{R_0} R_m$ is still a Noetherian local ring with the maximal ideal $\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m$. The powers $(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m)^n$, $n \in \mathbb{N}$, of the ideal $\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m$ form a descending sequence of finite-dimensional subspaces which eventually must become stationary. Let k_m be a positive integer such that

$$(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m)^{n+1} = (\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m)^n$$

for all $n \geq k_m$. Then by Nakayama's lemma, for all $n \geq k_m$,

$$(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0} R_m)^n = 0,$$

in other words, $\mathfrak{m}_{R_m}^n \subseteq \mathfrak{m}_{R_0} R_m$. ∎

PROPOSITION 3.4.2: *Let* m, n *and* l *be integers with* $m \geq 1$, $l \geq 1$ *and* $n \geq k_m - 1$ *where* k_m *is as stated in Lemma 3.4.1. If* $\gamma \in \Gamma_{n+m}$ *and if* $f \in R_m^{\mathrm{rig}}$, *then*

$$\|\gamma(f) - f\|_l \leq |\varpi|^{1/l}\|f\|_l.$$

Proof. Write $f = f_1 e_1 + \cdots + f_r e_r$ where $\{e_1, \ldots, e_r\}$ is a basis of R_m over R_0 and $f_i \in R_0^{\mathrm{rig}}$ for all $1 \leq i \leq r$. Let $x_i := \gamma(e_i) - e_i$. Then $x_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $1 \leq i \leq r$ by Theorem 2.2.6 and thus by Lemma 3.4.1, $x_i \in \mathfrak{m}_{R_0} R_m$ for all $1 \leq i \leq r$. Since $\|y\|_l \leq |\varpi|^{1/l}$ for any $y \in \mathfrak{m}_{R_0} = (\varpi, u_1, \ldots, u_{h-1})$, $\|x_i\|_l \leq |\varpi|^{1/l}$ for all $1 \leq i \leq r$. Now note that

$$
\begin{aligned}
\|\gamma(f) - f\|_l &\leq \max_{1 \leq i \leq r} \{\|\gamma(f_i e_i) - f_i e_i\|_l\} = \max_{1 \leq i \leq r} \{\|\gamma(f_i)\gamma(e_i) - f_i e_i\|_l\} \\
&= \max_{1 \leq i \leq r} \{\|\gamma(f_i)\gamma(e_i) - f_i \gamma(e_i) + f_i \gamma(e_i) - f_i e_i\|_l\} \\
&= \max_{1 \leq i \leq r} \{\|(\gamma(f_i) - f_i)\gamma(e_i) + (\gamma(e_i) - e_i)f_i\|_l\} \\
&= \max_{1 \leq i \leq r} \{\|(\gamma(f_i) - f_i)(e_i + x_i) + x_i f_i\|_l\}.
\end{aligned}
$$

Then Lemma 2.3.3 and Proposition 3.3.1 imply that for every $1 \leq i \leq r$,

$$
\begin{aligned}
\|(\gamma(f_i) - f_i)(e_i + x_i) + x_i f_i\|_l &\leq \max\{\|(\gamma(f_i) - f_i)(e_i + x_i)\|_l, \|x_i f_i\|_l\} \\
&\leq \max\{\|(\gamma(f_i) - f_i)\|_l, \|x_i\|_l \|f_i\|_l\} \\
&\leq \max\{|\varpi|^{(n+m)/l}\|f_i\|_l, |\varpi|^{1/l}\|f_i\|_l\} \\
&= |\varpi|^{1/l}\|f_i\|_l
\end{aligned}
$$

where we use that $e_i + x_i = \gamma(e_i) \in R_m$ has $\|\cdot\|_l$-norm less than or equal to 1. Therefore, $\|\gamma(f) - f\|_l \leq \max_{1 \leq i \leq r}\{|\varpi|^{1/l}\|f_i\|_l\} = |\varpi|^{1/l}\|f\|_l$. ∎

We now arbitrarily fix a level $m \geq 1$. As before, we have a uniform pro-p group Γ_o of rank t as an open subgroup of $\Gamma_{\mathbb{Q}_p}$. We also fix a positive integer $n \geq k_m - 1$ such that $\Gamma_{n+m} \subseteq \Gamma_o$. Then Γ_{n+m} contains $\Gamma_* := P_i(\Gamma_o)$ for some $i \geq 1$ which is also a uniform pro-p group of rank t.

Let $\{\gamma_1, \ldots, \gamma_t\}$ be an ordered basis of Γ_* and let $b_i := \gamma_i - 1 \in \Lambda(\Gamma_*)$. Then as before, we equip the $\breve{K}$-algebra $\Lambda(\Gamma_*)_{\breve{K}}$ with the sub-multiplicative norm $\|\cdot\|_l$ defined in (3.3.2) for every positive integer l. The natural inclusions $\Lambda(\Gamma_*)_{\breve{K}, l+1} \hookrightarrow \Lambda(\Gamma_*)_{\breve{K}, l}$ of $\breve{K}$-Banach completions endow the projective limit $D(\Gamma_*) = \varprojlim_l \Lambda(\Gamma_*)_{\breve{K}, l}$ with the structure of a $\breve{K}$-Fréchet algebra which is equal to the algebra of $\breve{K}$-valued locally K-analytic distributions on Γ_*.

PROPOSITION 3.4.3: *For any integer $l \geq 1$, the action of Γ_* on R_m^{rig} extends to $R_{m,l}^{\mathrm{rig}}$ and makes $R_{m,l}^{\mathrm{rig}}$ a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K}, l}$. The action of $\Gamma_{\mathbb{Q}_p}$ on R_m^{rig} extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p})$.*

Proof. The proof is similar to that of Theorem 3.3.4. First, we prove by induction on $|\alpha|$ that $\|b^\alpha(f)\|_l \leq \|b^\alpha\|_l \|f\|_l$ for any $f \in R_m^{\mathrm{rig}}$. This is clear if $|\alpha| = 0$. Let $|\alpha| > 0$ and let i be the minimal index such that $\alpha_i > 0$. Define $\beta_j := \alpha_j$ if $j \neq i$ and $\beta_i := \alpha_i - 1$. Then Proposition 3.4.2 and the induction hypothesis imply

$$\begin{aligned}
\|b^\alpha(f)\|_l &= \|((\gamma_i - 1)b^\beta)(f)\|_l = \|(\gamma_i - 1)(b^\beta(f))\|_l \leq |\varpi|^{1/l}\|b^\beta(f)\|_l \\
&\leq |\varpi|^{1/l}\|b^\beta\|_l\|f\|_l \leq |\varpi|^{1/l}|\varpi|^{|\beta|/l}\|f\|_l \\
&= |\varpi|^{(|\beta|+1)/l}\|f\|_l = \|b^\alpha\|_l\|f\|_l
\end{aligned}$$

as required. By Remark 2.2.8, this immediately gives $\|\mu(f)\|_l \leq \|\mu\|_l\|f\|_l$ for all $\mu \in \Lambda(\Gamma_*)_{\check{K}}$ and $f \in R_m[\frac{1}{\varpi}] = R_m \otimes_{\mathfrak{d}} \check{K}$. Hence the map

$$\Lambda(\Gamma_*)_{\check{K}} \times R_m\Big[\frac{1}{\varpi}\Big] \longrightarrow R_m\Big[\frac{1}{\varpi}\Big] \quad ((\mu, f) \mapsto \mu(f))$$

is continuous if $\Lambda(\Gamma_*)_{\check{K}}$ and $R_m[\frac{1}{\varpi}]$ are endowed with the respective $\|\cdot\|_l$-topologies and if the left hand side carries the product topology. Since $R_m[\frac{1}{\varpi}]$ is dense in $R_{m,l}^{\mathrm{rig}}$, we obtain a map $\Lambda(\Gamma_*)_{\check{K},l} \times R_{m,l}^{\mathrm{rig}} \longrightarrow R_{m,l}^{\mathrm{rig}}$ by passing to completions. By continuity, it gives $R_{m,l}^{\mathrm{rig}}$ the structure of a topological Banach module over the $\check{K}$-Banach algebra $\Lambda(\Gamma_*)_{\check{K},l}$.

Taking projective limits over l, we obtain a continuous map

$$D(\Gamma_*) \times R_m^{\mathrm{rig}} \longrightarrow R_m^{\mathrm{rig}}$$

giving R_m^{rig} the structure of a continuous module over $D(\Gamma_*)$. As $D(\Gamma_{\mathbb{Q}_p})$ is topologically isomorphic to the locally convex direct sum $\bigoplus_{\gamma\Gamma_* \in \Gamma_{\mathbb{Q}_p}/\Gamma_*} \delta_\gamma D(\Gamma_*)$, R_m^{rig} is a continuous module over $D(\Gamma_{\mathbb{Q}_p})$. ∎

We want to show that the $D(\Gamma_{\mathbb{Q}_p})$-action on R_m^{rig} factors through a continuous $D(\Gamma)$-action. As mentioned in the introduction, the idea is to use the local K-analyticity of the Γ-action at level $m = 0$ obtained in Theorem 3.3.4 and the étaleness property of the extension $R_m^{\mathrm{rig}}|R_0^{\mathrm{rig}}$.

For a ring homomorphism $A \to B$ of commutative unital rings, let $\mathrm{Der}_A(B, B)$ denote the B-module of A-linear derivations from B to B, and let $\Omega_{B/A}$ denote the B-module of differentials of B over A.

LEMMA 3.4.4: *Any $\check{K}$-linear derivation from R_0^{rig} to R_0^{rig} extends uniquely to a $\check{K}$-linear derivation from R_m^{rig} to R_m^{rig}.*

Proof. Since $R_m[\frac{1}{\varpi}]$ is étale over $R_0[\frac{1}{\varpi}]$ (cf. Theorem 2.1.2 (ii), [Str08ii]),

$$R_m^{\mathrm{rig}} \cong R_m\left[\frac{1}{\varpi}\right] \otimes_{R_0[\frac{1}{\varpi}]} R_0^{\mathrm{rig}}$$

is étale over R_0^{rig} by [Sta, Tag 00U0], and so is formally étale by [Sta, Tag 00UR]. Then, using [Sta, Tag 031K] and [Sta, Tag 00UO], we get that

$$\Omega_{R_m^{\mathrm{rig}}/\breve{K}} \cong \Omega_{R_0^{\mathrm{rig}}/\breve{K}} \otimes_{R_0^{\mathrm{rig}}} R_m^{\mathrm{rig}}.$$

Therefore

$$\begin{aligned}
\mathrm{Der}_{\breve{K}}(R_0^{\mathrm{rig}}, R_0^{\mathrm{rig}}) &\cong \mathrm{Hom}_{R_0^{\mathrm{rig}}}(\Omega_{R_0^{\mathrm{rig}}/\breve{K}}, R_0^{\mathrm{rig}}) \\
&\hookrightarrow \mathrm{Hom}_{R_0^{\mathrm{rig}}}(\Omega_{R_0^{\mathrm{rig}}/\breve{K}}, R_m^{\mathrm{rig}}) \\
&\cong \mathrm{Hom}_{R_m^{\mathrm{rig}}}(\Omega_{R_0^{\mathrm{rig}}/\breve{K}} \otimes_{R_0^{\mathrm{rig}}} R_m^{\mathrm{rig}}, R_m^{\mathrm{rig}}) \\
&\cong \mathrm{Hom}_{R_m^{\mathrm{rig}}}(\Omega_{R_m^{\mathrm{rig}}/\breve{K}}, R_m^{\mathrm{rig}}) \\
&\cong \mathrm{Der}_{\breve{K}}(R_m^{\mathrm{rig}}, R_m^{\mathrm{rig}}). \quad\blacksquare
\end{aligned}$$

THEOREM 3.4.5: *The action of $D(\Gamma_{\mathbb{Q}_p})$ on R_m^{rig} factors through a continuous action of $D(\Gamma)$ on R_m^{rig}. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(R_m^{\mathrm{rig}})'_b$ of R_m^{rig} is locally K-analytic.*

Proof. Recall the inclusion map $i : \mathfrak{g}_{\mathbb{Q}_p} \hookrightarrow D(\Gamma_p)$ from the discussion before Theorem 3.3.4. For every $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$, $i(\mathfrak{x}) \in D(\Gamma_{\mathbb{Q}_p})$ acts on R_m^{rig} as a $\breve{K}$-linear derivation from R_m^{rig} to R_m^{rig}. Let $\lambda \in K$ and $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$ be arbitrary, and consider the distribution

$$i(\lambda\mathfrak{x}) - \lambda i(\mathfrak{x}) \in D(\Gamma_{\mathbb{Q}_p}).$$

It gives rise to a zero derivation on R_0^{rig} by Theorem 3.3.4 and thus by Lemma 3.4.4, it is also zero on R_m^{rig}. This means that the action of $D(\Gamma_{\mathbb{Q}_p})$ on R_m^{rig} factors through a continuous action of $D(\Gamma)$ on R_m^{rig}.

As the $\breve{K}$-Fréchet space R_m^{rig} is topologically isomorphic to $\bigoplus_{i=1}^r R_0^{\mathrm{rig}}$, it follows from [Sch02], Proposition 19.7, that R_m^{rig} is nuclear. Therefore, [ST02ii], Corollary 3.4 implies that the locally convex $\breve{K}$-vector space $(R_m^{\mathrm{rig}})'_b$ is of compact type and that the action of Γ obtained by dualizing is locally K-analytic. $\quad\blacksquare$

Similar to the above, Theorem 3.4.5 can be generalized as follows. Fix $m \geq 1$, $n \geq k_m - 1$ and a uniform pro-p group $\Gamma_* \subseteq \Gamma_{2n+m+1}$ with k_m as in Lemma 3.4.1.

THEOREM 3.4.6: *The action of $\Gamma_{\mathbb{Q}_p}$ on M_m^s extends to a continuous action of the $\check{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p})$, which then factors through a continuous action of $D(\Gamma)$ on M_m^s. Hence the action of Γ on the strong continuous $\check{K}$-linear dual $(M_m^s)'_b$ of M_m^s is locally K-analytic for any $s \in \mathbb{Z}$.*

Proof. Using Theorem 2.2.7, Lemma 3.4.1 and Proposition 3.4.2, the proof of the first part of the assertion is similar to that of Theorem 3.3.5.

Observe that the isomorphism $\operatorname{Lie}(\mathbb{H}^{(m)}) \cong \operatorname{Lie}(\mathbb{H}^{(0)}) \otimes_{R_0} R_m$ is Γ-equivariant for the diagonal Γ-action on the right. Therefore, we have the following Γ-equivariant isomorphisms by (2.3.6):

$$
\begin{aligned}
M_m^s &\cong R_m^{\mathrm{rig}} \otimes_{R_m} \operatorname{Lie}(\mathbb{H}^{(m)}) \\
&\cong R_m^{\mathrm{rig}} \otimes_{R_0} \operatorname{Lie}(\mathbb{H}^{(0)}) \\
&\cong R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} R_0^{\mathrm{rig}} \otimes_{R_0} \operatorname{Lie}(\mathbb{H}^{(0)}) \\
&\cong R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s
\end{aligned}
$$

with Γ-acting diagonally on all the tensor products. As a consequence, the $\mathfrak{g}_{\mathbb{Q}_p}$-action on $f \otimes \delta \in M_m^s$ is given by

$$
\mathfrak{x}(f \otimes \delta) = \mathfrak{x}(f) \otimes \delta + f \otimes \mathfrak{x}(\delta).
$$

However, by Theorem 3.3.5 and Theorem 3.4.5, M_0^s and R_m^{rig} are not only $\mathfrak{g}_{\mathbb{Q}_p}$-modules but also $\mathfrak{g}$-modules. Thus, it follows that the $D(\Gamma_{\mathbb{Q}_p})$-action on M_m^s factors through a continuous $D(\Gamma)$-action as required. ∎

Remark 3.4.7: The method described above can also be used to show the local analyticity of the Γ-action on the dual space of the global sections of a large class of Drinfeld bundles over the Lubin–Tate moduli space (cf. [Koh11], Section 3). If V is a finite-dimensional $\check{K}$-linear smooth representation of Γ then, for any $m \geq 0$, the free R_m^{rig}-module $R_m^{\mathrm{rig}} \otimes_{\check{K}} V$ with the diagonal Γ-action induces a Γ-equivariant vector bundle over X_m^{rig} whose Fréchet space of global sections is $R_m^{\mathrm{rig}} \otimes_{\check{K}} V$ (cf. [Koh11], Theorem 1.2 and Corollary A.3). By choosing a basis $\{b_1, b_2, \ldots, b_d\}$ of V over $\check{K}$, the Fréchet topology of $R_m^{\mathrm{rig}} \otimes_{\check{K}} V$ can be given by the family of the norms $\|f_1 b_1 + f_2 b_2 + \cdots + f_d b_d\|_l := \max_{1 \leq i \leq d} \{\|f_i\|_l\}$ with $l \in \mathbb{N}$, where the l-norms on R_m^{rig} are as defined in Section 2.3. It then follows from Proposition 3.3.1 and Proposition 3.4.2 that if γ belongs to a small

enough open subgroup of Γ ($\subseteq$ intersection of stabilizers of b_i's) then

$$
\begin{aligned}
\|\gamma(f_1 b_1 + f_2 b_2 + \cdots + f_d b_d) &- (f_1 b_1 + f_2 b_2 + \cdots + f_d b_d)\|_l \\
&= \|(\gamma(f_1) - f_1)b_1 + (\gamma(f_2) - f_2)b_2 + \cdots + (\gamma(f_d) - f_d)b_d\|_l \\
&= \max_{1 \le i \le d}\{\|\gamma(f_i) - f_i\|_l\} \\
&\le |\varpi|^{1/l} \max_{1 \le i \le d}\{\|f_i\|_l\} \\
&= |\varpi|^{1/l}\|f_1 b_1 + f_2 b_2 + \cdots + f_d b_d\|_l.
\end{aligned}
$$

Therefore, similar to the the proof of Theorem 3.3.4, the Γ-action on $R_m^{\mathrm{rig}} \otimes_{\breve{K}} V$ extends to a continuous action of the Fréchet algebra $D(\Gamma_{\mathbb{Q}_p})$. Since R_m^{rig} is a $\mathfrak{g}$-module and V is annihilated by $\mathfrak{g}$, the $D(\Gamma_{\mathbb{Q}_p})$-action on $R_m^{\mathrm{rig}} \otimes_{\breve{K}} V$ factors through a continuous action of $D(\Gamma)$ making its strong topological dual a locally K-analytic Γ-representation.

4. Locally finite vectors in the global sections of equivariant vector bundles

This section is devoted to studying representation-theoretic aspects of the Γ-representations M_m^s which include a complete description of the Γ-locally finite (algebraic) vectors in M_m^s for all $s \in \mathbb{Z}$ and $m \ge 0$.

4.1. Locally finite vectors in the Γ-representations M_0^s.

Definition 4.1.1: Let G be a topological group and V be a vector space over a field F equipped with an F-linear G-action. We say that a vector $v \in V$ is **locally finite** (or **G-locally finite**) if there is an open subgroup H of G and a finite-dimensional H-stable subspace W of V containing v.[3] It follows easily that the set V_{lf} of all locally finite vectors of V forms a G-stable subspace. We call V a locally finite representation of G if $V_{\mathrm{lf}} = V$. If V and W are F-linear G-representations, and if $f : V \longrightarrow W$ is an F-linear G-equivariant map, then clearly $f(V_{\mathrm{lf}}) \subseteq W_{\mathrm{lf}}$.

---

[3] In [Eme17], Proposition-Definition 4.1.8, the notion of a locally finite vector is defined for the vector spaces over a complete non-archimedean field F, and requires locally finite vector v to be contained in a continuous finite-dimensional H-representation W for its natural Hausdorff topology as a finite-dimensional F-vector space. Since all the Γ-representations we are concerned with in this section are continuous representations on $\breve{K}$-Fréchet spaces, the continuity condition is automatically satisfied.

To calculate locally finite vectors, we make extensive use of the Lie algebra action. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of Γ over K. Note that $\mathfrak{g}$ is isomorphic to the Lie algebra associated with the associative K-algebra B_h. Thus,

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes_K K_h \cong \mathfrak{gl}_h(K_h).$$

Denote by $\mathfrak{x}_{ij} \in \mathfrak{gl}_h(K_h)$ the matrix with entry 1 at the place (i, j) and zero everywhere else. By Theorem 3.2.7, the Γ-representation

$$M_D^s \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$$

carries a continuous linear action of

$$U(\mathfrak{g}) \otimes_K \breve{K} \cong U(\mathfrak{gl}_h(K_h)) \otimes_{K_h} \breve{K} \hookrightarrow D(\Gamma).$$

Since $GL_h(K_h)$ acts on the projective coordinates $\varphi_0, \ldots, \varphi_{h-1}$ by fractional linear transformations, one can explicitly determine this Lie algebra action using the formula

$$\mathfrak{x}(f) = \frac{d}{dt}\exp(t\mathfrak{x})(f)|_{t=0}.$$

LEMMA 4.1.2: *Let i, j and s be integers with $0 \leq i, j \leq h - 1$. Put $w_0 := 1$. If $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ then*

$$(4.1.3) \qquad \mathfrak{x}_{ij}(f\varphi_0^s) = \begin{cases} w_i \frac{\partial f}{\partial w_j}\varphi_0^s, & \text{if } j \neq 0; \\ (sf - \sum_{l=1}^{h-1} w_l \frac{\partial f}{\partial w_l})\varphi_0^s, & \text{if } i = j = 0; \\ w_i(sf - \sum_{l=1}^{h-1} w_l \frac{\partial f}{\partial w_l})\varphi_0^s, & \text{if } i > j = 0. \end{cases}$$

Proof. This is exactly the same as [Koh14], Lemma 4.1, which treats the case $K = \mathbb{Q}_p$. ∎

Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}_h(K_h)$, and a $\breve{K}$-linear $\mathfrak{gl}_h(K_h)$-representation W, the $\breve{K}$-subspace of $\mathfrak{h}$**-invariants of** W is the subspace

$$\{w \in W | \mathfrak{x}(w) = 0 \text{ for all } \mathfrak{x} \in \mathfrak{h}\}$$

of W. Let us denote by $\mathfrak{n}$ the Lie subalgebra of $\mathfrak{gl}_h(K_h)$ consisting of strictly upper triangular matrices. For later use, we calculate the $\mathfrak{g}$-invariants and the $\mathfrak{n}$-invariants of $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ in the next lemma using the formulae (4.1.3).

LEMMA 4.1.4: $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{g}=0} = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0} = \breve{K}.$

Proof. Since $\breve{K} \subseteq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{g}=0} \subseteq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0}$, it suffices to show that the latter is $\breve{K}$. Now if $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0}$, then applying the formulae (4.1.3) we get

$$\mathfrak{x}_{0j}(f) = \frac{\partial f}{\partial w_j} = 0$$

for all $1 \leq j \leq h-1$. Therefore, f must be a constant power series. ∎

We now compute the space $(M_D^s)_{\mathrm{lf}}$ of locally finite vectors in the Γ-representation M_D^s. The key step is the following lemma based on Lemma 4.1.2:

LEMMA 4.1.5:   *The subspace $\breve{K}[w_1, \ldots, w_{h-1}]\varphi_0^s$ of M_D^s is contained in $(U(\mathfrak{g}) \otimes_K \breve{K})(f\varphi_0^s)$ for any non-zero homogeneous polynomial*

$$f \in \breve{K}[w_1, \ldots, w_{h-1}]$$

of total degree $d > s$.

Proof. Using (4.1.3), we have for all $0 < i, j \leq h-1$

$$\mathfrak{x}_{0j}(f\varphi_0^s) = \frac{\partial f}{\partial w_j}\varphi_0^s,$$

$$\mathfrak{x}_{i0}(f\varphi_0^s) = (s-d)w_i f\varphi_0^s.$$

To obtain $g\varphi_0^s$ with a monomial g of total degree $\leq s$, first reduce $f\varphi_0^s$ to φ_0^s by applying suitable $\mathfrak{x}_{0j}$ $(j \neq 0)$ to it iteratively and then apply appropriate $\mathfrak{x}_{i0}$ $(i > 0)$ to φ_0^s to get the desired element $g\varphi_0^s$. To obtain $g\varphi_0^s$ with a monomial g of total degree $> s$, reverse the procedure, i.e., first apply appropriate $\mathfrak{x}_{i0}$ $(i > 0)$ to $f\varphi_0^s$ and then reduce the result to $g\varphi_0^s$ by applying suitable $\mathfrak{x}_{0j}$ $(j \neq 0)$ to it.   ∎

For any $s \in \mathbb{Z}$, we define the $\breve{K}$-subspace of M_D^s by

$$V_s := \sum_{|\alpha| \leq s} \breve{K}w^\alpha \varphi_0^s.$$

Note that we have $V_s = 0$ if $s < 0$. For $s \geq 0$, it is easy to see that V_s is stable under the action of Γ. To see this, it is sufficient to prove that $\gamma(w^\alpha \varphi_0^s) \in V_s$ for any $w^\alpha \varphi_0^s$ with $|\alpha| \leq s$ and for any $\gamma = \sum_{i=0}^{h-1} \lambda_i \Pi^i \in \Gamma$. In this case, using the action of the matrix (3.1.8) on the projective coordinates $[\varphi_0 : \ldots : \varphi_{h-1}]$,

we find

$$
\begin{aligned}
\gamma(w^\alpha \varphi_0^s) &= \gamma(w_1^{\alpha_1} \dots w_{h-1}^{\alpha_{h-1}} \varphi_0^s) \\
&= \gamma(\varphi_1^{\alpha_1} \dots \varphi_{h-1}^{\alpha_{h-1}} \varphi_0^{s-|\alpha|}) \\
&= \gamma(\varphi_1)^{\alpha_1} \dots \gamma(\varphi_{h-1})^{\alpha_{h-1}} \gamma(\varphi_0)^{s-|\alpha|} \\
&= (\varpi\lambda_1\varphi_0 + \dots + \varpi\lambda_2^{\sigma^{h-1}}\varphi_{h-1})^{\alpha_1} \dots (\varpi\lambda_{h-1}\varphi_0 + \dots + \lambda_0^{\sigma^{h-1}}\varphi_{h-1})^{\alpha_{h-1}} \\
&\quad \times (\lambda_0\varphi_0 + \dots + \lambda_1^{\sigma^{h-1}}\varphi_{h-1})^{s-|\alpha|} \\
&= (\varpi\lambda_1 + \dots + \varpi\lambda_2^{\sigma^{h-1}}w_{h-1})^{\alpha_1} \dots (\varpi\lambda_{h-1} + \dots + \lambda_0^{\sigma^{h-1}}w_{h-1})^{\alpha_{h-1}} \\
&\quad \times (\lambda_0 + \dots + \lambda_1^{\sigma^{h-1}}w_{h-1})^{s-|\alpha|}\varphi_0^s \in V_s.
\end{aligned}
$$

THEOREM 4.1.6: *The Γ-representation M_D^s is topologically irreducible if $s < 0$, and if $s \geq 0$ then V_s is a topologically irreducible sub-representation of M_D^s with topologically irreducible quotient M_D^s/V_s.*

Proof. CASE $s < 0$: Let V be a non-zero closed Γ-stable subspace of M_D^s. Let $f_0\varphi_0^s \in V$ where

$$
f_0 = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \neq 0
$$

and $d\ (\geq 0)$ be the smallest natural number such that $c_\alpha \neq 0$ for some $\alpha \in \mathbb{N}_0^{h-1}$ with $|\alpha| = d$. Thus,

$$
f_0\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^\infty \sum_{|\alpha|=d+i} c_\alpha w^\alpha \right)\varphi_0^s \in V
$$

where

$$
\sum_{|\alpha|=d} c_\alpha w^\alpha \neq 0.
$$

For $n > 0$, define a sequence of elements of M_D^s inductively as follows:

$$
f_n\varphi_0^s := \frac{1}{n}((d + n - s)f_{n-1}\varphi_0^s + \mathfrak{x}_{00}(f_{n-1}\varphi_0^s)).
$$

Since V is closed and Γ-stable, V is stable under the action of the Lie algebra and thus $f_n \in V$ for all $n \in \mathbb{N}_0$.

We prove by induction on n that

$$
(4.1.7) \qquad f_n\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^\infty \sum_{|\alpha|=d+i} (-1)^n \binom{i-1}{n} c_\alpha w^\alpha \right)\varphi_0^s.
$$

Here the generalized binomial coefficients are defined by

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!}$$

for any $x \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The case $n = 0$ is true by definition. Assuming that (4.1.7) holds for $n - 1$, we compute using Lemma 4.1.2 that

$$
\begin{aligned}
f_n \varphi_0^s &= \frac{1}{n}\big((d + n - s) f_{n-1}\varphi_0^s + \mathfrak{x}_{00}(f_{n-1}\varphi_0^s)\big) \\
&= \frac{1}{n}\bigg((d + n - s)\bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^{n-1}\binom{i-1}{n-1}c_\alpha w^\alpha\bigg)\varphi_0^s \\
&\qquad + \mathfrak{x}_{00}\bigg(\bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^{n-1}\binom{i-1}{n-1}c_\alpha w^\alpha\bigg)\varphi_0^s\bigg)\bigg) \\
&= \frac{1}{n}\bigg((d + n - s)\bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^{n-1}\binom{i-1}{n-1}c_\alpha w^\alpha\bigg)\varphi_0^s \\
&\qquad + \bigg(\sum_{|\alpha|=d}(s-d)c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(s-(d+i))(-1)^{n-1}\binom{i-1}{n-1}c_\alpha w^\alpha\bigg)\varphi_0^s\bigg) \\
&= \bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(n-i)(-1)^{n-1}\frac{1}{n}\binom{i-1}{n-1}c_\alpha w^\alpha\bigg)\varphi_0^s \\
&= \bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^{n}\binom{i-1}{n}c_\alpha w^\alpha\bigg)\varphi_0^s.
\end{aligned}
$$

We now claim that the sequence $f_n \varphi_0^s$ converges to

$$\bigg(\sum_{|\alpha|=d} c_\alpha w^\alpha\bigg)\varphi_0^s$$

as n tends to ∞ with respect to the norm $\|\cdot\|_{M_D^s}$ defined in the proof of Theorem 3.2.7. As $f_0 \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, we know that given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}_0^{h-1}$ with $|\alpha| > N_\varepsilon$, we have

$$|c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} < \varepsilon.$$

Therefore,

$$\sup_{|\alpha|>N_\varepsilon} |c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} < \varepsilon.$$

Now for all $n > N_\varepsilon - d$, using (4.1.7), we have

$$
\left\| f_n \varphi_0^s - \left(\sum_{|\alpha|=d} c_\alpha w^\alpha \right) \varphi_0^s \right\|_{M_D^s} = \left\| \left(\sum_{i=1}^\infty \sum_{|\alpha|=d+i} (-1)^n \binom{i-1}{n} c_\alpha w^\alpha \right) \varphi_0^s \right\|_{M_D^s}
$$

$$
= \left\| \sum_{i=1}^\infty \sum_{|\alpha|=d+i} (-1)^n \binom{i-1}{n} c_\alpha w^\alpha \right\|_D
$$

$$
= \left\| \sum_{i=n+1}^\infty \sum_{|\alpha|=d+i} (-1)^n \binom{i-1}{n} c_\alpha w^\alpha \right\|_D
$$

$$
\leq \left\| \sum_{|\alpha|>d+n} c_\alpha w^\alpha \right\|_D
$$

$$
= \sup_{|\alpha|>d+n} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1 - \frac{i}{h})}
$$

$$
\leq \sup_{|\alpha|>N_\varepsilon} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1 - \frac{i}{h})} < \varepsilon.
$$

Hence, $f_n \varphi_0^s$ converges to $(\sum_{|\alpha|=d} c_\alpha w^\alpha) \varphi_0^s$ as $n \to \infty$ and $(\sum_{|\alpha|=d} c_\alpha w^\alpha) \varphi_0^s \in V$ because V is closed. Since $\sum_{|\alpha|=d} c_\alpha w^\alpha$ is a non-zero homogeneous polynomial of total degree $d \geq 0 > s$, Lemma 4.1.5 implies that $\check{K}[w_1, \ldots, w_{h-1}] \varphi_0^s \subseteq V$. Since $\check{K}[w_1, \ldots, w_{h-1}] \varphi_0^s$ is dense in $M_D^s = \mathcal{O}_{X_0^{\mathrm{rig}}}(D) \varphi_0^s$ and V is closed, it follows that $V = M_D^s$. Hence, M_D^s is topologically irreducible for all $s < 0$.

CASE $s \geq 0$: Let V be a non-zero closed Γ-stable subspace of V_s. Then V is stable under the action of the Lie algebra $\mathfrak{g}$ and thus it becomes a module over $U(\mathfrak{g}) \otimes_K \check{K}$. As mentioned in the proof of Lemma 4.1.5, any non-zero element $f\varphi_0^s$ of V can be reduced to φ_0^s by applying suitable $\mathfrak{r}_{0j}$ ($j \neq 0$) to it iteratively and then φ_0^s can be converted into any monomial of total degree $\leq s$ multiplied with φ_0^s by applying appropriate $\mathfrak{r}_{i0}$ ($i > 0$) to it. Therefore $V = V_s$ and V_s is topologically irreducible.

Now let $\phi \colon M_D^s \longrightarrow M_D^s / V_s$ be the canonical surjective map and let $W \subset M_D^s / V_s$ be a non-zero, closed Γ-stable subspace. Then $\phi^{-1}(W)$ is a non-zero, closed Γ-stable subspace of M_D^s not equal to V_s. Let $(\sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha) \varphi_0^s + V_s$ be a non-zero element of W. Then

$$
f_0 \varphi_0^s := \left(\sum_{|\alpha|>s} c_\alpha w^\alpha \right) \varphi_0^s \neq 0 \in \phi^{-1}(W).
$$

Let $d > s$ be the smallest natural number such that $c_\alpha \neq 0$ for some $\alpha \in \mathbb{N}_0^{h-1}$ with $|\alpha| = d$. Thus,

$$f_0\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty} \sum_{|\alpha|=d+i} c_\alpha w^\alpha \right) \varphi_0^s \in \phi^{-1}(W)$$

where $\sum_{|\alpha|=d} c_\alpha w^\alpha \neq 0$.

As in the case of $s < 0$, we define a sequence of elements in $\phi^{-1}(W)$ inductively for $n > 0$ as follows:

$$f_n\varphi_0^s := \frac{1}{n}((d + n - s)f_{n-1}\varphi_0^s + \mathfrak{r}_{00}(f_{n-1}\varphi_0^s)).$$

Using exactly the same proof in the previous case of $s < 0$, it can be shown that $f_n\varphi_0^s$ converges to $(\sum_{|\alpha|=d} c_\alpha w^\alpha)\varphi_0^s$ as $n \to \infty$ and $(\sum_{|\alpha|=d} c_\alpha w^\alpha)\varphi_0^s \in \phi^{-1}(W)$ because $\phi^{-1}(W)$ is closed. Since $\sum_{|\alpha|=d} c_\alpha w^\alpha$ is a non-zero homogeneous polynomial of total degree $d > s$, it follows from Lemma 4.1.5 that

$$\breve{K}[w_1, \ldots, w_{h-1}].\varphi_0^s \subseteq \phi^{-1}(W).$$

Since $\breve{K}[w_1, \ldots, w_{h-1}]\varphi_0^s$ is dense in $M_D^s = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$ and $\phi^{-1}(W)$ is closed, it follows that $\phi^{-1}(W) = M_D^s$. Hence $W = \phi(M_D^s) = M_D^s/V_s$. Thus M_D^s/V_s is topologically irreducible. ∎

COROLLARY 4.1.8: *For all $s \in \mathbb{Z}$, we have*

$$(M_D^s)_{\mathrm{lf}} = V_s.$$

Thus, $(M_D^s)_{\mathrm{lf}}$ is zero if $s < 0$ and is a finite-dimensional irreducible representation of Γ if $s \geq 0$.

Proof. Since Γ is compact, any $v \in (M_D^s)_{\mathrm{lf}}$ is contained in a finite-dimensional Γ-subrepresentation of M_D^s. Now the corollary immediately follows from Theorem 4.1.6 and the fact that M_D^s is not a finite-dimensional $\breve{K}$-vector space. ∎

Remark 4.1.9: For $s \geq 0$, the finite-dimensional Γ-representation V_s is also a $\mathfrak{gl}(K_h)$-module. Let $\mathfrak{t} \subset \mathfrak{sl}_h(K_h)$ be the Cartan subalgebra of $\mathfrak{sl}_h(K_h)$ consisting of diagonal matrices, and let $\{\varepsilon_1, \ldots, \varepsilon_{h-1}\}$ be the basis of the root system $(\mathfrak{sl}_h(K_h), \mathfrak{t})$ given by $\varepsilon_i(\mathrm{diag}(t_0, \ldots, t_{h-1})) := t_{i-1} - t_i$. Define the fundamental dominant weight

$$\chi_0 := \frac{1}{h} \sum_{i=1}^{h-1} (h - i)\varepsilon_i \in \mathfrak{t}^*.$$

Then, by the same proof as in [Koh14], Proposition 4.3, it follows that V_s is an irreducible $\mathfrak{sl}_h(K_h)$-representation of highest weight $s\chi_0$. Although this is stronger than saying that V_s is an irreducible Γ-representation, our result (Theorem 4.1.6) also gives information about the Γ-representation M_D^s when $s < 0$, and about the quotient M_D^s/V_s when $s \geq 0$.

Corollary 4.1.8 leads us to calculate locally finite vectors in the global sections M_0^s over X_0^{rig}. Recall from [BGR84], (9.3.4), Example 3, that the rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$ has a finite admissible covering by the $(h-1)$-dimensional closed unit polydiscs

$$V_i := \mathrm{Sp}\Big(\breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\Big\rangle\Big), \quad 0 \leq i \leq h-1.$$

If

$$V_{ij} := \mathrm{Sp}\Big(\breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, \Big(\frac{\varphi_j}{\varphi_i}\Big)^{-1}\Big\rangle\Big) \quad \text{for } 0 \leq i,j \leq h-1,$$

then gluing the V_i's along the identification $V_{ij} \cong V_{ji}$ of affinoid subdomains via $\breve{K}\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, (\frac{\varphi_j}{\varphi_i})^{-1}\rangle = \breve{K}\langle \frac{\varphi_0}{\varphi_j}, \ldots, \frac{\varphi_{h-1}}{\varphi_j}, (\frac{\varphi_i}{\varphi_j})^{-1}\rangle$ gives the rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$. The affinoid covering $\{V_i\}_{0 \leq i \leq h-1}$ allows us to describe the construction of the line bundles $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ on the rigid analytic projective space in a way analogous to the classical construction. For $s \geq 0$, define its sections over the affinoid space V_i

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(V_i) := \breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\Big\rangle \varphi_i^s$$

to be a free module of rank 1 generated by φ_i^s over $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(V_i) = \breve{K}\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\rangle$, and the transition functions $\psi_{ij} : V_{ij} \xrightarrow{\sim} V_{ji}$ induced by the homomorphisms of affinoid $\breve{K}$-algebras

$$\breve{K}\Big\langle \frac{\varphi_0}{\varphi_j}, \ldots, \frac{\varphi_{h-1}}{\varphi_j}, \Big(\frac{\varphi_i}{\varphi_j}\Big)^{-1}\Big\rangle \varphi_j^s \xrightarrow{\text{multiply by } \frac{\varphi_i^s}{\varphi_j^s}} \breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, \Big(\frac{\varphi_j}{\varphi_i}\Big)^{-1}\Big\rangle \varphi_i^s$$

for all $0 \leq i,j \leq h-1$. The above datum gives rise to a locally free $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}$-module $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ of rank 1. For $s < 0$, $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ turns out to be the $\mathcal{O}_{\mathbb{P}_{\breve{k}}^{h-1}}$-linear dual of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(-s)$. It then follows easily from the above description that the global sections of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ are the $\breve{K}$-vector space of homogeneous polynomials of degree s in the variables φ_i's if $s \geq 0$, and are 0 otherwise. The line bundles $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ carry a canonical action of Γ induced by its action on the projective space $\mathbb{P}_{\breve{K}}^{h-1}$.

Now for any $\mathcal{O}_{X_0^{\mathrm{rig}}}$-module $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}$-module $\mathcal{G}$, there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_{X_0^{\mathrm{rig}}}-\mathrm{mod}}(\Phi^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}-\mathrm{mod}}(\mathcal{G}, \Phi_*\mathcal{F}),$$

where $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\check{K}}^{h-1}$ is the Gross–Hopkins' period morphism. The morphism $\mathrm{id}_{\Phi^*\mathcal{G}}$ corresponds to the adjunction morphism $\mathrm{ad} : \mathcal{G} \longrightarrow \Phi_*\Phi^*\mathcal{G}$. Let $\mathcal{G} = \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)$ with $s \in \mathbb{Z}$. The period morphism Φ is constructed in such a way that

$$\Phi^*\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s) \cong (\mathcal{M}_0^s)^{\mathrm{rig}}$$

(cf. Remark 3.1.14). This gives us a map $\mathrm{ad} : \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s) \longrightarrow \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}$ of $\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}$-modules. Taking global sections, we get a homomorphism of Γ-representations

$$\mathrm{ad}_{\mathbb{P}_{\check{K}}^{h-1}} : \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\mathbb{P}_{\check{K}}^{h-1}) \longrightarrow \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\mathbb{P}_{\check{K}}^{h-1}) = (\mathcal{M}_0^s)^{\mathrm{rig}}(\Phi^{-1}(\mathbb{P}_{\check{K}}^{h-1}))$$
$$= (\mathcal{M}_0^s)^{\mathrm{rig}}(X_0^{\mathrm{rig}})$$
$$= M_0^s.$$

LEMMA 4.1.10: *The map* $\mathrm{ad}_{\mathbb{P}_{\check{K}}^{h-1}}$ *is injective.*

Proof. The period morphism Φ, when restricted to the affinoid subdomain D, is an isomorphism. Thus

$$(\mathcal{M}_0^s)^{\mathrm{rig}}(D) \cong \Phi^*\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(D) \cong \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\Phi(D)).$$

Also we have

$$(\mathcal{M}_0^s)^{\mathrm{rig}}(D) \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s \cong \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D))\varphi_0^s.$$

As a result, it follows from the preceding discussion on the line bundles that $\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\mathbb{P}_{\check{K}}^{h-1})$ maps bijectively onto $V_s \subset \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\Phi(D))$ under the restriction map. The lemma now follows from the following commutative diagram with vertical restriction maps:

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\mathbb{P}_{\check{K}}^{h-1}) & \xrightarrow{\mathrm{ad}_{\mathbb{P}_{\check{K}}^{h-1}}} & \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\mathbb{P}_{\check{K}}^{h-1}) = M_0^s \\
\Big\uparrow & & \Big\downarrow \\
\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(s)(\Phi(D)) & \xrightarrow[\mathrm{ad}_{\Phi(D)}]{\cong} & \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\Phi(D)) = M_D^s \quad\blacksquare
\end{array}
$$

COROLLARY 4.1.11: *For all $s \in \mathbb{Z}$, we have an isomorphism of Γ-representations*

$$(M_0^s)_{\mathrm{lf}} = \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1}) \cong V_s.$$

Thus, $(M_0^s)_{\mathrm{lf}}$ is zero if $s < 0$ and is a finite-dimensional irreducible representation of Γ if $s \geq 0$.

Proof. The inclusion $R_0^{\mathrm{rig}} \hookrightarrow \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ gives rise to a Γ-equivariant inclusion $M_0^s \hookrightarrow M_D^s \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D) \otimes_{R_0} \mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$ using (2.3.6) and the freeness of $\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$ as an R_0-module. As $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1})$ is a finite-dimensional $\breve{K}$-vector space, we have for $s \geq 0$, $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1}) \subseteq (M_0^s)_{\mathrm{lf}} \subseteq (M_D^s)_{\mathrm{lf}} = V_s$, where the first and the last $\breve{K}$-vector spaces are isomorphic as mentioned in the proof of the previous lemma. ∎

Remark 4.1.12: From now on, we identify the subrepresentation $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1})$ of M_0^s with V_s. For $s = 1$, the Γ-locally finite subrepresentation V_1 of M_0^1 is the representation $\mathbb{V}$ mentioned in the construction of the period morphism Φ (cf. the paragraph after Corollary 3.1.5), and thus is isomorphic to the h-dimensional Γ-representation $B_h \otimes_{K_h} \breve{K}$. Since $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1})$ is same as the s-th symmetric power $\mathrm{Sym}^s(\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(1)(\mathbb{P}_{\breve{K}}^{h-1}))$ of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(1)(\mathbb{P}_{\breve{K}}^{h-1})$, we obtain the isomorphism

$$(M_0^s)_{\mathrm{lf}} = V_s \cong \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$$

of Γ-representations for all $s \geq 0$.

4.2. LOCALLY FINITE VECTORS IN THE Γ-REPRESENTATIONS M_m^s WITH $m > 0$. We compute the locally finite vectors in two parts: $s \leq 0$ and $s > 0$. The idea here is to use the commuting actions of Γ and the finite group G_0/G_m on M_m^s.

PART I : $s \leq 0$

LEMMA 4.2.1: *Let G be a finite group acting on an integral domain R by ring automorphisms such that the subring of G-invariants R^G is a perfect field F. Then R is a field and the extension R/F is finite.*

Proof. If $\alpha \in R$ then $\prod_{\sigma \in G}(t - \sigma(\alpha))$ is a monic polynomial of degree $|G|$ with coefficients in $R^G = F$, and has α as a root. This implies that every nonzero α has a unique inverse, since R is an integral domain. The second assertion now follows from [Lang02], Chapter VI, Lemma 1.7. ∎

Let $\breve{K}_m$ denote the m-th Lubin–Tate extension of $\breve{K}$. This is a finite Galois extension of $\breve{K}$ obtained by adjoining ϖ^m-torsion points of any Lubin–Tate formal $\mathfrak{o}$-module over $\mathfrak{o}$ to it. It is a non-trivial result of M. Strauch (cf. [Str08i], Corollary 3.4 (ii)) that $\breve{K}_m \subset R_m^{\mathrm{rig}}$. In fact, $\breve{K}_m$ is stable under the actions of G_0/G_m and Γ on R_m^{rig}. For $g \in G_0/G_m$, $\gamma \in \Gamma$ and $\alpha \in \breve{K}_m$, these actions are given by

$$g(\alpha) = \det(g)^{-1}(\alpha) \quad \text{and} \quad \gamma(\alpha) = \mathrm{Nrd}(\gamma)(\alpha)$$

viewing $\breve{K}$ as a left $\mathfrak{o}^{\times}$-module via the map $\mathfrak{o}^{\times} \twoheadrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^{\times} \cong \mathrm{Gal}(\breve{K}_m/\breve{K})$ (cf. [Str08i], Theorem 4.4).

THEOREM 4.2.2: *For all $m \geq 0$, $(M_m^0)_{\mathrm{lf}} = (R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$.*

Proof. The kernel of the composition map $\Gamma \xrightarrow{\mathrm{Nrd}} \mathfrak{o}^{\times} \longrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^{\times}$ is an open subgroup of Γ which acts trivially on $\breve{K}_m$. Thus $\breve{K}_m \subseteq (R_m^{\mathrm{rig}})_{\mathrm{lf}}$. Notice that $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is a subring of R_m^{rig} and is stable under the action of G_0/G_m. To see the stability, let $f \in (R_m^{\mathrm{rig}})_{\mathrm{lf}}$ and V be a finite-dimensional H-subrepresentation of R_m^{rig} containing f for some open subgroup H of Γ. Let $g \in G_0/G_m$. Then the $\breve{K}$-vector space gV is H-stable since the actions of H and G_0/G_m on V commute. Thus gV is a finite-dimensional H-subrepresentation of R_m^{rig} containing gf implying that gf is locally finite. Now it follows from [Koh11] Theorem 1.4(i) and Corollary 4.1.11 that

$$(R_m^{\mathrm{rig}})_{\mathrm{lf}}^{G_0/G_m} = ((R_m^{\mathrm{rig}})^{G_0/G_m})_{\mathrm{lf}} = (R_0^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}.$$

As $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is an integral domain due to [Koh11], Theorem 1.2 (i) and G_0/G_m is finite, $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is a finite field extension of $\breve{K}$ by Lemma 4.2.1. So it is also finite over $\breve{K}_m$. However, Strauch's result that X_m^{rig} is geometrically connected over $\breve{K}_m$ implies that $\breve{K}_m$ is separably closed in R_m^{rig} (cf. [Koh11], Theorem 1.4). Therefore $(R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$. $\blacksquare$

Remark 4.2.3: By Theorem 3.4.5, we have a $\mathfrak{g}$-action on R_m^{rig}. The subspace of $\mathfrak{g}$-invariants $(R_m^{\mathrm{rig}})^{\mathfrak{g}=0}$ forms a subring of R_m^{rig}, and is stable under the action of G_0/G_m because the G_0/G_m-action on R_m^{rig} is continuous and commutes with that of Γ. As mentioned in the proof of Theorem 4.2.2, the kernel of the composition map $\Gamma \xrightarrow{\mathrm{Nrd}} \mathfrak{o}^{\times} \longrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^{\times}$ is an open subgroup of Γ which acts trivially on $\breve{K}_m$. Thus, $\breve{K}_m \subseteq (R_m^{\mathrm{rig}})^{\mathfrak{g}=0}$. Proceeding similarly to the above, we have $((R_m^{\mathrm{rig}})^{\mathfrak{g}=0})^{G_0/G_m} = ((R_m^{\mathrm{rig}})^{G_0/G_m})^{\mathfrak{g}=0} = (R_0^{\mathrm{rig}})^{\mathfrak{g}=0} = \breve{K}$ (cf. Lemma 4.1.4). Then by the same arguments as above, we get $(R_m^{\mathrm{rig}})^{\mathfrak{g}=0} = \breve{K}_m$.

For all integers s, the Γ-equivariant isomorphism

$$M_m^s \cong R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s$$

(cf. proof of Theorem 3.4.6) and the freeness of the R_0^{rig}-module M_0^s give rise to a Γ-equivariant inclusion $M_0^s \subset M_m^s$ of $\breve{K}$-vector spaces. Consequently, we have $(M_0^s)_{\mathrm{lf}} \subseteq (M_m^s)_{\mathrm{lf}}$. Using the above theorem, we see that $(M_m^s)_{\mathrm{lf}}$ is a module over $(R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$, and thus we obtain a natural map

$$\breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} \longrightarrow (M_m^s)_{\mathrm{lf}}$$

of $\breve{K}$-vector spaces. Our objective is to show that this map is an isomorphism of $\breve{K}[\Gamma]$-modules for all s.

LEMMA 4.2.4: *Suppose V and W are two representations of a topological group G over a field F such that one of them, say W, is finite-dimensional. Consider the representation $V \otimes_F W$ with diagonal G-action. Then*

$$(V \otimes_F W)_{\mathrm{lf}} = V_{\mathrm{lf}} \otimes_F W.$$

Proof. We omit the subscript F in $\otimes_F$, as all the tensor products are over F. The inclusion $V_{\mathrm{lf}} \otimes W \subseteq (V \otimes W)_{\mathrm{lf}}$ is clear. Let W^* be the F-linear dual of W equipped with the contragredient G-action, i.e., $(gf)(w) = f(g^{-1}w)$ for all $g \in G, w \in W$ and $f \in W^*$. Choose an F-basis $\{w_1, \ldots, w_d\}$ of W, and let $\{f_1, \ldots, f_d\}$ be the dual basis of W^* (i.e., $f_i(w_j) = \delta_{ij}$). Then the natural evaluation map

$$W \otimes W^* \longrightarrow F \quad (w \otimes f \mapsto f(w))$$

is G-equivariant for the diagonal G-action on the left and for the trivial G-action on the right. Tensoring both sides with V, we get a G-equivariant map $\phi : V \otimes W \otimes W^* \longrightarrow V$ for the diagonal G-action on the left, sending $v \otimes w \otimes f$ to $f(w)v$. Because of its G-equivariance, ϕ maps locally finite vectors to locally finite vectors. Now let $x \in (V \otimes W)_{\mathrm{lf}}$. Then x can be uniquely written as

$$x = \sum_{i=1}^{d} x_i \otimes w_i$$

for some $x_1, \ldots, x_d \in V$. Since W^* is finite-dimensional,

$$x \otimes f_i \in (V \otimes W)_{\mathrm{lf}} \otimes (W^*)_{\mathrm{lf}} \subseteq (V \otimes W \otimes W^*)_{\mathrm{lf}}$$

for all $1 \leq i \leq d$. Hence, $\phi(x \otimes f_i) = x_i \in V_{\mathrm{lf}}$ for all $1 \leq i \leq d$. Therefore, $x \in V_{\mathrm{lf}} \otimes W$. ∎

THEOREM 4.2.5: *For all $s < 0$ and for all $m \geq 0$, $(M_m^s)_{\mathrm{lf}} \cong \breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} = 0$.*

Proof. Recall from Corollary 3.1.5 that we have an isomorphism

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \cong R_m^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}$$

of Γ-representations. As a result, using Lemma 4.2.4 together with Theorem 4.2.2, we obtain locally finite vectors in the global sections of $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s}$,

$$(R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s})_{\mathrm{lf}} \cong \breve{K}_m \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}.$$

Then, since $s < 0$, the $(\Gamma \times (G_0/G_m))$-equivariant inclusion

$$M_m^s \subset R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s}$$

from (3.1.2) gives rise to a $(\Gamma \times (G_0/G_m))$-equivariant inclusion

$$(M_m^s)_{\mathrm{lf}} \subseteq \breve{K}_m \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}$$

of $\breve{K}$-vector spaces. As the action of $SL_h(\mathfrak{o}/\varpi^m\mathfrak{o}) \subset G_0/G_m$ on the right-hand side above is trivial, we get

$$(M_m^s)_{\mathrm{lf}} = (M_m^s)_{\mathrm{lf}}^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} = ((M_m^s)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})})_{\mathrm{lf}},$$

where the latter equality is due to the fact that both group actions on M_m^s commute. Therefore,

$$\begin{aligned}
(M_m^s)_{\mathrm{lf}} &= ((M_m^s)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})})_{\mathrm{lf}} \cong ((R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})})_{\mathrm{lf}} \\
&\cong ((R_m^{\mathrm{rig}})^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} \otimes_{R_0^{\mathrm{rig}}} M_0^s)_{\mathrm{lf}} \\
&\cong ((\breve{K}_m \otimes_{\breve{K}} R_0^{\mathrm{rig}}) \otimes_{R_0^{\mathrm{rig}}} M_0^s)_{\mathrm{lf}} \\
&\cong (\breve{K}_m \otimes_{\breve{K}} M_0^s)_{\mathrm{lf}} = \breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} = 0
\end{aligned}$$

where the second isomorphism holds because M_0^s is free over R_0^{rig} with trivial G_0/G_m-action, and the third isomorphism holds because $(R_m^{\mathrm{rig}})^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})}$ is Galois over R_0^{rig} with the Galois group isomorphic to

$$\frac{G_0/G_m}{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} \cong (\mathfrak{o}/\varpi^m\mathfrak{o})^{\times} \cong \mathrm{Gal}(\breve{K}_m/\breve{K})$$

and $\breve{K}_m \subseteq R_m^{\mathrm{rig}}$. For the second last equality in the above, we use Lemma 4.2.4 again. The final result then follows from Corollary 4.1.11. ∎

PART II : $s > 0$

To compute the locally finite vectors in M_m^s for $s > 0$, we make use of the action of the group

$$G^0 := \{g \in GL_h(K) | \det(g) \in \mathfrak{o}^\times\}$$

on the Lubin–Tate tower $(X_m^{\mathrm{rig}})_{m \in \mathbb{N}_0}$ described in [Str08ii], Section 2.2.2. Given $g \in G^0$ and $m \geq 0$, for every $m' \geq m$ sufficiently large (depending on g), there is a morphism $g_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$ of rigid analytic spaces satisfying the following properties:

(1) For all $g \in G^0$ and for all $n \geq m'' \geq m' \geq m$, we have

$$g_{n,m} = \pi_{m',m} \circ g_{m'',m'} \circ \pi_{n,m''},$$

where recall that $\pi_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$ denotes the covering morphism. In particular, if $g = e$, and if $m = m' = m''$, then we get $e_{n,m} = \pi_{n,m}$ for all $n \geq m$ because $e_{m,m} = \mathrm{id}_{X_m^{\mathrm{rig}}}$ by definition (cf. [Str08ii], Section 2.2.2).

(2) $(gh)_{m'',m} = g_{m',m} \circ h_{m'',m}$ for all $g, h \in G^0$ and for all $m'' \geq m' \geq m$.

(3) Set $\Phi_m := \Phi \circ \pi_{m,0} : X_m^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$. Then $\Phi_{m'} = \Phi_m \circ g_{m',m}$ for all $g \in G^0$, $m' \geq m$.

(4) All $g_{m',m}$ are Γ-equivariant morphisms.

(5) For $g \in GL_h(\mathfrak{o})$ and $m \geq 0$, $g_{m,m}$ is defined. The gives an action of $GL_h(\mathfrak{o})$ on X_m^{rig} which factors through $GL_h(\mathfrak{o}/\varpi^m\mathfrak{o}) = G_0/G_m$. The induced G_0/G_m-action coincides with the G_0/G_m-action introduced in Section 2.2.

Let $D_m := \pi_{m,0}^{-1}(D)$ where D is the Gross–Hopkins fundamental domain D in X_0^{rig}. The admissible open D_m is a Γ-stable affinoid subdomain because $\pi_{m,0}$ is a finite, Γ-equivariant morphism, and D is Γ-stable. For every $g \in G^0$ and $m \geq 0$, we define a g-translate of D_m as $gD_m := g_{m',m}(D_{m'})$ by choosing $m' \geq m$ large enough. Note that this definition is independent of the choice of m', since by property (1), for $m'' \geq m' \geq m$,

$$
\begin{aligned}
g_{m'',m}(D_{m''}) &= g_{m',m}(\pi_{m'',m'}(D_{m''})) \\
&= g_{m',m}(\pi_{m'',m'}(\pi_{m'',0}^{-1}(D))) \\
&= g_{m',m}(\pi_{m'',m'}(\pi_{m'',m'}^{-1}(\pi_{m',0}^{-1}(D)))) = g_{m',m}(D_{m'}),
\end{aligned}
$$

using that $\pi_{m'',m'}$ is surjective.

PROPOSITION 4.2.6: *The set $\{gD_m\}_{g \in G^0}$ forms an admissible affinoid covering of $\Phi_m^{-1}(\Phi(D))$ consisting of Γ-stable affinoid subdomains.*

Proof. This is a part of the cellular decomposition of the Lubin–Tate tower in [FGL08], Proposition I.7.1 relying on [GH94], Corollary 23.26. The Γ-stability of gD_m follows from (iv) and that of $D_{m'}$. ∎

LEMMA 4.2.7: *For all $g \in G^0$ and $m' \geq m$, the maps*

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$$

of affinoid $\breve{K}$-algebras induced by the morphisms $D_{m'} \xrightarrow{g_{m',m}} gD_m \xrightarrow{\Phi_m} \Phi(D)$ are injective.

Proof. By property (3), the composition

$$\Phi_m \circ g_{m',m} = \Phi_{m'} = \Phi \circ \pi_{m',0}$$

is flat because Φ and $\pi_{m',0}$ are flat. Hence the composition map

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$$

of affinoid $\breve{K}$-algebras is flat. Since

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \cong \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$$

is an integral domain (cf. [BGR84], (6.1.5), Proposition 2), we have that the map $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ is injective.

To show that the other map $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ is injective, choose $m'' \geq m'$ large enough so that $g_{m'',m'}^{-1} : X_{m''}^{\mathrm{rig}} \longrightarrow X_{m'}^{\mathrm{rig}}$ is defined. Using properties (1) and (2), we have

$$g_{m'',m'}^{-1}(gD_{m''}) = g_{m'',m'}^{-1}(g_{n,m''}(D_n)) = e_{n,m'}(D_n) = \pi_{n,m'}(D_n) = D_{m'}$$

and thus

$$gD_m = g_{m',m}(D_{m'}) = g_{m',m}(g_{m'',m'}^{-1}(gD_{m''})) = e_{m'',m}(gD_{m''}) = \pi_{m'',m}(gD_{m''}).$$

In other words,

$$(g_{m',m} \circ g_{m'',m'}^{-1})|_{gD_{m''}} = \pi_{m'',m}.$$

Hence the induced composition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'}) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}(gD_{m''})$ of the maps of affinoid $\breve{K}$-algebras is flat. Now it is not clear if the algebra $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ is an integral domain. However, we can decompose gD_m

into its finitely many disjoint connected components $gD_m = \bigsqcup_{i=1}^{r} U_i$ so that each $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ is an integral domain (cf. discussion after [BGR84], (9.1.4), Proposition 8 as well as [Con99], Lemma 2.1.4). This decomposition also gives a decomposition $gD_{m''} = \bigsqcup_{i=1}^{r}(\pi_{m'',m}|_{gD_{m''}})^{-1}(U_i)$ of $gD_{m''}$ into disjoint admissible open subsets. By the same argument as in the first paragraph, each map $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}((\pi_{m'',m}|_{gD_{m''}})^{-1}(U_i))$ is injective. As a consequence, the composition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}(gD_{m''})$ is injective since it is the finite direct product of all these maps. ∎

Remark 4.2.8: The affinoid subdomain D_m, by definition, is the same as the fibre product $X_m^{\mathrm{rig}} \times_{X_0^{\mathrm{rig}}} D$ for the maps

$$\pi_{m,0} : X_m^{\mathrm{rig}} \longrightarrow X_0^{\mathrm{rig}} \quad \text{and} \quad D \hookrightarrow X_0^{\mathrm{rig}}.$$

Thus, we have an isomorphism

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m) \cong R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$$

because $R_m^{\mathrm{rig}}|R_0^{\mathrm{rig}}$ is finite. The Galois group $G_0/G_m = \mathrm{Gal}(R_m^{\mathrm{rig}}|R_0^{\mathrm{rig}})$ acts on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)$ via

$$\sum_{i=1}^{r} f_i \otimes f_i' \mapsto \sum_{i=1}^{r} g(f_i) \otimes f_i'$$

for $g \in G_0/G_m$, which gives an action on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)$ by $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-linear automorphisms. Hence the extension $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)|\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is finite Galois with the Galois group G_0/G_m. Consequently, for all $m \geq 0$, the extension of coordinate rings $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)|\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D))$ induced by the map Φ_m is finite Galois with the same Galois group.

Remark 4.2.9: As both R_m^{rig} and $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ are $\mathfrak{g}$-modules (cf. Proposition 3.2.6, Theorem 3.4.5), we have a $\mathfrak{g}$-action on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m) \cong R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Namely, if $\mathfrak{x} \in \mathfrak{g}$ then on simple tensors,

$$\mathfrak{x}(f \otimes f') = \mathfrak{x}(f) \otimes f' + f \otimes \mathfrak{x}(f').$$

The $\mathfrak{g}$-action on $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ restricts to the subalgebra $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$, because by Remark 4.2.8, $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ is a Γ-stable submodule of the finitely generated $\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D))$-module $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ and hence is closed in $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ by [BGR84], (3.7.3), Proposition 1. Denoting by Ad_γ the adjoint automorphism of $\mathfrak{g}$ corresponding to $\gamma \in \Gamma$, we remark that the actions of Γ and $\mathfrak{g}$

on $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ are compatible in the sense that $\gamma(\mathfrak{r}(f)) = \mathrm{Ad}_\gamma(\mathfrak{r})(\gamma(f))$, since the Lie algebra action comes from the action of the distribution algebra $D(\Gamma)$ on R_m^{rig} and $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Using the isomorphism

$$(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m) \cong \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \otimes_{R_m^{\mathrm{rig}}} M_m^s$$

and Theorem 3.4.6, one obtains that $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m)$ carries compatible actions of Γ and $\mathfrak{g}$ for all $s \in \mathbb{Z}$.

PROPOSITION 4.2.10: *For all $g \in G^0$ and for all $m \geq 0$,*

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0}.$$

All these $\check{K}$-vector spaces are finite-dimensional.

Proof. Let $g \in G^0, m \geq 0$ be arbitrary, and $m' \geq m$ so that $g_{m',m}$ is defined. As seen in the proof of Lemma 4.2.7, the composition

$$\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D)) \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \hookrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$$

is induced by $\Phi_{m'}$. The Γ-equivariance of $g_{m',m}$ and of Φ_m yields the inclusions $\mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D))_{\mathrm{lf}} \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} \hookrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ of $\check{K}$-algebras. The Galois action on $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ commutes with the Γ-action. As a result, $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ is stable under the Galois action and

$$(\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}})^{G_0/G_{m'}} = (\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})^{G_0/G_{m'}})_{\mathrm{lf}}$$
$$= \mathcal{O}_{\mathbb{P}_{\check{K}}^{h-1}}(\Phi(D))_{\mathrm{lf}} = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)_{\mathrm{lf}} = \check{K}$$

(cf. Corollary 4.1.8). Since $G_0/G_{m'}$ is finite, $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ is integral over $\check{K}$, and thus $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}}$ is integral over $\check{K}$. As before, we write $gD_m = \bigsqcup_{i=1}^r U_i$ where U_is are the connected components of gD_m. Let Γ_i be the stabilizer of U_i in Γ; then each Γ_i has a finite index in Γ. Let $\Gamma_o \subseteq \Gamma$ be an open subgroup which is a uniform pro-p group. Then for every i, the intersection $\Gamma_i \cap \Gamma_o$ has a finite index in Γ_o, and thus is open in Γ_o by [DDMS03], Theorem 1.17. As a result, $\Gamma_i \cap \Gamma_o$ is open in Γ, and

$$\Gamma_i = \bigcup_{\bar{\gamma} \in \Gamma_i/\Gamma_i \cap \Gamma_o} \gamma(\Gamma_i \cap \Gamma_o)$$

implies that Γ_i is open in Γ for all i. Hence their intersection

$$\Gamma' := \bigcap_{i=1}^r \Gamma_i$$

is again an open subgroup of Γ.

Now the decomposition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \cong \prod_{i=1}^{r} \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ of $\breve{K}$-algebras is Γ'-equivariant for the componentwise Γ'-action on the right. Thus the compactness of Γ gives the decomposition

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\Gamma'-\mathrm{lf}} \cong \prod_{i=1}^{r} \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$$

of locally finite vectors. Denote by K_i the integral closure of $\breve{K}$ in the integral domain $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ for each i. It then follows that K_i is a field extension of $\breve{K}$. Since every projection $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ is a surjective $\breve{K}$-algebra homomorphism, the integrality of $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}}$ over $\breve{K}$ implies that $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ is integral over $\breve{K}$ for all i. Therefore, $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}} \subseteq K_i$ for all i. On the other hand, for each i, K_i is Γ'-stable as Γ' acts $\breve{K}$-linearly on $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$. Now for any classical point $x \in U_i$, the composition map

$$K_i \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i) \twoheadrightarrow \kappa(x)$$

is injective, and because $\kappa(x)|\breve{K}$ is finite, $K_i|\breve{K}$ is a finite extension. This gives the other inclusion $K_i \subseteq \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ for all i. Thus we have

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \prod_{i=1}^{r} K_i$$

with each K_i a finite field extension of $\breve{K}$.

We now claim that

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0} = \prod_{i=1}^{r} K_i.$$

Note that $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ is $\mathfrak{g}$-stable for all i because the projection map

$$\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \twoheadrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$$

of affinoid $\breve{K}$-algebras is surjective, continuous and Γ'-equivariant. Then all arguments in the last two paragraphs carry over to these cases since

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))^{\mathfrak{g}=0} = \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))^{\mathfrak{n}=0} = \breve{K}$$

(cf. Lemma 4.1.4). The only thing that remains to be shown is $K_i \subseteq \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)^{\mathfrak{g}=0}$ for all i: Write $K_i = \breve{K}[\alpha_i]$. Then the set $\{\gamma(\alpha_i)\}_{\gamma \in \Gamma'}$ is finite as Γ' takes α_i to its conjugates. Therefore, the stabilizer Γ'_i of α_i in Γ' has a finite index in Γ', and thus we obtain the open subgroup Γ'_i of Γ which acts trivially on K_i. ∎

The embedding $\Gamma \hookrightarrow GL_h(K_h)$ in (3.1.8) extends to an embedding $B_h^\times \hookrightarrow GL_h(K_h)$ of locally K-analytic groups via the same map. This yields an action of $B_h^\times$ on $\mathbb{P}_{\breve{K}}^{h-1}$. The Γ-action on X_m^{rig} is extended to the full group $B_h^\times$ by letting $b \in B_h^\times$ act by the action of $(1, b, \tilde{\sigma}^{-\mathrm{val}(\mathrm{Nrd}(b))}) \in GL_h(K) \times B_h^\times \times W_K$ given on page 20 of [Car90]. Here $\tilde{\sigma}$ denotes a lift of the Frobenius in the Weil group W_K and val is the normalized valuation of K. The maps Φ_m are equivariant for the extended $B_h^\times$-action for all m.

LEMMA 4.2.11: *The set $\{\Pi^i \Phi(D)\}_{0 \leq i \leq h-1}$ forms an admissible covering of $\mathbb{P}_{\breve{K}}^{h-1}$. Thus, X_m^{rig} has an admissible covering $\{\Pi^i \Phi_m^{-1}(\Phi(D))\}_{0 \leq i \leq h-1}$ for all $m \geq 0$.*

Proof. This is proved as a part of [GH94], Corollary 23.21. ∎

For $0 \leq i \leq h - 1$, $s \geq 0$, $m \geq 0$, define

$$N_m^s(i) := (\mathcal{M}_m^s)^{\mathrm{rig}}(\Pi^i \Phi_m^{-1}(\Phi(D)))$$

and

$$A_m(i) := N_m^0(i) = \mathcal{O}_{X_m^{\mathrm{rig}}}(\Pi^i \Phi_m^{-1}(\Phi(D))).$$

Note that each $\Pi^i \Phi_m^{-1}(\Phi(D))$ is Γ-stable because the conjugation by

$$\Pi^{-i} \quad (\gamma \mapsto \Pi^{-i} \gamma \Pi^i)$$

is an automorphism of Γ. Therefore, all $A_m(i)$ and $N_m^s(i)$ are Γ-representations. Moreover, they are also $\mathfrak{g}$-modules as explained below.

Because of Proposition 4.2.6, we have the exact diagram

$$A_m(0) \xrightarrow{r} \prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \underset{r_2}{\overset{r_1}{\rightrightarrows}} \prod_{g,g' \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m)$$

with maps given by $r(f) = (f|_{gD_m})_{g \in G^0}$, $r_1((f_g)_{g \in G^0}) = (f_g|_{gD_m \cap g'D_m})_{g,g' \in G^0}$, and $r_2((f_g)_{g \in G^0}) = (f_{g'}|_{gD_m \cap g'D_m})_{g,g' \in G^0}$. The continuity of the restriction maps $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m)$ between affinoid $\breve{K}$-algebras implies that the maps r_1 and r_2 are continuous for the product topology on their source and target. Remark 4.2.9 allows us to view $\prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ as a $\mathfrak{g}$-module with the componentwise $\mathfrak{g}$-action. Now, $A_m(0)$ can be identified with the kernel of the continuous map

$$r_1 - r_2 : \prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \prod_{g,g' \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m)$$

$$(f_g)_{g \in G^0} \longmapsto r_1((f_g)_{g \in G^0}) - r_2((f_g)_{g \in G^0}).$$

Hence, $A_m(0)$ is a closed Γ-stable subspace of $\prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ as r is Γ-equivariant. Consequently, $A_m(0)$ is stable under the induced $\mathfrak{g}$-action.

Observe that the isomorphism

$$(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m) \cong \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m) \otimes_{R_m^{\mathrm{rig}}} M_m^s$$

yields a $\mathfrak{g}$-action on $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m)$ (cf. Theorem 3.4.6 and Remark 4.2.9). The restriction maps $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m) \longrightarrow (\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m)$ are continuous for the topology of finitely generated Banach modules. Then by the similar argument as in the last paragraph, $N_m^s(0)$ carries a $\mathfrak{g}$-module structure. The $\mathfrak{g}$-action and the Γ-action on $A_m(0)$ and on $N_m^s(0)$ are compatible with each other (cf. Remark 4.2.9). The action of Π^i on $\Phi_m^{-1}(\Phi(D))$ induces an isomorphism $\Pi^i : N_m^s(i) \xrightarrow{\sim} N_m^s(0)$ of the sections of $(\mathcal{M}_m^s)^{\mathrm{rig}}$. The $\mathfrak{g}$-action on $N_m^s(i)$ is then given by $\mathfrak{r}(n) = \Pi^{-i}(\mathrm{Ad}_{\Pi^i}(\mathfrak{r})(\Pi^i(n)))$ for $\mathfrak{r} \in \mathfrak{g}, n \in N_m^s(i)$ and $1 \leq i \leq h-1$.

Now since M_m^s is generated over R_m^{rig} by V_s (cf. (3.1.2), Proposition 3.1.3), $N_m^s(i)$ is generated by V_s as an $A_m(i)$-module for all $0 \leq i \leq h-1$. Let $A_m(i)^{\mathfrak{g}=0}V_s$ and $A_m(i)^{\mathfrak{g}=0}\varphi_0^s$ denote the $A_m(i)^{\mathfrak{g}=0}$-submodules of $N_m^s(i)$ generated by V_s and φ_0^s respectively.

PROPOSITION 4.2.12: *For all $0 \leq i \leq h-1$, $s \geq 0$, $m \geq 0$, we have*

$$N_m^s(i)_{\mathrm{lf}} \subseteq A_m(i)^{\mathfrak{g}=0}V_s \quad and \quad (N_m^s(i)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq A_m(i)^{\mathfrak{g}=0}\varphi_0^s.$$

Proof. We first show that $N_m^s(0)_{\mathrm{lf}} \subseteq A_m(0)^{\mathfrak{g}=0}V_s$. Noticing

$$\varphi_0 \in \mathcal{O}_{\mathbb{P}_{\bar{K}}^{h-1}}(\Phi(D))^\times \hookrightarrow A_m(0)^\times$$

implies that φ_0^s alone generates $N_m^s(0)$ as a free $A_m(0)$-module of rank one. Now let $W \subseteq N_m^s(0)$ be a finite-dimensional Γ-stable subspace. As an $\mathfrak{sl}_h(K_h)$-representation, W decomposes as a direct sum of simple $\mathfrak{sl}_h(K_h)$-modules by Weyl's complete reducibility theorem. From highest weight theory, we know that each simple module in the decomposition is generated by an element annihilated by the subalgebra $\mathfrak{n}$ of strictly upper triangular matrices. Now

$$W^{\mathfrak{n}=0} \subseteq N_m^s(0)^{\mathfrak{n}=0} = (A_m(0)\varphi_0^s)^{\mathfrak{n}=0} = A_m(0)^{\mathfrak{n}=0}\varphi_0^s$$

because $\mathfrak{n}\varphi_0^s = 0$ (cf. Lemma 4.1.2). Let $f \in A_m(0)^{\mathfrak{n}=0}$; then

$$f|_{gD_m} \in \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0} \quad \text{for all } g \in G^0$$

by Proposition 4.2.10. The $\mathfrak{g}$-linear injection $A_m(0) \hookrightarrow \prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ of $\breve{K}$-algebras induces an equality

$$A_m(0)^{\mathfrak{g}=0} = A_m(0) \cap \prod_{g \in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0}.$$

Therefore, $f \in A_m(0)^{\mathfrak{g}=0}$, and hence

$$A_m(0)^{\mathfrak{g}=0} = A_m(0)^{\mathfrak{n}=0}.$$

This gives us $(N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq A_m(0)^{\mathfrak{g}=0}\varphi_0^s$. As explained earlier, $N_m^s(0)_{\mathrm{lf}}$ is generated as an $\mathfrak{sl}_h(K_h)$-module by its $\mathfrak{n}$-invariants $(N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}=0}$. Thus

$$N_m^s(0)_{\mathrm{lf}} = \mathfrak{sl}_h(K_h).(N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq \mathfrak{sl}_h(K_h).(A_m(0)^{\mathfrak{g}=0}\varphi_0^s) = A_m(0)^{\mathfrak{g}=0}V_s.$$

The last equality follows from the fact that φ_0^s is a highest weight vector in V_s (cf. Remark 4.1.9).

If the Γ-action on $\Phi_m^{-1}(\Phi(D))$ is changed via the automorphism $\gamma \mapsto \Pi^{-i}\gamma\Pi^i$, then the map $\Pi^i : \Phi_m^{-1}(\Phi(D)) \xrightarrow{\sim} \Pi^i\Phi_m^{-1}(\Phi(D))$ is a Γ-equivariant isomorphism. We note that the new Γ-action does not change the locally finite vectors in $N_m^s(0)$. Writing $\varphi_h := \varphi_0$ formally, we have an induced isomorphism

$$\Pi^i : (N_m^s(i))_{\mathrm{lf}} \xrightarrow{\sim} (N_m^s(0))_{\mathrm{lf}}$$

mapping φ_0^s to φ_{h-i}^s, and the $\mathfrak{n}$-invariants onto the $\mathfrak{n}_i := \mathrm{Ad}_{\Pi^i}(\mathfrak{n})$-invariants for all $0 \leq i \leq h-1$. Therefore,

$$\begin{aligned}
(N_m^s(i)_{\mathrm{lf}})^{\mathfrak{n}=0} = (\Pi^i)^{-1}((N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}_i=0}) &\subseteq (\Pi^i)^{-1}((A_m(0)^{\mathfrak{g}=0}V_s)^{\mathfrak{n}_i=0}) \\
&= (\Pi^i)^{-1}(A_m(0)^{\mathfrak{g}=0}V_s^{\mathfrak{n}_i=0}) \\
&= (\Pi^i)^{-1}(A_m(0)^{\mathfrak{g}=0}\varphi_{h-i}^s) \\
&= A_m(i)^{\mathfrak{g}=0}\varphi_0^s.
\end{aligned}$$

As before, this also implies $N_m^s(i)_{\mathrm{lf}} \subseteq A_m(i)^{\mathfrak{g}=0}V_s$ for all $0 \leq i \leq h-1$. ∎

THEOREM 4.2.13: *For all $s \geq 0$, $m \geq 0$, we have an isomorphism*

$$(M_m^s)_{\mathrm{lf}} \cong \breve{K}_m \otimes_{\breve{K}} V_s \cong \breve{K}_m \otimes_{\breve{K}} \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1})$$

$$\cong \breve{K}_m \otimes_{\breve{K}} \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$$

of Γ-representations for the diagonal Γ-action on the tensor products. The representation $(M_m^s)_{\mathrm{lf}}$ is a finite-dimensional semi-simple locally algebraic representation of Γ.

Proof. As before, $(M_m^s)_{\text{lf}}$ is generated as an $\mathfrak{sl}_h(K_h)$-module by its $\mathfrak{n}$-invariants. Let $x \in ((M_m^s)_{\text{lf}})^{\mathfrak{n}=0}$. Then, using the preceding proposition,

$$x|_{\Pi^i \Phi_m^{-1}(\Phi(D))} \in (N_m^s(i)_{\text{lf}})^{\mathfrak{n}=0} \subseteq A_m(i)^{\mathfrak{g}=0} \varphi_0^s \quad \text{for all } 0 \leq i \leq h-1.$$

Let $Y_i := \Pi^i \Phi_m^{-1}(\Phi(D))$, and write $x|_{Y_i} = f_i \varphi_0^s$ with $f_i \in A_m(i)^{\mathfrak{g}=0}$.

For all $0 \leq i, j \leq h-1$, we have

$$(f_i|_{Y_i \cap Y_j} - f_j|_{Y_i \cap Y_j})\varphi_0^s = x|_{Y_i \cap Y_j} - x|_{Y_i \cap Y_j} = 0.$$

Now M_m^s is free over the integral domain R_m^{rig}, and contains $\varphi_0^s \neq 0$. Hence the map $(r \mapsto r\varphi_0^s)$ from R_m^{rig} to M_m^s is injective and remains injective after any flat base change. In particular, the map

$$(r \mapsto r\varphi_0^s) : \mathcal{O}_{X_m^{\text{rig}}}(Y_i \cap Y_j) \longrightarrow (\mathcal{M}_m^s)^{\text{rig}}(Y_i \cap Y_j)$$

is injective, and thus $f_i|_{Y_i \cap Y_j} = f_j|_{Y_i \cap Y_j}$ for all $0 \leq i, j \leq h-1$. Therefore, by the sheaf axioms, the functions $(f_i)_i$ glue together to a global section $f \in R_m^{\text{rig}}$ and $x = f\varphi_0^s$. Since $f|_{Y_i} = f_i \in A_m(i)^{\mathfrak{g}=0}$ for all i, and the map $R_m^{\text{rig}} \hookrightarrow \prod_{i=0}^{h-1} A_m(i)$ is $\mathfrak{g}$-equivariant, $f \in (R_m^{\text{rig}})^{\mathfrak{g}=0} = \breve{K}_m$ (cf. Remark 4.2.3). Hence $x \in \breve{K}_m \varphi_0^s$. As a result, $(M_m^s)_{\text{lf}} = \mathfrak{sl}_h(K_h).(\breve{K}_m \varphi_0^s) = \breve{K}_m V_s$. The other inclusion $\breve{K}_m V_s \subseteq (M_m^s)_{\text{lf}}$ is easy to see as $(M_m^s)_{\text{lf}}$ is a module over $(R_m^{\text{rig}})_{\text{lf}} = \breve{K}_m$, and $V_s = (M_0^s)_{\text{lf}} \subseteq (M_m^s)_{\text{lf}}$.

Now to justify the isomorphism $\breve{K}_m \otimes_{\breve{K}} V_s \cong \breve{K}_m V_s$, it is enough to show that the natural map

$$\breve{K}_m \otimes_{\breve{K}} V_s \longrightarrow \breve{K}_m V_s$$

$$\sum_{0 \leq |\alpha| \leq s} c_\alpha (1 \otimes w^\alpha \varphi_0^s) \longmapsto \sum_{0 \leq |\alpha| \leq s} c_\alpha w^\alpha \varphi_0^s$$

is injective. Here the set $\{1 \otimes w^\alpha \varphi_0^s\}_{0 \leq |\alpha| \leq s}$ forms a $\breve{K}_m$-basis of $\breve{K}_m \otimes_{\breve{K}} V_s$. By Lemma 4.1.2, we have

$$\mathfrak{x}_{00}(w^\alpha \varphi_0^s) = (s - |\alpha|)w^\alpha \varphi_0^s \quad \text{and} \quad \mathfrak{x}_{ii}(w^\alpha \varphi_0^s) = \alpha_i w^\alpha \varphi_0^s$$

for all $1 \leq i \leq h-1$. Since $\mathfrak{g}$ annihilates $\breve{K}_m$, if

$$\sum_{0 \leq |\alpha| \leq s} c_\alpha w^\alpha \varphi_0^s = 0,$$

one can use the above actions of the diagonal matrices iteratively to deduce that each summand $c_\alpha w^\alpha \varphi_0^s$ is zero, and therefore $c_\alpha = 0$ for all $0 \leq |\alpha| \leq s$.

Unlike $(M_0^s)_{\mathrm{lf}} = V_s$, the space of locally finite vectors $(M_m^s)_{\mathrm{lf}} \cong \breve{K}_m \otimes_{\breve{K}} V_s$ at level $m > 0$ is not an irreducible Γ-representation as it properly contains the representation V_s. However, it is semi-simple and this can be seen as follows. The action of Γ on $\breve{K}_m$ factors through a finite group. As a result, $\breve{K}_m$ decomposes into a direct sum $\breve{K}_m \cong \bigoplus_{i=1}^n W_i$ of irreducible representations. This gives us a decomposition

$$(4.2.14) \qquad \breve{K}_m \otimes_{\breve{K}} V_s \cong \bigoplus_{i=1}^n (W_i \otimes_{\breve{K}} V_s).$$

Now we note that

$$V_s \cong \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$$

is an irreducible algebraic representation of $\Gamma \cong \mathfrak{o}_{B_h}^{\times}$ (cf. Theorem 4.1.6 and [Koh14], Remark 4.4), and $\breve{K}_m$ is a smooth representation of Γ by Remark 4.2.3. Thus every direct summand in (4.2.14) is a tensor product of a smooth irreducible representation and an irreducible algebraic representation of Γ. Such a product is an irreducible locally algebraic representation by [ST01], Appendix by Dipendra Prasad, Theorem 1. As a consequence, $(M_m^s)_{\mathrm{lf}}$ is a semi-simple locally algebraic representation of Γ and exhausts all locally algebraic vectors in M_m^s as every locally algebraic vector is locally finite by definition (cf. [Eme17], paragraph after Definition 4.2.1). ∎

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