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GELFAND–TSETLIN THEORY FOR RATIONAL GALOIS ALGEBRAS

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ABSTRACT

In the present paper we study Gelfand–Tsetlin modules defined in terms of BGG differential operators. The structure of these modules is described with the aid of the Postnikov–Stanley polynomials introduced in [PS09]. These polynomials are used to identify the action of the Gelfand–Tsetlin subalgebra on the BGG operators. We also provide explicit bases of the corresponding Gelfand–Tsetlin modules and prove a simplicity criterion for these modules. The results hold for modules defined over standard Galois orders of type A—a large class of rings that include the universal enveloping algebra of $\mathfrak{gl}(n)$ and the finite W-algebras of type A.

1. Introduction

The category of Gelfand–Tsetlin modules of the general linear Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ is an important category of modules that plays a prominent role in many areas of mathematics and theoretical physics. By definition, a Gelfand–Tsetlin module of $\mathfrak{gl}(n)$ is one that has a generalized eigenspace decomposition over a certain maximal commutative subalgebra (Gelfand–Tsetlin subalgebra) Γ of the universal enveloping algebra of $\mathfrak{gl}(n)$. This algebraic definition has a nice combinatorial flavor. The concept of a Gelfand–Tsetlin module generalizes the classical realization of the simple finite-dimensional representations of $\mathfrak{gl}(n)$ via the so-called Gelfand–Tsetlin tableaux introduced in [GT50]. The explicit nature of the Gelfand–Tsetlin formulas inevitably raises the question of what infinite-dimensional modules admit tableaux bases—a question that led to the systematic study of the theory of Gelfand–Tsetlin modules. This theory has attracted considerable attention in the last 30 years of the 20th century and has been studied in [DOF91, DFO94, Maz98, Maz01, Mol99, Zhe73], among others. Gelfand–Tsetlin bases and modules are also related to Gelfand–Tsetlin integrable systems that were first introduced for the unitary Lie algebra $\mathfrak{u}(n)$ by Guillemin and Sternberg in [GS83], and later for the general linear Lie algebra $\mathfrak{gl}(n)$ by Kostant and Wallach in [KW06a] and [KW06b].

Recently, the study of Gelfand–Tsetlin modules took a new direction after the theory of singular Gelfand–Tsetlin modules was initiated in [FGR16]. Singular Gelfand–Tsetlin modules are roughly those that have basis of tableaux whose entries may be zeros of the denominators in the Gelfand–Tsetlin formulas. For the last three years remarkable progress has been made towards the study of singular Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$. Important results in this direction were obtained in [FGR15, FGR16, FGR17, Zad17, Vis18, Vis17, RZ18]. In particular, explicit constructions of a Gelfand–Tsetlin module with a fixed singular Gelfand–Tsetlin character were obtained with algebro-combinatorial methods in [RZ18] and with geometric methods in [Vis17]. One notable property of these general constructions is their relations with Schubert calculus and reflection groups. As explained below, this relation is brought to a higher level in the present paper and new connections with Schubert polynomials and generalized Littlewood–Richardson coefficients are established. We hope that these new connections, combined with combinatorial results on skew Schubert polynomials, will allow us to better understand the structure of simple objects in the category of Gelfand–Tsetlin modules (a classification of simple Gelfand–Tsetlin modules was recently announced in [KTW+19, Web19]).

The study of Gelfand–Tsetlin modules is not limited to the cases of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$. Gelfand–Tsetlin subalgebras are part of a uniform algebraic theory, the theory of Galois orders. Galois orders are special types of rings that were introduced in [FO10] in an attempt to unify the representation theories of generalized Weyl algebras and the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{C})$. In addition to the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{C})$ examples of Galois orders include the n-th Weyl algebra, the quantum plane, the Witten–Woronowicz algebra, the q-deformed Heisenberg algebra, and finite W-algebras of type A (for details and more examples see [Har]).

The representation theory of Galois orders was initiated in [FO14]. In particular, the following finiteness theorem for Gelfand–Tsetlin modules of a Galois order U over an integral domain Γ was proven: given a maximal ideal $\mathfrak{m}$ of Γ there exist at least one but only finitely many non-isomorphic simple Gelfand–Tsetlin modules M such that $\mathfrak{m}$ annihilates some element of M. This theorem generalizes the finiteness theorem for $\mathfrak{gl}(n,\mathbb{C})$ obtained in [Ovs02]. Other important results of the Gelfand–Tsetlin theory of $\mathfrak{gl}(n,\mathbb{C})$ were extended to certain types of Galois orders in [EMV18, Har, Maz99]. One such important result is the

construction of a Gelfand–Tsetlin module with any fixed Gelfand–Tsetlin character over an orthogonal Gelfand–Tsetlin algebra obtained recently in [EMV18]. Another notable contribution is the new framework of rational and co-rational Galois orders established in [Har]. Examples of co-rational Galois orders are the universal enveloping algebra of $\mathfrak{gl}(n)$, restricted Yangians of $\mathfrak{gl}(n)$, orthogonal Gelfand–Tsetlin algebras, finite W-algebras of type A, among others.

The first goal of the present paper is to establish a closer connection of the singular Gelfand–Tsetlin theory with the theory of Schubert polynomials and reflection groups. We study a natural class of Γ-modules that consists of differential operators related to the polynomials introduced in [BGG73]. These Γ-modules are denoted by $\mathcal{D}(\Omega, v)$ parameterized by a base of roots Ω and an element v in the vector space V. The BGG differential operators have numerous applications in the cohomology theory of flag varieties. In the present paper, we use a particular aspect of these applications—the Postnikov–Stanley operators. Postnikov–Stanley polynomials were originally defined in [PS09] in order to express degrees of Schubert varieties in the generalized complex flag manifold G/B. The polynomials are given by weighted sums over saturated chains in the Bruhat order and have intimate relations with Schubert polynomials, harmonic polynomials, Demazure characters, and generalized Littlewood–Richardson coefficients. One of our main theorems can be written in the following non-technical terms.

THEOREM A: *The space of BGG differential operators $\mathcal{D}(\Omega, v)$ is a Γ-submodule of Γ^* and the action of Γ on $\mathcal{D}(\Omega, v)$ is given explicitly in terms of Postnikov–Stanley operators.*

Using the explicit action of Γ we prove the following useful result.

COROLLARY B: *Let $v \in V$ be standard and let $\gamma \in \Gamma$. Then the Jordan form of the endomorphism of $\mathcal{D}(\Omega, v)$ given by the action of γ consists of Jordan blocks of size at most $\ell(\omega_0^v) + 1$ and eigenvalue $\gamma(v)$. Furthermore, there is at most one block of this maximal size, and for a generic element γ of Γ there is exactly one such block.*

It turns out that to each $v \in V$ and each co-rational Galois order U we can associate a module spanned by certain BGG operators, which we denote by $V(\Omega, T(v))$. Since the action of Γ on BGG operators is locally finite this is a Gelfand–Tsetlin module. We summarize our main results regarding this module in the following (minuscule elements are defined prior to Proposition 7.2).

THEOREM C: *If U is a co-rational Galois order then the module of BGG differential operators $V(\Omega, T(v))$ is a Gelfand–Tsetlin module over U. Furthermore, under mild conditions on v, an explicit basis of BGG operators for $V(\Omega, T(v))$ can be provided. Finally, if U is generated by minuscule elements, then the matrix coefficients coming from the U-action of the generators on this basis are rational functions expressed in terms of Postnikov–Stanley operators.*

The detailed statements that are included in Theorem C are Theorem 6.4, Proposition 7.1 and Proposition 7.2. All the examples given in [Har] are generated by minuscule elements, so Theorem C applies to a large class of algebras. The explicitness of the bases and the action is in the spirit of Gelfand and Tsetlin's original paper and will be useful when Gelfand–Tsetlin character formulas are studied. In particular, we use this explicitness in our very recent work [FGRZ20], where the Gelfand–Tsetlin support of simple Gelfand–Tsetlin modules and Verma modules of $\mathfrak{gl}(n)$ are described.

One further application is our last result, Corollary 7.3, which is a sufficient condition for our U-modules to be simple in certain special cases. This simplicity criterion generalizes the criterion for orthogonal Gelfand–Tsetlin algebras obtained in [EMV18]. It is worth noting that, as an immediate corollary, our result provides new examples of simple modules of any finite W-algebra of type A.

The organization of the paper is as follows. Preliminary results on reflection groups, BGG differential operators, and Postnikov–Stanley differential operators are collected in Section 2. Definitions and properties of Galois orders and Gelfand–Tsetlin modules are included in Section 3. In Section 4 we discuss generalities on rational Galois orders. The Γ-module of BGG operators is defined in Section 5, where we study its structure with the aid of Postnikov–Stanley operators. In this section we also give an upper bound for the size of a Jordan block of any γ of Γ considered as an endomorphism of the Γ-module of BGG differential operators. The U-action on the U-module of (a larger space of) BGG differential operators is studied in Section 6. In Section 7 we provide a basis of this U-module, prove that it is a Gelfand–Tsetlin module, and provide a sufficient condition for its simplicity.

We finish the introduction with a few notational conventions, which will be used throughout the paper. Unless otherwise stated, the ground field will be $\mathbb{C}$. By $\mathbb{N}$ we denote the set of positive integer numbers. A reflection group will always be a finite group isomorphic to a subgroup of $\mathsf{O}(n, \mathbb{R})$ for some $n \in \mathbb{N}$

and generated by reflections. Given a ring R and a monoid $\mathcal{M}$ acting on R by ring morphisms, by $R\#\mathcal{M}$ we denote the smash product of R and $\mathcal{M}$, i.e., the free R-module with basis $\mathcal{M}$ and product given by

$$r_1 m_1 \cdot r_2 m_2 = r_1 m_1(r_2) m_1 m_2$$

for any $r_1, r_2 \in R$ and any $m_1, m_2 \in \mathcal{M}$.

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2. Preliminaries on Schubert calculus

We recall some basic facts and fix notation on root systems and reflection groups. Our definition of root system is slightly different from the classical one, but is easily seen to be equivalent.

2.1. Root systems and reflection groups. Let V be a finite-dimensional complex vector space with a fixed inner product which we denote by $(-,-)$. We use this inner product to identify V with its dual V^* and for each $\alpha \in V^*$ we denote by v_α the unique element of V such that $\alpha(v') = (v', v_\alpha)$ for all $v' \in V$. Given $\alpha \in V^*$ we denote by s_α the orthogonal reflection through the hyperplane $\ker \alpha$, and by s_α^* the corresponding endomorphism of V^*. In this article a **finite root system** over V will be a finite set $\Phi \subset V^*$ such that for each $\alpha \in \Phi$ we have

(R1) $\Phi \cap \mathbb{C}\alpha = \{\pm\alpha\}$ and
(R2) $s_\alpha^*(\Phi) \subset \Phi$.

In classical references such as [Hum90] and [Hil82] root systems are defined as subsets of a Euclidean vector space $V_\mathbb{R}$ with $\mathbb{R}$ instead of $\mathbb{C}$ in (R1). Taking $V = \mathbb{C} \otimes_\mathbb{R} V_\mathbb{R}$ for an adequate $V_\mathbb{R}$ our definition is equivalent to theirs. We use the definition above since we work with complex vector spaces endowed with the action of a reflection group.

We now review the basic features of the theory of root systems. For more details we refer the reader to the two references above. Fix a root system Φ. The Weyl group associated to Φ is the group $W(\Phi)$ generated by $\{s_\alpha \mid \alpha \in \Phi\}$. Since we do not assume that the root systems are reduced or crystallographic, nor that Φ generates $V^*_{\mathbb{R}}$, the group $W(\Phi)$ is a finite reflection group which may be decomposable, and its action on V may have a nontrivial stabilizer. Any reflection group $G \subset \mathsf{GL}(V)$ is the Weyl group of some root system $\Phi \subset V^*$ [Hil82, §1.2].

Just as in the case of root systems for Lie algebras, for each root system Φ we can choose a linearly independent subset $\Sigma \subset \Phi$ which is a basis of the $\mathbb{R}$-span of Φ such that the coefficients of each root of Φ in this basis are either all nonnegative or all nonpositive. Such sets are called **bases** or **simple systems**, and their elements are called **simple roots**. Each choice of a base defines a partition $\Phi = \Phi^+ \cup -\Phi^+$, where Φ^+ is the set of all **positive** roots, i.e., those whose coordinates over Σ are nonnegative. If we fix a base Σ then the set S of reflections corresponding to simple roots is a minimal generating set of the reflection group $W = W(\Phi)$, and hence (W, S) is a finite Coxeter system in the sense of [Hum90, 1.9]. Each $s \in W$ of order two is of the form s_α for some $\alpha \in \Phi^+$ [Hum90, Proposition 2.14], and given $s \in W$ of order two we denote by α_s the corresponding positive root.

Fixing a base Σ, or equivalently, a minimal generating set $S \subset W$, we define the **length** $\ell(\sigma)$ of $\sigma \in W$ as the least positive integer ℓ such that σ can be written as a composition of ℓ reflections in S. Any sequence $s_1, \dots, s_{\ell(\sigma)}$ such that $\sigma = s_1 \cdots s_{\ell(\sigma)}$ is called a **reduced decomposition**; notice that reduced decompositions are not unique. The group W acts faithfully and transitively on Φ. Furthermore,

$$\ell(\sigma) = |\sigma(\Phi^+) \cap -\Phi^+|,$$

so W has a unique longest element whose length equals $|\Phi|$. We will denote this element by $\omega_0(W)$, or simply by ω_0 if the group W is clear from the context.

For the rest of this section we fix a root system Φ with base Σ and denote by (W, S) the corresponding Coxeter system.

2.2. Subsystems, subgroups and stabilizers. In this subsection we follow [Hum90, 1.10], where the reader can find most proofs. Given $\Omega \subset \Sigma$ we denote by $\Phi(\Omega)$ the root subsystem generated by Ω. We will call such subsystems **standard**. If $\Psi \subset \Phi$ is an arbitrary subsystem, then we can choose a base $\Omega \subset \Psi$

which can be extended to a base $\overline{\Omega}$ of Φ. By [Hum90, 1.4 Theorem] W acts transitively on the set of all bases of Φ, so for some $\sigma \in W$ we have $\sigma(\overline{\Omega}) = \Sigma$ and hence $\sigma(\Psi)$ is standard.

Let $\theta \subset S$ and denote by W_θ the subgroup of W generated by θ. Then (W_θ, θ) is also a Coxeter system and it determines a standard root system $\Phi_\theta \subset \Phi$ with simple roots $\Sigma_\theta = \{\alpha_s \mid s \in \theta\}$. We will refer to subgroups of the form W_θ as **standard parabolic** subgroups. A parabolic subgroup is any subgroup of W that is conjugate to a standard parabolic subgroup.

If $\sigma \in W_\theta$, then we can compute its length as an element of W with respect to the generating set S or as an element of W_θ with respect to the generating set θ. Both lengths turn out to be equal and will be denoted by $\ell(\sigma)$. Since W_θ is also a Coxeter group it has a unique element of maximal length which we will denote by $\omega_0(\theta)$. The set

$$W^\theta = \{\sigma \in W \mid \ell(\sigma s) > \ell(\sigma) \text{ for all } s \in \theta\}$$

is a set of representatives of the classes in the quotient W/W_θ, and for each $\sigma \in W$ there exist unique elements $\sigma^\theta \in W^\theta$ and $\sigma_\theta \in W_\theta$ such that $\sigma = \sigma^\theta \sigma_\theta$ with $\ell(\sigma) = \ell(\sigma^\theta) + \ell(\sigma_\theta)$. The element σ^θ is the element of minimal length in the coclass σW_θ. It follows that $(\omega_0)_\theta = \omega_0(\theta)$ and therefore $\omega_0^\theta = \omega_0 \omega_0(\theta)^{-1}$.

Given $v \in V$ we denote by $\Phi_0(v)$ the set of all roots in Φ such that $\alpha(v) = 0$, which is clearly a root subsystem of Φ. We also denote by W_v the stabilizer of v in W. We will say that v is **Σ-standard**, or just **standard** when Σ is fixed or clear from the context, if $\Phi_0(v)$ is a Σ-standard subsystem of Φ. It is easy to check that v is standard if and only if W_v is a standard parabolic subgroup, and $W_v = W(\Phi_0(v))$. Since $W_{\sigma(v)} = \sigma W_v \sigma^{-1}$ and $\Phi_0(\sigma(v)) = \sigma(\Phi_0(v))$ for all $\sigma \in W$, it follows that for every $v \in V$ there exists $\sigma \in W$ such that $\sigma(v)$ is standard and hence $W_{\sigma(v)}$ is a standard parabolic subgroup. If v is standard, then we denote by W^v the set of minimal length representatives of the left coclasses W/W_v.

2.3. Divided differences. From this point on V is a fixed finite-dimensional complex vector space, $\Lambda = S(V)$, and L is the fraction field of Λ. Note that following the convention of [PS09], we write $S(V)$ for $\mathrm{Sym}(V^*)$. Also, we fix a finite root system Φ with base Σ, and set $W = W(\Phi)$ to be the corresponding reflection group with minimal generating set S. Thus W acts on Λ and L, and we set $\Gamma = \Lambda^W$ and $K = L^W$.

Since W acts on L we can form the smash product $L\#W$. Recall that the product in this complex algebra is given over generators by $f\sigma \cdot g\tau = f\sigma(g)\sigma\tau$ for all $f, g \in L$ and all $\sigma, \tau \in W$. Dedekind's theorem on linear independence of field homomorphisms implies that the algebra morphism $L\#W \hookrightarrow \mathrm{End}_{\mathbb{C}}(L)$ defined by mapping $l\sigma \in L\#W$ to the endomorphism $f \mapsto l\sigma(f)$ is an embedding. We identify $L\#W$ with its image, and so must be careful to distinguish the result of applying the endomorphism $l\sigma$ to f, whose result is $l\sigma(f)$, and the product of $l\sigma$ and f in $L\#W$, which is $l\sigma \cdot f = l\sigma(f)\sigma$.

For $s \in W$ we set

$$\nabla_s = \frac{1}{\alpha_s}(1 - s) \in L\#W.$$

It is easy to show that for each $f, g \in L$,

$$\nabla_s(fg) = \nabla_s(f)g + s(f)\nabla_s(g)$$

so ∇_s is a twisted derivation of L. Notice that $\ker \nabla_s$ is exactly $L^{\langle s \rangle}$ and so ∇_s is $L^{\langle s \rangle}$-linear. Also it follows from the definition that $\nabla_s(\Lambda) \subset \Lambda$.

Example: Suppose $V = \mathbb{C}^2$ and let $\{x, y\} \subset (\mathbb{C}^2)^*$ be the dual basis to the canonical basis. Let s be the reflection given by $s(z_1, z_2) = (z_2, z_1)$, so $\alpha_s = x - y$. Then for each $f(x, y) \in \mathbb{C}[x, y]$ we have

$$\nabla_s(f)(x, y) = \frac{f(x, y) - f(y, x)}{x - y}.$$

Notice that this quotient is always a polynomial, since $f(x, y) - f(y, x)$ is an antisymmetric polynomial and hence divisible by $x - y$.

Given $\sigma \in W$ we take a reduced decomposition $\sigma = s_1 \cdots s_\ell$ and set

$$\partial_\sigma = \nabla_{s_1} \circ \cdots \circ \nabla_{s_\ell};$$

this element is called the **divided difference** corresponding to σ and does not depend on the chosen reduced decomposition [Hil82, Chapter IV (1.6)]. Notice though that the definition of ∂_σ does depend on the choice of a base $\Sigma \subset \Phi$.

By definition, an $L\#W$-module Z is an L-vector space endowed with a W-module structure such that the action of L on Z is W-equivariant. A simple induction on the length of σ shows that the divided difference ∂_σ defines a K-linear map over any $L\#W$-module Z. In particular L is such a module, and since $\nabla_s(\Lambda) \subset \Lambda$ for any $s \in S$, it follows that Λ is closed under the action of divided differences.

2.4. Coinvariant spaces and Schubert polynomials. The algebra Λ is $\mathbb{Z}_{\geq 0}$-graded with $\Lambda_1 = V^*$ and Γ is a graded subalgebra of Λ . We denote by I_W the ideal of Λ generated by the elements of Γ of positive degree. By the Chevalley–Shephard–Todd theorem Γ is isomorphic to a polynomial algebra in $\dim V$ variables and Λ is a free Γ-module of rank $|W|$. Also, a set $B \subset \Lambda$ is a basis of the Γ-module Λ if and only if its image in the quotient Λ/I_W is a $\mathbb{C}$-basis. Furthermore, Λ/I_W is naturally a graded W-module isomorphic to the regular representation of W with Hilbert series $\sum_{\sigma\in W} t^{\ell(\sigma)}$. For proofs we refer the reader to [Hil82, Chapter II, Section 3].

We now recall the construction of the basis of Schubert polynomials of Λ/I_W. This construction is due to Bernstein, Gelfand and Gelfand [BGG73] and Demazure [Dem74] in the case when W is a Weyl group, and to Hiller [Hil82, Chapter IV] in the case of arbitrary Coxeter groups. Set

$$\Delta(\Phi) = \prod_{\alpha\in\Phi^+} \alpha,$$

and for each $\sigma \in W$ set

$$\mathfrak{S}_\sigma^\Sigma = \frac{1}{|W|}\partial_{\sigma^{-1}\omega_0}\Delta(\Phi).$$

We will often write $\mathfrak{S}_\sigma$ instead of $\mathfrak{S}_\sigma^\Sigma$ when the base Σ is clear from the context. Notice that by definition $\deg \mathfrak{S}_\sigma = \ell(\sigma)$. The polynomials $\{\mathfrak{S}_\sigma \mid \sigma \in W\}$ are known as **Schubert polynomials**, and they form a basis of Λ as a Γ-module, so the projection of this set is a basis of Λ/I_W as a complex vector space. With a slight abuse of notation, we will denote the projections of the schubert polynomials by the same letters. Since $K = L^W$ we know that L is a K-vector space of dimension $|W|$ and so $\{\mathfrak{S}_\sigma \mid \sigma \in W\}$ is also a basis of L over K. Given $f \in L$ we will denote by $f_{(\sigma)}$ the coefficient of $\mathfrak{S}_\sigma$ in the expansion of f relative to this basis, so $f = \sum_{\sigma\in W} f_{(\sigma)}\mathfrak{S}_\sigma$.

Since Schubert polynomials form a basis of Λ/I_W, for all $\sigma, \tau, \rho \in W$ there exists $c_{\sigma,\tau}^\rho \in \mathbb{C}$ defined implicitly by the equation

$$\mathfrak{S}_\sigma\mathfrak{S}_\tau = \sum_{\rho\in W} c_{\sigma,\tau}^\rho \mathfrak{S}_\rho$$

The coefficients $c_{\sigma,\tau}^\rho$ are the **generalized Littlewood–Richardson coefficients** relative to the base Σ. It follows from the definition that $c_{\sigma,\tau}^\rho = 0$ unless $\ell(\sigma) + \ell(\tau) = \ell(\rho)$. If $\theta \subset S$, then the space of W_θ-invariants $(\Lambda/I_W)^{W_\theta}$ is generated by the set $\{\mathfrak{S}_\sigma \mid \sigma \in W^\theta\}$ [Hil82, Chapter IV (4.4)]. In particular, if $\sigma, \tau \in W^\theta$ then $c_{\sigma,\tau}^\rho \neq 0$ implies that $\rho \in W^\theta$.

2.5. Postnikov–Stanley operators. Throughout this paragraph we fix a root system Φ with base Σ and Weyl group W. All references to Schubert polynomials are with respect to these data.

We denote by Λ° the algebra of polynomial differential operators on Λ. If we fix an orthonormal basis $x_1, \dots, x_n$ of V^*, then

$$\Lambda = \mathbb{C}[x_1, \dots, x_n] \quad \text{and} \quad \Lambda^\circ = \mathbb{C}\Big[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big].$$

There is a natural pairing $(-,-) : \Lambda^\circ \times \Lambda \longrightarrow \mathbb{C}$ given by $(D, p) = D(p)(0)$, which allows us to identify Λ° with the graded dual of Λ. Furthermore, with this identification the coproduct of Λ° is the adjoint of the multiplication of Λ.

For every graded ideal $I \subset \Lambda$ we write

$$\mathcal{H}_I = \{D \in \Lambda^\circ \mid (D, f) = 0 \text{ for all } f \in I\}.$$

Since the pairing $(-,-)$ is non-degenerate, the space $\mathcal{H}_I$ is naturally isomorphic to the graded dual of Λ/I. We denote by $\mathfrak{D}_\sigma$ the unique element in $\mathcal{H}_{I_W}$ such that $(\mathfrak{D}_\sigma, \mathfrak{S}_\tau) = \delta_{\sigma,\tau}$. It follows that the set $\{\mathfrak{D}_\sigma \mid \sigma \in W\}$ is a graded basis of $\mathcal{H}_{I_W}$, dual to the Demazure basis of Λ/I_W. Also, for each $\theta \subset S$ the set $\{\mathfrak{D}_\sigma \mid \sigma \in W^\theta\}$ is a graded basis of the dual of $(\Lambda/I_W)^{W_\theta}$.

In [PS09], Postnikov and Stanley introduce new operators indexed by pairs of elements $\sigma, \tau \in W$ and given by

$$\mathfrak{D}_{\tau,\sigma} = \sum_{\rho \in W} c^\sigma_{\tau,\rho} \mathfrak{D}_\rho.$$

Notice that $\mathfrak{D}_{e,\sigma} = \mathfrak{D}_\sigma$, that $\mathfrak{D}_{\sigma,\sigma} = 1$, and that $\mathfrak{D}_{\tau,\sigma} = 0$ unless $\tau \leq \sigma$ in the Bruhat order of W. These operators have the property that

$$\Delta(\mathfrak{D}_\sigma) = \sum_\tau \mathfrak{D}_{\tau,\sigma} \otimes \mathfrak{D}_\tau = \sum_\tau \mathfrak{D}_\tau \otimes \mathfrak{D}_{\tau,\sigma}$$

which can be checked by evaluating $\Delta(\mathfrak{D}_\sigma)$ in the set $\{\mathfrak{S}_\sigma \otimes \mathfrak{S}_\tau \mid \sigma, \tau \in W\}$.

Given $D \in \Lambda^\circ$ we denote by D^0 the map $(D, -) : \Lambda \longrightarrow \mathbb{C}$. The formula for the coproduct of the Demazure operators is equivalent to the fact that for any $f, g \in \Lambda$

$$\mathfrak{D}^0_\sigma(fg) = \sum_\tau \mathfrak{D}^0_{\tau,\sigma}(f)\mathfrak{D}^0_\tau(g) = \sum_\tau \mathfrak{D}^0_\tau(f)\mathfrak{D}^0_{\tau,\sigma}(g).$$

It follows from the definition that each operator $\mathfrak{D}^0_\sigma$ is Λ^W-linear. By a slight abuse of notation, we denote by $\mathfrak{D}^0_\sigma$ the extension of this operator to the algebra L of rational functions regular at 0.

PROPOSITION 2.1: *Let $f \in L$ be regular at zero and let $\sigma \in W$. Then*

$$\mathfrak{D}_\sigma(f)(0) = (\partial_\sigma f)(0).$$

Furthermore, if $g \in L$ is also regular at 0 then

$$\mathfrak{D}^0_\sigma(fg) = \sum_{\rho \le \sigma} \mathfrak{D}^0_{\rho,\sigma}(f)\mathfrak{D}^0_\rho(g) = \sum_{\rho \le \sigma} \mathfrak{D}^0_\rho(f)\mathfrak{D}^0_{\rho,\sigma}(g).$$

Proof. Let us first prove that the result holds when f, g are polynomials. Let $I_{\sigma^{-1}}$ be the operator adjoint to ∂_σ with respect to the pairing $(-,-)$. As shown in [PS09, Theorem 6.5] $\mathfrak{D}_\sigma = I_{\sigma^{-1}}(1)$, so $(\mathfrak{D}_\sigma, f) = (1, \partial_\sigma f) = \partial_\sigma(f)(0)$. We have already discussed the product formula in the preamble to this Proposition. Let $A = I_W^{-1}\Gamma$. We have seen that the differential operators $\mathfrak{D}^0_\sigma$ and the divided differences ∂_σ are Γ-linear, and hence they are also A-linear. Since $L = A\Gamma$ the result follows. ∎

3. Galois orders and Gelfand–Tsetlin modules

Throughout this section Γ is a noetherian integral domain, K is its field of fractions, and L is a finite Galois extension of K with Galois group G. Hence $K = L^G$.

3.1. GALOIS ORDERS. We first recall the notion of a Galois ring (order), that was introduced in [FO10]. Let $\mathcal{M}$ be a monoid acting on L by ring automorphisms, such that for all $t \in \mathcal{M}$ and all $\sigma \in G$ we have $\sigma \circ t \circ \sigma^{-1} \in \mathcal{M}$. Then the action of G extends naturally to an action on the smash product $L\#\mathcal{M}$. We assume that the monoid $\mathcal{M}$ is K-**separating**, that is, given $m, m' \in \mathcal{M}$, if $m|_K = m'|_K$ then $m = m'$.

Definition 3.1: Set $\mathcal{K} = L\#\mathcal{M}$.

(i) A **Galois ring over** Γ is a finitely generated Γ-subring $U \subset (L\#\mathcal{M})^G$ such that $UK = KU = \mathcal{K}$.

(ii) Set $S = \Gamma \setminus \{0\}$. A Galois ring U over Γ is a **right** (respectively, **left**) **Galois order**, if for any finite-dimensional right (respectively left) K-subspace $W \subset U[S^{-1}]$ (respectively, $W \subset [S^{-1}]U$), the set $W \cap U$ is a finitely generated right (respectively, left) Γ-module. A Galois ring is **Galois order** if it is both a right and a left Galois order.

We will always assume that Galois rings are complex algebras. In this case we say that a Galois ring is a Galois algebra over Γ.

3.2. Principal and co-principal Galois orders. Notice that $L\#\mathcal{M}$ acts on L, where for each $X = \sum_{m\in\mathcal{M}} l_m m \in L\#M$ we define its action on $f \in L$ by

$$X(f) = \sum_m l_m m(f).$$

As an example of a Galois order, Hartwig introduced the **standard Galois Γ-order** in $\mathcal{K}$ defined as

$$\mathcal{K}_\Gamma = \{X \in \mathcal{K} \mid X(\Gamma) \subset \Gamma\};$$

see [Har, Theorem 2.21]. In this article the term "standard Galois order" has a different meaning, and for sake of clarity will refer to the algebra above as the **left Hartwig order** of $\mathcal{K}$. A **principal Galois order** is any Galois order $U \subset \mathcal{K}_\Gamma$. By restriction Γ is a left U-module for any principal Galois order, and hence its complex dual Γ^* is a right U-module.

Denote by $\mathcal{M}^{-1}$ the monoid formed by the inverses of the elements in $\mathcal{M}$. Following [Har], we define an anti-isomorphism $-^\dagger : L\#\mathcal{M} \longrightarrow L\#\mathcal{M}^{-1}$ by

$$(lm)^\dagger = m^{-1}\cdot l = m^{-1}(l)m^{-1}$$

for any $l \in L, m \in \mathcal{M}$. The **right Hartwig order** is thus defined as

$${}_\Gamma\mathcal{K} = \{X \in \mathcal{K} \mid X^\dagger(\Gamma) \subset \Gamma\},$$

and a **co-principal Galois order** is any Galois order contained in ${}_\Gamma\mathcal{K}$. Thus Γ^* is a left U-module for any co-principal Galois order, with action given by

$$X \cdot \chi = \chi \circ X^\dagger$$

for any $X \in U$ and $\chi \in \Gamma^*$.

3.3. Gelfand–Tsetlin modules. Let U be a Galois order over Γ and let M be any U-module. Given $\mathfrak{m} \in \operatorname{Specm}\Gamma$ we set

$$M[\mathfrak{m}] = \{x \in M \mid \mathfrak{m}^k x = 0 \text{ for } k \gg 0\}.$$

Since ideals in $\operatorname{Specm}\Gamma$ are in one-to-one correspondence with characters $\chi : \Gamma \longrightarrow \mathbb{C}$ we also set

$$M[\chi] = \{x \in M \mid (\gamma - \chi(\gamma))^k x = 0 \text{ for all } \gamma \in \Gamma \text{ and } k \gg 0\}.$$

If χ is given by the natural projection $\Gamma \longrightarrow \Gamma/\mathfrak{m} \cong \mathbb{C}$, then $M[\mathfrak{m}] = M[\chi]$.

Definition 3.2: A **Gelfand–Tsetlin module** is a finitely generated U-module M such that its restriction $M|_\Gamma$ to Γ can be decomposed as a direct sum

$$M|_\Gamma = \bigoplus_{\mathfrak{m}\in \operatorname{Specm}\Gamma} M[\mathfrak{m}].$$

A U-module M is a Gelfand–Tsetlin module if and only if for each $x \in M$ the cyclic Γ-module $\Gamma \cdot x$ is finite dimensional over $\mathbb{C}$ [DFO94, §1.4], which easily implies the following result.

LEMMA 3.3: *A U-submodule N of a Gelfand–Tsetlin module M is again a Gelfand–Tsetlin module. If $v \in N$, then its projection to $N[\mathfrak{m}]$ lies in N for each $\mathfrak{m} \in \operatorname{Specm}\Gamma$.*

For every maximal ideal $\mathfrak{m}$ of Γ we denote by $\varphi(\mathfrak{m})$ the number of non-isomorphic simple Gelfand–Tsetlin modules M for which $M[\mathfrak{m}] \neq 0$. Sufficient conditions for the number $\varphi(\mathfrak{m})$ to be nonzero and finite were established in [FO14].

Consider the integral closure $\overline{\Gamma}$ of Γ in L. It is a standard fact that if Γ is finitely generated as a complex algebra, then any character of Γ has finitely many extensions to characters of $\overline{\Gamma}$. Let $\overline{\mathfrak{m}}$ be any lifting of $\mathfrak{m}$ to $\overline{\Gamma}$, and $\mathcal{M}_\mathfrak{m}$ be the stabilizer of $\overline{\mathfrak{m}}$ in $\mathcal{M}$. Note that the monoid $\mathcal{M}_\mathfrak{m}$ is defined uniquely up to G-conjugation. Thus the cardinality of $\mathcal{M}_\mathfrak{m}$ does not depend on the choice of the lifting. We denote this cardinality by $|\mathfrak{m}|$.

THEOREM 3.4 ([FO14, Main Theorem and Theorem 4.12]): *Let Γ be a commutative domain which is finitely generated as a complex algebra and let*

$$U \subset (L\#\mathcal{M})^G$$

be a right Galois order over Γ. Let also $\mathfrak{m} \in \operatorname{Specm}\Gamma$ be such that $|\mathfrak{m}|$ is finite. Then the following hold:

(i) *The number $\varphi(\mathfrak{m})$ is nonzero.*
(ii) *If U is a Galois order over Γ, then the number $\varphi(\mathfrak{m})$ is finite and uniformly bounded.*
(iii) *If U is a Galois order over Γ and Γ is a normal noetherian algebra, then for every simple Gelfand–Tsetlin module M the set $\dim M[\mathfrak{m}]$ is uniformly bounded.*

4. Rational Galois orders

Recall that V is a complex vector space with an inner product. We set

$$\Lambda = S(V) \quad \text{and} \quad L = \mathrm{Frac}(\Lambda).$$

Recall that an element $g \in \mathsf{GL}(V)$ is called a **pseudo-reflection** if it has finite order and fixes a hyperplane of codimension 1. By definition every reflection is a pseudo-reflection, and the converse holds over $\mathbb{R}$ but not over $\mathbb{C}$, which is why finite groups generated by pseudo-reflections are called **pseudo-reflection groups** or **complex reflection groups**. We fix $G \subset \mathsf{GL}(V)$ a pseudo-reflection group. As usual the action of G on V induces actions on Λ and L, and we denote by Γ the algebra of G-invariant elements of Λ and set $K = L^G$.

4.1. Let $L \hookrightarrow \mathrm{End}_{\mathbb{C}}(L)$ be the $\mathbb{C}$-algebra morphism that sends any rational function $f \in L$ to the $\mathbb{C}$-linear map $m_f : f' \in L \mapsto ff' \in L$. Although $\mathrm{End}_{\mathbb{C}}(L)$ is not an L-algebra, it is an L-vector space with $f \cdot \varphi = m_f \circ \varphi$ for all $\varphi \in \mathrm{End}_{\mathbb{C}}(L)$. Also, G acts on $\mathrm{End}_{\mathbb{C}}(L)$ by conjugation and

$$\sigma \cdot m_f = \sigma \circ m_f \circ \sigma^{-1} = m_{\sigma(f)}$$

for each $\sigma \in G$, so the map $f \mapsto m_f$ is G-equivariant. For simplicity we will write f for the operator m_f.

Given $v \in V$ we define a map $a_v : V \longrightarrow V$ given by

$$a_v(v') = v' + v.$$

This in turn induces an endomorphism of Λ, which we denote by t_v, given by

$$t_v(f) = f \circ a_v;$$

we sometimes write $f(x + v)$ for $t_v(f)$. Each map t_v can be extended to a $\mathbb{C}$-linear operator on L and $t_v \circ t_{v'} = t_{v+v'}$, so V acts on L by automorphisms and we can form the smash product $L\#V$. Once again there is an algebra morphism $L\#V \to \mathrm{End}_{\mathbb{C}}(L)$, and the definitions imply that this map is G-equivariant.

Lemma 4.1: *Let G, V, and L be as above, and let $Z \subset V$ be an arbitrary subset. Then the set $\{t_z \mid z \in Z\} \subset \mathrm{End}_{\mathbb{C}}(L)$ is linearly independent over L, and the map $L\#V \longrightarrow \mathrm{End}_{\mathbb{C}}(L)$ is injective.*

Proof. Put

$$T = \sum_{i=1}^{N} f_i t_{z_i}$$

where $f_i \in L^\times$ and each $z_i \in Z$, and assume $T = 0$. Given $p \in \Lambda$ it follows from $T(p) = 0$ that

$$p \sum_i f_i = \sum_i [p(x + z_i) - p(x)] f_i.$$

Let $v \in V$ be arbitrary and choose a polynomial p of positive degree such that $p(v) = p(v + z_j)$ for all $j \neq i$ but $p(v) + 1 = p(v + z_i)$. Then $0 = p(v + z_i) f_i(v)$ so $f_i(v) = 0$. Since v is arbitrary this implies that $f_i = 0$ so the set $\{t_z \mid z \in Z\}$ is L-linearly independent. Since the morphism $L\#V \longrightarrow \mathrm{End}_{\mathbb{C}}(L)$ is L-linear and sends an L-basis of $L\#V$ to a linearly independent subset, it must be injective. ∎

4.2. Co-rational Galois orders. Given a character $\chi : G \longrightarrow \mathbb{C}^\times$ the space of relative invariants

$$\Lambda_\chi^G = \{f \in \Lambda \mid \sigma \cdot p = \chi(\sigma) p \text{ for all } \sigma \in G\} \subset \Lambda$$

is a Λ^G-submodule of Λ. By a theorem of Stanley [Hil82, 4.4 Proposition] Λ_χ^G is a free Λ^G-module of rank 1. Furthermore,

$$d_\chi = \prod_{H \in \mathcal{A}(G)} (\alpha_H)^{a_H}$$

is a generator of Λ_χ^G, where $\mathcal{A}(G)$ is the set of hyperplanes that are fixed pointwise by some element of G, each α_H is a linear form such that $\ker \alpha_H = H$, and $a_H \in \mathbb{Z}_{\geq 0}$ is minimal with the property $\det[s_H^*]^{a_H} = \chi(s_H)$ for an arbitrary generator s_H of the stabilizer of H in G. Note that a_H is independent of the choice of s_H, and that if G is a Coxeter group then a_H is either 1 or 0.

Definition 4.2 ([Har, Definition 4.3]): A **rational Galois order**, resp. **co-rational Galois order**, is a subalgebra $U \subset \mathrm{End}_{\mathbb{C}}(L)$ generated by Γ and a finite set of operators $\mathcal{X} \subset (L\#V)^G$ such that for each $X \in \mathcal{X}$ there exists $\chi \in \hat{G}$ with $d_\chi X \in \Lambda\#V$, resp. $X d_\chi \in \Lambda\#V$

As shown in [Har] every rational Galois order is isomorphic to a co-rational Galois order, and for technical reasons we will restrict to the latter. Examples of co-rational Galois orders include the enveloping, quantized enveloping and W-algebras of $\mathfrak{gl}(n, \mathbb{C})$ for all $n \geq 1$.

4.3. Given $X \in L\#V$ we define its **support** as the set of all $v \in V$ such that t_v appears with nonzero coefficient in X. Note that the support is well-defined since the set $\{t_v \mid v \in V\}$ is free over L. We denote the support of X by $\operatorname{supp} X$. Given a co-rational Galois order $U \subset (L\#V)^G$ we denote by $\mathsf{Z}(U)$ the additive submonoid of V generated by the supports of all its elements. By [Har, Theorem 4.2] U is a co-principal Galois order in $(L\#\mathsf{Z}(U))^G$.

Let $v \in V$, let $\mathsf{ev}_v : \Gamma \longrightarrow \mathbb{C}$ be the character given by evaluation at v, and let $\mathfrak{m} = \ker \mathsf{ev}_v$. Then the cyclic U-module $U \cdot \mathsf{ev}_v \subset \Gamma^*$ is a Gelfand–Tsetlin module [Har, Theorem 3.3], and since $\mathsf{ev}_v \in U \cdot \mathsf{ev}_v[\mathfrak{m}]$ we have a new proof that $\varphi(\mathfrak{m}) \neq 0$ for rational Galois orders. In the following sections we show that this module is spanned by BGG operators, and give an explicit presentation in special cases including all of Hartwig's examples.

5. Structure of Γ-modules associated to BGG operators

Throughout this section we fix a complex vector space V and a root system Φ. We also fix a root subsystem $\Psi \subset \Phi$ with base $\Omega \subset \Psi$. We denote by G the Weyl group associated to Φ and by W the one associated to Ψ. Like before, $\Lambda = S(V)$, $L = \operatorname{Frac}(\Lambda)$, $\Gamma = \Lambda^G$, and $K = L^G$. Since $W \subset G$, the group W also acts on the vector spaces Λ, Γ, etc. All Schubert polynomials, Postnikov–Stanley operators, standard elements, etc. are defined with respect to the subsystem Ψ and the base Ω unless otherwise stated.

5.1. For every $v \in V$ and any $\sigma \in W$ we denote by $\mathfrak{D}^v_\sigma$ the differential operator sending a rational function f to $\mathfrak{D}_\sigma(f)(v)$, whenever the latter is defined. We denote by $\mathcal{D}^v_\sigma$ the restriction of this differential operator to Γ. If v is Ω-standard then we set

$$\mathcal{D}(\Omega, v) = \langle \mathcal{D}^v_\sigma \mid \sigma \in W \rangle_{\mathbb{C}} \subset \Gamma^*.$$

We call $\mathcal{D}(\Omega, v)$ the **Γ-space of BGG differential operators** associated to Ω and v.

LEMMA 5.1: *Let $v \in V$ and let $\pi^W : \Lambda \longrightarrow \Lambda/I_W$ be the natural projection.*

(a) $\pi^W(t_v(\Gamma)) = (\Lambda/I_W)^{W_v}$.

(b) *If v is Ω-standard then the set $\{\mathcal{D}^v_\sigma \mid \sigma \in W^v\}$ is a basis of $\mathcal{D}(\Omega, v)$.*

Proof. Recall that K is the fixed field of G in L, and hence the fraction field of Γ. Since the extension $L^W \subset L$ is a Galois extension with Galois group W, the field $L^W t_v(K) \subset L$ must be the fixed field of a subgroup $\widetilde{W} \subset W$. For $\sigma \in W$ we have that

$$\sigma \circ t_v|_K = t_{\sigma(v)} \circ \sigma|_K = t_{\sigma(v)}|_K.$$

Since $\sigma \in \widetilde{W}$ if and only if $\sigma \circ t_v = t_v$, it follows that $\widetilde{W} = W_v$. Thus

$$L^W t_v(K) = L^{W_v}$$

which implies that $\Lambda^W t_v(\Gamma) = \Lambda^{W_v}$.

Since all non-constant polynomials in Λ^W are in the kernel of π^W we see that

$$\pi^W(\Lambda^W t_v(\Gamma)) = \pi^W(t_v(\Gamma)),$$

so this last space equals $\pi^W(\Lambda^{W_v}) = (\Lambda/I_W)^{W_v}$. This proves the first part of the lemma. It follows that for each $\sigma \in W^v$ there exists $\gamma_\sigma \in \Gamma$ such that

$$\pi^W(t_v(\gamma_\sigma)) = \mathfrak{S}_\sigma,$$

so $\mathcal{D}^v_\tau(\gamma_\sigma) = \delta_{\sigma,\tau}$, and hence the set in the statement of part (b) is linearly independent. ∎

As we saw before, the set $\{\mathfrak{D}^0_\sigma \mid \sigma \in W^v\}$ is a basis for the dual of the algebra $A = (\Lambda/I_W)^{W_v}$. Now A^* has the structure of an A-module, and hence Γ acts on A^* through the map $\pi^W \circ t_v : \Gamma \longrightarrow A$. The element $\mathfrak{D}^0_\sigma \cdot \gamma$ is by definition the map $a \mapsto \mathfrak{D}^0_\sigma(t_v(\gamma)a)$, so using Proposition 2.1 we have

$$\mathfrak{D}^0_\sigma(t_v(\gamma)a) = \sum_{\tau \le \sigma} \mathfrak{D}^0_{\tau,\sigma}(t_v(\gamma))\mathfrak{D}^0_\tau(a) = \gamma(v)\mathfrak{D}^0_\sigma(a) + \sum_{\tau < \sigma} \mathfrak{D}^v_{\tau,\sigma}(\gamma)\mathfrak{D}^0_\tau(a),$$

where we have used that $\mathfrak{D}^0_\sigma \circ t_v = \mathfrak{D}^v_\sigma$. Since $a = t_v(\gamma')$ for some $\gamma' \in \Gamma$, we can rewrite this as

$$(\gamma \cdot \mathfrak{D}^v_\sigma)(\gamma') = \gamma(v)\mathfrak{D}^v_\sigma(\gamma') + \sum_{\tau < \sigma} \mathfrak{D}^v_{\tau,\sigma}(\gamma)\mathfrak{D}^v_\tau(\gamma').$$

This shows that $\mathcal{D}(\Omega, v)$ is a Γ-submodule of Γ^*, isomorphic to the pullback of A^* as A-module through the map $\pi^W \circ t_v$. The above discussion implies Theorem A, and more precisely, the following theorem.

THEOREM 5.2: *Let $v \in V$. Then $\mathcal{D}(\Omega, v)$ is a Γ-submodule of Γ^* and for each $\gamma \in \Gamma$*

$$\gamma \cdot \mathcal{D}^v_\sigma = \gamma(v)\mathcal{D}^v_\sigma + \sum_{\tau < \sigma} \mathfrak{D}^v_{\tau,\sigma}(\gamma)\mathcal{D}^v_\tau.$$

5.2. The structure of $\mathcal{D}(\Omega, v)$ as Γ-module. The modules $\mathcal{D}(\Omega, v)$ will appear as Γ-eigenspaces of certain Gelfand–Tsetlin modules, so it is important that we record some facts regarding their Γ-module structure. We thank the reviewer for the observation that this structure is given by the action of a Frobenius algebra on its dual, which allowed for a more streamlined presentation.

Recall that by definition a finite-dimensional algebra A is Frobenius if its dual is isomorphic to A as a right or left A-module. By [Lam99, (16.55)], a commutative Frobenius algebra has a non-degenerate symmetric bilinear form. Also recall that a finite-dimensional graded algebra

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_d$$

is said to satisfy the hard Lefschetz condition if there exists an element $l \in A_1$ such that for each $i \leq d/2$, multiplication by l^{d-2i} induces an isomorphism between A_i and A_{d-i}.

Recall that W^v is the set of minimal length representatives of the left W_v-cosets.

Lemma 5.3: *Let $v \in V$ be Ω-standard, let $A = (\Lambda/I_W)^{W_v}$, and let ω_0^v be the longest element in W^v. Then A is a Frobenius algebra with the hard Lefschetz property. Furthermore, the non-degenerate bilinear form of A is given by*

$$(a, b) \in A \times A \mapsto \mathfrak{D}^0_{\omega_0^v}(ab) \in \mathbb{C}x.$$

Proof. The hard Lefschetz property is a classical result in the case when W is a Weyl group, and for all finite Coxeter groups is established in [MNW11, McD11].

The following argument was suggested by D. Speyer in [Spe17]. By the Chevalley–Shephard–Todd theorem Λ^W and Λ^{W_v} are polynomial algebras, generated by algebraically independent sets $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ respectively. Clearly $p_i \in \Lambda^{W_v}$ and

$$A = \mathbb{C}[q_1, \ldots, q_s]/J,$$

where J is the ideal generated by the p_i's. This implies that A is a finite-dimensional complete intersection, hence a graded artinian self-injective ring, and hence a commutative Frobenius algebra. Since its top degree component is spanned over $\mathbb{C}$ by $\mathfrak{S}_{\omega_0^v}$, this bilinear form is given by sending (a, b) to the coefficient of $\mathfrak{S}_{\omega_0^v}$ in the product ab, which is equal to $\mathfrak{D}^0_{\omega_0^v}(ab)$. ∎

PROPOSITION 5.4: *Suppose that $v \in V$ is Ω-standard and let*

$$x = \sum_{\sigma \in W^v} a_\sigma \mathcal{D}^v_\sigma.$$

(a) *The element x is a cyclic generator of $\mathcal{D}(\Omega, v)$ if and only if $a_{\omega_0^v} \neq 0$.*
(b) *The element x is an eigenvector of Γ if and only if $a_\sigma = 0$ for all $\sigma \neq e$.*
(c) *For each $\gamma \in \Gamma$ the element $\gamma - \gamma(v)$ acts as a nilpotent operator on $\mathcal{D}(\Omega, v)$. Its nilpotency order is at most*

$$r = \ell(\omega_0^v) + 1,$$

and generically equals r.
(d) *Let $v' \in V$. The space $\mathcal{D}(\Omega, v) \cap \mathcal{D}(\Omega, v')$ is nonzero if and only if v' is in the G-orbit of v. Furthermore, if v' is in the W-orbit of v then*

$$\mathcal{D}(\Omega, v) = \mathcal{D}(\Omega, v').$$

Proof. Set

$$A = (\Lambda / I_W)^{W_v}.$$

By construction $\mathcal{D}(\Omega, v)$ is isomorphic as Γ-module to the pullback of A^* as A-module through the map $\pi^W \circ t_v$. It follows from Lemma 5.3 that A is isomorphic to A^*, and the isomorphism maps a to $a \cdot \mathfrak{D}^0_{\omega_0^v}$. Since A is cyclically generated by any element whose projection to A_0 is nonzero, A^* is cyclically generated by any element whose projection to the top component is a nonzero multiple of $\mathfrak{D}^0_e$. Part (a) is a restatement of the latter in terms of the action of Γ on $\mathcal{D}(\Omega, v)$. Also, since the only eigenvector of A acting on itself is $\mathfrak{S}_{\omega_0^v}$, the only eigenvector of the action of A on A^* is $\mathfrak{D}^0_e$, and this is equivalent to part (b).

Since $A \cong A^*$, an element of A acts by zero on A^* if and only if it is zero. Clearly for any $a \in A_{\geq 1}$ we have $a^r = 0$. Now the elements such that $a^{r-1} = 0$ form a Zarisky closed subset of A, and again by Lemma 5.3 there is an element $l \in A_1$ such that $l^{r-1} \neq 0$. It follows that the elements $a \in A_{\geq 1}$ such that $a^r \neq 0$ are a dense open Zarisky set of A. Since

$$\pi^W(t_v(\gamma - \gamma(v))) \in A_{\geq 1},$$

it follows that $(\gamma - \gamma(v))^r$ acts by zero on $\mathcal{D}(\Omega, v)$, and generically $(\gamma - \gamma(v))^{r-1}$ acts by a nonzero linear transformation. This proves part (c).

Each element in $\mathcal{D}(\Omega, v)$ is a generalized eigenvector of γ with eigenvalue $\gamma(v)$. Thus if $\mathcal{D}(\Omega, v) \cap \mathcal{D}(\Omega, v') \neq 0$ we must have $\gamma(v) = \gamma(v')$ for all $\gamma \in \Gamma$ which implies that $v' \in G \cdot v$. Now if $v' = \tau(v)$ for some $\tau \in W$ then

$$\mathcal{D}_\sigma^{\tau(v)} = \mathfrak{D}_\sigma^0 \circ t_{\tau(v)} \mid_\Gamma = \mathfrak{D}_\sigma^0 \circ \tau \circ t_v \circ \tau^{-1} \mid_\Gamma = \mathfrak{D}_\sigma^0 \circ \tau \circ t_v \mid_\Gamma .$$

Since $\mathfrak{D}_\sigma^0 \circ \tau$ lies in $\mathcal{H}_W$, for each $\rho \in W$ there exist $c_\rho \in \mathbb{C}$ such that

$$\mathfrak{D}_\sigma^0 \circ \tau = \sum_\rho c_\rho \mathfrak{D}_\rho^0.$$

Hence, $\mathcal{D}_\sigma^{\tau(v)} = \sum_\rho c_\rho \mathcal{D}_\rho^v$, which proves part (d). ∎

Part (c) of the last proposition implies Corollary B in the Introduction.

6. Action of a co-rational Galois order

In this section G is a reflection group acting on V, and hence on $\Lambda = S(V)$ and on its field of rational functions $L = \mathrm{Frac}(\Lambda)$. We fix a co-rational Galois order $U \subset (L\#V)^G$ and denote by $\mathsf{Z} \subset V$ the additive monoid generated by $\operatorname{supp} U$. The algebra $L\#V$ has an anti-automorphism, given by $(ft_v)^\dagger = t_v f$ for each $f \in L$ and $v \in V$. The action of U on Γ^* is given by

$$X \cdot \varphi = \varphi \circ X^\dagger$$

for each $X \in U$ and each $\varphi \in \Gamma^*$. Thus U acts by composition with elements $X^\dagger \in L\#V$ such that $d_\chi X^\dagger \in \Lambda\#V$ for some $\chi \in \hat{G}$.

We assume again that $\Phi \subset V$ is a root system with base Σ and $G = W(\Phi)$. We denote by Ψ a standard subsystem with base $\Omega \subset \Sigma$ and set $W = W(\Psi)$. All Schubert polynomials, BGG and Postnikov–Stanley operators appearing in this section are defined with respect to Ω unless otherwise stated.

6.1. Recall that for each $\sigma \in G$ we introduced a divided difference operator as an element of the smash product $L\#G$. Since $\mathrm{End}_\mathbb{C}(L)$ is an $(L\#G)$-module, given $X \in \mathrm{End}_\mathbb{C}(L)$ and $\sigma \in G$, we obtain a new operator on L by taking $\partial_\sigma(X)$. For example, if $s \in S$ then

$$\partial_s(X) = \frac{1}{\alpha_s}(X - s \circ X \circ s^{-1})$$

Notice that, in general, this operator is different from the composition of ∂_σ (regarded as an element of $\mathrm{End}_\mathbb{C}(L)$) and X. In the following lemma we collect some properties of these operators.

LEMMA 6.1: *Let $X \in \mathrm{End}_{\mathbb{C}}(L)$.*

(a) *For each $\sigma \in G$ we have $\partial_\sigma(X)|_K = \partial_\sigma \circ X|_K$.*

(b) *Let $v \in V$ be Ω-standard. If $\sigma \in W^v$ and $\tau \in W_v$ then*

$$\mathfrak{D}^v_\sigma \circ \partial_\tau = \begin{cases} \mathfrak{D}^v_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

(c) *Let $\widetilde{\Psi} \subset \Psi$ be a standard subsystem, $W_\theta \subset W$ be the corresponding parabolic subgroup, ω_0^θ be the longest word in W^θ, and*

$$\Delta(\Psi)^\theta := \Delta(\Psi)/\Delta(\widetilde{\Psi}).$$

If $X \in \mathrm{End}_{\mathbb{C}}(L)^{W_\theta}$, then

$$\sum_{\sigma \in W} \sigma \cdot X = |W_\theta| \partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta).$$

Proof. Item (a) is clear for $\sigma = s$ from the formula for $\partial_s(X)$ given above. The result follows by a simple induction in $\ell(\sigma)$.

We now prove part (b). The fact that $\tau \in W_v$ implies that $t_v \circ \partial_\tau = \partial_\tau \circ t_v$. Now recall from Proposition 2.1 that $\mathfrak{D}^0_\sigma = \mathsf{ev}_0 \circ \partial_\sigma$, so

$$\begin{aligned} \mathfrak{D}^v_\sigma \circ \partial_\tau &= \mathfrak{D}^0_\sigma \circ \partial_\tau \circ t_v \\ &= \mathsf{ev}_0 \circ \partial_\sigma \circ \partial_\tau \circ t_v = \begin{cases} \mathsf{ev}_0 \circ \partial_{\sigma\tau} \circ t_v = \mathfrak{D}^v_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally we prove part (c). The statement of [Hil82, Chapter IV (1.6)] implies that

$$\partial_{\omega_0} = \frac{1}{\Delta(\Phi)} \sum_{\sigma \in G} (-1)^{\ell(\sigma)} \sigma$$

as operators on L, and since the map $L\#G \longrightarrow \mathrm{End}_{\mathbb{C}}(L)$ is injective, the identity holds in $L\#G$. Using that and the fact that $\sigma \cdot \Delta(\Phi) = (-1)^{\ell(\sigma)}\Delta(\Phi)$ we deduce that

$$\sum_{\sigma \in G} \sigma \cdot X = \partial_{\omega_0}(X\Delta(\Phi))$$

for any $X \in \mathrm{End}_{\mathbb{C}}(L)$. Certainly, the analogous identity holds if we replace G by any subgroup and Φ by the corresponding root subsystem.

Let ω_0 and ω_1 be the longest elements of W and W_θ, respectively. Then $\omega_0\omega_1^{-1} \in \omega_0 W_\theta$ and its length equals $\ell(\omega_0) - \ell(\omega_1)$, the smallest possible length of an element in the coset $\omega_0 G_\theta$. Thus $\omega_0^\theta = \omega_0\omega_1^{-1}$ and

$$\sum_{\sigma\in W} \sigma \cdot X = \partial_{\omega_0}(X\Delta(\Psi)) = \partial_{\omega_0^\theta}\partial_{\omega_1}(X\Delta(\widetilde{\Psi})\Delta(\Psi)^\theta).$$

Now both $\Delta(\Psi)^\theta$ and X are W_θ-invariant, so the last expression equals

$$\partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta\partial_{\omega_1}(\Delta(\widetilde{\Psi}))) = |W_\theta|\partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta),$$

which completes the proof. ∎

The following lemma shows that the span of the BGG operators is stable under the action of certain elements of $L\#V$.

LEMMA 6.2: *Let $v \in V$ be Ω-standard, let $\sigma \in V^v, \tau \in W_v$ and let $F_z \in L$ be regular at v. Then*

$$\mathcal{D}^v_\sigma \cdot \partial_\tau(F_z t_z) = \begin{cases} \sum_{\nu\le\sigma\tau} \mathfrak{D}^v_{\nu,\sigma\tau}(F_z)\mathcal{D}^{v+z}_\nu & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As mentioned above, the fact that $\tau \in W_v$ implies that $t_v \circ \partial_\tau = \partial_\tau \circ t_v$. Thus using parts (a) and (b) of Lemma 6.1 we get

$$\begin{aligned} \mathfrak{D}^v_\sigma \circ \partial_\tau(F_z t_z) \mid_\Gamma &= \mathfrak{D}^0_\sigma \circ \partial_\tau \circ (t_v(F_z)t_{v+z}) \mid_\Gamma \\ &= \begin{cases} \mathfrak{D}^0_{\sigma\tau} \circ (t_v(F_z)t_{v+z}) \mid_\Gamma & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The result now follows by evaluation at $\gamma \in \Gamma$ using Proposition 2.1. ∎

6.2. *U*-SUBMODULE OF Γ^* ASSOCIATED TO v. Recall that to each $v \in V$ we associate the character $\mathsf{ev}_v : \Gamma \longrightarrow \mathbb{C}$ given by evaluation at v. Since Γ consists of G-symmetric polynomials, $\mathsf{ev}_v = \mathsf{ev}_{\sigma(v)}$ for any $\sigma \in G$, so we can assume that v is Ω-standard. Furthermore, note that $\mathsf{ev}_v = \mathcal{D}^v_e$ in $\mathcal{D}(\Omega, v) \subset \Gamma^*$.

Definition 6.3: Let $v \in V$ be standard. We denote by $V(\Omega, T(v))$ the complex vector subspace of Γ spanned by the set $\{\mathcal{D}^{v+z}_\sigma \mid z \in \mathsf{Z}(U),\ \sigma \in W\}$. We call $V(\Omega, T(v))$ the U**-module of BGG differential operators** associated to Ω and v (see Theorem 6.4).

Recall from part (d) of Proposition 5.4 that $\mathcal{D}_\sigma^{v+z}$ is a linear combination of operators $\mathcal{D}_\nu^{\tau(v+z)}$ with $\tau(v+z)$ the unique Ω-standard element in the W-orbit of $v+z$. Since two Ω-standard elements can be G-conjugates, the generating set given above is not necessarily a basis.

Recall that $\Phi_0(v)$ is the set of all roots in Φ such that $\alpha(v)=0$. The following theorem shows that under certain conditions the space $V(\Omega, T(v))$ is a U-module. This theorem generalizes [EMV18, Theorem 10] and [RZ18, 5.6 Theorem] to rational Galois orders.

THEOREM 6.4: *Let $v \in V$ be standard and assume that $\Phi_0(v+z) \subset \Psi$ for each $z \in \mathsf{Z}$. Then $V(\Omega, T(v)) \subset \Gamma^*$ is a Gelfand–Tsetlin U-module.*

To prove this theorem, we will first show that the generators of U can be expressed as operators of the form presented in Lemma 6.2 in a suitable way. Recall that for each $z \in V$ there exists some Ω-standard element in the orbit $W \cdot z$. Thus, given $Z \subset V$ that is stable by the action of W, we can choose a set of Ω-standard representatives of Z/W.

PROPOSITION 6.5: *Let $X \in (L\#V)^G$ and assume that there exists $\chi \in \hat{G}$ such that $Xd_\chi \in \Lambda\#V$.*

(a) *For each $z \in \operatorname{supp} X$ there exists $f_z \in \Lambda^{G_z}$ such that*

$$X = \sum_{z \in \operatorname{supp} X} \frac{f_z}{d_\chi^z} t_z,$$

where d_χ^z is the product of all $\alpha \in \Phi^+$ dividing d_χ such that $\alpha(z) \neq 0$.

(b) *Let Y be a set of Ω-standard representatives of $\operatorname{supp} X/W$, and for each $y \in Y$ denote by ω_0^y the longest element in W^y, and by $\Delta(\Psi)^y$ the product of all roots in Ψ^+ with $\alpha(y) \neq 0$. Then*

$$X = \sum_{y \in Y} \frac{1}{|W_y|} \partial_{\omega_0^y}\Big(\frac{f_y \Delta(\Psi)^y}{d_\chi^y} t_y\Big).$$

Proof. Fix $z \in \operatorname{supp} X$ and let h be the coefficient of t_z in X, which is well defined by Lemma 4.1. Since X is G-invariant we know that $\sigma \cdot X = X$ for any $\sigma \in G_z$, so $\sigma(h) = h$. Writing $h = \frac{g}{d_\chi}$ we have

$$\frac{g}{d_\chi} = \sigma \cdot \frac{g}{d_\chi} = \frac{\sigma \cdot g}{\chi(g) d_\chi}.$$

Therefore, $\sigma \cdot g = \chi(\sigma) g$ for all $\sigma \in G_z$.

Denote by χ' the restriction of χ to G_z. Observe that G_z is the reflection group generated by the reflections fixing z and it acts on Λ by restriction. Thus, by Stanley's theorem, the space of relative invariants $\Lambda^{G_z}_{\chi'}$ is generated over Λ^{G_z} by $d_{\chi'}$, and this polynomial is the product of all roots $\alpha \in \Phi^+$ dividing d_χ such that $\alpha(z) = 0$. Therefore, $g = f_z d_\chi$ for some $f_z \in \Lambda^{G_z}$, which implies that

$$\frac{g}{d_\chi} = \frac{f_z}{d_\chi/d_{\chi'}} = \frac{f_z}{d^z_\chi}.$$

This proves part (a).

Since X is G-invariant, it is clear that

$$X = \frac{1}{|W|}\sum_{\sigma\in W} \sigma \cdot X = \sum_{y\in Y} \frac{1}{|W|}\sum_{\sigma\in W} \sigma \cdot \Big(\frac{f_y}{d^y_\chi} t_y\Big).$$

As we mentioned before, the coefficient of t_y is G_y-invariant, and hence it is W_y-invariant. After applying Lemma 6.1(c) to W, we obtain

$$\sum_{\sigma\in W} \sigma \cdot \Big(\frac{f_y}{d^y_\chi} t_y\Big) = |W^y| \partial_{\omega_0^y}\Big(\frac{f_y \Delta(\Psi)^y}{d^y_\chi} t_y\Big)$$

and the result follows. ∎

Proof of Theorem 6.4. By Theorem 5.2, the action of Γ on $V(\Omega, T(v))$ is locally finite, so we only need to show that it is a U-submodule of Γ^*. Since

$$V(\Omega, T(v + z)) = V(\Omega, T(v))$$

for each $z \in \mathsf{Z}(U)$, it is enough to show that $\mathcal{D}^v_\sigma \cdot X^\dagger$ lies in $V(\Omega, T(v))$ for any generator X of U.

Set $\widetilde{W} = W_v$, and denote by $\widetilde{\Psi}$ the corresponding parabolic subgroup, by $\widetilde{\Omega}$ its base and by $\widetilde{\omega}_0$ the longest word in $\widetilde{W}$. According to part (b) of Proposition 6.5, $X^\dagger$ can be rewritten as a sum of operators of the form $\partial_{\widetilde{\omega}_0^y}(F_y t_y)$, where

$$F_y = f_y \Delta(\widetilde{\Psi})^y / d^y_\chi$$

and y is $\widetilde{\Omega}$-standard. Thus it is enough to show that

$$\mathcal{D}^v_\sigma \cdot \partial_{\widetilde{\omega}_0^y}(F_y t_y) \in \mathcal{D}(\Omega, v)$$

for y a $\widetilde{\Omega}$-standard element. By Lemma 6.2 it is enough to show that F_y is regular at v.

Recall that d_χ^y is the product of all roots α_s such that $\chi(s) = -1$ and $\alpha_s(y) \neq 0$. If one of this factors is such that $\alpha_s(v) = 0$, then $\alpha_s \in \Phi_0(v)$. Since $\Phi_0(v) \subset \Psi$ by hypothesis, it follows that $\Phi_0(v) \subset \Psi$, and hence α_s is also a factor of $\Delta(\widetilde{\Psi})^y$. Thus the term $\Delta(\widetilde{\Psi})^y$ in the numerator cancels out all the linear terms in the denominator which are zero at v and hence F_y is regular at v. ∎

7. Explicit bases and formulas for BGG modules

As mentioned earlier, our original aim was to produce modules over co-rational Galois orders containing an arbitrary character ev_v. In the previous section we showed that the module $U \cdot \mathsf{ev}_v$ is contained in the span of certain BGG operators and the action of the generators of U can be described in terms of Postnikov–Stanley operators. We now show how in some cases we can give a basis for these spaces and produce explicit formulas for the action of the generators of U. In particular, the next two propositions give bases and explicit formulas for all modules $V(\Omega, T(v))$ over the co-rational orders presented in [Har, Sections 4, 5].

We fix a co-rational Galois algebra U and denote by $\mathsf{Z} = \mathsf{Z}(U)$ the monoid generated by its support. As in the previous paragraph, we fix a subset of simple roots $\Omega \subset \Sigma$ and denote by Ψ the corresponding standard parabolic root system and by W the corresponding standard parabolic subgroup of G.

7.1. We begin by finding a basis for some modules of the form $V(\Omega, T(v))$. Since Z is stable by the action of G and acts on V by translations, the smashed product $G\#\mathsf{Z}$ acts on V and we say that $\overline{v} \in V$ is a *seed* for Z if it is Σ-standard and its stabilizer in $G\#\mathsf{Z}$ is equal to $G_{\overline{v}}$. We denote by $\mathbb{D}(G_{\overline{v}}, \mathsf{Z})$ the set of all elements $z \in \mathsf{Z}$ such that $\alpha(z) \geq 0$ for all roots in the root system associated to $G_{\overline{v}}$, that is, z lies in the positive chamber of $G_{\overline{v}}$.

PROPOSITION 7.1: *Suppose $W \subset G$ contains all elements $\sigma \in G$ acting non-trivially on Z, and let $\overline{v}$ be a seed with respect to Z. Then we have a direct sum decomposition*

$$V(\Omega, T(\overline{v})) = \bigoplus_{z \in \mathbb{D}(G_{\overline{v}}, \mathsf{Z})} \mathcal{D}(\Omega, \overline{v} + z).$$

In particular, the set $\{\mathcal{D}_\sigma^{\overline{v}+z} \mid z \in \mathbb{D}(G_{\overline{v}}, \mathsf{Z}), \sigma \in W_{\overline{v}}^z\}$ is a basis of $V(\Omega, T(\overline{v}))$.

Proof. Given $z' \in \mathsf{Z}$ there exists a unique element $z \in G_{\overline{v}} \cdot z'$ lying in the positive chamber of $G_{\overline{v}}$; furthermore, $z \in W_{\overline{v}} z'$ since W contains all elements acting nontrivially on Z. Thus by part (d) of Proposition 5.4, for each $\sigma \in W$ the BGG operator $\mathcal{D}_\sigma^{\overline{v}+z'}$ lies in the space $\mathcal{D}(\Omega, \overline{v}+z)$.

Given $z, w \in \mathbb{D}(G_{\overline{v}}, \mathcal{L})$ and $\sigma \in G$ such that $\overline{v}+z = \sigma(\overline{v}+w)$, the fact that $\overline{v}$ is a seed implies that $\sigma \in G_{\overline{v}}$. Since there is a unique element in $G_{\overline{v}} \cdot z \cap \mathbb{D}(G_{\overline{v}}, \mathcal{L})$, it follows that $z = w$. Again by Proposition 5.4, if $z \neq w$ the corresponding BGG operator spaces are eigenspaces of Γ with different eigenvalues, and hence the sum in the statement is direct. ∎

This proposition raises the question of how common the seeds for a given Z are. We leave as an exercise to the reader the following statement: if W is a crystallographic group and the projection of Z to V/V^G covers the weight lattice of W, then for every $v \in V$ there is a seed in $(G\#\mathsf{Z})v$. Thus in this case every module $V(\Omega, T(v))$ is covered by the proposition.

7.2. Let us say that $e \in V$ is **minuscule** if $\alpha(e) \in \{1, 0, -1\}$ for every root α. This is analogous to the minuscule weights in Lie theory. Given $f \in \Lambda$ and minuscule $e \in V$, we set

$$X(f, e) = \mathrm{sym}_W \left(t_e \frac{f}{\Delta(\Psi)^e} \right).$$

All the examples of co-rational Galois orders from [Har, Sections 4, 5] are generated by elements of this form. In particular, Z is generated by minuscule elements and hence $\alpha(\mathsf{Z}) \subset \mathbb{Z}$ for any root α. In this case we have explicit formulas for the action of $X(f, e)$ on a module of the form $V(\Omega, T(\overline{v}))$ in the basis given by Proposition 7.1.

PROPOSITION 7.2: *Suppose $W \subset G$ contains all elements $\sigma \in G$ acting nontrivially on Z, and let $\overline{v}$ be a seed with respect to Z. Suppose furthermore that $\alpha(\mathsf{Z}) \subset \mathbb{Z}$ for all $\alpha \in \Psi$.*

Let $f \in \Lambda$ and let $e \in V$ be a minuscule element. Then for all $z \in \mathbb{D}(G_{\overline{v}}, Z)$ and all $\sigma \in W_{\overline{v}}^z$ we have

$$X(f, e) \cdot \mathcal{D}_\sigma^{\overline{v}+z} = \sum_{y \in Y} \sum_{\tau \leq \sigma \omega_0^y} \mathfrak{D}_{\tau, \sigma\omega_0^y}^{\overline{v}+z} \left(\frac{f \Delta(\widetilde{\Psi})^y}{\Delta(\Psi)^y} \right) \mathcal{D}_\tau^{\overline{v}+z+y}$$

where $\widetilde{\Psi}$ is the root system corresponding to $W_{\overline{v}+z}$ and ω_0 is its longest element, and Y is the set elements in $W \cdot e$ lying in the positive chamber of $W_{\overline{v}+z}$ and such that

$$\ell(\sigma\omega_0^y) = \ell(\sigma) + \ell(\omega_0^y).$$

The BGG operators on the right-hand side are in the basis presented in Proposition 7.1.

Proof. The set Y in the statement is a choice of representatives of $\operatorname{supp} X$ modulo the base $\widetilde{\Omega} \subset \Omega$ of the standard parabolic root system $\widetilde{\Psi}$, so the proof follows the same reasoning as the one of Theorem 6.4.

To see that $\overline{v} + z + y$ lies in the positive chamber of W, notice that if $\alpha \in \Psi$ is such that $\alpha(\overline{v} + z) > 0$, then the fact that e is minuscule implies that $\alpha(\overline{v} + z + y) \geq 0$ for all $y \in Y$. On the other hand, if $\alpha(\overline{v} + z) = 0$, then by the choice of y we have

$$\alpha(\overline{v} + z + y) = \alpha(y) \geq 0. \qquad \blacksquare$$

7.3. We conclude this article with a simplicity criterion for modules over a special class of Galois orders. Again, this covers the examples given above. This criterion was originally formulated for OGZ algebras by Early, Mazorchuk and Vishnyakova in [EMV18, Theorem 4.5]. With the aid of Propositions 5.4, 7.1 and 7.2, the proof reduces to the same combinatorial argument given in the reference so we omit it.

Corollary 7.3: *Suppose that*

$$V = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_r}$$

and that

$$G = S_{n_1} \times \cdots \times S_{n_r}$$

with $n_i \in \mathbb{N}$. Let $e_{k,i}$ be the i-th element in the canonical basis of $\mathbb{C}^{n_i}$ and let U be the co-rational Galois order generated by elements

$$X_i = X(f_i, e_{k,1}); \quad Y_i = Y(g_i, e_{k,n_k}) \quad (1 \leq i \leq k \leq r).$$

If $\overline{v}$ is a seed and f_i, g_i are never zero in $\overline{v} + \mathsf{Z}$, then $V(\Omega, T(\overline{v}))$ is a simple module.

For a refined version of this corollary for $U = U(\mathfrak{gl}(n, \mathbb{C}))$ see [FGRZ20, Section 4].

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