ISRAEL JOURNAL OF MATHEMATICS **238** (2020), 639–661
DOI: 10.1007/s11856-020-2034-8

PROOF OF THE KALAI–MESHULAM CONJECTURE

BY

MARIA CHUDNOVSKY*

Department of Mathematics, Fine Hall, Washington Road, Princeton University, Princeton, NJ 08544, USA
e-mail: mchudnov@math.princeton.edu

AND

ALEX SCOTT**

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK
e-mail: scott@maths.ox.ac.uk

AND

PAUL SEYMOUR† AND SOPHIE SPIRKL

Department of Mathematics, Fine Hall, Washington Road, Princeton University, Princeton, NJ 08544, USA
e-mail: pds@math.princeton.edu, sspirkl@math.princeton.edu

ABSTRACT

Let G be a graph, and let f_G be the sum of $(-1)^{|A|}$, over all stable sets A. If G is a cycle with length divisible by three, then $f_G = \pm 2$. Motivated by topological considerations, G. Kalai and R. Meshulam [8] made the conjecture that, if no induced cycle of a graph G has length divisible by three, then $|f_G| \leq 1$. We prove this conjecture.

* Supported by NSF Grant DMS-1763817. This material is based upon work supported in part by the U.S. Army Research Laboratory and the U.S. Army Research Office under grant number W911NF-16-1-0404.
** Supported by a Leverhulme Trust Research Fellowship.
† Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.
Received July 17, 2018 and in revised form April 29, 2019

1. Introduction

In the late 1990's, G. Kalai and R. Meshulam [8] made an intriguing sequence of conjectures about the connections between induced cycle lengths, chromatic number, and the number of stable sets of different parities in a graph.

A graph is **ternary** if no induced cycle has length a multiple of three; thus, ternary graphs have no triangles. (All graphs in this paper are finite and have no loops or parallel edges.) First, Kalai and Meshulam conjectured:

1.1: *There exists c such that every ternary graph is c-colourable.*

This was proved by Bonamy, Charbit and Thomassé [1], for some large constant c (although it may be that all ternary graphs are 3-colourable, and this remains open). A much stronger result was later proved by two of us [9]: that for all integers $p, q \geq 0$, every graph with bounded clique number and with no induced cycle of length p modulo q has bounded chromatic number.

Second, Kalai and Meshulam conjectured:

1.2: *For every ternary graph, the number of stable sets with even cardinality and the number with odd cardinality differ by at most one.*

This has remained open, and we prove it in this paper.

Two further conjectures of Kalai and Meshulam were proved in [9]. The stronger of these conjectures stated that for all k there exists c, such that, if for every induced subgraph of G the number of even stable sets and the number of odd ones differ by at most k, then G is c-colourable. This follows from a generalization of the strengthening of 1.1 mentioned above.

A final Kalai–Meshulam conjecture concerns Betti numbers and ternary graphs. The **independence complex** $I(G)$ of a graph G is the simplicial complex whose faces are the stable sets of vertices of G. Let b_i denote the ith Betti number of $I(G)$ and let $b(G)$ denote the sum of the Betti numbers.

Conjecture 1.3: *A graph G is ternary if and only if $b(H) \leq 1$ for every induced subgraph H.*

Let $f_G(\emptyset)$ denote the number of even stable sets in G minus the number of odd ones. If $b(H) \leq 1$ for every induced subgraph H, then G has no induced cycle of length divisible by 3, since $b(H) = 2$ for every cycle H of length divisible by three. For the converse, suppose G has no such induced cycle. Then by 1.2, $|f_G(\emptyset)| \leq 1$, but we need to prove that $b(G) \leq 1$. Now $f_G(\emptyset)$ is the Euler characteristic of $I(G)$, and in particular there is a connection between $f_G(\emptyset)$ and $b(G)$. It is a basic theorem from homology theory that the Euler characteristic of $I(G)$ is the alternating sum of the Betti numbers of $I(G)$ (see [6]). It follows that $|f_G(\emptyset)| \leq b(G)$; but this inequality is in the wrong direction for us, and the conjecture remains open.

We mention a few other related results:

- Chen and Saito [3] proved that every non-null graph with no cycle of length divisible by three (not just induced cycles) has a vertex of degree at most two (and so all such graphs are 3-colourable).
- G. Gauthier [5] found an explicit construction for all graphs with no cycle of length divisible by three.
- D. Král' asked (unpublished): is it true that in every ternary graph with an edge, there is an edge e such that the graph obtained by deleting e is also ternary? This would have implied that all ternary graphs are 3-colourable, but has very recently been disproved; a counterexample was found by M. Wrochna. (Take the disjoint union of a 5-cycle and a 10-cycle, and join each vertex of the 5-cycle to two opposite vertices of the 10-cycle, in order.)
- The difference between the numbers of odd and even stable sets has also appeared in statistical physics. Let us define the polynomial

$$I_G(z) = \sum_I z^{|I|},$$

  where the sum is over stable sets I in G. This polynomial is known in combinatorics as the **independent set polynomial** and in statistical physics as the **partition function of the hard-core lattice gas** (see, for instance, [10]). We see that $I_G(-1)$ is the number of even stable sets minus the number of odd stable sets. The question of when $|I_G(-1)| \leq 1$ has been the focus of considerable study, particularly on the square lattice (see [2, 4, 7]).

If G is a graph, and X, Y are disjoint subsets of $V(G)$, let $f_G(X,Y)$ be the sum of $(-1)^{|A|}$, summed over all stable sets A in G that include X and are disjoint from Y. Our main theorem states:

1.4: *If G is ternary then $|f_G(\emptyset, \emptyset)| \leq 1$.*

The proof of 1.4 is by induction on $|V(G)|$, and it follows easily that if G is a minimum counterexample then $f_G(\emptyset, \emptyset) = \pm 2$. It is very helpful to know the value of $f_G(\emptyset, \emptyset)$, and so the proof breaks into two cases, depending whether this value is 2 or -2. The proof for the second is obtained from the first proof by negating f_G throughout, and we would like to say "we may assume that $f_G(\emptyset, \emptyset) = 2$ without loss of generality"; but this gives us a difficulty, because negating f_G does not give a function that equals f_H for some graph H. We overcome this as follows.

Let G be a graph, and with f_G as before, let us say the functions f_G and $-f_G$ are **counters** on G. We will prove that if G is ternary and g is a counter on G, then $|g(\emptyset, \emptyset)| \leq 1$. Now we are free to replace g by its negative if that is convenient.

We will frequently need to talk about $g(X, Y)$ when $Y = \emptyset$; so often that it is worthwhile to make a special convention for it. We define $g(X) = g(X, \emptyset)$ (and the same for f_G).

If g is a counter on G, we say g is a *good* counter if for all disjoint $X, Y \subseteq V(G)$ with $X \cup Y \neq \emptyset$:

- $|g(X, Y)| \leq 1$; and
- $|g(X \cup \{u\}, Y) - g(X \cup \{v\}, Y)| \leq 1$ for all $u, v \in V(G) \setminus (X \cup Y)$.

In Section 3, we show that:

1.5: *If g is a good counter on a graph G, then $|g(\{u\}) - g(\{v\})| \leq 1$ for all $u, v \in V(G)$.*

Then in Section 4, we show that:

1.6: *If g is a good counter on a ternary graph G, then $|g(\emptyset)| \leq 1$.*

Proof of 1.4, assuming 1.5 and 1.6. We prove by induction on $|V(G)|$ that for every ternary graph G, if g is a counter on G, then $|g(\{u\}) - g(\{v\})| \leq 1$ for all $u, v \in V(G)$, and $|g(\emptyset)| \leq 1$. Thus we may assume that these two statements hold for every proper induced subgraph of G. Now g is a counter on G, and so $g = \pm f_G$. If the result holds for $-g$ then it holds for g; so we may assume that $g = f_G$, by replacing g by $-g$ if necessary.

(1) *If $X, Y \subseteq V(G)$ are disjoint, with $X \cup Y \neq \emptyset$, then $|f_G(X, Y)| \leq 1$.*

We may assume that X is a stable set. Let H be the graph obtained from G by deleting $X \cup Y$ and deleting all vertices with a neighbour in X. Thus, if A

is a stable set of G including X and disjoint from Y, then $A \setminus X$ is a stable set of H; and conversely, if B is a stable set of H, then $X \cup B$ is a stable set of G including X and disjoint from Y. In particular, $f_H(\emptyset) = (-1)^{|X|} f_G(X, Y)$; but from the inductive hypothesis, $|f_H(\emptyset)| \leq 1$, and so $|f_G(X, Y)| \leq 1$. This proves (1).

(2) *If $X, Y \subseteq V(G)$ are disjoint, with $X \cup Y \neq \emptyset$, and $u, v \in V(G) \setminus (X \cup Y)$, then*

$$|f_G(X \cup \{u\}, Y) - f_G(X \cup \{v\}, Y)| \leq 1.$$

We may assume that X is stable. Suppose first that u has a neighbour in X. Then $f_G(X \cup \{u\}, Y) = 0$ (because $X \cup \{u\}$ is not a subset of any stable set). Also $|f_G(X \cup \{v\}, Y)| \leq 1$, by (1), and the claim follows. So we may assume that u and similarly v has no neighbour in X; and so $u, v \in V(H)$, if we define H as before. Thus

$$f_G(X \cup \{u\}, Y) = (-1)^{|X|} f_H(\{u\}) \quad \text{and} \quad f_G(X \cup \{v\}, Y) = (-1)^{|X|} f_H(\{v\});$$

and from the inductive hypothesis, $|f_H(\{u\}) - f_H(\{v\})| \leq 1$. It follows that

$$|f_G(X \cup \{u\}, Y) - f_G(X \cup \{v\}, Y)| \leq 1.$$

This proves (2).

From (1) and (2), g is a good counter on G. From 1.6 and 1.5, it follows that $|g(\{u\}) - g(\{v\})| \leq 1$ for all $u, v \in V(G)$, and $|g(\emptyset)| \leq 1$. This completes the inductive proof; and 1.4 follows. ∎

2. Some lemmas

Here are a few useful lemmas. First, we observe:

2.1: *Let g be a counter on G, let $X, Y \subseteq V(G)$ be disjoint, and let $Y' \subseteq Y$. Then*

$$g(X, Y) = \sum_{Z \subseteq Y \setminus Y'} (-1)^{|Z|} g(X \cup Z, Y').$$

Proof. We may assume that $g = f_G$, by replacing g by $-g$ if necessary. If A is a stable set of G including X and disjoint from Y', define n_A to be

$$\sum_{Z \subseteq A \cap Y} (-1)^{|A| - |Z|}.$$

Thus $n_A = 0$ unless $A \cap Y = \emptyset$, in which case $n_A = (-1)^{|A|}$. But

$$\sum_{Z \subseteq Y \setminus Y'} (-1)^{|Z|} f_G(X \cup Z, Y')$$

is the sum of n_A, over all stable sets A of G including X and disjoint from Y'. It follows that $\sum_{Z \subseteq Y \setminus Y'} (-1)^{|Z|} f_G(X \cup Z, Y')$ is the sum of $(-1)^{|A|}$ over all stable sets of G that include X and are disjoint from Y. But this sum equals $f_G(X, Y)$. This proves 2.1. ∎

In evaluating an expression given by 2.1, it often happens that for some number ℓ, $g(X \cup Z) = \ell$ for "most" subsets $Z \subseteq Y$, and if so the following is helpful:

2.2: *Let g be a counter on G, let $X, Y \subseteq V(G)$ be disjoint, with $Y \neq \emptyset$, and let ℓ be some number. Then*

$$g(X, Y) = \sum_{Z \subseteq Y} (-1)^{|Z|} (g(X \cup Z) - \ell).$$

Proof. By 2.1,

$$g(X, Y) = \sum_{Z \subseteq Y} (-1)^{|Z|} g(X \cup Z),$$

and $\sum_{Z \subseteq Y} (-1)^{|Z|} (-\ell) = 0$ since $Y \neq \emptyset$. This proves 2.2. ∎

2.3: *Let g be a good counter on G, let $X, Y \subseteq V(G)$ be disjoint, and let $v \in V(G) \setminus (X \cup Y)$. Then*

$$|g(X, Y) - g(X \cup \{v\}, Y)| \leq 1 \quad \textit{and} \quad |g(X, Y) - g(X, Y \cup \{v\})| \leq 1.$$

Proof. We may assume that $g = f_G$. Every stable set including X and disjoint from Y either includes $X \cup \{v\}$ or is disjoint from $Y \cup \{v\}$, and not both. Consequently

$$g(X, Y) = g(X \cup \{v\}, Y) + g(X, Y \cup \{v\}).$$

But $|g(Y \cup \{v\})| \leq 1$ since g is a good counter, and therefore

$$|g(X, Y) - g(X \cup \{v\}, Y)| \leq 1;$$

and the second claim follows similarly. ∎

For $X \subseteq V(G)$, let $N[X]$ denote the set of vertices in G that either belong to X or have a neighbour in X. We observe that

2.4: *Let g be a counter on G. If $X, Y \subseteq V(G)$ are disjoint with $g(X,Y) \neq 0$, and $v \in V(G) \setminus (N[X] \cup Y)$, then v has a neighbour in $V(G) \setminus (N[X] \cup Y)$.*

Proof. We may assume that $g = f_G$, by replacing g by $-g$ if necessary. The stable sets of G that include X and are disjoint from Y are obtained from the stable sets of $G \setminus (N[X] \cup Y)$ ($= H$ say) by adding the set X to each such stable set; and so $f_H(\emptyset) \neq 0$. But $f_K(\emptyset) = 0$ for every graph K with a vertex of degree zero, and so H has no vertex of degree zero. The result follows. ∎

2.5: *Let g be a good counter on G, let $X, Y \subseteq V(G)$ be disjoint, and let $u, v \in V(G) \setminus (X \cup Y)$. If $g(X,Y) = g(X \cup \{u,v\}, Y) \neq 0$, then*

$$g(X,Y) = g(X \cup \{v\}, Y).$$

Proof. We proceed by induction on $|V(G) \setminus (X \cup Y)|$. By replacing g by $-g$ if necessary we may assume that $g(X,Y) > 0$. For all disjoint

$$A, B \subseteq V(G) \setminus (X \cup Y),$$

let $h(A,B) = g(X \cup A, Y \cup B)$ (and $h(A)$ means $h(A, \emptyset)$). Since g is a good counter it follows that $|h(\{u,v\})| \leq 1$, and so

$$h(\{u,v\}) = h(\emptyset) = 1.$$

We suppose for a contradiction that $h(\{v\}) \neq 1$. Hence $u \neq v$, and $X \cup \{u,v\}$ is stable. By 2.3, it follows that $h(\{v\}) = 0$. Since $|h(\emptyset, \{u,v\})| \leq 1$, 2.1 implies that

$$h(\emptyset) - h(\{u\}) - h(\{v\}) + h(\{u,v\}) \leq 1.$$

Consequently $h(\{u\}) \geq 1$, and so $h(\{u\}) = 1$. From 2.4, v has a neighbour w.

Now

$$h(\emptyset, \{v\}) = h(\emptyset) - h(\{v\}) = 1,$$

and

$$h(\{u\}, \{v\}) = h(\{u\}) - h(\{u,v\}) = 0,$$

and so from the inductive hypothesis, $h(\{u,w\}, \{v\}) \neq 1$. Consequently

$$h(\{u,w\}) - h(\{u,v,w\}) \neq 1,$$

and since $h(\{u,v,w\}) = 0$, it follows that $h(\{u,w\}) \neq 1$. By 2.3, $h(\{u,w\}) = 0$. Thus $h(\{u\},\{w\}) = 1$ by 2.1, since $h(\{u\}) = 1$. Since

$$h(\{v\},\{w\}) = 0 \quad \text{and} \quad h(\{u,v\},\{w\}) = 1$$

by 2.1 (the first since $h(\{v,w\}) = 0$ and $h(\{v\}) = 0$, and the second since $h(\{u,v,w\}) = 0$ and $h(\{u,v\}) = 1$), it follows from the inductive hypothesis that $h(\emptyset, \{w\}) \neq 1$, and so $h(\emptyset, \{w\}) = 0$ by 2.3. Hence $h(\emptyset) - h(\{w\}) = 0$ by 2.1, and so $h(\{w\}) = 1$. But then $h(\{w\}, \{u\}) = 1$, because $h(\{u,w\}) = 0$; and $h(\{v\}, \{u\}) = -1$, since $h(\{v\}) = 0$ and $h(\{u,v\}) = 1$. This contradicts that g is good, and so proves 2.5. ∎

The next result has been independently discovered several times.

2.6: *Let G be a non-null graph and let A_1, A_2, A_3 be the classes of a 3-colouring of G. Suppose that for $i = 1,2,3$, every vertex in A_i has a neighbour in A_{i+1}, where A_4 means A_1. Then G is not ternary.*

Proof. Throughout we read subscripts modulo 3. For $i = 1,2,3$, direct each edge of G between A_i and A_{i+1} from A_i to A_{i+1}. Since each vertex has positive outdegree, the digraph we form has a directed cycle, and hence an induced directed cycle. But such a cycle is an induced cycle of G, and has length a multiple of three. ∎

2.7: *Let $\mathcal{H}$ be a set of subsets of some set V, all of the same cardinality k; and suppose that for every subset $X \subseteq V$ with $|X| = k+1$, if X includes a member of $\mathcal{H}$ then it includes at least two such members. Then there is a partition $P_1, \ldots, P_n$ of V with $P_1, \ldots, P_n$ all nonempty, such that for all distinct $u, v \in V$, either there exists $i \in \{1, \ldots, n\}$ with $u, v \in P_i$, or there exists $B \in \mathcal{H}$ with $u, v \in B$, and not both.*

Proof. Say two vertices $u, v \in V$ are **equivalent** if either $u = v$, or:

- there is no member of $\mathcal{H}$ containing both u, v; and
- for each $C \subseteq V \setminus \{u,v\}$, $C \cup \{u\} \in \mathcal{H}$ if and only if $C \cup \{v\} \in \mathcal{H}$.

We claim that this is an equivalence relation. To see this, we may assume that $u, v, w \in V(G)$ are distinct, and v is equivalent to both u and w; and we must show that u, w are equivalent. If there exists $B \in \mathcal{H}$ containing u, w, then $v \notin B$ (since u, v are equivalent) and so $(B \setminus \{u\}) \cup \{v\} \in \mathcal{H}$ (since $(B \setminus \{u\}) \cup \{u\} \in \mathcal{H}$ and u, v are equivalent), and so this is a member of $\mathcal{H}$ containing v, w, a contradiction. Thus there is no such B. Let $C \subseteq V \setminus \{u, w\}$, with $C \cup \{u\} \in \mathcal{H}$. Consequently $v \notin C$, and $C \cup \{v\} \in \mathcal{H}$ (because u, v are equivalent), and consequently $C \cup \{w\} \in \mathcal{H}$ (since v, w are equivalent). Similarly $C \cup \{u\} \in \mathcal{H}$ if and only if $C \cup \{w\} \in \mathcal{H}$. This proves that equivalence is indeed an equivalence relation.

We claim that for all distinct $u, v \in V$, if they do not belong to the same equivalence class then some member of $\mathcal{H}$ contains both u, v. To see this, since u, v are not equivalent, if no member of $\mathcal{H}$ contains both u and v, then we may assume (exchanging u, v if necessary) that there exists $C \subseteq V \setminus \{u, v\}$ such that $C \cup \{u\} \in \mathcal{H}$ and $C \cup \{v\} \notin \mathcal{H}$. Thus $|C| = k - 1$, and since $C \cup \{u, v\}$ includes a member of $\mathcal{H}$, by hypothesis it includes at least two members. But since no member of $\mathcal{H}$ contains both u, v, and $C \cup \{v\} \notin \mathcal{H}$, this is impossible. This proves 2.7. ∎

3. The value on distinct vertices

In this section we prove 1.5. Thus, throughout this section, let g be a good counter on a graph G. For $i = -1, 0, 1$ let A_i be the set of vertices v of G such that $g(\{v\}) = i$. Thus A_{-1}, A_0, A_1 are disjoint and have union $V(G)$. We need to show that one of A_{-1}, A_1 is empty, and so we assume for a contradiction that they are both nonempty. We will prove a series of statements about G, g. We begin with:

3.1: *The following hold:*

- $g(\emptyset) = 0$;
- G *is connected;*
- A_1, A_{-1} *are both stable sets;*
- *there is not both an edge between* A_1, A_0 *and an edge between* A_{-1}, A_0.

Proof. Since there exists $v \in A_1$, and hence with $g(\{v\}) = 1$, we deduce from 2.3 that $g(\emptyset) \geq 0$, and similarly $g(\emptyset) \leq 0$. This proves the first statement.

For the second statement, we may assume (replacing g by $-g$ if necessary) that $g = f_G$. By assumption, there exist $u_i \in V(G)$ with $g(\{u_i\}) = i$, for $i \in \{1, -1\}$. Suppose that G is not connected, and let G_1 be a component of G containing u_1, and let G_2 be obtained from G by deleting G_1. Write g_i for $f_{G_i} (i = 1, 2)$. Thus for disjoint $X, Y \subseteq V(G)$,

$$g(X, Y) = g_1(X \cap V(G_1), Y \cap V(G_1))g_2(X \cap V(G_2), Y \cap V(G_2)),$$

and in particular, $g_1(X) = g(X, V(G_2))$ for $X \subseteq V(G_1)$, and $g_2(X) = g(X, V(G_1))$ for $X \subseteq V(G_2)$. Since $0 = g(\emptyset) = g_1(\emptyset)g_2(\emptyset)$, one of $g_1(\emptyset), g_2(\emptyset)$ is zero.

Since $g(\{u_1\}) = g_1(\{u_1\})g_2(\emptyset)$, it follows that $g_2(\emptyset) \neq 0$, and so $g_1(\emptyset) = 0$. In particular, G_1 is the unique component C of G such that $f_C(\emptyset) = 0$, and so $u_{-1} \in V(G_1)$. Thus

$$g(\{u_{-1}\}) = g_1(\{u_{-1}\})g_2(\emptyset),$$

and so one of $g_1(\{u_1\}), g_1(\{u_{-1}\})$ equals 1 and the other equals -1, contradicting that g is good. This proves the second statement.

For the third, suppose that $u, v \in A_1$ are adjacent. By 2.1,

$$g(\emptyset, \{u, v\}) = g(\emptyset) - g(\{u\}) - g(\{v\}) + g(\{u, v\});$$

but the last term is zero since u, v are adjacent, and since $u, v \in A_1$ and $g(\emptyset) = 0$, we deduce that $g(\emptyset, \{u, v\}) = -2$, contradicting that g is good.

For the fourth statement, suppose that $u_1 \in A_1$ is adjacent to $v_1 \in A_0$, and $u_{-1} \in A_{-1}$ is adjacent to $v_{-1} \in A_0$. Suppose first that $g(\{v_1, u_{-1}\}) = 0$. Then by two applications of 2.1,

$$g(\{u_{-1}\}, \{v_1\}) = g(\{u_{-1}\}) - g(\{u_{-1}, v_1\}) = -1,$$

and

$$g(\{u_1\}, \{v_1\}) = g(\{u_1\}) - g(\{u_1, v_1\}) = 1$$

(since u_1, v_1 are adjacent), contradicting that g is good. This proves that $g(\{v_1, u_{-1}\}) \neq 0$, and so

$$g(\{v_1, u_{-1}\}) = -1$$

by 2.3. Similarly

$$g(\{v_{-1}, u_1\}) = 1$$

(and in particular, $v_1 \neq v_{-1}$). But by 2.1,

$$g(\{v_1\}, \{u_1, u_{-1}\}) = g(\{v_1\}) - g(\{v_1, u_1\}) - g(\{v_1, u_{-1}\}) + g(\{v_1, u_1, u_{-1}\});$$

and since $g(\{v_1\}) = 0$ and $g(\{v_1, u_1\}) = g(\{v_1, u_1, u_{-1}\}) = 0$ (since u_1, v_1 are adjacent) it follows that

$$g(\{v_1\}, \{u_1, u_{-1}\}) = 1.$$

Similarly

$$g(\{v_{-1}\}, \{u_1, u_{-1}\}) = -1,$$

contradicting that g is good. This proves 3.1. ∎

In the same notation, because of the fourth statement of 3.1, we may assume (replacing g by $-g$ if necessary) that there are no edges between A_{-1} and A_0. Let B_1 be the set of vertices $v \in A_0$ such that $g(\{u, v\}) = 1$ for each $u \in A_1$ and $g(\{u, v\}) = 0$ for each $u \in A_{-1}$; and let B_{-1} be the set of vertices $v \in A_0$ such that $g(\{u, v\}) = 0$ for each $u \in A_1$ and $g(\{u, v\}) = -1$ for each $u \in A_{-1}$.

3.2: *Every vertex in A_0 belongs to one of B_1, B_{-1}.*

Proof. Let $v \in A_0$, and for $i \in \{1, -1\}$ let $u_i \in A_i$. Not both $g(\{v, u_1\}) = 1$ and $g(\{v, u_{-1})) = -1$, since g is good. Suppose that neither of these holds. Then $g(\{v, u_1\}) = 0$ and $g(\{v, u_{-1})) = 0$, by 2.3. Then by two applications of 2.1,

$$g(\{u_1\}, \{v\}) = g(\{u_1\}) - g(\{u_1, v\}) = 1,$$

and

$$g(\{u_{-1}\}, \{v\}) = g(\{u_{-1}\}) - g(\{u_{-1}, v\}) = -1,$$

contradicting that g is good. It follows that either $g(\{v, u_1\}) = 1$ and $g(\{v, u_{-1}\}) = 0$, or $g(\{v, u_1\}) = 0$ and $g(\{v, u_{-1}\}) = -1$. Since this holds for all u_1, u_{-1}, it follows that $v \in B_1 \cup B_{-1}$. This proves 3.2. ∎

3.3: *A_0 is empty.*

Proof. Suppose that $A_0 \neq \emptyset$. Since G is conected by 3.1, and by assumption there are no edges between A_{-1} and A_0, it follows that there is an edge between A_0 and A_1, say between $b \in A_0$ and $a_1 \in A_1$. Consequently $g(\{a_1, b\}) = 0$, and so $b \notin B_1$ from the definition of B_1; and so $b \in B_{-1}$ by 3.2. Choose $a_{-1} \in A_{-1}$. By three applications of 2.1,

$$\begin{aligned} g(\emptyset, \{a_1\}) =& g(\emptyset) - g(\{a_1\}) = -1, \\ g(\{b\}, \{a_1\}) =& g(\{b\}) - g(\{b, a_1\}) = 0, \text{ and} \\ g(\{b, a_{-1}\}, \{a_1\}) =& g(\{b, a_{-1}\}) - g(\{b, a_1, a_{-1}\}) = -1, \end{aligned}$$

contrary to 2.5. Thus $A_0 = \emptyset$. This proves 3.3. ∎

Now we prove 1.5, which we restate:

3.4: *If g is a good counter on a graph G, then*

$$|g(\{u\}) - g(\{v\})| \leq 1$$

for all $u, v \in V(G)$.

Proof. As all through this section, we assume that G, g is a counterexample. In the previous notation, 3.3 and 3.1 imply that G is bipartite, and (A_1, A_{-1}) is a bipartition. We recall that $g(\emptyset) = 0$.

(1) *Every vertex of G has degree at least two.*

Since G is connected by 3.1, all vertices have degree at least one; suppose that $v \in A_1$ has only one neighbour $u \in A_{-1}$ say. Since G is connected and $|V(G)| \geq 3$, u has another neighbour $v' \in A_1$. Now $g(\{v'\}) = 1$, and since $v \in V(G) \setminus N[\{v'\}]$, 2.4 implies that v has a neighbour in $V(G) \setminus N[\{v'\}]$, a contradiction. This proves (1).

(2) *For $i = 1, -1$ there is a subset $X \subseteq A_i$ with $g(X) = 0$.*

Choose $v \in A_i$, and let $X = A_i \setminus \{v\}$. Since $v \in V(G) \setminus N[X]$, and v has no neighbour in $V(G) \setminus N[X]$ (by (1)), 2.4 implies that $g(X) = 0$. This proves (2).

For $i \in \{1, -1\}$ let $k_i > 0$ be minimum such that some subset B of A_i with cardinality k_i satisfies $g(B) \neq i$. Thus $k_i \geq 2$; and by 2.3, $g(B) = 0$ or i for each subset $B \subseteq A_i$ with $|B| = k_i$.

(3) *For $i \in \{1, -1\}$, k_i is odd.*

Choose $B \subseteq A_i$ with cardinality k_i such that $g(B) \neq i$, and hence $g(B) = 0$. Since g is good, $|g(\emptyset, B)| \leq 1$; and so by 2.2,

$$\left| \sum_{Z \subseteq B} (-1)^{|Z|}(g(Z) - i) \right| \leq 1.$$

But $g(Z) = i$ for all $Z \subseteq B$ with $Z \neq B, \emptyset$, and $g(Z) = 0$ if $Z = B, \emptyset$; and consequently

$$| - i - i(-1)^{k_i}| \leq 1,$$

and so k_i is odd. This proves (3).

Let $\mathcal{H}_i$ be the set of all subsets B of A_i such that $|B| = k_i$ and $g(B) = 0$. Thus $\mathcal{H}_i \neq \emptyset$.

(4) *For every subset X of A_i with cardinality $k_i + 1$, if X includes a member of $\mathcal{H}_i$ then it includes at least two such members.*

Let $X = \{v_0, \ldots, v_{k_i}\}$, and suppose that $\{v_1, \ldots, v_{k_i}\}$ is the only member of $\mathcal{H}_i$ included in X. Then $g(X) \neq i$, by 2.5, and $g(X) \neq -i$ by 2.3; so $g(X) = 0$.

Let $Y = \{v_2, \dots, v_{k_i}\}$. By 2.2 and (3):

$$\begin{aligned} g(\emptyset, Y) &= \sum_{Z \subseteq Y} (-1)^{|Z|}(g(Z) - i) = -i, \\ g(\{v_0\}, Y) &= \sum_{Z \subseteq Y} (-1)^{|Z|}(g(Z \cup \{v_0\}) - i) = 0, \\ g(\{v_0, v_1\}, Y) &= \sum_{Z \subseteq Y} (-1)^{|Z|}(g(Z \cup \{v_0, v_1\}) - i) = -(-1)^{|Y|} i = -i, \end{aligned}$$

contrary to 2.5. This proves (4).

(5) *There exist $B_i \in \mathcal{H}_i$ for $i \in \{1, -1\}$, such that there are two edges of G between B_1 and B_{-1} with no end in common.*

By (4) and 2.7, there is a partition $P_1, \dots, P_m$ of A_1 such that every two vertices in A_1 either belong to the same P_i or to some member of $\mathcal{H}_1$, and not both; and let $Q_1, \dots, Q_n \subseteq A_{-1}$ be defined analogously. For $i = 1, 2$, since $\mathcal{H}_i \neq \emptyset$, and $k_i \geq 2$, it follows that $m, n \geq 2$. Say P_i, Q_j are adjacent if there is an edge in G between a vertex in P_i and a vertex in Q_j. Since $m, n \geq 2$ and each P_i is adjacent to some Q_j and vice versa, there are distinct P_1, P_2 (say) and distinct Q_1, Q_2 such that P_1 is adjacent to Q_1 and P_2 to Q_2. Choose $p_i \in P_i$ and $q_i \in Q_i (i = 1, 2)$ such that $p_1 q_1$ and $p_2 q_2$ are edges of G. Since p_1, p_2 do not belong to the same one of $P_1, \dots, P_m$, there exists $B_1 \in \mathcal{H}_1$ containing p_1, p_2; and similarly there exists $B_{-1} \in \mathcal{H}_{-1}$ containing q_1, q_2. This proves (5).

For $i \in \{1, -1\}$ choose B_i as in (5).

(6) *For $i \in \{1, -1\}$, let $X_i \subseteq B_i$ with $\emptyset \neq X_i \neq B_i$. Then $g(X_1 \cup X_{-1}) = 0$.*

Suppose not, and for $i \in \{1, -1\}$ choose $X_i \subseteq B_i$ with $\emptyset \neq X_i \neq B_i$, with $X_1 \cup X_{-1}$ minimal such that $g(X_1 \cup X_{-1}) \neq 0$. We may assume that $g(X_1 \cup X_{-1}) = 1$, by replacing g by $-g$ if necessary. By 2.1 and the minimality of $X_1 \cup X_{-1}$,

$$g(X_1, X_{-1}) = g(X_1) + (-1)^{|X_{-1}|} g(X_1 \cup X_{-1}) = 1 + (-1)^{|X_{-1}|},$$

and so $|X_{-1}|$ is odd; and similarly $|X_1|$ is even. Choose $u \in X_1$ and $v \in X_{-1}$. Then by three applications of 2.1,

$$\begin{aligned} g(X_1 \setminus \{u\}, X_{-1} \setminus \{v\}) =& g(X_1 \setminus \{u\}) = 1, \\ g((X_1 \cup \{v\}) \setminus \{u\}, X_{-1} \setminus \{v\}) =& 0, \\ g(X_1 \cup \{v\}, X_{-1} \setminus \{v\}) =& (-1)^{|X_{-1} \setminus \{v\}|} g(X_1 \cup X_{-1}) = 1, \end{aligned}$$

contrary to 2.5. This proves (6).

Choose $C_1 \subseteq B_1$ maximal such that either $C_1 = \emptyset$ or $g(C_1 \cup B_{-1}) \neq 0$, and choose $C_{-1} \subseteq B_{-1}$ maximal such that either $C_{-1} = \emptyset$ or $g(C_{-1} \cup B_1) \neq 0$. It follows that $|C_i| \leq k_i - 2$ for $i \in \{1, -1\}$, since there is a 2-edge matching between B_1, B_{-1}. For $i \in \{1, -1\}$ let $D_i = B_i \setminus C_i$, and let $C = C_1 \cup C_{-1}$ and $D = D_1 \cup D_{-1}$.

(7) *If $C_1 \neq \emptyset$ then $g(C_1 \cup B_{-1}) = 1$; and if $C_{-1} \neq \emptyset$ then $g(C_{-1} \cup B_1) = -1$.*

Since $g(C_1, B_{-1}) \neq 2$ (because g is good), and $g(C_1 \cup Z) = 0$ for all $Z \subseteq B_{-1}$ with $Z \neq \emptyset, B_{-1}$ by (6), 2.1 implies that $g(C_1)+(-1)^{|k_{-1}|}g(C_1 \cup B_{-1}) \leq 1$. But $g(C_1) = 1$ (since $C_1 \neq \emptyset$), and k_1 is odd, and so $g(C_1 \cup B_{-1}) = 1$. Similarly if $C_{-1} \neq \emptyset$ then $g(C_{-1} \cup B_1) = -1$. This proves (7).

(8) *One of C_1, C_{-1} is empty.*

Suppose they are both nonempty. By 2.1,

$$g(C, D) = \sum_{Z \subseteq D} (-1)^{|Z|} g(C \cup Z).$$

But for $Z \subseteq D$, $g(C \cup Z) \neq 0$ only if Z includes one of D_1, D_{-1} by (6), and only if one of $Z \cap B_1, Z \cap B_{-1}$ is empty (from the definition of C_1, C_{-1}); that is, only if Z is one of D_1, D_{-1}. These two sets are distinct, since they are nonempty. Consequently

$$g(C, D) = (-1)^{|D_1|} g(B_1 \cup C_{-1}) + (-1)^{|D_{-1}|} g(B_{-1} \cup C_1)$$

and so by (7), $g(C, D) = (-1)^{|D_1|+1} + (-1)^{|D_{-1}|}$. Since $|g(C, D)| \leq 1$ (because g is good) it follows that $|D_1|, |D_{-1}|$ have the same parity.

Choose $u \in D_1$ and $v \in D_{-1}$. Then by 2.1,

$$g(C \cup \{u\}, D \setminus \{u, v\}) = \sum_{Z \subseteq D \setminus \{u,v\}} (-1)^{|Z|} g(C \cup \{u\} \cup Z).$$

But for $Z \subseteq D \setminus \{u, v\}$, $g(C \cup \{u\} \cup Z) \neq 0$ only if $Z = D_1 \setminus \{u\}$ (by (6) and the definition of C_1, C_{-1}) and so

$$g(C \cup \{u\}, D \setminus \{u, v\}) = (-1)^{|D_1 \setminus \{u\}|} g(B_1 \cup C_{-1}) = (-1)^{|D_1|}.$$

Similarly

$$g(C \cup \{v\}, D \setminus \{u, v\}) = (-1)^{|D_2 \setminus \{v\}|} g(B_{-1} \cup C_1) = (-1)^{|D_{-1}|+1}.$$

Since $|D_1|, |D_{-1}|$ have the same parity, one of $g(C \cup \{u\}, D \setminus \{u, v\})$, $g(C \cup \{v\}, D \setminus \{u, v\})$ equals 1 and the other equals -1, contradicting that g is good. This proves (8).

From (8) we may assume that $C_{-1} = \emptyset$ (replacing g by $-g$ if necessary).

(9) $|D_1|$ *is odd.*

To prove this, we may assume that $C_1 \neq \emptyset$, since $|B_1|$ is odd. By 2.1,

$$g(C_1, B_{-1} \cup D_1) = \sum_{Z \subseteq B_{-1} \cup D_1} (-1)^{|Z|} g(C_1 \cup Z).$$

But, by (6), for $Z \subseteq B_{-1} \cup D_1$, $g(C_1 \cup Z)$ is nonzero only if $Z \subseteq D_1$ or $Z = B_{-1}$; and then it has value 1 if $Z \subseteq D_1$ and $Z \neq D_1$; 0 if $Z = D_1$; and 1 if $Z = B_{-1}$. Thus $g(C_1, B_{-1} \cup D_1) = (-1)^{|D_1|+1} + (-1)^{|B_{-1}|}$ and since $|B_{-1}|$ is odd by (5), and $|g(C_1, B_{-1} \cup D_1)| \leq 1$ since g is good, it follows that $|D_1|$ is odd. This proves (9).

Now $|C_1| \leq |B_1| - 2$ as we saw. Choose $u \in D_1$ and $v \in B_{-1}$, and let $W = (D_1 \cup B_{-1}) \setminus \{u, v\}$. By 2.1,

$$g(C_1 \cup \{u\}, W) = \sum_{Z \subseteq W} (-1)^{|Z|} g(C_1 \cup \{u\} \cup Z).$$

But for $Z \subseteq W$, $g(C_1 \cup \{u\} \cup Z)$ is nonzero only if $Z \subseteq D_1$, and in that case it has value 1 if $Z \neq D_1 \setminus \{u\}$, and 0 if $Z = D_1 \setminus \{u\}$. Since $|D_1| \geq 2$, it follows that

$$g(C_1 \cup \{u\}, W) = (-1)^{|D_1|} = -1$$

since $|D_1|$ is odd by (9). On the other hand, by 2.1,

$$g(C_1 \cup \{v\}, W) = \sum_{Z \subseteq W} (-1)^{|Z|} g(C_1 \cup \{v\} \cup Z).$$

We claim that $g(C_1 \cup \{v\}, W) = 1$. To see this there are two cases, depending whether $C_1 \neq \emptyset$ or not. First, suppose that $C_1 \neq \emptyset$. Then for $Z \subseteq W$, $g(C_1 \cup \{v\} \cup Z)$ is nonzero only if $Z = B_{-1} \setminus \{v\}$, by (6) and the maximality of C_1; so

$$g(C_1 \cup \{v\}, W) = (-1)^{|B_1|-1} g(C_1 \cup B_{-1}) = 1,$$

by (7) and (3), contradicting that g is good. Now suppose that $C_1 = \emptyset$. Then, again by (6), for $Z \subseteq W$, $g(C_1 \cup \{v\} \cup Z)$ is nonzero only if $Z \subsetneq B_{-1} \setminus \{v\}$, and in that case it has value -1. Consequently

$$g(C_1 \cup \{v\}, W) = (-1)^{|B_{-1} \setminus \{v\}|} = 1,$$

again contradicting that g is good. This proves 3.4. ∎

4. The value on the null set

In this section we prove 1.6, thereby completing the inductive proof of 1.4. We need to show that if g is a good counter on a ternary graph G, then $|g(\emptyset)| \leq 1$. The proof is divided into several steps. We may assume the statement is false, for a contradiction; and by replacing g by $-g$ if necessary, we may assume that $g(\emptyset) \geq 2$. Throughout this section, G is a counterexample to 1.6, and g is a good counter on G, with $g(\emptyset) \geq 2$.

4.1: *The following hold:*

- $g(\emptyset) = 2$;
- $g(\{v\}) = 1$ *for every vertex* $v \in V(G)$; *and*
- G *is connected.*

Proof. Let $v \in V(G)$; since g is good, it follows that $|g(\{v\})| \leq 1$, and so 2.3 implies that $g(\{v\}) = 1$ and $g(\emptyset) = 2$. This proves the first two statements.

Suppose that G is not connected, let G_1 be a component of G and let G_2 be obtained from G by deleting $V(G_1)$. Since $f_{G_1}(\emptyset) = \pm g(\emptyset, V(G_2))$, and g is good, it follows that $|f_{G_1}(\emptyset)| \leq 1$, and similarly $|f_{G_2}(\emptyset)| \leq 1$. But

$$g(\emptyset) = \pm f_G(\emptyset) = \pm f_{G_1}(\emptyset) f_{G_2}(\emptyset),$$

a contradiction. This proves the third statement, and so proves 4.1. ∎

In particular, if $u, v \in V(G)$ are distinct, then since $g(\{u\}) = 1$ by the second statement of 4.1, it follows that $g(\{u, v\}) \in \{0, 1\}$ by 2.3. Let H be the graph with vertex set $V(G)$ in which distinct u, v are adjacent if $g(\{u, v\}) = 1$.

4.2: *Every component of H is a complete graph, and H has at least two and at most four components.*

Proof. Suppose the first statement is false. Then there are three distinct vertices $u, v, w \in V(H)$ such that $uv, vw \in E(H)$ and $uw \notin E(H)$. From 2.3, $g(\{u, w\}) = 0$. Now

$$\begin{aligned} g(\emptyset, \{w\}) =& g(\emptyset) - g(\{w\}) = 1, \\ g(\{v\}, \{w\}) =& g(\{v\}) - g(\{v, w\}) = 0 \\ g(\{u, v\}, \{w\}) =& g(\{u, v\}) - g(\{u, v, w\}); \end{aligned}$$

and by 2.5, $g(\{u,v\},\{w\}) \neq 1$. Consequently $g(\{u,v,w\}) = 1$. But then

$$g(\{w\}) = 1, \quad g(\{u,w\}) = 0 \quad \text{and} \quad g(\{u,v,w\}) = 1,$$

contrary to 2.5. This proves that every component of H is a complete graph.

Since each edge of H joins two vertices that are nonadjacent in G, it follows that H has at least two components. Suppose it has at least five. Since G is connected, there is a vertex of H that has neighbours (in G) in at least two components of H. Thus we can choose $v_1, \dots, v_5 \in V(G)$, all in different components of H, where v_1 is adjacent (in G) to v_2, v_3. Let $a, b, c \in \{v_1, \dots, v_5\}$ be distinct. Since $|g(\emptyset, \{a,b,c\})| \leq 1$, and $g(\{a,b\}) = 0$ (because $g(\{a,b\}) \neq 1$ since a, b belong to different components of H, and $g(\{a,b\}) \neq -1$ by 2.3), and the same for $\{a,c\}$ and $\{b,c\}$, it follows from 2.1 that

$$|2 - 3 + 0 - g(\{a,b,c\})| \leq 1,$$

and so $g(\{a,b,c\}) \neq 1$. Hence $g(\{a,b,c\}) \in \{0,-1\}$ for every triple a, b, c of distinct members of $\{v_1, \dots, v_5\}$.

Note that since $v_1v_2, v_1v_3 \in E(G)$, it follows that $g(\{v_1, v_2, v_i\}) = 0$ for every $i \in \{3,4,5\}$ and $g(\{v_1, v_3, v_j\}) = 0$ for every $j \in \{2,4,5\}$. Let $\mathcal{T}$ be the set of all subsets $T \subseteq \{v_1, \dots, v_5\}$ with $|T| = 3$ and $g(T) = -1$. Thus $g(T) = 0$ for all triples $T \notin \mathcal{T}$. Since $|g(\emptyset, \{v_1, v_2, v_3, v_4\})| \leq 1$, it follows from 2.1 that $\{v_2, v_3, v_4\} \in \mathcal{T}$, and similarly $\{v_2, v_3, v_5\} \in \mathcal{T}$.

Suppose that $\{v_1, v_4, v_5\} \notin \mathcal{T}$. Now 2.1 implies that

$$g(\emptyset, \{v_1, v_2, v_4, v_5\}) = 2 - 4 + 0 - g(\{v_2, v_4, v_5\}),$$

and so $\{v_2, v_4, v_5\} \in \mathcal{T}$, and similarly $\{v_3, v_4, v_5\} \in \mathcal{T}$. But then

$$g(\{v_5\}, \{v_2, v_3, v_4\}) = -2 - g(v_2, v_3, v_4, v_5) \leq -1$$

and $g(\{v_1\}, \{v_2, v_3, v_4\}) = 1$, contradicting that g is good. Thus $\{v_1, v_4, v_5\} \in \mathcal{T}$.

If also $\{v_2, v_4, v_5\} \in \mathcal{T}$ then

$$g(\{v_4, v_5\}, \{v_1, v_2\}) = 2,$$

contradicting that g is good; so $\{v_2, v_4, v_5\} \notin \mathcal{T}$, and similarly $\{v_3, v_4, v_5\} \notin \mathcal{T}$. Since $g(\{v_2, v_3\}, \{v_4, v_5\}) \leq 1$, it follows that $g(\{v_2, v_3, v_4, v_5\}) = -1$. But then

$$g(\{v_4\}, \{v_2\}) = 1, \quad g(\{v_4, v_5\}, \{v_2\}) = 0 \quad \text{and} \quad g(\{v_3, v_4, v_5\}, \{v_2\}) = 1,$$

contrary to 2.5. This proves 4.2. ∎

4.3: *Let C_1, C_2 be distinct components of H, and let $X \subseteq C_1 \cup C_2$. Suppose that*

- $X \cap C_1, X \cap C_2 \neq \emptyset$;
- $g(X) \neq 0$; *and*
- *for all $X' \subseteq X$, if $g(X') \neq 0$ then either $X' = X$ or $X' \subseteq C_1$ or $X' \subseteq C_2$.*

If $|X \cap C_1| > 1$ then there is a subset $B \subseteq X \cap C_1$ with $g(B) = 0$.

Proof. Let $X_i = X \cap C_i$ for $i = 1, 2$; and suppose there is no $B \subseteq X_1$ with $g(B) = 0$. From 2.3 it follows that $g(B) = 1$ for all nonempty subsets B of X_1, and in particular, $g(X_1) = 1$. Let $g(X) = i = \pm 1$. Because of the third bullet of the hypothesis, 2.1 implies that

$$g(X_1, X_2) = \sum_{Z \subseteq X_2} (-1)^{|Z|} g(X_1 \cup Z) = g(X_1) + (-1)^{|X_2|} i;$$

and since $g(X_1, X_2) \leq 1$, it follows that $(-1)^{|X_2|} i = -1$, that is, $|X_2|$ is odd if $i = 1$, and even if $i = -1$. Choose $u \in X_1$ and $v \in X_2$; then by 2.1,

$$g(X_1 \setminus \{u\}, X_2 \setminus \{v\}) = 1$$

(since $|X_1| > 1$),

$$g(X_1 \cup \{v\} \setminus \{u\}, X_2 \setminus \{v\}) = 0,$$

and by 2.1,

$$g(X_1 \cup \{v\}, X_2 \setminus \{v\}) = \sum_{Z \subseteq X_2 \setminus \{v\}} (-1)^{|Z|} g(X_1 \cup Z \setminus \{v\}) = (-1)^{|X_2|-1} g(X) = 1,$$

contrary to 2.5. This proves 4.3. ∎

Let C be a component of H, and let $D \subseteq C$. We say that $B \subseteq D$ is a **base** of D if $g(B) \neq 1$ and there is no $B' \subseteq D$ with $|B'| < |B|$ and with $g(B') \neq 1$.

4.4: *Let C be a component of H, and let $D \subseteq C$.*

- *If there is a vertex v of G such that all its neighbours belong to D, then D has a base.*
- *If B is a base of D then $g(B) = 0$, and $|B|$ is even and at least four.*
- *If D has a base, of cardinality k say, then every subset of D of cardinality $k + 1$ includes two bases of D, and so every vertex of D belongs to a base of D.*

- *If D has a base, of cardinality k, then there is a partition of D into nonempty sets $D_1, \dots, D_n$, such that for all distinct $u, v \in D$, there is a base of D containing both u, v if and only if u, v do not belong to the same set D_i; and consequently $n \geq k$.*

Proof. For the first statement, suppose that all neighbours of v belong to D. If $V(G) = C \cup \{v\}$, then v is adjacent to all other vertices (since no vertex has degree zero, by 2.4), contradicting that $g(\emptyset) = 2$. Thus we may choose $u \notin C \cup \{v\}$. By 2.4, $g(\{u\}, D) = 0$, but $g(\{u\}) = 1$, and so by 2.1, there exists a nonempty subset $Z \subseteq D$ such that $g(Z \cup \{u\}) \neq 0$. Since C is the vertex set of a component of H, it follows that $|Z| \geq 2$. From 4.3, there exists $B \subseteq Z$ with $g(B) = 0$. This proves the first statement.

For the second, let B be a base of D. Then $g(B) \neq 1$ by hypothesis, and in particular $|B| \geq 3$, since $B \subseteq C$. For every $B' \subseteq B$ with $B' \neq \emptyset, B$, we have

$$g(B') = 1,$$

and since there is such a choice of B' with $|B'| = |B| - 1$, 2.3 implies that $g(B) \neq -1$; and hence $g(B) = 0$ since g is good. But by 2.2,

$$g(\emptyset, B) = \sum_{Z \subseteq B} (-1)^{|Z|}(g(Z) - 1) = (g(\emptyset) - 1) + (-1)^{|B|}(g(B) - 1) = 1 - (-1)^{|B|},$$

and so $|B|$ is even. This proves the second statement.

For the third, let B be a base of D, with $|B| = k$ say; it suffices to prove that for all $v \in D \setminus B$, $B \cup \{v\}$ includes at least two bases of D. Let $X = B \cup \{v\}$, and choose $u \in B$. Thus

$$g(X \setminus \{u, v\}) = 1 \quad \text{and} \quad g(X \setminus \{v\}) = 0,$$

so by 2.5, $g(X) \neq 1$. We may assume that $g(X \setminus \{u\}) = 1$, and so by 2.3, $g(X) = 0$. By 2.1,

$$g(\emptyset, X \setminus \{u, v\}) = 1 \quad \text{and} \quad g(\{u, v\}, X \setminus \{u, v\}) = 1,$$

so by 2.5, $g(\{v\}, X \setminus \{u, v\}) = 1$. Hence by 2.1, since $|X| \geq 3$, there exists $Z \subseteq X \setminus \{u, v\}$ with $g(Z \cup \{v\}) \neq 1$. Then $|Z| \leq |B|$, and since B is a base for D, it follows that Z is minimal with $g(Z) \neq 1$, and hence Z is another base for D. This proves the third statement.

The fourth statement follows from 2.7. This proves 4.4. ∎

We call a partition $D_1, \ldots, D_n$ as in the fourth statement of 4.4 the **induced** partition of D, and the sets $D_1, \ldots, D_n$ are called its **classes**. (If the partition exists then it is unique, as is easily seen.)

4.5: *Let C_1, C_2 be distinct components of H, and for $i = 1, 2$, let $D_i \subseteq C_i$, including a base for D_i. Then for one of $i = 1, 2$, there is a class of the induced partition of D_i that meets all edges between D_1 and D_2.*

Proof. Let the induced partition of D_1 have classes $P_1, \ldots, P_m$, and let the induced partition of D_2 have classes $Q_1, \ldots, Q_n$. We may assume that there is no $i \in \{1, \ldots, m\}$ such that all edges between D_1, D_2 have an end in P_i, and there is no $j \in \{1, \ldots, n\}$ similarly. By König's theorem, there exist distinct $i_1, i_2 \in \{1, \ldots, m\}$ and distinct $j_1, j_2 \in \{1, \ldots, n\}$ such that there is an edge between P_{i_1} and Q_{j_1}, and an edge between P_{i_2} and Q_{j_2}. Hence there is a base B_1 for D_1 and a base B_2 for D_2, such that there are two edges of G between B_1, B_2 with no end in common.

(1) *Suppose that there exists $M_1 \subseteq B_1$ with $g(B_2 \cup M_1) \neq 0$, and choose M_1 maximal with this property. Then $|M_1| \leq |B_1| - 2$, and $g(B_2 \cup M_1) = -1$, and $|M_1|$ is odd.*

Since there are two edges of G between B_1, B_2 with no end in common, and both have an end in B_2, it follows that neither has an end in M_1, and so $|M_1| \leq |B_1| - 2$. Let $A_1 = B_1 \setminus M_1$. By 2.1,

$$g(M_1, B_2) = \sum_{Z \subseteq B_2} (-1)^{|Z|} g(M_1 \cup Z).$$

But for $Z \subseteq B_2$, $g(M_1 \cup Z) \neq 0$ only if $Z = \emptyset$ or $Z = B_2$, by 4.3. Consequently $g(M_1, B_2) = g(M_1) + (-1)^{|B_2|} g(M_1 \cup B_2)$. But $g(M_1) = 1$ and $|B_2|$ is even, so $g(M_1 \cup B_2) = -1$ since g is good. Now by 2.1,

$$g(M_1, A_1 \cup B_2) = \sum_{Z \subseteq A_1 \cup B_2} (-1)^{|Z|} g(M_1 \cup Z).$$

But for $Z \subseteq A_1 \cup B_2$, $g(M_1 \cup Z) \neq 0$ only if $Z \subseteq A_1$ or $Z = B_2$; and so

$$g(M_1, A_1 \cup B_2) = (-1)^{|A_1|}(g(B_1) - 1) + (-1)^{|B_2|} g(M_1 \cup B_2).$$

Since $|B_2|$ is even, $g(B_1) = 0$ and $g(M_1 \cup B_2) = -1$, it follows that

$$g(M_1, A_1 \cup B_2) = (-1)^{|A_1|+1} - 1,$$

and so $|A_1|$ is odd, and therefore so is $|M_1|$. This proves (1).

(2) *There do not exist* $M_1 \subseteq B_1$ *and* $M_2 \subseteq B_2$ *with* $g(B_2 \cup M_1), g(B_1 \cup M_2) \neq 0$ *and with* M_1, M_2 *both nonempty.*

Suppose such sets M_1, M_2 exist and choose them maximal. Let $A_i = B_i \setminus M_i$ for $i = 1, 2$. By (1), $g(B_2 \cup M_1), g(B_1 \cup M_2) = -1$, and $|M_1|, |M_2|$ are odd. Thus $|A_1|$ and $|A_2|$ are odd, and so $g(M_1 \cup M_2, A_1 \cup A_2) = 2$ by 2.1, a contradiction, This proves (2).

(3) $g(X) = 0$ *for all* $X \subseteq B_1 \cup B_2$ *with* $X \cap B_1, X \cap B_2$ *both nonempty.*

Suppose not; then from 4.3, and by exchanging C_1, C_2 if necessary, we may assume that there exists $M_1 \subseteq B_1$, nonempty, with $g(B_2 \cup M_1) \neq 0$. Choose M_1 maximal. By (1), $g(B_2 \cup M_1) = -1$ and $|M_1|$ is odd. Let $A_1 = B_1 \setminus M_1$, and choose $u \in A_1$. Choose $v \in B_2$. Then by 2.1, since $A_1 \setminus \{u\} \neq \emptyset$, it follows that

$$g(M_1 \cup \{u\}, (A_1 \cup B_2) \setminus \{u, v\}) = -1 \quad \text{and} \quad g(M_1 \cup \{v\}, (A_1 \cup B_2) \setminus \{u, v\}) = 1,$$

contradicting that g is good. This proves (3).

From (3), 2.1 implies that

$$g(\emptyset, B_1 \cup B_2) = -2,$$

a contradiction. This proves 4.5. ∎

4.6: *Let* C_1, C_2 *be distinct components of* H*, and suppose there is a base for* C_2*. Let* $D_1, \ldots, D_n$ *be the induced partition of* C_2*. Then there is no* $i \in \{D_1, \ldots, D_n\}$ *such that every edge of* G *between* C_2 *and* $V(G) \setminus (C_1 \cup C_2)$ *has an end in* D_i*.*

Proof. Suppose there is such a value of i, say $i = 1$. Let A_1 be the set of vertices in C_1 with neighbours in C_2. Now $n \geq 4$ (by the second and last statements of 4.4); choose $v \in D_2$. Thus all neighbours of v belong to C_1, and hence to A_1. By the first statement of 4.4, there is a base for A_1. By 4.5, there is a set X that meets all edges between A_1 and C_2, and X is either a class of the induced partition of C_2 or a class of the induced partition of A_1. The first is impossible since there are at least four classes of the induced partition of C_2, and each such class different from D_1 meets an edge between C_2 and A_1 (because it meets some edge, and it has no edge to $V(G) \setminus (C_1 \cup C_2)$ from the choice of D_1). Also the second is impossible, since each class of the induced partition of A_1 has an edge to C_2, from the definition of A_1. This proves 4.6. ∎

Now we complete the proof of 1.6, which we restate:

4.7: *If g is a good counter on a ternary graph G, then $|g(\emptyset)| \leq 1$.*

Proof. In the same notation as before, we know that H has two, three or four components. Suppose it has only two, say C_1, C_2. By the first statement of 4.4, there are bases for C_1 and for C_2, contrary to 4.6.

Now suppose that H has exactly three components C_1, C_2, C_3. By 2.6 we may assume that some vertex $v \in C_2$ has no neighbour in C_1, and so by 4.4, there is a base for C_3. Suppose that there is also a base for C_2. By 4.5, by exchanging C_2, C_3 if necessary, we may assume that there is a class of the induced partition of C_2 that meets all edges between C_2, C_3, contrary to 4.6. Thus, neither of C_1, C_2 have bases. By 4.4, every vertex in $C_1 \cup C_2$ has a neighbour in C_3. We recall that $v \in C_2$ has no neighbour in C_1. Since C_1 has no base, it follows that $g(C_1) = 1$, and so by 2.4, v has a neighbour, u say, with no neighbour in C_1. But then all neighbours of u are in C_2, and so by 4.4, there is a base for C_2, a contradiction.

Finally, suppose that H has four components $C_1, \ldots, C_4$. Let K be the graph with vertex set $\{1, \ldots, 4\}$ in which distinct i, j are adjacent if there is an edge of G between C_i, C_j. Since G is connected, it follows that every vertex of K has nonzero degree. Suppose that K has a 2-edge matching; then by renumbering $C_1, \ldots, C_4$ we may assume that there exist $u_i \in C_i$ for $1 \leq i \leq 4$ such that $u_1u_2, u_3u_4 \in E(G)$. But then $g(\emptyset, \{u_1, u_2, u_3, u_4\}) = -2$ by 2.1, a contradiction. Thus K has no 2-edge matching, and since every vertex of K has nonzero degree, we may assume that every edge of K is incident with 1, and so all edges of G have an end in C_1.

For $i = 2, 3, 4$, let X_i be the set of vertices in C_1 with no neighbour in C_i. By the first statement of 4.4, there is a base for C_1. By 4.6, there is no base for C_2, and similarly none for C_3, C_4; and so by the first statement of 4.4, every vertex of C_1 has neighbours in at least two of C_2, C_3, C_4. In particular, for all distinct $i, j \in \{2, 3, 4\}$ every vertex in X_i has a neighbour in C_j.

Since $g(C_2) \neq 0$, 2.4 implies that for all distinct $i, j \in \{2, 3, 4\}$, every vertex in C_i has a neighbour in X_j. Make a digraph J with vertex set $C_2 \cup C_3 \cup C_4$ in which for $i = 2, 3, 4$ and $u \in C_i$ and $v \in C_{i+1}$ (where C_5 means C_2), there is an edge of J from u to v if u, v has a common neighbour in X_{i-1} (where X_1 means X_4). Every vertex has positive outdegree in J, and so J has an induced directed cycle. Let K be such a cycle, with vertices (in order):

$$a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_k, b_k, c_k, a_1$$

where $a_1, \dots, a_k \in C_2$, $b_2, \dots, b_k \in C_3$ and $c_1, \dots, c_k \in C_4$. For each i with $1 \le i \le k$, there exist $x_i \in X_4$ adjacent in G to a_i, b_i, and $y_i \in X_2$ adjacent to b_i, c_i, and $z_i \in X_3$ adjacent to c_i, a_{i+1} (where a_{k+1} means a_1). Also, for each such i, x_i has no other neighbours in $V(K)$; it is nonadjacent to each a_j because $x_i \in X_4$, and nonadjacent to the remaining vertices of $V(K)$ since K is induced. A similar statement holds for the y_i's and z_i's. Consequently the subgraph of G induced on

$$\{a_i, b_i, c_i, x_i, y_i, z_i : 1 \le i \le k\}$$

is an induced cycle of length $6k$, contradicting that G is ternary. This proves that H does not have four components, and so proves 4.7 and hence 1.4. ∎

References

[1] M. Bonamy, P. Charbit and S. Thomassé, *Graphs with large chromatic number induce 3k-cycles*, https://arxiv.org/abs/1408.2172.

[2] M. Bousquet-Mélou, S. Linusson and E. Nevo, *On the independence complex of square grids*, Journal of Algebraic Combinatorics **27** (2008), 423–450.

[3] G. Chen and A. Saito, *Graphs with a cycle of length divisible by three*, Journal of Combinatorial Theory, Series B **60** (1994), 277–292.

[4] P. Fendley, K. Schoutens and H. van Eerten, *Hard squares with negative activity*, Journal of Physics. A **38** (2005), 315–322.

[5] G. Gauthier, *Graphs with no cycle length divisible by three*, Ph. D. thesis, Princeton University, 2017.

[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[7] J. Jonsson, *Hard squares with negative activity and rhombus tilings of the plane*, Electronic Journal of Combinatorics **13** (2006), Article no. R67.

[8] G. Kalai, *When do a few colors suffice?*, https://gilkalai.wordpress.com/2014/12/19/when-a-few-colors-suffice.

[9] A. Scott and P. Seymour, *Induced subgraphs of graphs with large chromatic number. X. Holes with specific residue*, Combinatorica **39** (2019), 1105–1132.

[10] A. Scott and A. D. Sokal, *The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma*, Journal of Statistical Physics **118** (2005), 1151–1261.