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ASYMPTOTIC BEHAVIOUR OF pTH MEANS OF ANALYTIC AND SUBHARMONIC FUNCTIONS IN THE UNIT DISC AND ANGULAR DISTRIBUTION OF ZEROS

BY

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In memory of Professor Anatolii Grishin

ABSTRACT

We propose a new approach for studying asymptotic behaviour of pth means of the logarithmic potential and classes of analytic and subharmonic functions in the unit disc. In particular, we generalize a criterion due to G. MacLane and L. Rubel of boundedness of the L_2-norm of $\log|B|$, where B is a Blaschke product, in several directions. We describe growth and decrease of pth means, $p \in (1,\infty)$, for nonpositive subharmonic functions in the unit disc. As a consequence, we obtain a complete description of the asymptotic behaviour of pth logarithmic means of bounded analytic functions in the unit disc in terms of its zeros and the boundary measure. We also prove sharp upper estimates of pth means of analytic and subharmonic functions of finite order in the unit disc.

1. Introduction and main results

In the present paper we investigate an interplay between the zero distribution and the growth of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Especially we are interested in the asymptotic behaviour of logarithmic means of such functions. We consider them as a special case of subharmonic functions. Our approach is based on a Martin-type represetation (cf. [3]) and the concept

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of the complete measure of a subharmonic function (see [6], [15], [16], [20]) which takes into account both the Riesz measure (divisor) and the boundary measure of the function.

The paper is organized as follows. Existing results and the main results of the paper are discussed further in Section 1. Section 2 contains some notation and auxiliary statements mostly connected to the case of a function of finite order of the growth. Proofs of the theorems are contained in Sections 3, while examples showing sharpness of our results are presented in Section 4.

1.1. Some results on growth and angular distribution of zeros of Blaschke products. Given a sequence (a_n) in $\mathbb{D}$ such that $\sum_n(1-|a_n|) < \infty$, we consider the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}(a_n - z)}{|a_n|(1 - z\overline{a_n})}. \tag{1.1}$$

It was A. Zygmund (see [33]) who asked to describe those sequences (a_n) in $\mathbb{D}$ such that

$$I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log|B(re^{i\theta})|)^2 \, d\theta$$

is bounded. In [33] G. Maclane and L. Rubel answered this question using a Fourier series method.

Theorem A ([33, Theorem 1]): *A necessary and sufficient condition that $I(r)$ be bounded is that $J(r)$ be bounded, where*

$$J(r) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left| (r^k - r^{-k}) \sum_{|a_n|\le r} \bar{a}_n^k + r^k \sum_{|a_n|>r} (\bar{a}_n^k - a_n^{-k}) \right|^2.$$

Since it was difficult to check the boundedness of $J(r)$ they gave also the following sufficient condition.

Let $n(r, B)$ be the number of zeros in the closed disc $\overline{D}(0, r)$; here and in what follows $D(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$.

Theorem B ([33]): *If*

$$n(r, B) = O((1 - r)^{-\frac{1}{2}}), \quad r \in (0, 1), \tag{1.2}$$

then $I(r)$ is bounded.

They also noted that (1.2) is equivalent to the condition

$$\sum_{|a_n|>r}(1-|a_n|)=O(\sqrt{1-r}).$$

MacLane and Rubel also proved that (1.2) becomes necessary if all zeros lie on finitely many rays emanating from the origin, but it is not the case in general. After that C. N. Linden ([30, Corollary 1]) generalized this showing that it is sufficient to require that the zero sequence is contained in a finite number of Stolz angles with vertices on $\partial\mathbb{D}$.

For a subharmonic function u in $\mathbb{D}$ and $p\ge 1$ we define

$$m_p(r,u)=\left(\frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}},\quad 0<r<1,$$

$$\rho_p[u]=\limsup_{r\uparrow 1}\frac{\log^+ m_p(r,u)}{-\log(1-r)}.$$

The growth of $m_p(r,\log|f|)$ was studied in many papers, for instance [31], [32], [34], [37], [17], [38], [5], [9].

The classes defined by the growth of $m_p(r,\log|f|)$, where f is an anlytic function in $\mathbb{D}$, are closely related to the generalized Nevanlinna classes $\mathbb{A}_{p,\alpha}$, $1\le p<\infty$, $0<\alpha<\infty$ defined by the condition

$$\|f\|_{p,\alpha}=\frac{1}{\pi}\int_{\mathbb{D}}|\log|f(z)||^p(1-|z|^2)^\alpha\,dA(z)<\infty$$

where $dA(z)$ is the planar Lebesgue measure. It is clear that the inequality $\rho_p[\log|f|]<\frac{\alpha+1}{p}$ implies $f\in\mathbb{A}_{p,\alpha}$, but not vice versa. Factorization and zeros of these classes have been studied by E. Beller [1, 2], A. Heilper [24], J. Bruna and J. Ortega-Cerdá [4], Ch. Horowitz [25] and others. In particular, in [4], sequences (a_n) which are the zero sequences for some analytic functions f from $\mathbb{A}_{p,0}$ are described. We just mention here that in the case $p=1$ the zeros can be described in terms of the counting function $n(r,f)$, or, in other words, the angular distribution of zeros (the Riesz measure) is not important (see, e.g., [9] and references therein). In the case $p=\infty$ the reader should consult recent works of B. Khabibullin [26]–[28].

Nevertheless, to the best of our knowledge, only one paper, namely [34], contains criteria of boundedness of pth means $m_p(r,u)$ when $u=\log|B|$. Proofs of the results announced in [34] have been published recently in [40, Chap. 3].

Note that Ya. V. Mykytyuk and Ya. V. Vasyl'kiv used methods of functional analysis, and we propose another, more elementary, approach.

In [34] Ya. V. Mykytyuk and Ya. V. Vasyl'kiv introduced two auxiliary functions defined on $\partial\mathbb{D}$ by (a_n):

$$\psi_r(\zeta) = \sum_{r\le |a_n|<1} \frac{(1-|a_n|)^2}{|\zeta - a_n|^2}, \quad \zeta\in\partial\mathbb{D},\ r\in[0,1),$$

and $\varphi(\zeta)$, which satisfies the relation

$$\varphi(\zeta) \asymp \Phi(\zeta) := \#\{a_n : |1-a_n\bar\zeta| < 2(1-|a_n|)\},$$

i.e., the number of zeros in the Stolz angle with the vertex ζ. They established that ψ_0 and Φ belong to the same classes $L^p(\partial\mathbb{D})$, $p\in[1,\infty)$, and $\psi_0\log|\psi_0|$ and $\Phi\log|\Phi|$ belong to $L^1(\partial\mathbb{D})$, simultaneously. Moreover, for a branch of $\log B$ in $\mathbb{D}$ with radial cuts $[a_k, \frac{a_k}{|a_k|})$ the following statement holds.

Theorem C ([34]): *Let B be a Blaschke product, and $p\in(1,\infty)$. Then:*

1. (1) *$m_p(r,\log B)$ is bounded on $[0,1)$ if and only if $\psi_0\in L^p(\partial\mathbb{D})$.*
2. (2) *$m_1(r,\log B)$ is bounded if and only if $\psi_0\log^+\psi_0\in L^1(\partial\mathbb{D})$.*
3. (3) *$m_p(r,\log|B|)$ is bounded on $[0,1)$ if and only if*

$$\sup_{0<r<1}\int_0^{2\pi}\left(\int_0^{2\pi}\frac{1-r^2}{|re^{i\theta}-e^{i\varphi}|^2}\psi_r(e^{i\theta})d\theta\right)^p d\varphi<\infty.$$

4. (4) $\psi_0\in L^p(\partial\mathbb{D}) \Rightarrow \sup_{0<r<1} m_p(r,\log|B|)<\infty$.
5. (5) $n(r,f)=O((1-r)^{-\frac1p}) \Rightarrow \sup_{0<r<1} m_p(r,\log|B|)<\infty$.

Relations between conditions on the zeros of a Blaschke product B and the belonging of $\arg B(e^{i\theta})$ to L^p spaces $0<p\le\infty$ were investigated by A. Rybkin ([35]).

The following tasks arise naturally:

- (i) Describe the asymptotic behaviour of pth means of $\log|f|$ where f is a bounded analytic function in $\mathbb{D}$, $1<p<\infty$.
- (ii) Find 'geometric' conditions on the zero distribution providing a prescribed growth of $m_p(r,\log|f|)$.
- (iii) Extend the description to functions of finite order of growth.

In this paper we accomplish these tasks.

1.2. Complete measure and main results. Our method is based on a concept of the so-called complete measure of a subharmonic function introduced by A. Grishin in the case of the half-plane (see [20], [15]). For the unit disc a similar approach was used by N. Govorov [18, §1, Theorem 1.8]. A similar notion, called the related measure, was introduced by S. Gardiner (see [16]) for the real half-space case. As mentioned there, this concept allows one to obtain a very simple representation for a subharmonic function of finite order and defines this function up to a harmonic summand in the closure of the domain. These concepts are closely related to the notions of the Martin boundary and Martin's representations [3].

Let SH^∞ be the class of subharmonic functions in $\mathbb{D}$ bounded from above. In particular, $\log|f| \in SH^\infty$ if $f \in H^\infty$, the space of bounded analytic functions in $\mathbb{D}$. In this case (cf. [22, Ch. 3.7]), i.e., for $u \in SH^\infty(\mathbb{D})$,

$$u(z) = \int_{\mathbb{D}} \log \frac{|z-\zeta|}{|1-z\bar{\zeta}|} d\mu_u(\zeta) - \frac{1}{2\pi}\int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\psi(\zeta) + C, \tag{1.3}$$

where ψ is a positive Borel measure, μ_u is the Riesz measure of u ([22]), and $\int_{\mathbb{D}}(1-|\zeta|)d\mu_u(\zeta) < \infty$. The **complete measure** λ_u of u in the **sense of Grishin** is defined [20, 15] by the boundary measure and the Riesz measure of $u(z)$. But, since ([11])

$$\lim_{r\uparrow 1} \int_{\theta_1}^{\theta_2} \int_{\mathbb{D}} \log \frac{|re^{i\theta}-\zeta|}{|1-re^{i\theta}\bar{\zeta}|} d\mu_u(\zeta) d\theta = 0, \quad -\pi \le \theta_1 < \theta_2 \le \pi,$$

i.e., the boundary values of the first integral from (1.3) do not contribute to the boundary measure, we can define λ_u of a Borel set $M \subset \overline{\mathbb{D}}$ by ([6])

$$\lambda_u(M) = \int_{\mathbb{D}\cap M} (1-|\zeta|)\, d\mu_u(\zeta) + \psi(M \cap \partial\mathbb{D}). \tag{1.4}$$

The measure $\lambda = \lambda_u$ has the following properties:

(1) λ is finite on $\overline{\mathbb{D}}$;
(2) λ is non-negative;
(3) λ is a zero measure outside $\overline{\mathbb{D}}$;
(4) $d\lambda|_{\partial\mathbb{D}}(\zeta) = d\psi(\zeta)$;
(5) $d\lambda|_{\mathbb{D}}(\zeta) = (1-|\zeta|)\, d\mu_u(\zeta)$.

If B is a Blaschke product of form (1.1), then

$$\lambda_{\log|B|}(M) = \sum_{a_n \in M} (1-|a_n|).$$

Let

$$\mathcal{C}(\varphi,\delta)=\{\zeta\in\overline{\mathbb{D}}:|\zeta|\ge 1-\delta,|\arg\zeta-\varphi|\le\pi\delta\}$$

be the Carleson box based on the arc $[e^{i(\varphi-\pi\delta)},e^{i(\varphi+\pi\delta)}]$.

The following theorem describes the growth of integral means for $u\in SH^\infty$.

THEOREM 1.1: *Let $u\in SH^\infty$, $\gamma\in(0,2)$, $p\in(1,\infty)$. Let λ be the complete measure of u. A necessary and sufficient condition for*

$$m_p(r,u)=O((1-r)^{\gamma-1}),\quad r\uparrow 1, \tag{1.5}$$

to hold is that

$$\left(\int_0^{2\pi}\lambda^p(\mathcal{C}(\varphi,\delta))\,d\varphi\right)^{\frac{1}{p}}=O(\delta^\gamma),\quad \delta\downarrow 0. \tag{1.6}$$

THEOREM 1.2: *Let $f\in H^\infty$, $\gamma\in(0,2)$, $p\in(1,\infty)$. Let λ be the complete measure of* $\log|f|$. *A necessary and sufficient condition for*

$$m_p(r,\log|f|)=O((1-r)^{\gamma-1}),\quad r\uparrow 1, \tag{1.7}$$

to hold is (1.6).

Remark 1.3: It was proved in [5] that if $\operatorname{supp}\lambda\subset\partial\mathbb{D}$, i.e., u is harmonic, $\gamma\in(0,1)$, then (1.6) is equivalent to (1.5).

Remark 1.4: Though Theorems 1.1 and 1.2 look like Carleson-type results, we cannot use standard tools (e.g., [14, Chap. 9]) here, because u and $\log|f|$ have logarithmic singularities.

Remark 1.5: It follows from Example 4.1 that the assumption $\gamma<2$ in Theorem 1.1 cannot be relaxed. For $\gamma=1$, (1.6) is the boundedness condition of $m_p(r,u)$. If $\gamma\in(1,2)$, it defines the rate of decrease to 0.

Remark 1.6: One can prove 'o'-analogs of Theorems 1.1 and 1.2 using the same method (see [10]).

Lemma 3.1 plays the crucial role in the proof of the sufficiency. In order to prove necessity of Theorems 1 and 2 we essentially use the fact that the kernels in representation (1.3) preserve the sign. The method allows one to spread the sufficient part of Theorems 1.1 and 1.2 to functions of finite order of growth (see Theorems 1.10, 1.11 below).

Under additional assumptions on the zero location of a Blaschke product (the support of the Riesz measure of the Green potential) (1.6) could be simplified.

THEOREM 1.7: *Let*

$$u(z) = \int_{\mathbb{D}} \log \frac{|z-\zeta|}{|1-z\bar{\zeta}|} d\mu_u(\zeta), \quad \int_{\mathbb{D}} (1-|\zeta|) d\mu_u(\zeta) < \infty, \tag{1.8}$$

$\alpha \in [0,1)$, $p \in (1,\infty)$, $\alpha + \frac{1}{p} < 1$. *Suppose that* $\operatorname{supp} \mu_u$ *is contained in a finite number of Stolz angles with vertices on* $\partial\mathbb{D}$. *A necessary and sufficient condition for*

$$m_p(r,u) = O((1-r)^{-\alpha}), \quad r \uparrow 1, \tag{1.9}$$

to hold is that

$$n(r,u) := \mu_u(\overline{D(0,r)}) = O((1-r)^{-\alpha-\frac{1}{p}}), \quad r \uparrow 1. \tag{1.10}$$

Similar to the case $\alpha = 0$, the growth condition (1.10) appears to be sufficient for (1.9) when u is of finite order (see Theorem 1.15 below).

Remark 1.8: Taking $u = \log|B|$, we obtain a generalization of MacLane and Rubel's, and Linden's results mentioned in Subsection 1.1.

Remark 1.9: If $\alpha + \frac{1}{p} \geq 1$, bounds (1.9) and (1.10) become trivial, see the remark after Theorem 1.11.

1.3. GROWTH AND ZERO DISTRIBUTION OF FUNCTIONS OF FINITE ORDER. In order to formulate results on the angular distribution of zeros for unbounded analytic functions we need some growth characteristics. The standard characteristics are the maximum modulus $M(r,f) = \max\{|f(z)| : |z| = r\}$, and the Nevanlinna characteristic ([21])

$$T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta, \quad x^+ = \max\{x, 0\}.$$

Note that both of them are bounded for $f = B$. Note that the order defined by $T(r,f)$ coincides with $\rho_1[\log|f|]$.

It follows from results of C. Linden [29] that $M(r,f)$ does take into account the angular distribution of the zeros when it grows sufficiently fast, namely, when the order of growth

$$\rho_M[f] = \limsup_{r\uparrow 1} \frac{\log^+ \log^+ M(r,f)}{-\log(1-r)} \geq 1.$$

To be more precise, consider the canonical product

$$\mathcal{P}(z,(a_k),s) := \prod_{k=1}^{\infty} E(A(z,a_k),s),$$

where

$$E(w,s) = (1-w)\exp\{w + w^2/2 + \cdots + w^s/s\}, \quad s \in \mathbb{Z}_+,$$

is the Weierstrass primary factor,

$$A(z,\zeta) = \frac{1-|\zeta|^2}{1-z\bar{\zeta}}, \quad z \in \mathbb{D},\ \zeta \in \overline{\mathbb{D}}.$$

Let

$$\mathcal{R}(re^{i\varphi},\sigma) = \Big\{\zeta : r \le |\zeta| \le \frac{1+r}{2}, |\arg\zeta - \varphi| \le \sigma\Big\},$$

$$Q(re^{i\varphi}) := \mathcal{R}\Big(re^{i\varphi}, \frac{1-r}{2}\Big).$$

Let $\nu(re^{i\varphi})$ be the number of zeros of $\mathcal{P}$ in $Q(re^{i\varphi})$. We define

$$\nu_1(\varphi) = \limsup_{r\uparrow 1} \frac{\log^+ \nu(re^{i\varphi},\mathcal{P})}{-\log(1-r)}, \quad \nu[\mathcal{P}] = \sup_{\varphi} \nu_1(\varphi). \tag{1.11}$$

With the notation above we have ([29, Theorem V])

$$\rho_M[\mathcal{P}] \begin{cases} = \nu[\mathcal{P}], & \rho_M[\mathcal{P}] \ge 1, \\ \le \nu[\mathcal{P}] \le 1, & \rho_M[\mathcal{P}] < 1. \end{cases} \tag{1.12}$$

We now consider subharmonic counterparts of a canonical product. Given a Borel measure μ on $\mathbb{D}$ satisfying $0 \notin \operatorname{supp}\mu$ and

$$\int_{\mathbb{D}} (1-|\zeta|)^{s+1}\, d\mu_f(\zeta) < \infty, \quad s \in \mathbb{N}\cup\{0\}, \tag{1.13}$$

define the canonical integral as

$$U(z;\mu,s) := \int_{\mathbb{D}} \log|E(A(z,\zeta),s)|\, d\mu(\zeta). \tag{1.14}$$

Let $(q > -1)$

$$S_q(z) = \Gamma(1+q)\Big(\frac{2}{(1-z)^{q+1}} - 1\Big), \quad P_q(z) = \operatorname{Re} S_q(z), \quad S_q(0) = \Gamma(q+1).$$

Note that S_0 and P_0 are the Schwarz and Poisson kernels, respectively.

Let u be a subharmonic function in $\mathbb{D}$ of the form

$$u(z) = U(z;\mu,s) - \int\limits_{\partial\mathbb{D}} P_s(z\bar{\zeta})\, d\psi(\zeta) + C, \tag{1.15}$$

where ψ is a finite (signed) Borel measure on $\partial\mathbb{D}$ and μ is the Riesz measure of u satisfying (1.13). Note that every subharmonic function u of finite order in $\mathbb{D}$, i.e., satisfying

$$\log\max\{u(z) : |z| = r\} = O\Big(\log\frac{1}{1-r}\Big) \quad (r \uparrow 1),$$

can be represented in the form (1.15) for an appropriate $s \in \mathbb{N} \cup \{0\}$ ([22], [12, Chap. 9], [13]).

Let M be Borel's subset of $\overline{\mathbb{D}}$. Let u be a subharmonic function in $\mathbb{D}$ of the form (1.15). We set

$$\lambda_u(M) = \int_{\mathbb{D}\cap M} (1 - |\zeta|)^{s+1}\, d\mu_u(\zeta) + \psi(M \cap \partial\mathbb{D}), \tag{1.16}$$

where μ_u is the Riesz measure of u. Note that in the case $u = \log|f|$ we have $\mu_{\log|f|}(\zeta) = \sum_n \delta(\zeta - a_n)$, where (a_n) is the zero sequence of f.

Let $|\lambda|$ denote the total variation of λ.

Theorem 1.10: *Let u be a subharmonic function in $\mathbb{D}$ of the form* (1.15), *harmonic in some neighbourhood of the origin, $\gamma \in (0, s+1)$, and $p \in (1, \infty)$. Let λ be defined by* (1.16). *If*

$$\Big(\int_0^{2\pi} |\lambda|^p(\mathcal{C}(\varphi,\delta))\, d\varphi\Big)^{\frac{1}{p}} = O(\delta^\gamma), \quad \delta \downarrow 0, \tag{1.17}$$

holds, then

$$m_p(r, u) = O((1 - r)^{\gamma - s - 1}), \quad r \uparrow 1. \tag{1.18}$$

Theorem 1.11: *Let f be of the form*

$$f(z) = C_q z^\nu \mathcal{P}(z, (a_k), q) \exp\Big\{\int_{\partial\mathbb{D}} S_q(z\bar\zeta)d\psi(\zeta)\Big\}, \tag{1.19}$$

where ψ is a finite (signed) Borel measure on $\partial\mathbb{D}$, (a_k) is the zero sequence of f such that $\sum_k (1 - |a_k|)^{q+1} < +\infty$, $\nu \in \mathbb{Z}_+$, $C_q \in \mathbb{C}$. Let $\gamma \in (0, s+1)$, $p \in (1, \infty)$. Let λ be defined by (1.16) *for $u = \log|f|$. If*

$$\Big(\int_0^{2\pi} |\lambda|^p(\mathcal{C}(\varphi,\delta))\, d\varphi\Big)^{\frac{1}{p}} = O(\delta^\gamma), \quad \delta \downarrow 0, \tag{1.20}$$

holds, then

$$m_p(r, \log|f|) = O((1 - r)^{\gamma - s - 1}), \quad r \uparrow 1. \tag{1.21}$$

Remark 1.12: Theorem 1.11 follows from Theorem 1.10 applied for

$$u = \log|f(z)| - \nu \log|z|.$$

We also note that an arbitrary analytic function in $\mathbb{D}$ of finite order of the growth can be represented in the form (1.19).

The next proposition characterizes smoothness of an arbitrary periodic measure.

PROPOSITION 1.13: *Suppose that μ is a 2π-periodic measure on $\mathbb{R}$ finite on the compact Borel sets, and $p \geq 1$. Then*

$$\left(\int_0^{2\pi} (\mu((x-\delta, x+\delta)))^p \, dx\right)^{\frac{1}{p}} = O(\delta^{\frac{1}{p}}), \quad \delta \in (0, 2\pi).$$

Remark 1.14: It follows from representation (1.15) that $\rho_1[u] \leq s$. Then, by Proposition 1.13, (1.18) implies

$$\rho_p[u] \leq s + 1 - \frac{1}{p}.$$

It is known that this is a sharp inequality ([31, 32]), in general. However, Theorems 1.10 and 1.11 characterize classes where $\rho_p[u]$ takes a particular value.

Examples 4.2 and 4.3 in Section 4 show that the assertion of Theorem 1.11 is sharp.

The following theorem provides a sharp estimate for means of canonical integrals or products in terms of the growth of their Riesz measures.

THEOREM 1.15: *Suppose that u is of the form* (1.14), $s \in \mathbb{N} \cup \{0\}$, $p \in (1, \infty)$, *and $\alpha > 0$ are such that $\alpha + \frac{1}{p} < s + 1$. If* (1.10) *holds, then* (1.9) *is valid.*

2. Kernels $K_s(z, \zeta)$ and representation of functions of finite order

We define (cf. [18, p.16])

$$K(z, \zeta) = \frac{G(z, \zeta)}{1 - |\zeta|} = \frac{1}{1 - |\zeta|} \log\left|\frac{1 - z\bar{\zeta}}{z - \zeta}\right|, \quad z \in \mathbb{D}, \ \zeta \in \mathbb{D}, \ z \neq \zeta,$$

where $G(z, \zeta)$ is the Green function for $\mathbb{D}$. We have the following properties of $K(z, \zeta)$, $z = re^{i\varphi}$, $\zeta = \rho e^{i\theta}$.

Proposition 2.1: *The following hold:*

(a) $K(z,0) = -\log|z|$.

(b) $0 \le K(z,\zeta) \le \frac{1-|z|^2}{|z-\zeta|^2}$.

(c) *If $D \Subset \mathbb{D}$, then uniformly in $z \in D$*

$$\lim_{\rho\uparrow 1} K(z,\rho e^{i\theta}) = \frac{1-|z|^2}{|\rho e^{i\theta}-z|^2} = P_0(ze^{-i\theta}).$$

(d) $|K(z,\zeta)| \ge \frac{1}{12}\frac{1-|z|^2}{|z-\zeta|^2}$, *for* $1-|\zeta| \le \frac{1}{2}(1-|z|)$.

Proof of Proposition 2.1. (b) We have

$$\begin{aligned}0 \le& K(re^{i\varphi},\rho e^{i\theta})\\ =&\frac{1}{2(1-\rho)}\log\frac{1-2r\rho\cos(\varphi-\theta)+r^2\rho^2}{r^2-2r\rho\cos(\varphi-\theta)+\rho^2}\\ =&\frac{1}{2(1-\rho)}\log\Big(1+\frac{(1-r^2)(1-\rho^2)}{r^2-2r\rho\cos(\varphi-\theta)+\rho^2}\Big)\\ \le&\frac{1}{2(1-\rho)}\frac{(1-r^2)(1-\rho^2)}{r^2-2r\rho\cos(\varphi-\theta)+\rho^2} \le \frac{1-r^2}{|re^{i\varphi}-\rho e^{i\theta}|^2}.\end{aligned}$$

(c) The assertion easily follows from the equality

$$K(z,\zeta) = \frac{1}{2(1-|\zeta|)}\log\Big(1+\frac{(1-|z|^2)(1-|\zeta|^2)}{|z-\zeta|^2}\Big);$$

see (b).

(d) It is proved in [9]. ∎

Due to (d), we set

$$K(z,e^{i\theta}) := P_0(ze^{-i\theta})$$

preserving continuity of K on $\partial\mathbb{D}$ with respect to the second variable.

Let $s \in \mathbb{N}$. We write

$$K_s(z,\zeta) = -\frac{\log|E(A(z,\zeta),s)|}{(1-|\zeta|)^{s+1}}, \quad \zeta \in \overline{\mathbb{D}},\ z \in \mathbb{D},\ z \ne \zeta,$$

i.e.,

$$K_0(z,\zeta) = K(z,\zeta) + \frac{\log\frac{1}{|\zeta|}}{1-|\zeta|};$$

we set $K_s(z,z) = +\infty$ and $K_s(z,0) = +\infty$, $z \in \mathbb{D}$.

Let $D^*(z,\sigma) = \{\zeta : |\frac{z-\zeta}{1-z\bar{\zeta}}| < \sigma\}$ be the pseudohyperbolic disc with center z and radius $\sigma \in (0,1]$.

Proposition 2.2: *Let $s \in \mathbb{N}$, $r_0 \in (0,1)$. The following hold:*

(i)

$$|K_s(z,\zeta)| \le \frac{C(s)}{|1-z\bar\zeta|^{s+1}}, \quad \zeta \notin D^*\Big(z,\frac{1}{7}\Big) \cup D(0,r_0). \tag{2.1}$$

(ii)

$$|K_s(z,\zeta)| \le \frac{C}{(1-|\zeta|)^{s+1}} \log\left|\frac{1-z\bar\zeta}{z-\zeta}\right|, \quad \zeta \in D^*\Big(z,\frac{1}{7}\Big) \setminus D(0,r_0). \tag{2.2}$$

(iii) *If $z \in D \Subset \mathbb{D}$, then the following uniform limit exists:*

$$K_s(z,\rho e^{i\theta}) \rightrightarrows \frac{2^{s+1}}{s+1} \operatorname{Re} \frac{1}{(1-ze^{-i\theta})^{s+1}} = \frac{2^s P_s(ze^{-i\theta})}{(s+1)!} + C(s), \quad \rho \uparrow 1.$$

Proof of Proposition 2.2. The lower estimate for $K_s(z,\zeta)$,

$$K_s(z,\zeta) \ge -\frac{2^{s+2}}{(1-|\zeta|)^{s+1}}|A(z,\zeta)|^{s+1} \ge -\frac{2^{2s+3}}{|1-z\bar\zeta|^{s+1}}, \quad z \in \mathbb{D}, \zeta \in \overline{\mathbb{D}}, \tag{2.3}$$

follows from the known upper estimate of the primary factor (e.g., [39, Chap. V.10]). Also

$$K_s(z,\zeta) = \frac{\operatorname{Re}\sum_{j=s+1}^{\infty} \frac{1}{j}(A(z,\zeta))^j}{(1-|\zeta|)^{s+1}}, \tag{2.4}$$

provided that $|A(z,\zeta)| < \frac{1}{2}$, so

$$|K_s(z,\zeta)| \le \frac{2|A(z,\zeta)|^{s+1}}{(s+1)(1-|\zeta|)^{s+1}} \le \frac{2^{s+2}}{s+1}\frac{1}{|1-z\bar\zeta|^{s+1}}.$$

Hence, it remains to consider the case when $|A(z,\zeta)| \ge \frac{1}{2}$.

Since for all $z \in \mathbb{D}$, $\zeta \in \overline{\mathbb{D}}$, $|A(z,\zeta)| \le 2$, we have for $\zeta \notin D^*(z,\frac{1}{7}) \cup D(0,r_0)$

$$\begin{aligned} K_s(z,\zeta)(1-|\zeta|)^{s+1} &= \log\left|\frac{1-z\bar\zeta}{(z-\zeta)\bar\zeta}\right| - \operatorname{Re}\sum_{j=1}^{s}\frac{A(z,\zeta)}{j} \\ &\le \log\frac{7}{r_0} + \sum_{j=1}^{s}\frac{2^j}{j} \\ &= C(s,r_0) \le C(s,r_0)2^{s+1}|A(z,\zeta)|^{s+1}. \end{aligned}$$

Hence,

$$K_s(z,\zeta) \le \frac{C(s,r_0)}{|1-z\bar\zeta|^{s+1}}, \quad \zeta \notin D^*(z,\frac{1}{7}) \cup D(0,r_0), \tag{2.5}$$

and (i) follows.

Similar arguments give (ii).

Let us prove (iii). If $z \in D \Subset \mathbb{D}$, then it follows from the representation (2.4) that

$$K_s(z, \rho e^{i\theta}) \rightrightarrows \frac{2^{s+1}}{s+1} \operatorname{Re} \frac{1}{(1-ze^{-i\theta})^{s+1}} = \frac{2^s P_s(ze^{-i\theta})}{(s+1)!} + C(s), \quad \rho \uparrow 1. \qquad \blacksquare$$

Due to properties of $K_s(z, \zeta)$ the representation (1.15) could be rewritten in the form (cf. [20], [19, Part II])

$$u(z) = -\int_{\overline{\mathbb{D}}} K_s(z, \zeta) d\lambda(\zeta) + C, \tag{2.6}$$

where

$$\frac{2^s}{(s+1)!} d\lambda(e^{i\theta}) = \frac{1}{2\pi} d\psi^*(\theta),$$

and $\lambda = \lambda_u$ is defined by (1.16).

Similarly, for any $u \in \mathrm{SH}^\infty$ we have the following representation (cf. (1.3)):

$$u(z) = -\int_{\bar{\mathbb{D}}} K(z, \zeta)\, d\lambda_u(\zeta) + C. \tag{2.7}$$

Remark 2.3: The idea of such representation goes back to results of Martin [3, Chap. XIV].

3. Proofs

Proof of the sufficiency of Theorem 1.1. We write

$$u_1(z) = -\int_{D^*(z, \frac{1}{7})} K(z, \zeta)\, d\lambda(\zeta), \quad u_2(z) = -\int_{\bar{\mathbb{D}} \setminus D^*(z, \frac{1}{7})} K(z, \zeta)\, d\lambda(\zeta).$$

Let us estimate $I_1 = \int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p \, d\varphi$.

By the Hölder inequality

$$\begin{aligned} |u_1(re^{i\varphi})| &= \int\limits_{D^*(re^{i\varphi}, \frac{1}{7})} \log \left| \frac{1 - re^{i\varphi}\bar{\zeta}}{re^{i\varphi} - \zeta} \right| d\mu(\zeta) \\ &\le \Bigg(\int\limits_{D^*(re^{i\varphi}, \frac{1}{7})} \left(\log \left| \frac{1 - re^{i\varphi}\bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right)^p d\mu(\zeta) \Bigg)^{\frac{1}{p}} \left(\mu\left(D^*\left(re^{i\varphi}, \frac{1}{7} \right) \right) \right)^{\frac{p-1}{p}}, \end{aligned}$$

hence

$$I_1 \le \int_{-\pi}^{\pi} \Bigg(\int\limits_{D^*(re^{i\varphi},\frac{1}{7})} \left| \log \left| \frac{1-re^{i\varphi}\bar{\zeta}}{re^{i\varphi}-\zeta} \right| \right|^p d\mu(\zeta)\, \mu^{p-1}\Big(D^*\Big(re^{i\varphi}, \frac{1}{7}\Big)\Big) \Bigg) d\varphi.$$

Since $D^*(z, \frac{h}{2+h}) \subset D(z, (1-|z|)h)$ ([7]), with $h = \frac{1}{3}$ we get

$$D^*\Big(z, \frac{1}{7}\Big) \subset D\Big(z, (1-|z|)\frac{1}{3}\Big) \subset \Box\Big(z, \frac{1}{3}(1-|z|)\Big),$$

where

$$\Box(re^{i\varphi}, \sigma) = \{\rho e^{i\theta} : |\rho - r| \le \sigma, |\theta - \varphi| \le \sigma\}.$$

Therefore, using Fubini's theorem, we deduce

$$\begin{aligned}
I_1 &\le \int_{-\pi}^{\pi} \Bigg(\int_{\Box(z,\frac{1}{3}(1-r))} \Big(\log \Big| \frac{1-re^{i\varphi}\bar{\zeta}}{re^{i\varphi}-\zeta} \Big| \Big)^p \Bigg) \mu^{p-1}\Big(\Box\Big(z, \frac{1}{3}(1-r)\Big)\Big)\, d\mu(\zeta) d\varphi \\
&\le \int_{-\pi}^{\pi} \Bigg(\int_{\Box(z,\frac{1}{3}(1-r))} \Big(\log \Big| \frac{1-re^{i\varphi}\bar{\zeta}}{re^{i\varphi}-\zeta} \Big| \Big)^p \Bigg) \mu^{p-1}\Big(\Box v\Big(re^{i\arg\zeta}, \frac{2}{3}(1-r)\Big)\Big)\, d\mu(\zeta) d\varphi \\
&\le \iint\limits_{\substack{-\pi - \frac{1-r}{3} \le \theta \le \pi + \frac{1-r}{3} \\ |\rho - r| \le \frac{1-r}{3} \\ |\theta - \varphi| \le \frac{1-r}{3}}} \Big(\log \Big| \frac{1-r\rho e^{i(\varphi-\theta)}}{re^{i\varphi}-\rho e^{i\theta}} \Big| \Big)^p \mu^{p-1}\Big(\Box\Big(re^{i\theta}, \frac{2}{3}(1-r)\Big)\Big)\, d\mu(\rho e^{i\theta}) d\varphi \\
&\le 2 \int\limits_{||\zeta|-r| \le \frac{1}{3}(1-r)} \mu^{p-1}\Big(\Box\Big(re^{i\arg\zeta}, \frac{2}{3}(1-r)\Big)\Big) \int_{-\pi}^{\pi} \Big(\log \Big| \frac{1-re^{i\varphi}\bar{\zeta}}{re^{i\varphi}-\zeta} \Big| \Big)^p d\varphi\, d\mu(\zeta).
\end{aligned}$$

We know ([8]) that for any $a, b \in \mathbb{C}$ and $p > 1$

$$\int_{-\pi}^{\pi} \left| \log \left| \frac{a - e^{i\theta}}{b - e^{i\theta}} \right| \right|^p d\theta \le C(p)|a-b|$$

holds. Using this inequality we obtain ($r \in (\frac{1}{2}, 1)$)

$$I_1 \le 4C(p)(1-r) \int\limits_{||\zeta|-r| \le \frac{1}{3}(1-r)} \mu^{p-1}\Big(\Box\Big(re^{i\arg\zeta}, \frac{2}{3}(1-r)\Big)\Big) d\mu(\zeta). \tag{3.1}$$

In order to proceed we need the following lemma. A multidimensional analog of this lemma can be found in [10].

Lemma 3.1: *Let ν be a 2π periodic positive Borel measure on $\mathbb{R}$, $p \ge 1$, $\delta \in (0, \pi)$. Then*

$$\int_{[-\pi,\pi)} \nu^{p-1}((\theta-\delta, \theta+\delta)) d\nu(\theta) \le \frac{2^{p+1}}{\delta} \int_{[-\pi,\pi)} \nu^p((\theta-\delta, \theta+\delta)) d\theta. \tag{3.2}$$

Proof of Lemma 3.1. First, we prove (3.2) for $p = 1$.[1]

We have

$$\begin{aligned}\int_{[-\pi,\pi)} d\nu(\theta) &= \int_{[-\pi,\pi)} \frac{1}{\delta}\int_{\theta-\frac{\delta}{2}}^{\theta+\frac{\delta}{2}} dx d\nu(\theta)\\ &\le \int_{[-\pi-\frac{\delta}{2},\pi+\frac{\delta}{2})} dx \int_{[x-\frac{\delta}{2},x+\frac{\delta}{2})} \frac{1}{\delta} d\nu(\theta)\\ &= \int_{[-\pi-\frac{\delta}{2},\pi+\frac{\delta}{2})} \frac{\nu([x-\frac{\delta}{2},x+\frac{\delta}{2}))}{\delta} dx \qquad (3.3)\\ &\le 2\int_{[-\pi,\pi)} \frac{\nu([x-\frac{\delta}{2},x+\frac{\delta}{2}))}{\delta} dx\\ &\le 2\int_{[-\pi,\pi)} \frac{\nu((x-\delta,x+\delta))}{\delta} dx.\end{aligned}$$

We now consider $p > 1$. Applying (3.3) with $d\nu_1(\theta) = \nu^{p-1}((\theta-\delta,\theta+\delta))d\nu(\theta)$, we get

$$\begin{aligned}&\int_{[-\pi,\pi)} \nu^{p-1}((\theta-\delta,\theta+\delta))d\nu(\theta)\\ &= \int_{[-\pi,\pi)} d\nu_1(\theta) \le 2 \int_{[-\pi,\pi)} \frac{\nu_1([x-\frac{\delta}{2},x+\frac{\delta}{2}))}{\delta} dx\\ &= 2 \int_{[-\pi,\pi)} \int_{[x-\frac{\delta}{2},x+\frac{\delta}{2})} \nu^{p-1}((\theta-\delta,\theta+\delta))d\nu(\theta)dx\\ &\le 2\int_{[-\pi,\pi)} \frac{\nu^{p-1}((x-\frac{3\delta}{2},x+\frac{3\delta}{2}))\nu([x-\frac{\delta}{2},x+\frac{\delta}{2}))}{\delta} dx\\ &\le 2\int_{[-\pi,\pi)} \frac{\nu^{p}((x-\frac{3\delta}{2},x+\frac{3\delta}{2}))}{\delta} dx\\ &\le 2\int_{[-\pi,\pi)} \frac{\nu^{p}((x-\frac{3\delta}{2},x)\cup[x,x+\frac{3\delta}{2}))}{\delta} dx\\ &\le 2^p\int_{[-\pi,\pi)} \frac{\nu^{p}((x-\frac{3\delta}{2},x))}{\delta} dx + 2^p\int_{[-\pi,\pi)} \frac{\nu^{p}([x,x+\frac{3\delta}{2}))}{\delta} dx\\ &\le 2^{p+1}\int_{[-\pi,\pi)} \frac{\nu^{p}((x-\delta,x+\delta))}{\delta} dx.\end{aligned}$$

The lemma is proved. ∎

[1] The author thanks Prof. Sergii Favorov for the idea of the proof of this lemma.

Let us continue the proof of the sufficiency. Define the nondecreasing function

$$N_r(\theta) = \lambda\Big(\Big\{\rho e^{i\alpha} : |r-\rho| \le \frac{2}{3}(1-r), -\pi \le \alpha \le \theta\Big\}\Big), \quad \theta \in [-\pi,\pi).$$

We extend it on the real axis preserving monotonicity by

$$N_r(x+2\pi) - N_r(x) = N_r(2\pi) - N_r(0), \quad x \in \mathbb{R}.$$

Let ν_r be the corresponding Stieltjes measure on $\mathbb{R}$. Since $|\zeta - r| \le \frac{1}{3}(1-r)$ implies $1-|\zeta| \asymp 1-r$, estimate (3.1) can be written in the form

$$\begin{aligned} I_1 \le & \frac{C}{(1-r)^{p-1}} \int\limits_{||\zeta|-r|\le\frac{1}{3}(1-r)} \lambda^{p-1}\Big(\Box\Big(re^{i\arg\zeta}, \frac{2}{3}(1-r)\Big)\Big) d\lambda(\zeta) \\ = & \frac{C}{(1-r)^{p-1}} \int_{-\pi}^{\pi} \nu_r^{p-1}\Big(\Big[\theta - \frac{2}{3}(1-r), \theta + \frac{2}{3}(1-r)\Big]\Big) d\nu_r(\theta) \\ \le & 2^{p+1}\frac{3C}{2(1-r)^p} \int_{-\pi}^{\pi} \nu_r^{p}\Big(\Big[\theta - \frac{2}{3}(1-r), \theta + \frac{2}{3}(1-r)\Big]\Big) d\theta \\ \le & \frac{C(p)}{(1-r)^p} \int_{-\pi}^{\pi} \lambda^p\Big(\Box\Big(re^{i\theta}, \frac{2}{3}(1-r)\Big)\Big)\, d\theta \\ \le & \frac{C}{(1-r)^p}(1-r)^{p\gamma}. \end{aligned}$$

We have used Lemma 3.1 and the assumption of the theorem on the complete measure.

Thus, we have

$$\Big(\int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p\, d\varphi\Big)^{\frac{1}{p}} \le C(p)(1-r)^{\gamma-1}. \tag{3.4}$$

Let us estimate

$$u_2(z) = -\int_{\bar{\mathbb{D}}} K(z,\zeta) d\tilde{\lambda}(\zeta),$$

where

$$d\tilde{\lambda}(\zeta) = \chi_{\bar{\mathbb{D}}\setminus D^*(z,\frac{1}{7})}(\zeta) d\lambda(\zeta).$$

Since $\operatorname{supp}\tilde{\lambda} \cap D^*(z,\frac{1}{7}) = \varnothing$, for $\zeta \notin D^*(z,\frac{1}{7})$ we have by Proposition 2.1 that

$$K(z,\zeta) \le \frac{49(1-|z|^2)}{|1-z\bar{\zeta}|^2}. \tag{3.5}$$

Let $E_n = E_n(re^{i\varphi}) = \mathcal{C}(\varphi, 2^n(1-r))$, $n \in \mathbb{N}$, $E_0 = \varnothing$. Since

$$2\arcsin(2^{n-1}(1-r)) \le \pi 2^{n-1}(1-r),$$

we have $D(e^{i\varphi}, 2^{n-1}(1-r)) \subset \mathcal{C}(\varphi, 2^n(1-r))$ provided that $2^n(1-r) \le \pi$. Then for $\zeta = \rho e^{it} \in \mathbb{D} \setminus E_n(z)$, $n \ge 1$, we obtain

$$(3.6)\quad |1-\rho r e^{i(\varphi-t)}| \ge |1-\rho e^{i(\varphi-t)}| - \rho(1-r) \ge 2^n(1-r)-(1-r) \ge 2^{n-1}(1-r),$$

and $|1-\rho r e^{i(\varphi-t)}| \ge 1 - r\rho \ge 1-r$ for $\zeta \in E_1(z)$. Therefore

$$\begin{aligned}|u_2(re^{i\varphi})| \le& \Bigg(\sum_{n=1}^{[\log_2 \frac{1}{1-r}]} \int_{E_{n+1}\setminus E_n} + \int_{E_1}\Bigg) \frac{49(1-r^2)}{|1-re^{i\varphi}\bar{\zeta}|^2} d\tilde{\lambda}(\zeta)\\ \le& 49 \sum_{n=1}^{[\log_2 \frac{1}{1-r}]} \int_{E_{n+1}\setminus E_n} \frac{2(1-r)}{(2^{n-1}(1-r))^2} d\tilde{\lambda}(\zeta) + \int_{E_1} \frac{2}{1-r} d\tilde{\lambda}(\zeta)\\ \le& \frac{100}{1-r}\Bigg(\sum_{n=1}^{[\log_2 \frac{1}{1-r}]} \frac{\tilde{\lambda}(E_{n+1}(z))}{2^{2n-2}} + \tilde{\lambda}(E_1(z))\Bigg)\\ \le& \frac{400}{1-r} \sum_{n=1}^{[\log_2 \frac{1}{1-r}]+1} \frac{\tilde{\lambda}(E_n(z))}{2^{2n}}.\end{aligned}$$

Fix any $\alpha \in (\gamma, 2)$. By Hölder's inequality $(\frac{1}{p} + \frac{1}{p'} = 1)$

$$(3.7)\quad \begin{aligned}|u_2(re^{i\varphi})|^p \le& \Big(\frac{400}{1-r}\Big)^p \sum_{n=1}^{[\log_2 \frac{1}{1-r}]+1} \frac{\tilde{\lambda}^p(E_n(z))}{2^{\alpha n p}} \Bigg(\sum_{n=1}^{\infty} \frac{1}{2^{(2-\alpha)np'}}\Bigg)^{\frac{p}{p'}}\\ \le& \frac{C(p,\gamma)}{(1-r)^p} \sum_{n=1}^{\infty} \frac{\tilde{\lambda}^p(E_n(z))}{2^{\alpha n p}}.\end{aligned}$$

It follows from the latter inequalities and the assumption of the theorem that $(r \in [\frac{1}{2}, 1))$

$$\begin{aligned}\int_0^{2\pi} |u_2(re^{i\varphi})|^p\, d\varphi \le& \frac{C(p,\gamma)}{(1-r)^p} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\tilde{\lambda}^p(E_n(re^{i\varphi}))}{2^{\alpha n p}}\, d\varphi\\ \le& \frac{C(p,\gamma)}{(1-r)^p} \sum_{n=1}^{\infty} \frac{(2^n(1-r))^{p\gamma}}{2^{\alpha n p}}\\ =& \frac{C(p,\gamma)}{(1-r)^{p(1-\gamma)}} \sum_{n=1}^{\infty} 2^{np(\gamma-\alpha)} = \frac{C'(p,\gamma)}{(1-r)^{p(1-\gamma)}}.\end{aligned}$$

Hence

$$\Bigg(\int_0^{2\pi} |u_2(re^{i\varphi})|^p\, d\varphi\Bigg)^{\frac{1}{p}} \le \frac{C''(\gamma,p)}{(1-r)^{1-\gamma}}, \quad r \in [0,1).$$

The sufficiency of Theorem 1.1 is proved.

Necessity. Using property (d) of Proposition 2.1, we obtain

$$|u(re^{i\theta})| \geq \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} K(re^{i\varphi}, \zeta)\, d\lambda(\zeta) \geq \frac{1}{12} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} \frac{1-r^2}{|re^{i\varphi} - \zeta|^2}\, d\lambda(\zeta).$$

Elementary geometric arguments show that

$$|re^{i\varphi} - \rho e^{i\theta}| \leq |re^{i\varphi} - e^{i\theta}|$$

for $1 > \rho \geq r \geq 0$. Since $|re^{i\varphi} - e^{i\theta}|^2 = (1-r)^2 + 4r \sin^2 \frac{\varphi - \theta}{2}$, it then follows that

$$\begin{aligned} |u(re^{i\theta})| \geq & \frac{1}{12} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} \frac{1-r^2}{|re^{i\varphi} - e^{i\theta}|^2} d\lambda(\rho e^{i\theta}) \\ \geq & \frac{1}{12(\frac{\pi^2}{4}+1)} \frac{1-r^2}{(1-r)^2} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} d\lambda(\rho e^{i\theta}) \\ \geq & \frac{\lambda(\mathcal{C}(\varphi, \frac{1-r}{2}))}{12(\frac{\pi^2}{4}+1)(1-r)}. \end{aligned}$$

By the assumption of the theorem we deduce that

$$\frac{C}{(1-r)^{(1-\gamma)p}} \geq \int_0^{2\pi} |u(re^{i\varphi})|^p\, d\varphi \geq C \frac{\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi, \frac{1-r}{2}))\, d\varphi}{(1-r)^p}.$$

Hence

$$\int_0^{2\pi} \lambda^p\Big(\mathcal{C}\Big(\varphi, \frac{1-r}{2}\Big)\Big)\, d\varphi = O((1-r)^{\gamma p})$$

as $r \uparrow 1$. This completes the proof of necessity. ∎

Proof of Theorem 1.10. Due to (2.6) we write

$$\begin{aligned} u(z) &= -\int_{\bar{\mathbb{D}}} K_s(z,\zeta)\, d\lambda(\zeta) \\ &= -\int_{D^*(z,\frac{1}{7})} K_s(z,\zeta)\, d\lambda(\zeta) - \int_{\bar{\mathbb{D}} \setminus D^*(z,\frac{1}{7})} K_s(z,\zeta)\, d\lambda(\zeta) \equiv u_1 + u_2. \end{aligned}$$

According to (2.2)

$$|u_1(z)| \leq C(s) \int_{D^*(z,\frac{1}{7})} \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta).$$

Its estimate repeats that for the case $s = 0$.

Let us estimate pth means of $u_2(z)$. Using Proposition 2.2 and (3.6) we deduce (cf. proof of Theorem 1.1)

$$\begin{aligned}|u_2(re^{i\varphi})|^p \le & \Bigg(\Bigg(\sum_{n=1}^{[\log_2 \frac{1}{1-r}]} \int_{E_{n+1}\setminus E_n} + \int_{E_1}\Bigg) \frac{C(s)}{|1-re^{i\varphi}\bar\zeta|^{s+1}} |d\tilde\lambda(\zeta)|\Bigg)^p \\ \le & \frac{C}{(1-r)^{(s+1)p}} \Bigg(\sum_{n=1}^{[\log_2 \frac{1}{1-r}]+1} \frac{|\tilde\lambda(E_n(z))|}{2^{n(s+1)}}\Bigg)^p \\ \le & \frac{C}{(1-r)^{(s+1)p}} \sum_{n=1}^{[\log_2 \frac{1}{1-r}]+1} \frac{|\tilde\lambda|^p(E_n(z))}{2^{(s+1-\frac{\eta}{2})np}} \Bigg(\sum_{n=1}^{\infty} \frac{1}{2^{\frac{n\eta p'}{2}}}\Bigg)^{\frac{p}{p'}} \\ \le & \frac{C}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \frac{|\tilde\lambda|^p(E_n(z))}{2^{(s+1-\frac{\eta}{2})np}},\end{aligned}$$

where $\eta = s+1-\gamma$. It follows from the latter inequalities and the assumption of the theorem that

$$\begin{aligned}\int_0^{2\pi} |u_2(re^{i\varphi})|^p \, d\varphi \le & \frac{C(p)}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{|\tilde\lambda|(E_n(re^{i\varphi}))}{2^{(s+1-\frac{\eta}{2})np}} \, d\varphi \\ \le & \frac{C(p)}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \frac{(2^n(1-r))^{p\gamma}}{2^{(s+1-\frac{\eta}{2})np}} \\ = & \frac{C(p,\gamma)}{(1-r)^{p(s+1-\gamma)}}, \quad r \in \Big[\frac{1}{2}, 1\Big).\end{aligned}$$

Finally,

$$\Big(\int_0^{2\pi} |u_2(re^{i\varphi})|^p \, d\varphi\Big)^{\frac{1}{p}} \le \frac{C(\gamma,p)}{(1-r)^{s+1-\gamma}}, \quad r \in \Big[\frac{1}{2}, 1\Big). \qquad \blacksquare$$

Proof of Theorem 1.7. Without loss of generality we assume that

$$\operatorname{supp} \mu_u \subset \{z \in \overline{\mathbb{D}} : |1-z| < 2(1-|z|)\} =: \triangle.$$

Necessity. Note that $\mathcal{R}(1-\delta, \pi\delta) \subset \mathcal{C}(\varphi, 2\delta)$ for $\varphi \in [-\delta, \delta]$. Applying Theorem 1.1 we obtain

$$\Big(\int_{-\delta}^{\delta} \lambda^p(\mathcal{R}(1-\delta, \pi\delta)) \, d\varphi\Big)^{\frac{1}{p}} = O(\delta^{1-\alpha}), \quad 0 < \delta < 1,$$

or

$$\mu_u(\mathcal{R}(1-\delta, \pi\delta)) = O(\delta^{-\alpha-\frac{1}{p}}), \quad 0 < \delta < 1.$$

Since

$$\triangle \subset \overline{D\Big(0, \frac{1}{2}\Big)} \cup \bigcup_{n=1}^{\infty} \mathcal{R}(1-2^{-n}, \pi 2^{-n}) \tag{3.8}$$

we deduce

$$n(1-2^{-k}, u) \le C \sum_{n=1}^{k} 2^{n(\alpha+\frac{1}{p})} + C = O(2^{k(\alpha+\frac{1}{p})}), \quad k \in \mathbb{N},$$

and the assertion follows.

Sufficiency. It follows from the assumptions that

$$\lambda(\mathcal{R}((1-\delta)e^{i\varphi}, 4\delta)) = O(\delta^{1-\alpha-\frac{1}{p}}), \quad \delta \downarrow 0.$$

Then

$$\begin{aligned}\lambda(\mathcal{C}(\varphi,\delta)) \le& \lambda\Big(\bigcup_{n=0}^{\infty} \mathcal{R}\Big(1-\frac{\delta}{2^n}e^{i\varphi}, \frac{4\delta}{2^n}\Big)\Big) \\ \le& C\sum_{n=0}^{\infty}\Big(\frac{\delta}{2^n}\Big)^{1-\alpha-\frac{1}{p}} = O(\delta^{1-\alpha-\frac{1}{p}}), \delta \downarrow 0.\end{aligned}$$

Since $\operatorname{supp}\mu_u \subset \triangle$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \lambda^p(\mathcal{C}(\varphi,\delta))\,d\varphi =& \int_{-2\pi\delta}^{2\pi\delta} \lambda^p(\mathcal{C}(\varphi,\delta))\,d\varphi \\ =& O(\delta\delta^{p(1-\alpha-\frac{1}{p})}) = O(\delta^{p(1-\alpha)}), \delta \downarrow 0.\end{aligned}$$

It remains to apply Theorem 1.1. The sufficiency is proved.

In the case of a general Stolz angle of opening $\beta < \pi$, one should choose an appropriate factor depending on β in the second argument of $\mathcal{R}$ in (3.8) instead of π in the proof of the necessity, and make similar changes in the integral bounds in the proof of the sufficiency. ∎

Proof of Theorem 1.15. We confine ourselves to the case $s = 0$. We keep the notation from the proof of Theorem 1.1. It follows from estimate (3.1) that

$$\begin{aligned}\int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p\,d\varphi \le& (1-r)n^{p-1}\Big(r+\frac{2}{3}(1-r), u\Big)n\Big(r+\frac{1}{2}(1-r), u\Big) \\ =& O((1-r)^{-\alpha p}).\end{aligned} \tag{3.9}$$

Let us estimate the pth mean of u_2. We use estimate (3.5), the integral of Minkowski's inequality ([36, §A1]), and integration by parts to obtain

$$\begin{aligned}
\Big(\int_{-\pi}^{\pi}|u_2(re^{i\varphi})|^p\,d\varphi\Big)^{\frac{1}{p}} \leq& C\Big(\int_{-\pi}^{\pi}\Big(\int_{\mathbb{D}}\frac{1-r^2}{|1-re^{i\varphi}\bar{\zeta}|^2}\,d\lambda(\zeta)\Big)^p\,d\varphi\Big)^{\frac{1}{p}}\\
\leq& C\int_{\mathbb{D}}\Big(\int_{-\pi}^{\pi}\Big(\frac{1-r^2}{|1-re^{i\varphi}\bar{\zeta}|^2}\Big)^p\,d\varphi\Big)^{\frac{1}{p}}\,d\lambda(\zeta)\\
\leq& C\int_{\mathbb{D}}\frac{1-r}{(1-r|\zeta|)^{2-\frac{1}{p}}}\,d\lambda(\zeta)\\
=& C(1-r)\int_0^1\frac{(1-t)dn(t,u)}{(1-rt)^{2-\frac{1}{p}}}\\
\leq& C(1-r)\Big(\int_0^r\frac{dn(t,u)}{(1-t)^{1-\frac{1}{p}}}+\int_r^1\frac{(1-t)dn(t,u)}{(1-r)^{2-\frac{1}{p}}}\Big)\\
\leq& \frac{C}{(1-r)^{1-\frac{1}{p}}}\int_r^1 n(t,u)dt\\
=& O((1-r)^{-\alpha}),\quad r\uparrow 1.
\end{aligned}$$

Here we have used the well-known estimate

$$\int_{-\pi}^{\pi}\frac{d\varphi}{|1-re^{i\varphi}\bar{\zeta}|^{2p}}\leq\frac{C(p)}{(1-r|\zeta|)^{2p-1}}$$

(see, e.g., ([23]). Taking into account (3.9), we obtain the desired estimate. ∎

Proof of Proposition 1.13. Assume that $\mu([0,2\pi))=C$. Then by Fubini's theorem

$$\begin{aligned}
\int_0^{2\pi}(\mu((x-\delta,x+\delta)))^p\,dx \leq& (2C)^{p-1}\int_0^{2\pi}\mu((x-\delta,x+\delta))\,dx\\
=& (2C)^{p-1}\int_0^{2\pi}\int_{(x-\delta,x+\delta)}d\mu(y)\,dx\\
\leq& (2C)^{p-1}\int_{-\delta}^{2\pi+\delta}d\mu(y)\int_{(y-\delta,y+\delta)}dx\\
\leq& (2C)^{p-1}2\delta\mu((-\delta,2\pi+\delta))\\
\leq& 3(2C)^p\delta. \quad ∎
\end{aligned}$$

4. Examples

We finish the paper with several examples. The subharmonic function constructed in Example 4.1 shows that the assumption $\gamma < 2$ in Theorem 1.1 cannot be relaxed. In its turn, Examples 4.2 and 4.3 show sharpness of the estimates given by Theorem 1.11 for infinitely many values of parameters. While in Example 4.2 a canonical product is constructed, in Example 4.3 we have an analytic function without zeros, so the complete measure coincides with the Stieltjes measure.

Example 4.1: Let

$$d\mu(z) = \frac{dA(z)}{1-|z|},$$

where A is the planar Lebesgue measure, and $u(z) = U(z;\mu,0)$ be the Green potential. Then the complete measure λ coincides with A, thus $\lambda(C(\varphi,\delta)) \asymp \delta^2$, and consequently,

$$\left(\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi,\delta))\, d\varphi\right)^{\frac{1}{p}} \asymp \delta^2.$$

On the other hand, using the equality

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\varphi}-a|d\varphi = \log^+|a|,$$

we deduce that

$$\begin{aligned} u(re^{i\theta}) =& \int_0^1\int_0^{2\pi} \log\frac{|re^{i\theta}-\rho e^{i\varphi}|}{|1-re^{i\theta}\rho e^{-i\varphi}|}d\varphi\frac{\rho d\rho}{1-\rho} \\ =&2\pi\int_0^1 \Big(\log r + \log^+\frac{\rho}{r} - \log^+(r\rho)\Big)\frac{d\rho}{1-\rho} \\ =&2\pi\int_0^r \log r\frac{\rho d\rho}{1-\rho} + 2\pi\int_r^1 \log\rho\frac{\rho d\rho}{1-\rho} \\ =&2\pi\log r\Big(\log\frac{1}{1-r} - r\Big) + O(1-r) \\ =& -(2\pi+o(1))(1-r)\log\frac{1}{1-r}, \quad r\uparrow 1. \end{aligned}$$

Therefore,

$$m_p(r,u) = (1+o(1))2\pi(1-r)\log\frac{1}{1-r}, \quad r\uparrow 1.$$

Hence, the assertion of Theorem 1.1 does not hold for $\gamma = 2$.

Example 4.2: Following Linden [32, Lemma 1], given $\alpha \ge 1$, $\beta \in [0,1]$, we consider the sequence of complex numbers

$$a_{k,m} = (1-2^{-k})e^{im2^{-k}}, \quad 1 \le m \le [2^{k\beta}], \tag{4.1}$$

where each of the numbers (4.1) is counted $[2^{\alpha k}]$ times. Then for

$$\mathcal{P}(z) = \mathcal{P}(z, (a_{k,m}), s),$$

where $s = \min\{q \in \mathbb{N} : q > \alpha + \beta - 1\}$, we have (see [32])

$$n(r, \mathcal{P}) \asymp \Big(\frac{1}{1-r}\Big)^{\alpha+\beta}, \quad \nu(r, \mathcal{P}) \asymp \Big(\frac{1}{1-r}\Big)^{\alpha}, \quad r \uparrow 1.$$

Therefore, by Theorem A [29], $\rho_M[\mathcal{P}] = \alpha$. In [32] it is proved that

$$\rho_p[\log|\mathcal{P}|] = \alpha + \frac{\beta-1}{p}.$$

We are going to prove that

$$\Big(\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi,\delta))\,d\varphi\Big)^{\frac{1}{p}} \ge C(\delta^{s+1-\alpha-\frac{\beta-1}{p}}), \quad \delta \downarrow 0. \tag{4.2}$$

It would imply that restriction (1.17) could not be weakened.

We first assume that $\beta \in (0,1)$. Given $\delta \in (0, \delta_0)$ we define $\varphi_\delta = \delta^{1-\beta} - \pi\delta$, where δ_0 is chosen such that $\varphi_\delta > 0$. Note that $\varphi_\delta \sim \delta^{1-\beta}$, $\delta \downarrow 0$. According to the definition of $\mathcal{C}(\varphi,\delta)$, $a_{k,m} \in \mathcal{C}(\varphi,\delta)$ if and only if

$$1 - |a_{k,m}| = 2^{-k} \le \delta, \quad \varphi - \pi\delta \le m2^{-k} \le \varphi + \pi\delta. \tag{4.3}$$

Let $G(\varphi,\delta)$ denote the set of (k,m) such that (4.3) is valid. It is easy to check that for $\varphi \in (0, \varphi_\delta)$ the set $G(\varphi,\delta)$ is not empty. Let

$$k_1(\varphi) = \min\{k : 2^{-k}[2^{\beta k}] \le \varphi + \pi\delta\},$$

where $\varphi \in (0, \varphi_\delta)$. Since $k_1(\varphi)$ tends to infinity uniformly with respect to $\varphi \in (0,\varphi_\delta)$ as $\delta \downarrow 0$, one can choose δ_1 so small that for all $\delta \in (0,\delta_1)$, $\varphi \in (0,\varphi_\delta)$ and $k \ge k_1(\varphi)$ the inequality $\frac{2^{-\beta k}}{(1-\beta)(1-2^{\beta k})\log 2} \le 1$ holds. Under this assumption we deduce subsequently from the definition of $k_1 = k_1(\varphi)$ that

$$\begin{gathered} |2^{k_1}(\varphi+\pi\delta) - 2^{\beta k_1}| < 1, \\ \frac{1-2^{-\beta k_1}}{\varphi+\pi\delta} < 2^{k_1(1-\beta)} < \frac{1+2^{-\beta k_1}}{\varphi+\pi\delta}, \\ |k_1 - \frac{1}{1-\beta}\log_2\frac{1}{\varphi+\pi\delta}| < 1. \end{gathered} \tag{4.4}$$

It follows from the definition of φ_δ and (4.4) that

$$2^{k_1} > \frac{(1-2^{-k_1\beta})^{\frac{1}{1-\beta}}}{\delta} > \frac{2}{\pi\delta}, \quad 0<\varphi<\varphi_\delta. \tag{4.5}$$

Then, according to (4.3), (4.5) for $\delta \in (0, \min\{\delta_0, \delta_1\})$ and $\varphi \in (\frac{1}{2}\varphi_\delta, \varphi_\delta)$, $\delta \downarrow 0$,

$$\begin{aligned} \lambda(\mathcal{C}(\varphi,\delta)) &= \sum_{(k,m)\in G(\varphi,\delta)} [2^{\alpha k}]2^{-k(s+1)} \\ &\ge \sum_{m=[2^{k_1}(\varphi-\pi\delta)]+1}^{[2^{k_1}(\varphi+\pi\delta)]} [2^{\alpha k_1}]2^{-k_1(s+1)} \\ &\ge [2^{\alpha k_1}]2^{-k_1(s+1)}(2^{k_1}2\pi\delta - 2) \ge [2^{\alpha k_1}]2^{-k_1 s}\pi\delta \\ &\ge \frac{\pi\delta}{2}2^{(\alpha-s)k_1} \sim \frac{\pi\delta}{2}\left(\frac{1}{\varphi}\right)^{\frac{\alpha-s}{1-\beta}}. \end{aligned} \tag{4.6}$$

It follows from the last estimate that

$$\begin{aligned} \left(\int_0^{2\pi} (\lambda(\mathcal{C}(\varphi,\delta)))^p d\varphi\right)^{\frac{1}{p}} &\ge \frac{\pi\delta}{2}\left(\int_{\varphi_\delta/2}^{\varphi_\delta} (\varphi^{\frac{s-\alpha}{1-\beta}})^p d\varphi\right)^{\frac{1}{p}} \\ &= \frac{\pi}{2(\frac{s-\alpha}{1-\beta}p+1)}\delta\varphi^{\frac{s-\alpha}{1-\beta}+\frac{1}{p}}\Big|_{\varphi_\delta/2}^{\varphi_\delta} \\ &\sim C(s,\alpha,p)\delta^{1+s-\alpha-\frac{\beta-1}{p}}, \quad \delta\downarrow 0. \end{aligned}$$

In the case $\beta=1$ the arguments could be simplified. By the choice of s, $s>\alpha$. For $0<\varphi\le\frac{1}{2}$, according to (4.3) we deduce

$$\begin{aligned} \lambda(\mathcal{C}(\varphi,\delta)) &= \sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty} \sum_{m=[2^k(\varphi-\pi\delta)]+1}^{[2^k(\varphi+\pi\delta)]} [2^{\alpha k}]2^{-k(s+1)} \\ &\ge \sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty} [2^{\alpha k}]2^{-k(s+1)}(2^k 2\pi\delta-2) \\ &\ge \sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty} [2^{\alpha k}]2^{-ks}\pi\delta \\ &\ge \frac{\pi\delta}{2}\sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty} 2^{(\alpha-s)k} \asymp \delta^{1+s-\alpha}. \end{aligned}$$

Hence

$$\left(\int_0^{2\pi}(\lambda(C(\varphi,\delta)))^p d\varphi\right)^{\frac{1}{p}} \geq C(s,\alpha,p)\delta^{1+s-\alpha}, \quad \delta\downarrow 0.$$

If $\beta=0$, then all zeros $a_k=1-2^{-k}$ are located on $[0,1)$, and $s>\alpha-1$. For $\varphi\in(-\pi\delta,\pi\delta)$, we then have according to (4.3)

$$\lambda(C(\varphi,\delta)) = \sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty}[2^{\alpha k}]2^{-k(s+1)} \geq C\sum_{k=[\log_2\frac{1}{\delta}]+1}^{\infty}2^{(\alpha-s-1)k}\asymp\delta^{1+s-\alpha}.$$

Then

$$\begin{aligned}\left(\int_0^{2\pi}(\lambda(C(\varphi,\delta)))^p d\varphi\right)^{\frac{1}{p}} &\geq C\delta^{1+s-\alpha}\left(\int_{-\pi\delta}^{\pi\delta}d\varphi\right)^{\frac{1}{p}}\\ &= C(s,\alpha,p)\delta^{1+s-\alpha+\frac{1}{p}}, \quad \delta\downarrow 0\end{aligned}$$

as required.

Example 4.3: Let $f(z)=\exp\left\{\left(\frac{1}{1-z}\right)^{q+1}\right\}, \quad q>-1, \quad f(0)=e$. In this case $f(z)$ is of the form (1.19) with $(a_k)=\varnothing$, $\psi^*(\theta)=H(\theta)m_0$, $m_0>0$, where $H(\theta)$ is the Heaviside function, i.e., $\lambda(\zeta)=m_0\delta(\zeta-1)$, where $\delta(\cdot)$ denotes the Dirac function. It is easy to check that

$$\left(\int_0^{2\pi}(\lambda(C(\varphi,\delta)))^p d\varphi\right)^{\frac{1}{p}} = m_0(2\pi\delta)^{\frac{1}{p}},$$

and $m_p(r,\log|f|)\asymp(1-r)^{\frac{1}{p}-q-1}$.

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