

ON RIEMANN SURFACES OF GENUS g WITH $4g - 4$ AUTOMORPHISMS

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ABSTRACT

In this article we study compact Riemann surfaces with a non-large group of automorphisms of maximal order, namely, compact Riemann surfaces of genus g with a group of automorphisms of order $4g - 4$. Under the assumption that $g - 1$ is prime, we provide a complete classification of them and determine isogeny decompositions of the corresponding Jacobian varieties.

1. Introduction and statement of the results

The classification of finite groups of automorphisms of compact Riemann surfaces is a classical and interesting problem which has attracted considerable interest, going back to contributions of Wiman, Klein, Schwartz and Hurwitz, among others.

It is classically known that the full automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite, and that its order is bounded by $84g - 84$. This bound is sharp for infinitely many values of g , and a Riemann surface with this maximal number of automorphisms is characterized as a branched regular cover of the projective line with three branch values, marked with 2, 3 and 7.

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A result due to Wiman asserts that the largest cyclic group of automorphisms of a Riemann surface of genus $g \geq 2$ has order at most $4g + 2$. Moreover, the curve

$$(1.1) \quad y^2 = x^{2g+1} - 1$$

shows that this upper bound is sharp for each genus; see [46] and [18].

Besides, Accola and Maclachlan independently proved that for fixed $g \geq 2$ the maximum among the orders of the full automorphism group of Riemann surfaces of genus g is at least $8g + 8$. Moreover, the curve

$$(1.2) \quad y^2 = x^{2g+2} - 1$$

shows that this lower bound is sharp for each genus; see [1] and [32].

An interesting problem is to understand the extent to which the order of the full automorphism group determines the Riemann surface. In this regard, Kulkarni proved that for g sufficiently large the curve (1.1) is the unique Riemann surface with an automorphism of order $4g + 2$, and for $g \not\equiv -1 \pmod{4}$ sufficiently large the curve (1.2) is the unique Riemann surface with $8g + 8$ automorphisms; see [25].

Let $a, b \in \mathbb{Z}$. Following [26], the sequence $ag + b$ for $g = 2, 3, \dots$ is called **admissible** if for infinitely many values of g there is a compact Riemann surface of genus g with a group of automorphisms of order $ag + b$.

In addition to the mentioned admissible sequences $84g - 84$, $4g + 2$ and $8g + 8$, very recently the cases $4g + 4$ and $4g$ have been studied; see [8], [15] and also [37].

Let $a \geq 7$. The admissible sequence $ag - a$ has been considered by Belolipetsky and Jones in [3]. Concretely, they succeeded in proving that under the assumption that $g - 1$ is a sufficiently large prime number, a compact Riemann surface of genus g with a group of automorphisms of order $ag - a$ lies in one of six infinite well-described sequences of examples. The cases $a = 5$ and $a = 6$ have been recently classified by Izquierdo and the author in [22].

All the aforementioned cases are examples of compact Riemann surfaces possessing a so-called **large** group of automorphisms: namely, a group of automorphisms of order greater than $4g - 4$, where g is the genus. In this case, it is known that the Riemann surface is either quasiplatonic (it does not admit non-trivial deformations in the moduli space with its automorphisms) or belongs to a complex one-dimensional family in such a way that the signature of the

action is

$$(0; 2, 2, 2, n) \text{ for } n \geq 3 \quad \text{or} \quad (0; 2, 2, 3, n) \text{ for } 3 \leq n \leq 5.$$

Riemann surfaces with large groups of automorphisms have been considered from different points of view; see, for example, [13], [27], [30], [34], [44], [45] and [46].

In this article we consider compact Riemann surfaces admitting a non-large group of automorphisms of maximal order; concretely, we study and classify those compact Riemann surfaces of genus g with a group of automorphisms of order $4g - 4$, under the assumption that $g - 1$ is a prime number.

THEOREM 1: *Let $g \geq 8$ such that $g - 1$ is prime, and let S be a compact Riemann surface of genus g with a group of automorphisms of order $4g - 4$.*

If $g \equiv 0 \pmod 4$ then S belongs to $\bar{\mathcal{F}}_g^2$, where $\bar{\mathcal{F}}_g^2$ is the complex two-dimensional equisymmetric family of compact Riemann surfaces of genus g with a group of automorphisms G_2 isomorphic to the dihedral group

$$\langle r, s : r^{2(g-1)} = s^2 = (sr)^2 = 1 \rangle,$$

such that the signature of the action of G_2 is $(0; 2, 2, 2, 2, 2)$.

If $g \equiv 2 \pmod 4$ then S belongs to either $\bar{\mathcal{F}}_g^2$ or $\bar{\mathcal{F}}_g^1$, where $\bar{\mathcal{F}}_g^1$ is the complex one-dimensional equisymmetric family of compact Riemann surfaces of genus g with a group of automorphisms G_1 isomorphic to

$$\langle a, b : a^{g-1} = b^4 = 1, bab^{-1} = a^r \rangle,$$

where r is a primitive 4-th root of the unity in the field of $g - 1$ elements, such that the signature of the action of G_1 is $(0; 2, 2, 4, 4)$.

By virtue of Dirichlet’s prime number theorem, the congruences of Theorem 1 are satisfied for infinitely many values. The genera $g = 3, 4$ and 6 are exceptional in the sense that, in addition to the families introduced earlier, there appear finitely many quasiplatonic Riemann surfaces (see [2], [5], [7], [11], [14], [28] and [33]).

As proved by Costa and Izquierdo in [15], the largest order of the full automorphism group of a complex one-dimensional family of Riemann surfaces of genus g , appearing in all genera, is $4g + 4$. On the other hand, $4g - 4$ is the maximal possible order of the full automorphism group of a complex two-dimensional family of compact Riemann surfaces of genus g . It is worth mentioning that the

existence of the family $\bar{\mathcal{F}}_g^2$ shows that this upper bound is attained in each genus, with $g - 1$ not necessarily prime.

Let $t = 1, 2$. The family $\bar{\mathcal{F}}_g^t$ is a closed algebraic subvariety of the moduli space of compact Riemann surfaces of genus g ; we shall denote by \mathcal{F}_g^t its interior. Based on classical results due to Singerman [43] and on Belolipetsky and Jones' classification [3], we are able to describe both the interior \mathcal{F}_g^t and (up to possibly finitely many exceptional cases in small genera) the set $\bar{\mathcal{F}}_g^t \setminus \mathcal{F}_g^t$.

More precisely:

THEOREM 2: *Let $g \geq 8$ such that $g - 1$ is prime.*

For $g \equiv 2 \pmod 4$, the interior \mathcal{F}_g^1 of the family $\bar{\mathcal{F}}_g^1$ consists of those Riemann surfaces for which G_1 is the full automorphism group. Moreover, there is a positive integer ϵ_1 such that if $g \geq \epsilon_1$ then

$$\bar{\mathcal{F}}_g^1 \setminus \mathcal{F}_g^1 = \begin{cases} \{X_1, X_2\} & \text{if } g \equiv 2 \pmod 8, \\ \emptyset & \text{if } g \not\equiv 2 \pmod 8, \end{cases}$$

where X_1 and X_2 are the two non-isomorphic compact Riemann surfaces of genus g with full automorphism group of order $8g - 8$.

The interior \mathcal{F}_g^2 of the family $\bar{\mathcal{F}}_g^2$ consists of those Riemann surfaces for which G_2 is the full automorphism group. Moreover, there is a positive integer ϵ_2 such that if $g \geq \epsilon_2$ then

$$\bar{\mathcal{F}}_g^2 \setminus \mathcal{F}_g^2 = \begin{cases} \{Y_1, Y_2\} & \text{if } g \equiv 2 \pmod 3, \\ \emptyset & \text{if } g \not\equiv 2 \pmod 3, \end{cases}$$

where Y_1 and Y_2 are the two non-isomorphic compact Riemann surfaces of genus g with full automorphism group of order $12g - 12$.

We should mention that the existence, uniqueness and description of the full automorphism groups of the compact Riemann surfaces X_1, X_2 and Y_1, Y_2 mentioned above have been determined in [3].

We recall that the Jacobian variety JS of a compact Riemann surface S of genus g is an irreducible principally polarized abelian variety of dimension g . The relevance of the Jacobian varieties lies in the well-known Torelli's theorem, which establishes that two compact Riemann surfaces are isomorphic if and only if the corresponding Jacobian varieties are isomorphic as principally polarized abelian varieties. See, for example, [4] and [16].

If H is a group of automorphisms of S , then the associated regular covering map $\pi : S \rightarrow S_H$ given by the action of H on S induces a homomorphism

$$\pi^* : JS_H \rightarrow JS$$

between the corresponding Jacobians; the set $\pi^*(JS_H)$ is an abelian subvariety of JS which is isogenous to JS_H . We keep the same notation as in Theorem 1.

THEOREM 3: *Let $g \geq 8$ such that $g - 1$ is prime.*

If $S \in \bar{\mathcal{F}}_g^1$ then the Jacobian variety JS decomposes, up to isogeny, as the product

$$JS \sim JS_{(a)} \times (JS_{(b)})^4.$$

If $S \in \bar{\mathcal{F}}_g^2$ then the Jacobian variety JS decomposes, up to isogeny, as the product

$$JS \sim JS_{(r)} \times JS_{(s)} \times JS_{(sr)}.$$

The article is organized as follows.

- (1) In Section 2 we shall review the basic background: namely, Fuchsian groups, actions on compact Riemann surfaces, the equisymmetric stratification of the moduli space and the decomposition of Jacobian varieties.
- (2) In Section 3 we shall prove the existence of the families $\bar{\mathcal{F}}_g^1$ and $\bar{\mathcal{F}}_g^2$.
- (3) Theorems 1, 2 and 3 will be proved in Sections 4, 5 and 6 respectively.
- (4) Finally, and for the sake of completeness, in Section 7 two examples will be constructed to show that Theorem 1 is false if $g - 1$ is not prime.

2. Preliminaries

2.1. FUCHSIAN GROUPS AND GROUP ACTIONS ON RIEMANN SURFACES. By a **Fuchsian group** we mean a discrete group of automorphisms of the upper-half plane

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

If Δ is a Fuchsian group and the orbit space \mathbb{H}_Δ given by the action of Δ on \mathbb{H} is compact, then the algebraic structure of Δ is determined by its **signature**:

$$(2.1) \quad \sigma(\Delta) = (h; m_1, \dots, m_l),$$

where h is the genus of the quotient surface \mathbb{H}_Δ and m_1, \dots, m_l are the branch indices in the universal canonical projection $\mathbb{H} \rightarrow \mathbb{H}_\Delta$. If $l = 0$ then Δ is called a **surface Fuchsian group**.

Let Δ be a Fuchsian group of signature (2.1). Then:

- (1) Δ has a canonical presentation given by generators $a_1, \dots, a_h, b_1, \dots, b_h, x_1, \dots, x_l$ and relations

$$(2.2) \quad x_1^{m_1} = \dots = x_l^{m_l} = \prod_{i=1}^h [a_i, b_i] \prod_{i=1}^l x_i = 1,$$

where $[u, v]$ stands for the commutator $uvu^{-1}v^{-1}$,

- (2) the elements of Δ of finite order are conjugate to powers of x_1, \dots, x_l ,
- (3) the Teichmüller space of Δ is a complex analytic manifold homeomorphic to the complex ball of dimension $3h - 3 + l$, and
- (4) the hyperbolic area of each fundamental region of Δ is

$$\mu(\Delta) = 2\pi \left[2h - 2 + \sum_{i=1}^l \left(1 - \frac{1}{m_i} \right) \right].$$

Let Γ be a group of automorphisms of \mathbb{H} . If Δ is a subgroup of Γ of finite index, then Γ is also Fuchsian and they are related by the Riemann–Hurwitz formula

$$\mu(\Delta) = [\Gamma : \Delta] \cdot \mu(\Gamma).$$

Let S be a compact Riemann surface and let $\text{Aut}(S)$ denote its full automorphism group. It is said that a finite group G acts on S if there is a group monomorphism

$$\psi : G \rightarrow \text{Aut}(S).$$

The space of orbits S_G of the action of G on S is endowed with a Riemann surface structure such that the projection

$$S \rightarrow S_G$$

is holomorphic.

Compact Riemann surfaces and group actions can be understood in terms of Fuchsian groups as follows. By the uniformization theorem (see, for example, [17, p. 203]), there is a (uniquely determined, up to conjugation) surface Fuchsian group Γ such that S and \mathbb{H}_Γ are isomorphic. Moreover, G acts on $S \cong \mathbb{H}_\Gamma$ if and only if there is a Fuchsian group Δ containing Γ together with a group epimorphism

$$\theta : \Delta \rightarrow G \quad \text{such that } \ker(\theta) = \Gamma.$$

In this case, it is said that G acts on S with signature $\sigma(\Delta)$ and that this action is **represented** by the epimorphism θ . If G is a subgroup of G' , then the action of G on S is said to **extend** to an action of G' on S if:

- (1) there is a Fuchsian group Δ' containing Δ ,
- (2) the Teichmüller spaces of Δ and Δ' have the same dimension, and
- (3) there exists an epimorphism

$$\Theta : \Delta' \rightarrow G' \quad \text{in such a way that } \Theta|_{\Delta} = \theta.$$

An action is termed **maximal** if it cannot be extended. A complete list of signatures of Fuchsian groups Δ and Δ' for which it might be possible to have an extension as before was determined by Singerman in [43].

2.2. ACTIONS AND EQUISYMMETRIC STRATIFICATION. Let $\text{Hom}^+(S)$ denote the group of orientation preserving self-homeomorphisms of S . Two actions $\psi_i : G \rightarrow \text{Aut}(S)$ are **topologically equivalent** if there exist $\omega \in \text{Aut}(G)$ and $f \in \text{Hom}^+(S)$ such that

$$(2.3) \quad \psi_2(g) = f\psi_1(\omega(g))f^{-1} \quad \text{for all } g \in G.$$

Each homeomorphism f satisfying (2.3) yields an automorphism f^* of Δ where $\mathbb{H}_{\Delta} \cong S_G$. If \mathcal{B} is the subgroup of $\text{Aut}(\Delta)$ consisting of them, then $\text{Aut}(G) \times \mathcal{B}$ acts on the set of epimorphisms defining actions of G on S with signature $\sigma(\Delta)$ by

$$((\omega, f^*), \theta) \mapsto \omega \circ \theta \circ (f^*)^{-1}.$$

Two epimorphisms $\theta_1, \theta_2 : \Delta \rightarrow G$ define topologically equivalent actions if and only if they belong to the same $(\text{Aut}(G) \times \mathcal{B})$ -orbit (see [7]; also [19] and [31]). We remark that if the genus $h = g_{S_G}$ of S_G is zero and

$$\Delta = \langle x_1, \dots, x_l : x_1^{m_1} = \dots = x_l^{m_l} = x_1 \cdots x_l = 1 \rangle,$$

then \mathcal{B} is generated by the **braid transformations** $\Phi_i \in \text{Aut}(\Delta)$ defined by

$$x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_{i+1}^{-1}x_i x_{i+1} \quad \text{and} \quad x_j \mapsto x_j \quad \text{when } j \neq i, i + 1$$

for each $i \in \{1, \dots, l - 1\}$. See, for example, [24, p. 31].

Let \mathcal{M}_g denote the moduli space of compact Riemann surfaces of genus $g \geq 2$. It is well-known that \mathcal{M}_g is endowed with a structure of complex analytic space of dimension $3g - 3$, and that for $g \geq 4$ its singular locus $\text{Sing}(\mathcal{M}_g)$ agrees with the set of points representing compact Riemann surfaces with non-trivial automorphisms.

Following [6], the singular locus $\text{Sing}(\mathcal{M}_g)$ admits an **equisymmetric stratification** $\{\mathcal{M}_g^{G,\theta}\}$, where each **equisymmetric stratum** $\mathcal{M}_g^{G,\theta}$, if nonempty, corresponds to one topological class of maximal actions. More precisely:

- (1) the closure $\bar{\mathcal{M}}_g^{G,\theta}$ of $\mathcal{M}_g^{G,\theta}$ consists of those Riemann surfaces of genus g admitting an action of the group G with fixed topological class given by θ ,
- (2) $\bar{\mathcal{M}}_g^{G,\theta}$ is a closed irreducible algebraic subvariety of \mathcal{M}_g ,
- (3) if the stratum $\mathcal{M}_g^{G,\theta}$ is nonempty, then it is a smooth, connected, locally closed algebraic subvariety of \mathcal{M}_g which is Zariski dense in $\bar{\mathcal{M}}_g^{G,\theta}$,
- (4) there are finitely many distinct strata, and

$$\text{Sing}(\mathcal{M}_g) = \bigcup_{G \neq 1, \theta} \bar{\mathcal{M}}_g^{G,\theta}.$$

Let $\bar{\mathcal{F}}$ be a (closed) family of compact Riemann surfaces of genus g such that each of its members has a group of automorphisms isomorphic to G . The family $\bar{\mathcal{F}}$ is termed **equisymmetric** if its interior \mathcal{F} consists of only one stratum.

2.3. DECOMPOSITION OF JACOBIANS. If a finite group G acts on a compact Riemann surface S , then it is known that this action induces an isogeny decomposition

$$(2.4) \quad JS \sim A_1 \times \cdots \times A_r$$

which is G -equivariant; see [10] and [29]. The factors A_j in (2.4) are in bijective correspondence with the rational irreducible representations of G . If the factor A_1 is associated with the trivial representation of G , then

$$A_1 \sim JS_G.$$

The decomposition of Jacobians with group actions has been extensively studied, going back to Wirtinger, Schottky and Jung (see, for example, [41] and [47]). For decompositions of Jacobians with respect to special groups, we refer to the articles [9], [20], [21], [35], [36] and [39].

Assume that G acts on a compact Riemann surface S with signature (2.1), and that this action is determined by the epimorphism $\theta : \Delta \rightarrow G$, where Δ is written with its canonical presentation (2.2). We define \mathfrak{J}_θ as the set of complex irreducible representations V of G characterized as follows:

- (1) the trivial representation belongs to \mathfrak{J}_θ if and only if $h \neq 0$, and
- (2) a non-trivial representation V belongs to \mathfrak{J}_θ if and only if

$$d_V(h - 1) + \frac{1}{2} \sum_{i=1}^l (d_V - d_V^{\langle \theta(x_i) \rangle}) \neq 0,$$

where d_V is the degree of V and $d_V^{\langle \theta(x_i) \rangle}$ is the dimension of the subspace of V fixed under the action of the subgroup of G generated by $\theta(x_i)$.

Let H_1, \dots, H_t be groups of automorphisms of S such that G contains H_i for each i . Following [38] (and using [40, Theorem 5.12]), the collection $\{H_1, \dots, H_t\}$ is termed **G -admissible** if

$$d_V^{H_1} + \dots + d_V^{H_t} \leq d_V \quad \text{for each } V \in \mathfrak{J}_\theta,$$

and is called **admissible** if it is G -admissible for some group G . If $\{H_1, \dots, H_t\}$ is admissible then, by [38], JS decomposes, up to isogeny, as

$$JS \sim \prod_{i=1}^t JS_{H_i} \times P$$

for some abelian subvariety P of JS . See also [23].

NOTATION. Let $n \geq 2$ be an integer. Throughout this article we shall denote by C_n the cyclic group of order n and by \mathbf{D}_n the dihedral group of order $2n$.

3. Existence of the families $\bar{\mathcal{F}}_g^1$ and $\bar{\mathcal{F}}_g^2$

PROPOSITION 1: *Let $g \geq 6$ such that $g - 1$ is a prime number and $g \equiv 2 \pmod{4}$. There exists a complex one-dimensional equisymmetric family $\bar{\mathcal{F}}_g^1$ of compact Riemann surfaces of genus g with a group of automorphisms isomorphic to*

$$\langle a, b : a^{g-1} = b^4 = 1, bab^{-1} = a^r \rangle = C_{g-1} \rtimes_4 C_4,$$

where r is a primitive 4-th root of the unity in the field of $g - 1$ elements, such that the signature of the action is $(0; 2, 2, 4, 4)$.

Proof. Set $q = g - 1$, and let Δ be a Fuchsian group of signature $\sigma = (0; 2, 2, 4, 4)$ with canonical presentation

$$\Delta = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^4 = x_4^4 = x_1x_2x_3x_4 = 1 \rangle.$$

The epimorphism $\Theta : \Delta \rightarrow C_q \rtimes_4 C_4$ defined by

$$\Theta(x_1) = b^2, \quad \Theta(x_2) = ab^2, \quad \Theta(x_3) = ab, \quad \Theta(x_4) = b^3$$

guarantees the existence of a complex one-dimensional family $\bar{\mathcal{F}}_g^1$ of compact Riemann surfaces S of genus g with a group of automorphisms G isomorphic to $C_q \rtimes_4 C_4$ acting on S with signature σ .

To prove that $\bar{\mathcal{F}}_g^1$ consists of only one stratum, we first notice that the involutions of $C_q \rtimes_4 C_4$ are $a^l b^2$ and the elements of order 4 are $a^l b$ and $a^l b^3$ for $1 \leq l \leq q$. Then, up to a permutation, an epimorphism $\theta : \Delta \rightarrow C_q \rtimes_4 C_4$ representing an action of G on S is defined by

$$\theta(x_1) = a^{l_1} b^2, \quad \theta(x_2) = a^{l_2} b^2, \quad \theta(x_3) = a^{l_3} b, \quad \theta(x_4) = a^{l_4} b^3,$$

for some $1 \leq l_1, \dots, l_4 \leq q$. After applying an inner automorphism of G , we can assume that $l_4 \equiv 0 \pmod q$ and therefore $l_2 \equiv l_1 + l_3 \pmod q$. Note that if $l_1 \equiv l_3 \equiv 0 \pmod q$ then θ is not surjective; thus, without loss of generality, we can assume $l_3 \not\equiv 0 \pmod q$. Now, consider the automorphism of G given by $a \mapsto a^{t_3}, b \mapsto b$, where $l_3 t_3 \equiv 1 \pmod q$, to obtain that θ is equivalent to the epimorphism θ_n defined by

$$\theta_n(x_1) = a^n b^2, \quad \theta_n(x_2) = a^{n+1} b^2, \quad \theta_n(x_3) = ab, \quad \theta_n(x_4) = b^3$$

for some $1 \leq n \leq q$. Finally, as $\Phi_1 \cdot \theta_n = \theta_{n+1}$, each θ_n is equivalent to $\theta_0 = \Theta$. ■

PROPOSITION 2: *There exists a complex two-dimensional family $\bar{\mathcal{F}}_g^2$ of compact Riemann surfaces of genus $g \geq 2$ with a group of automorphisms isomorphic to the dihedral group of order $4g - 4$ such that the signature of the action is $(0; 2, 2, 2, 2, 2)$. If, in addition, $g - 1$ is prime then the family is equisymmetric.*

Proof. Set $q = g - 1$, and let Δ be a Fuchsian group of signature $\sigma = (0; 2, 2, 2, 2, 2)$ with canonical presentation

$$\Delta = \langle x_1, x_2, x_3, x_4, x_5 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_1 x_2 x_3 x_4 x_5 = 1 \rangle.$$

The epimorphism $\Theta : \Delta \rightarrow \mathbf{D}_{2q} = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle$ defined by

$$\Theta(x_1) = s, \quad \Theta(x_2) = s, \quad \Theta(x_3) = sr^{q+1}, \quad \Theta(x_4) = sr, \quad \Theta(x_5) = r^q$$

guarantees the existence of a complex two-dimensional family $\bar{\mathcal{F}}_g^2$ of compact Riemann surfaces S of genus g with a group of automorphisms G isomorphic to \mathbf{D}_{2q} acting on S with signature σ .

We now assume q to be prime and proceed to prove that $\bar{\mathcal{F}}_g^2$ is equisymmetric. Let $\theta : \Delta \rightarrow \mathbf{D}_{2q}$ be an epimorphism representing an action of G on S . Note that the involutions of G are r^q and sr^t for $1 \leq t \leq 2q$, and if some $\theta(x_i)$ equals r^q then, after considering suitable braid automorphisms, it can be supposed that $i = 5$.

We claim that one and only one among the elements $\theta(x_i)$ equals r^q . Indeed, if l denotes the number of elements $\theta(x_i)$ which are equal to r^q , then clearly l is different from 4 and 5 because otherwise θ is not surjective. If $l = 3$ then it can be supposed $\theta(x_1x_2) = r^q$. If we write $\theta(x_i) = sr^{t_i}$ then $t_2 - t_1 \equiv q \pmod{2q}$, showing that $t_2 - t_1$ is odd and therefore, after considering the automorphism of G given by $r \mapsto r, s \mapsto sr$, we can assume that t_1 is even and that t_2 is odd. Now, we apply an appropriate inner automorphism of G to suppose that $t_1 \equiv 0 \pmod{2q}$ and $t_2 \equiv q \pmod{2q}$. The contradiction is obtained by noticing that $\langle s, r^q \rangle \cong C_2^2$. Similarly, if $l = 2$ then we can suppose $\theta(x_1x_2x_3) = 1$. If we write $\theta(x_i) = sr^{t_i}$ then $sr^{t_1-t_2+t_3} = 1$, which is not possible.

It follows that, up to equivalence, the epimorphism $\theta : \Delta \rightarrow \mathbf{D}_{2q}$ is given by

$$\theta(x_1) = sr^{t_1}, \quad \theta(x_2) = sr^{t_2}, \quad \theta(x_3) = sr^{t_3}, \quad \theta(x_4) = sr^{t_4}, \quad \theta(x_5) = r^q$$

for some $1 \leq t_1, \dots, t_4 \leq 2q$ which satisfy $t_2 - t_1 + t_4 - t_3 \equiv q \pmod{2q}$. Now, after considering, if necessary, braid automorphisms and the automorphism of G given by $r \mapsto r, s \mapsto sr$, we can suppose t_1, t_2, t_3 to be even and t_4 to be odd. Furthermore, after applying a suitable inner automorphism of G , we can assume that $t_1 = 0$.

If $t_4 \neq q$, then we apply the automorphism of G given by $r \mapsto r^{l_4}, s \mapsto s$, where $t_4 l_4 \equiv 1 \pmod{2q}$, to see that θ is equivalent to the epimorphism θ_n given by

$$\theta_n(x_1) = s, \quad \theta_n(x_2) = sr^n, \quad \theta_n(x_3) = sr^{n+q+1}, \quad \theta_n(x_4) = sr, \quad \theta_n(x_5) = r^q$$

for some $1 \leq n \leq 2q$ even. The equality $\Phi_2^2 \cdot \theta_n = \theta_{n+2}$ shows that θ_n is equivalent to $\theta_0 = \Theta$. Similarly, if now $t_4 = q$ then $t_2 \equiv t_3 \pmod{2q}$; we write $t = t_2 = t_3$. We apply the inner automorphism of G induced by $sr^{t/2}$ and then the braid automorphism $\Phi_2 \circ \Phi_1$ to see that θ is equivalent to the epimorphism θ_t defined by

$$\theta_t(x_1) = s, \quad \theta_t(x_2) = s, \quad \theta_t(x_3) = sr^t, \quad \theta_t(x_4) = sr^{t+q}, \quad \theta_t(x_5) = r^q$$

where $t \not\equiv 0 \pmod{2q}$. Finally, consider the automorphism of G given by $r \mapsto r^l, s \mapsto s$, where $(q + t)l \equiv 1 \pmod{2q}$, to see that θ_t is equivalent to Θ . ■

4. Proof of Theorem 1

Set $q = g - 1$ and let S be a compact Riemann surface of genus $g \geq 8$ with a group of automorphisms G of order $4q$ where q is prime. By the Riemann-Hurwitz formula the possible signatures of the action of G on S are

$$(1; 2), \quad (0; 2, 2, 4, 4) \quad \text{and} \quad (0; 2, 2, 2, 2, 2).$$

By the classical Sylow’s theorems, if $q \equiv 3 \pmod 4$ then G is isomorphic to either

$$C_{4q}, \quad C_q \times C_2^2, \quad \mathbf{D}_{2q} \quad \text{or} \quad \langle a, b : a^q = b^4 = 1, bab^{-1} = a^{-1} \rangle = C_q \rtimes_2 C_4,$$

and if $q \equiv 1 \pmod 4$ then, in addition to these groups, G can be isomorphic to

$$\langle a, b : a^q = b^4 = 1, bab^{-1} = a^r \rangle = C_q \rtimes_4 C_4$$

where r is a primitive 4-th root of the unity in the field of q elements.

The proof of Theorem 1 is a consequence of the following three claims.

CLAIM 1: *The following statements are equivalent.*

- (1) S is a compact Riemann surface of genus g with a group of automorphisms of order $4q$ acting on S with signature $\sigma = (0; 2, 2, 4, 4)$.
- (2) $g \equiv 2 \pmod 4$ and $S \in \overline{\mathcal{F}}_g^1$.

Let us assume that S is a compact Riemann surface of genus g with a group of automorphisms G of order $4q$ acting on S with signature σ . Let Δ be a Fuchsian group of signature σ with canonical presentation

$$\Delta = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^4 = x_4^4 = x_1x_2x_3x_4 = 1 \rangle,$$

and assume the action of G on S to be represented by the epimorphism $\theta: \Delta \rightarrow G$.

First of all, note that G cannot be isomorphic to $C_q \times C_2^2$ or \mathbf{D}_{2q} because they do not have elements of order 4, and cannot be isomorphic to C_{4q} because otherwise C_{4q} would be generated by elements of order 2 and 4. We claim that G cannot be isomorphic to $C_q \rtimes_2 C_4$ either. Indeed, as b^2 is the unique involution of $C_q \rtimes_2 C_4$ and as its elements of order 4 are $a^t b$ and $a^t b^3$ for $1 \leq t \leq q$, after considering the automorphism of $C_q \rtimes_2 C_4$ given by $a \mapsto a, b \mapsto b^3$, the epimorphism θ could only be defined either by

- (1) $\theta(x_1) = b^2, \theta(x_2) = b^2, \theta(x_3) = a^{t_1} b, \theta(x_4) = a^{t_2} b$, or
- (2) $\theta(x_1) = b^2, \theta(x_2) = b^2, \theta(x_3) = a^{t_1} b, \theta(x_4) = a^{t_2} b^3$,

for some $1 \leq t_1, t_2 \leq q$. The former case cannot yield an action because the image of $x_1x_2x_3x_4$ is different from 1 for each possible choice of t_1 and t_2 . Similarly, the latter case could give rise to an action only if $t_1 \equiv t_2 \pmod q$; however, in this case θ would not be surjective.

All of the above shows that $g \equiv 2 \pmod 4$ and that G is isomorphic to $C_q \rtimes_4 C_4$; consequently $S \in \mathcal{F}_g^1$. The converse is direct, and the proof of the claim is completed.

CLAIM 2: *There is no compact Riemann surface S of genus g with a group of automorphisms of order $4q$ acting on it with signature $(1; 2)$.*

Let G be a group of order $4q$ and let Δ be a Fuchsian group of signature $(1; 2)$. For every torsion-free kernel epimorphism

$$\Delta = \langle a_1, b_1, x_1 : [a_1, b_1]x_1 = x_1^2 = 1 \rangle \rightarrow G,$$

the image of x_1 must belong to the commutator subgroup of G . Thus, to conclude it suffices to note that the commutator subgroup of each group of order $4q$ does not contain involutions.

CLAIM 3: *The following statements are equivalent.*

- (1) *S is a compact Riemann surface of genus g with a group of automorphisms of order $4q$ acting on S with signature $\sigma = (0; 2, 2, 2, 2, 2)$.*
- (2) *$S \in \mathcal{F}_g^2$.*

Let us assume that S is a compact Riemann surface of genus g with a group of automorphisms G of order $4q$ acting on S with signature σ . Let Δ be a Fuchsian group of signature σ with canonical presentation

$$\Delta = \langle x_1, x_2, x_3, x_4, x_5 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_1x_2x_3x_4x_5 = 1 \rangle,$$

and assume the action of G on S to be represented by the epimorphism $\theta: \Delta \rightarrow G$.

The group G cannot be isomorphic to $C_q \rtimes_2 C_4$ because it has a unique involution and therefore every homomorphism $\Delta \rightarrow C_q \rtimes_2 C_4$ is not surjective. Similarly, the group G cannot be isomorphic to $C_q \rtimes_4 C_4$ because it has exactly q involutions and all of them are contained in the proper subgroup $\langle a, b^2 \rangle \cong \mathbf{D}_q$. Finally, if G were abelian then G would be isomorphic to a subgroup of C_{2q}^4 ; but this is not possible.

All the above ensures that G is isomorphic to the dihedral group of order $4q$ and therefore $S \in \mathcal{F}_g^2$. The converse is direct, and the proof of the claim is completed.

5. Proof of Theorem 2

Let $g \geq 8$ such that $q = g - 1$ is prime. Assume $g \equiv 2 \pmod 4$ and let S be a compact Riemann surface lying in the family \mathcal{F}_g^1 . We recall that S has a group of automorphisms G_1 isomorphic to

$$\langle a, b : a^q = b^4 = 1, bab^{-1} = a^r \rangle = C_q \rtimes_4 C_4,$$

where r is a primitive 4-th root of the unity in the field of q elements, and that the action is represented by the epimorphism $\Theta : \Delta \rightarrow C_q \rtimes_4 C_4$ defined by

$$\Theta(x_1) = b^2, \quad \Theta(x_2) = ab^2, \quad \Theta(x_3) = ab, \quad \Theta(x_4) = b^3$$

where

$$\Delta = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^4 = x_4^4 = x_1x_2x_3x_4 = 1 \rangle.$$

By classical results due to Singerman [43], the action of G_1 on a (generic) member S of \mathcal{F}_g^1 can be possibly extended to only an action of a group of order $8q$ with signature $\hat{\sigma} = (0; 2, 2, 2, 4)$. However, as proved in [3] for $g \geq 18$ and in [12] for $g = 14$, there is no compact Riemann surfaces of genus g with a group of automorphisms of order $8q$ acting with signature $\hat{\sigma}$. It follows that:

- (1) the interior \mathcal{F}_g^1 of the family \mathcal{F}_g^1 consists of those Riemann surfaces for which G_1 agrees with the full automorphism group, and
- (2) the set

$$\mathcal{F}_g^1 \setminus \mathcal{F}_g^1 = \{X \in \mathcal{F}_g^1 : G_1 \subsetneq \text{Aut}(X)\}$$

consists of finitely many points.

Note that for each $X \in \mathcal{F}_g^1 \setminus \mathcal{F}_g^1$ its full automorphism group has order μq where 4 divides μ . Then, following [3], there exists a positive integer ϵ_1 such that if $g \geq \epsilon_1$ then either

- (1) $\text{Aut}(X) \cong C_q \rtimes_8 C_8$ acting with signature $(0; 2, 8, 8)$, for $g \equiv 2 \pmod 8$;
or
- (2) $\text{Aut}(X) \cong (C_q \rtimes_6 C_6) \times C_2$ acting with signature $(0; 2, 6, 6)$, for $g \equiv 2 \pmod 3$.

The latter case is not possible because $(C_q \rtimes_6 C_6) \times C_2$ does not have elements of order 4, showing that if $g \not\equiv 2 \pmod 8$ then $\bar{\mathcal{F}}_g^1 \setminus \mathcal{F}_g^1$ is empty. Let us now assume that $g \equiv 2 \pmod 8$, and let

$$\Delta' = \langle y_1, y_2, y_3 : y_1^2 = y_2^8 = y_3^8 = y_1 y_2 y_3 = 1 \rangle$$

be a Fuchsian group of signature $(0; 2, 8, 8)$. Again, following [3], there are exactly two non-isomorphic Riemann surfaces X_1 and X_2 of genus $g \geq 18$ such that

$$\text{Aut}(X_i) \cong \langle \alpha, \beta : \alpha^q = \beta^8 = 1, \beta \alpha \beta^{-1} = \alpha^u \rangle = C_q \rtimes_8 C_8,$$

where u is a primitive 8-th root of the unity in the field of q elements, and the action of $\text{Aut}(X_i)$ on X_i is determined by the epimorphisms $\Theta_i : \Delta' \rightarrow C_q \rtimes_8 C_8$,

$$\begin{aligned} \Theta_1(y_1) &= \beta^4, & \Theta_1(y_2) &= \alpha^{-u} \beta, & \Theta_1(y_3) &= \alpha \beta^3, \\ \Theta_2(y_1) &= \beta^4, & \Theta_2(y_2) &= \alpha^u \beta, & \Theta_2(y_3) &= \alpha \beta^7. \end{aligned}$$

The subgroup of Δ' generated by the elements

$$\hat{x}_1 = y_1, \quad \hat{x}_2 = y_1, \quad \hat{x}_3 = y_3^2, \quad \hat{x}_4 = y_3^6$$

is isomorphic to Δ , and

$$\begin{aligned} \Theta_i(\hat{x}_1) &= \beta^4, & \Theta_i(\hat{x}_2) &= \beta^4, \\ \Theta_i(\hat{x}_3) &= \alpha^{1+(-1)^{i+1}u^3} \beta^6, & \Theta_i(\hat{x}_4) &= \alpha^{u^3+(-1)^i u^2} \beta^7. \end{aligned}$$

Thus, for $i = 1, 2$, the restriction of Θ_i to $\Delta \cong \langle \hat{x}_1, \dots, \hat{x}_4 \rangle$,

$$\Delta \rightarrow \langle \beta^4, \alpha^{1 \pm u^3} \beta^6 \rangle = \langle \alpha, \beta^2 \rangle = C_q \rtimes_4 C_4,$$

defines an action $C_q \rtimes_4 C_4$ on X_i with signature $(0; 2, 2, 4, 4)$, showing that $\bar{\mathcal{F}}_g^1 \setminus \mathcal{F}_g^1$ agrees with $\{X_1, X_2\}$.

Now, let S be a compact Riemann surface lying in the family $\bar{\mathcal{F}}_g^2$. We recall that S has a group of automorphisms G_2 isomorphic to

$$\langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle = \mathbf{D}_{2q},$$

and that the action of G_2 on S is represented by the epimorphism $\Theta : \Delta \rightarrow \mathbf{D}_{2q}$,

$$\Theta(x_1) = s, \quad \Theta(x_2) = s, \quad \Theta(x_3) = sr^{q+1}, \quad \Theta(x_4) = sr, \quad \Theta(x_5) = r^q$$

where $\Delta = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_1 x_2 x_3 x_4 x_5 = 1 \rangle$.

By [43] the action of G_2 on a generic member S of $\bar{\mathcal{F}}_g^2$ cannot be extended. Thus:

- (1) the interior \mathcal{F}_g^2 of the family $\bar{\mathcal{F}}_g^2$ consists of those Riemann surfaces for which G_2 agrees with the full automorphism group, and
- (2) the set

$$\bar{\mathcal{F}}_g^2 \setminus \mathcal{F}_g^2 = \{Y \in \bar{\mathcal{F}}_g^2 : G \not\subseteq \text{Aut}(Y)\}$$

consists of finitely many points and finitely many one-dimensional families.

By [3], there exists ϵ_2 such that if $g \geq \epsilon_2$ and $Y \in \bar{\mathcal{F}}_g^2 \setminus \mathcal{F}_g^2$ then either

- (1) $\text{Aut}(Y) \cong C_q \rtimes_8 C_8$ acting with signature $(0; 2, 8, 8)$, for $g \equiv 2 \pmod 8$,
or
- (2) $\text{Aut}(Y) \cong (C_q \rtimes_6 C_6) \times C_2$ acting with signature $(0; 2, 6, 6)$, for $g \equiv 2 \pmod 3$.

The former case is not possible because $C_q \rtimes_8 C_8$ does not have elements of order $2q$; thus, if $g \not\equiv 2 \pmod 3$ then $\bar{\mathcal{F}}_g^2 \setminus \mathcal{F}_g^2$ is empty. Let us now assume that $g \equiv 2 \pmod 3$, and let Δ' be a Fuchsian group of signature $(0; 2, 6, 6)$ with canonical presentation

$$\Delta' = \langle y_1, y_2, y_3 : y_1^2 = y_2^6 = y_3^6 = y_1 y_2 y_3 = 1 \rangle.$$

Again, following [3], for $g \geq \epsilon_2$ there are exactly two non-isomorphic Riemann surfaces Y_1 and Y_2 of genus g with full automorphism group isomorphic to

$$\langle \alpha, \beta, \gamma : \alpha^q = \beta^6 = \gamma^2 = [\gamma, \alpha] = [\gamma, \beta] = 1, \beta\alpha\beta^{-1} = \alpha^u \rangle = (C_q \rtimes_6 C_6) \times C_2,$$

where u is a primitive 6-th root of the unity in the field of q elements, and the action of $\text{Aut}(Y_i)$ on Y_i is determined by the epimorphisms $\Theta_i: \Delta' \rightarrow (C_q \rtimes_6 C_6) \times C_2$,

$$\begin{aligned} \Theta_1(y_1) &= \beta^3, & \Theta_1(y_2) &= \alpha^{-u}\beta\gamma, & \Theta_1(y_3) &= \alpha\beta^2\gamma, \\ \Theta_2(y_1) &= \beta^3\gamma, & \Theta_2(y_2) &= \alpha^{-u^2}\beta^2\gamma, & \Theta_2(y_3) &= \alpha\beta. \end{aligned}$$

The subgroup of Δ' generated by

$$\tilde{x}_1 = y_3^3, \quad \tilde{x}_2 = y_1, \quad \tilde{x}_3 = y_2 y_1 y_2^{-1}, \quad \tilde{x}_4 = y_2^2 y_1 y_2^{-2}, \quad \tilde{x}_5 = y_2^3$$

is isomorphic to Δ , and

$$\begin{aligned} \Theta_1(\tilde{x}_1) &= \gamma, & \Theta_1(\tilde{x}_2) &= \beta^3, & \Theta_1(\tilde{x}_3) &= \alpha^{-2u}\beta^3, \\ \Theta_1(\tilde{x}_4) &= \alpha^{2-4u}\beta^3, & \Theta_1(\tilde{x}_5) &= \alpha^{2-2u}\beta^3\gamma, \\ \Theta_2(\tilde{x}_1) &= \alpha^{2u}\beta^3, & \Theta_2(\tilde{x}_2) &= \beta^3\gamma, & \Theta_2(\tilde{x}_3) &= \alpha^{-2u^2}\beta^3\gamma, \\ \Theta_2(\tilde{x}_4) &= \alpha^2\beta^3\gamma, & \Theta_2(\tilde{x}_5) &= \gamma. \end{aligned}$$

Thus, for $i = 1, 2$, the restriction of Θ_i to $\Delta \cong \langle \tilde{x}_1, \dots, \tilde{x}_4 \rangle$,

$$\Delta \rightarrow \langle \alpha\gamma, \beta^3 \rangle = \mathbf{D}_{2q},$$

defines an action \mathbf{D}_{2q} on Y_i with signature $(0; 2, 2, 2, 2, 2)$, showing that $\bar{\mathcal{F}}_g^2 \setminus \mathcal{F}_g^2$ agrees with $\{Y_1, Y_2\}$.

The proof is complete.

6. Proof of Theorem 3

Let $g \geq 8$ such that $q = g - 1$ is prime.

We recall the well-known fact that the dihedral group

$$\langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle = \mathbf{D}_{2q}$$

has, up to equivalence, 4 complex irreducible representations of degree one, namely,

$$U_1^\pm : r \mapsto 1, s \mapsto \pm 1, \quad U_2^\pm : r \mapsto -1, s \mapsto \pm 1,$$

and $q - 1$ complex irreducible representations of degree two; namely,

$$V_j : r \mapsto \text{diag}(\omega_{2q}^j, \bar{\omega}_{2q}^j), s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $1 \leq j \leq q - 1$ and $\omega_t = \exp(\frac{2\pi i}{t})$. See, for example, [42, p. 36].

Following the notation introduced in Subsection 2.3, the trivial representation U_1^+ does not belong to \mathfrak{J}_θ , where θ represents the action of \mathbf{D}_{2q} on each member S of the family $\bar{\mathcal{F}}_g^2$. The following table summarizes the dimension of the vector subspaces of the non-trivial complex irreducible representations of \mathbf{D}_{2q} fixed under the action of the subgroups $\langle r \rangle$, $\langle s \rangle$ and $\langle sr \rangle$:

	U_1^-	U_2^+	U_2^-	V_j
$\langle s \rangle$	0	1	0	1
$\langle r \rangle$	1	0	0	0
$\langle sr \rangle$	0	0	1	1

It follows that the collection $\{\langle r \rangle, \langle s \rangle, \langle sr \rangle\}$ is admissible and therefore, by [38], if $S \in \bar{\mathcal{F}}_g^2$ then there exists an abelian subvariety P of JS such that

$$JS \sim JS_{\langle s \rangle} \times JS_{\langle r \rangle} \times JS_{\langle sr \rangle} \times P.$$

The covering maps given by the action of $\langle s \rangle$, $\langle r \rangle$ and $\langle sr \rangle$ ramify over six, two and two values respectively; then, the Riemann–Hurwitz formula implies that

$$g_{S_{\langle s \rangle}} = \frac{q - 1}{2}, \quad g_{S_{\langle r \rangle}} = 1 \quad \text{and} \quad g_{S_{\langle sr \rangle}} = \frac{q + 1}{2}.$$

It follows that $P = 0$ and the desired decomposition is obtained.

Remark 1:

- (a) Note that if $S \in \bar{\mathcal{F}}_g^2$ then JS contains an elliptic curve.
- (b) For the sake of completeness, we mention that in [22, Theorem 2] it was proved that the Jacobian variety of the Riemann surfaces Y_1 and Y_2 of Theorem 2 also admit an isogeny decomposition in terms of an abelian surface and of an abelian variety of dimension $\frac{q-1}{6}$ with multiplicity 6.

We now assume $g \equiv 2 \pmod 4$. Let r be a primitive 4-th root of the unity in the field of q elements, write $m = \frac{q-1}{4}$ and choose $1 \leq k_1, \dots, k_m \leq q - 1$ such that

$$\{1, \dots, q - 1\} = \sqcup_{j=1}^m \{\pm k_j, \pm r k_j\},$$

where the symbol \sqcup stands for disjoint union. Then

$$\langle a, b : a^q = b^4 = 1, bab^{-1} = a^r \rangle = C_q \rtimes_4 C_4$$

has, up to equivalence, m complex irreducible representations of degree 4, given by

$$V_j : a \mapsto \text{diag}(\omega_q^{k_j}, \omega_q^{k_j r}, \omega_q^{-k_j}, \omega_q^{-k_j r}), \quad b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } \omega_t = \exp\left(\frac{2\pi i}{t}\right),$$

and four complex irreducible representations of degree 1, given by

$$U_l : a \mapsto 1, \quad b \mapsto \omega_4^l, \quad \text{for } 0 \leq l \leq 3$$

(see, for example, [42, p. 62]). Choose four pairwise different integers

$$t_1, t_2, t_3, t_4 \in \{1, \dots, q - 1\},$$

and consider the following subgroups of $C_q \rtimes_4 C_4$:

$$\langle a \rangle = C_q \quad \text{and} \quad \langle a^{t_i} b \rangle = C_4 \quad \text{for } 1 \leq i \leq 4.$$

The trivial representation U_0 does not belong to \mathfrak{J}_θ , where θ represents the action of $C_q \rtimes_4 C_4$ on each member S of the family $\bar{\mathcal{F}}_g^1$. The dimension of the vector subspaces of the non-trivial complex irreducible representations of $C_q \rtimes_4 C_4$ fixed under the action of the subgroups $\langle a \rangle$ and $\langle a^{t_i} b \rangle$ is

$$U_l^{(a)} = V_j^{(a^{t_i} b)} = 1 \quad \text{and} \quad U_l^{(a^{t_i} b)} = V_j^{(a)} = 0.$$

Thus $\{\langle a \rangle, \langle a^{t_1} b \rangle, \dots, \langle a^{t_4} b \rangle\}$ is admissible and therefore, by [38], if $S \in \bar{\mathcal{F}}_q^1$ then

$$JS \sim JS_{\langle a \rangle} \times \prod_{i=1}^4 JS_{\langle a^{t_i} b \rangle} \times Q \cong JS_{\langle a \rangle} \times (JS_{\langle b \rangle})^4 \times Q,$$

for some abelian subvariety Q of JS , where the isomorphism follows after noticing that each $\langle a^{t_i} b \rangle$ and $\langle b \rangle$ are conjugate. The covering map $S \rightarrow S_{\langle a \rangle}$ is unbranched, and the covering map $S \rightarrow S_{\langle b \rangle}$ ramifies over four values, two marked with 2 and two marked with 4. Then, the Riemann–Hurwitz formula implies that

$$g_{S_{\langle a \rangle}} = 2 \quad \text{and} \quad g_{S_{\langle b \rangle}} = m.$$

Thereby, $Q = 0$ and the decomposition of JS stated in Theorem 3 is complete.

7. The case $g - 1$ not prime

Example 1: Let $n \geq 3$ be an integer, and consider the group

$$\langle x, y, z : x^4 = z^n = 1, x^2 = y^2, yxy^{-1} = x^3, [x, z] = [y, z] = 1 \rangle = \mathbf{Q}_8 \times C_n$$

where \mathbf{Q}_8 denotes the quaternion group, and let Δ be a Fuchsian group of signature $(1; 2)$ with canonical presentation

$$\Delta = \langle a_1, b_1, x_1 : [a_1, b_1]x_1 = x_1^2 = 1 \rangle.$$

For each n odd, the epimorphism $\Delta \rightarrow \mathbf{Q}_8 \times C_n$ given by

$$a_1 \mapsto x, \quad b_1 \mapsto yz, \quad x_1 \mapsto y^2$$

guarantees the existence of a complex one-dimensional family of compact Riemann surfaces of genus $g = 1 + 2n$ with a group of automorphisms of order $8n = 4g - 4$ isomorphic to $\mathbf{Q}_8 \times C_n$ acting with signature $(1; 2)$.

Example 2: Let $n \geq 3$. Choose $m \in \{\pm 1, 2^{n-1} \pm 1\}$, consider the group

$$\langle r, s, t : r^{2^n} = s^2 = (sr)^2 = t^2 = 1, trt = r^m, tst = s \rangle = \mathbf{D}_{2^n} \rtimes C_2,$$

and let Δ be a Fuchsian group of signature $\sigma = (0; 2, 2, 2, 2, 2)$ with presentation

$$\Delta = \langle x_1, x_2, x_3, x_4, x_5 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_1x_2x_3x_4x_5 = 1 \rangle.$$

The epimorphism $\Delta \rightarrow \mathbf{D}_{2^n} \rtimes C_2$ given by

$$x_1 \mapsto sr, \quad x_2 \mapsto sr, \quad x_3 \mapsto s, \quad x_4 \mapsto t, \quad x_5 \mapsto st$$

guarantees the existence of a complex two-dimensional family of compact Riemann surfaces of genus $g = 2^n + 1$, with a group of automorphisms of order $2^{n+2} = 4g - 4$ isomorphic to $\mathbf{D}_{2^n} \rtimes C_2$ acting with signature σ . Two different choices of m yield non-isomorphic groups, showing that if $g - 1 = 2^n$ then there exist at least four pairwise non-isomorphic groups of order $4g - 4$ acting on compact Riemann surfaces of genus g with the same signature σ .

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