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THE HARDY INEQUALITY AND FRACTIONAL HARDY INEQUALITY FOR THE DUNKL LAPLACIAN

BY

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ABSTRACT

We prove the L^p Hardy inequality and L^p fractional Hardy inequality for the Dunkl Laplacian on $\mathbb{R}^N$. Further, we prove the same kind of inequalities for a half-space and cone.

1. Introduction

The Hardy inequality is of fundamental importance in many areas of mathematical analysis and mathematical physics. A general Hardy inequality is of the form

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \Big(\frac{|N-p|}{p}\Big)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx,$$

for $u \in C_0^\infty(\mathbb{R}^N)$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ respectively with respect to $1 \leq p < N$ or $p > N$. It is known that the constant $(\frac{|N-p|}{p})^p$ is sharp and never attained in the corresponding spaces $\dot{W}_p^1(\mathbb{R}^N)$ or $\dot{W}_p^1(\mathbb{R}^N \setminus \{0\})$ respectively. A lot of work concerning the fractional Hardy inequality has been developed in the literature. A remarkable work on the same was done by R. L. Frank and R. Seiringer in [3].

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They have proven the sharp Hardy inequality with sharp constants as follows: for $p \geq 1$, $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx,$$

where the constant $C_{N,s,p}$ is sharp. Also they proved the fractional Hardy inequality with remainder term. That is, for $p \geq 2$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \\ \geq c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{(N-ps)/2}} \frac{dy}{|y|^{(N-ps)/2}},\end{aligned}$$

where $v := |x|^{(N-ps)/2} u$ and c_p is as in (3.18).

The same authors have proven the fractional Hardy inequality in half-spaces $\mathbb{R}^N_+$ with and without remainder terms in [4], where

$$\mathbb{R}^N_+ = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}.$$

They have proven that, for some sharp constant $D_{N,p,s}$,

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq D_{N,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}}\, dx,$$

for all $u \in \dot{W}^s_p(\mathbb{R}^N)$ with $ps \neq 1$. Similar to the case of $\mathbb{R}^N$ they obtained an improved fractional Hardy inequality which states, for $p \geq 2$, that

$$\begin{aligned}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - D_{N,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}}\, dx \\ \geq c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{x_N^{(1-ps)/2}} \frac{dy}{y_N^{(1-ps)/2}},\end{aligned}$$

where $v := x_N^{(1-ps)/p} u$ and c_p is given in (3.18).

Our aim in this paper is to prove both the Hardy and fractional Hardy inequality in a Dunkl setting. We cite a few papers in which authors studied some of the related inequalities in a Dunkl setting. Pitts inequalities for the fractional Dunkl operator is studied by D. V. Gorbachev et al. in [5]. F. Soltani et al. have proven certain inequalities, namely the Stein–Weiss inequality, Hardy–Littlewood–Sobolev inequality, uncertainty principles and some Pitts inequalities in the Dunkl setting in the papers [12, 13, 14]. In [1] Óscar Ciaurri et al. studied the Hardy-type inequalities for the Dunkl Hermite operator. We mainly adapt the techniques used in [3] to prove the Hardy and fractional Hardy inequalities.

The paper is organized as follows. In Section 3 we prove a generalized version of the classical L^p Hardy inequality in the Dunkl setting. We use the 'ground state substitution' technique to achieve it. For $p \geq 2$ we obtain an improved version of the Hardy inequality in (3.20). In Section 4 we obtain an optimal fractional Hardy inequality for the Dunkl Laplacian. As in Section 3 we obtain a fractional Hardy inequality with a remainder term for $p \geq 2$. Section 5 and Section 6 deal with a similar type of fractional Hardy inequalities on a half-space and cone respectively.

2. The general Dunkl setting

In this section we give some basics on Dunkl theory which we will be using in the coming sections. We suggest readers consult [2, 10, 15, 16] for details of Fourier analysis related to the Dunkl operator. Let $\langle \cdot , \cdot \rangle$ denote the standard inner product on $\mathbb{R}^N$ and $| \cdot | := \sqrt{\langle \cdot , \cdot \rangle}$. For a non-zero element α in $\mathbb{R}^N$ the reflection in the hyperplane $\langle \alpha \rangle^{\perp}$ is defined as

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha.$$

A finite subset R of $\mathbb{R}^N$ is said to be a reduced root system if, for $\alpha \in R$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$. Each root system can be written as a disjoint union of its subsets, say R_+ and $(-R_+)$, which are separated by a hyperplane passing through the origin. The subset R_+ of R is called the positive roots of R. The subgroup G of $O(N)$ which is generated by the reflections $\{\sigma_\alpha : \alpha \in R\}$ is called the reflection group with root system R or the Coxeter group. For the convenience of the calculations we assume that R is normalized, that is $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$. A G-invariant function k defined on R, i.e., $k(g\alpha) = k(\alpha)$ for all $g \in G$, is called a multiplicity function. For $j \in \{1, 2, \dots, N\}$ the differential-difference operators T_j (the Dunkl operators) is defined by

$$T_j f(x) := \partial_j f(x) + E_j f(x),$$

where E_j is the difference part of T_j and is given by

$$E_j = \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}$$

with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. The Dunkl operators T_j are a generalization of the partial differential operator in the classical analysis. As in the classical case we can define the Dunkl gradient by $\nabla_k = (T_1, T_2, \ldots, T_N)$ and the Dunkl Laplacian Δ_k by $\Delta_k = \sum_{j=1}^{N} T_j^2$.

One of the important properties of the Dunkl operators is that they commute, that is $T_iT_j = T_jT_i$. Also, for every $f, g \in C^1(\mathbb{R}^N)$ and for every $1 \leq j \leq N$, one can see that $T_j(fg) = T_j(f)g + fT_j(g)$ when at least one of the functions is G-invariant.

Fix a reflection group G and a multiplicity function k. We can define the G-invariant homogeneous weight function $h_k^2(x)$ of degree $\gamma_k := \sum_{\alpha \in R_+} k(\alpha)$ by

$$h_k^2(x) = \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2k(\alpha)}.$$

Throughout the paper we assume that $k(\alpha) \geq 0$ and denote the weighted measure $h_k^2(x)dx$ by $d\mu_k(x)$. Further, we use the notations $d_k := N + 2\gamma_k$ and $\lambda_k := \frac{d_k - 2}{2}$.

Let $\mathcal{S}(\mathbb{R}^N)$ be the space of Schwartz class functions. If $g \in \mathcal{S}(\mathbb{R}^N)$ and if f is a bounded function with $f \in C^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} T_j f(x) g(x) d\mu_k(x) = -\int_{\mathbb{R}^N} f(x) T_j g(x) d\mu_k(x).$$

It is known that there exists a unique real analytic solution $f = E_k(., y)$ for the system $T_i f = y_i f$, $1 \leq i \leq N$ satisfying $f(0) = 1$ with $y \in \mathbb{R}^N$. The kernel $E_k(x, y)$ is called the Dunkl kernel and it is clearly a generalization of the exponential functions $e^{<x,y>}$.

The Dunkl Fourier transform is a generalization of the Fourier transform. For $u \in L^1(\mathbb{R}^N, d\mu_k(x))$, its Dunkl Fourier transform is defined by

$$\mathcal{F}_k u(\xi) = c_k^{-1} \int_{\mathbb{R}^N} u(x) E_k(-i\xi, x) d\mu_k(x),$$

where $c_k := (\int_{\mathbb{R}^N} e^{-\|x\|^2/2} d\mu_k(x))^{-1}$. The Dunkl translation $\tau_y^k f$ of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\mathcal{F}_k(\tau_y^k f)(\xi) = E_k(iy, \xi)\mathcal{F}_k f(\xi).$$

It also makes sense for all $f \in L^2(\mathbb{R}^N, d\mu_k(x))$ as $E_k(iy, \xi)$ is a bounded function and the Dunkl Fourier transform is a unitary operator on $L^2(\mathbb{R}^n, d\mu_k(x))$. Dunkl translation has the property $\tau_y^k f(x) = \tau_{-x}^k f(-y)$.

3. Hardy inequality

In this section we prove the optimal L^p Hardy inequality for $1 \leq p < \infty$ and an improved Hardy inequality for $p \geq 2$ for a G-invariant real-valued smooth function having compact support. Also, we will prove a generalized L^p Hardy inequality with optimal constant for the same function space. However, we can relax the condition on the G-invariant function for certain cases. We define the p-Dunkl Laplacian $\Delta_{k,p}$ by

$$\Delta_{k,p} f = \mathrm{div}_k(|\nabla_k f|^{p-2}\nabla_k f),$$

where $\mathrm{div}_k(f_1, f_2, \ldots, f_N) = \sum_{j=1}^{N} T_j f_j$. We will compute $\Delta_{k,p} w$ for a radial function w which is needed to prove the Hardy inequality. For a radial function w

$$\begin{aligned}
&\mathrm{div}_k(|\nabla_k w|^{p-2}\nabla_k w)\\
&\quad= \sum_{j=1}^{N} T_j \Big(|w'(r)|^{p-2} w'(r) \frac{x_j}{r}\Big)\\
&\quad= \sum_{j=1}^{N} (\partial_j + E_j)\Big(|w'(r)|^{p-2} w'(r) \frac{x_j}{r}\Big)\\
&\quad= \sum_{j=1}^{N} \Big((p-1)|w'(r)|^{p-2} w''(r) \Big(\frac{x_j}{r}\Big)^2 + |w'(r)|^{p-2} w'(r) \Big(\frac{1}{r} - \frac{1}{r^2}\frac{x_j^2}{r}\Big)\Big)\\
&\qquad + \frac{|w'(r)|^{p-2} w'(r)}{r} \sum_{j=1}^{N} E_j(x_j)\\
&\quad= (p-1)|w'(r)|^{p-2} w''(r) + \Big(\frac{N-1}{r} + 2\gamma_k\Big)|w'(r)|^{p-2} w'(r).
\end{aligned}$$

Hence for a radial function w we have

$$\Delta_{k,p} w = (p-1)|w'(r)|^{p-2} w''(r) + \Big(\frac{d_k - 1}{r}\Big)|w'(r)|^{p-2} w'(r). \tag{3.1}$$

Theorem 3.1: *Let $1 \leq p < \infty$. Let u be a real-valued G-invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$, and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$, then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geq \Big|\frac{d_k - p}{p}\Big|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} d\mu_k(x). \tag{3.2}$$

The constant $|\frac{d_k-p}{p}|^p$ given in the inequality is optimal.

Proof. Let w be a positive radial function and let v be a G-invariant real-valued function with $u = vw$. Use the inequality for real numbers a and b and for $p \geq 1$, $|a+b|^p \geq |a|^p + p|a|^{p-2}a.b$, so we obtain

$$\begin{aligned} |\nabla_k u|^p =& |\nabla_k (vw)|^p \\ =& |v\nabla_k w + w\nabla_k v|^p \\ \geq& |v|^p|\nabla_k w|^p + p|v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w. \end{aligned} \tag{3.3}$$

Since w is radial we write $w(x) = w(r)$ with $r = |x|$ and denote the derivatives as $w'(r) = \frac{dw}{dr}$ and $w''(r) = \frac{d^2w}{dr^2}$. First we will prove an inequality of the form

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geq \int_{\mathbb{R}^N} V|u|^p d\mu_k(x) \tag{3.4}$$

for the given radial function w and a function V, where w is a weak solution of the following equation:

$$\mathrm{div}_k(|\nabla_k w|^{p-2}\nabla_k w) + Vw^{p-1} = 0. \tag{3.5}$$

After proving the inequality (3.4) for the functions which satisfy (3.5), we will look for some explicit V and w which provide us the Hardy inequality.

In order to estimate the integral $\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x)$ we estimate the integral of each term on the right-hand side of (3.3).

We start with

$$\begin{aligned} \int_{\mathbb{R}^N} |v|^p|\nabla_k w|^p d\mu_k(x) =& \int_{\mathbb{R}^N} |v|^p|\nabla_k w|^{p-2}\left(\sum_{j=1}^N T_j w T_j w\right) d\mu_k(x) \\ =& \sum_{j=1}^N \int_{\mathbb{R}^N} |v|^p|\nabla_k w|^{p-2} T_j w T_j w d\mu_k(x) \\ =& -\sum_{j=1}^N \int_{\mathbb{R}^N} wT_j(|v|^p|\nabla_k w|^{p-2}T_j w) d\mu_k(x). \end{aligned} \tag{3.6}$$

Let ∇_0 be the Eucledian gradient. Calculating $T_j(|v|^p|\nabla_k w|^{p-2}T_j w)$ separately, we obtain

$$\begin{aligned} &T_j(|v|^p|\nabla_k w|^{p-2}T_j w) \\ &\quad= (\partial_j + E_j)(|v|^p|\nabla_0 w|^{p-2}\partial_j w) \\ &\quad= (p|v|^{p-1}\partial_j v)|\nabla_o w|^{p-2}\partial_j w + |v|^p\partial_j(|\nabla_0 w|^{p-2}\partial_j w) \\ &\qquad + E_j\left(|v|^p|w'(r)|^{p-2}\frac{w'(r)}{r}x_j\right). \end{aligned} \tag{3.7}$$

Since $\frac{|w'(r)|^{p-2}w'(r)}{r}$ is radial we can write

$$E_j\Big(\frac{|w'(r)|^{p-2}w'(r)}{r}|v|^p x_j\Big) = \frac{|w'(r)|^{p-2}w'(r)}{r}E_j(|v|^p x_j). \tag{3.8}$$

Using the definition of E_j and reflection one can easily calculate

$$\sum_{j=1}^{N} E_j(|v|^p x_j) = \sum_{\alpha\in R_+} k(\alpha)[|v(x)|^p + |v(\sigma_\alpha(x))|^p]. \tag{3.9}$$

Substituting (3.7), (3.8) and (3.9) in (3.6) and denoting the Euclidean divergence as div_0,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x)\\
&\quad = -\, p\int_{\mathbb{R}^N} w|v|^{p-1}|\nabla_0 w|^{p-2}\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\qquad - \int_{\mathbb{R}^N} w|v|^p \,\mathrm{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) d\mu_k(x)\\
&\qquad - \sum_\alpha k(\alpha)\int_{\mathbb{R}^N} \frac{w(r)|w'(r)|^{p-2}w'(r)}{r}(|v(x)|^p + |v(\sigma_\alpha x)|^p) d\mu_k(x).
\end{aligned} \tag{3.10}$$

Since radial functions and the Dunkl measure are invariant under reflection, a change of variable in the third integral on the right-hand side gives us

$$\begin{aligned}
\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x) = &-\, p\int_{\mathbb{R}^N} w|v|^{p-2}v|\nabla_0 w|^{p-2}\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&- \int_{\mathbb{R}^N} w|v|^p \,\mathrm{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) d\mu_k(x)\\
&- 2\gamma_k\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-2}w'(r)w(r)}{r}|v(x)|^p d\mu_k(x).
\end{aligned} \tag{3.11}$$

Since w is radial we can write from (3.1)

$$\mathrm{div}_k(|\nabla_k w|^{p-2}\nabla_k w) = \mathrm{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) + 2\gamma_k\frac{|w'(r)|^{p-2}w'(r)}{r}.$$

Now we can write the above equation (3.11) as

$$\begin{aligned}
\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x) = &-\, p\int_{\mathbb{R}^N} w|v|^{p-2}v\nabla_0 v.\nabla_0 w|\nabla_0 w|^{p-2} d\mu_k(x)\\
&- \int_{\mathbb{R}^N} w(x)|v(x)|^p \,\mathrm{div}_k(|\nabla_k w|^{p-2}\nabla_k w) d\mu_k(x).
\end{aligned}$$

Consider the second term on the right-hand side of (3.3) and integrate:

$$\begin{aligned} p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w d\mu_k(x) \\ =& p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x) \\ &+ p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_0 w|^{p-2} vw\frac{w'(r)}{r}\Big(\sum_{j=1}^N E_j(v)x_j\Big)d\mu_k(x). \end{aligned}$$

Using the definition of E_j we find that

$$\sum_{j=1}^N E_j(v)x_j = \sum_{\alpha\in R_+} k(\alpha)(v(x)-v(\sigma_\alpha x)).$$

Since v is G-invariant we can write

$$\begin{aligned} &p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w d\mu_k(x) \\ =&p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_0 w|^{p-2} vw\nabla_0 v.\nabla_k w d\mu_k(x) \\ &+ p\int_{\mathbb{R}^N} |v|^{p-2}||\nabla_0 w|^{p-2} vw\frac{w'(r)}{r}\sum_{\alpha\in R_+}(k(\alpha)(v(x)-v(\sigma_\alpha x))d\mu_k(x) \\ =&p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x). \end{aligned} \tag{3.12}$$

Substituting all the above calculated estimations and integrals into inequality (3.3),

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_k(vw)|^p d\mu_k(x) \geq &- p\int_{\mathbb{R}^N} w|v|^{p-2}v\nabla_0 w.\nabla_0 v|\nabla_0 w|^{p-2}d\mu_k(x) \\ &- \int_{\mathbb{R}^N} w(x)|v(x)|^p \operatorname{div}_k(|\nabla_k w|^{p-2}\nabla_k w)d\mu_k(x) \\ &+ p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x). \end{aligned}$$

That is, we end up with

$$\int_{\mathbb{R}^N} |\nabla_k(vw)|^p d\mu_k(x) \geq -\int_{\mathbb{R}^N} w(x)|v(x)|^p \operatorname{div}_k(|\nabla_k w|^{p-2}\nabla_k w)d\mu_k(x). \tag{3.13}$$

Now if w is a weak solution of the equation

$$\mathrm{div}_k(|\nabla_k w|^{p-2}\nabla_k w) + Vw^{p-1} = 0$$

for some function V, the above inequality (3.13) becomes

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geq \int_{\mathbb{R}^N} V|u|^p d\mu_k(x).$$

Now we choose a w and V explicitly to obtain the desired Hardy inequality.

Let us choose

$$w(x) = |x|^{-(d_k-p)/p},$$

that is $w(r) = r^{-(d_k-p)/p}$. By a straightforward calculation we get

$$w'(r) = -\frac{(d_k-p)}{p} r^{-(d_k-p)/p-1}$$

and

$$w''(r) = \Big(\frac{(d_k-p)}{p}\Big)\Big(\frac{(d_k-p)}{p}+1\Big) r^{-((d_k-p))/p)-2}.$$

Using the Dunkl p-Laplacian for radial functions given in (3.1) we find that for $r \neq 0$

$$\Delta_{k,p} w(r) = -\Big|\frac{d_k-p}{p}\Big|^p r^{-((\frac{(d_k-p)}{p})(p-1)+p)}.$$

Choose

$$V(x) = \Big|\frac{d_k-p}{p}\Big|^p |x|^{-p};$$

then w is a weak solution of $\Delta_{k,p} w = -Vw^{p-1}$. Substituting V and w in (3.4) we obtain the desired Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geq \Big|\frac{d_k-p}{p}\Big|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x).$$

To prove the optimality, consider the functions u_ϵ below and take the limit as $\epsilon \to 0$:

$$u_\epsilon(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ |x|^{-\frac{|d_k-p|}{p}-\epsilon}, & \text{if } |x| > 1. \end{cases} \qquad \blacksquare$$

Remark 3.1: (1) We assumed that the function u in Theorem 3.1 is G-invariant. Assume that $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and $u = vw$ with some v and a radial function w

with $w'(r) \geq 0$. Now by using the Hölder inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^{p-2} v(x) v(\sigma_\alpha x) \frac{w(r)w'(r)}{r} |\nabla_0 w|^{p-2} d\mu_k(x)\\
&= \int_{\mathbb{R}^N} |v|^{p-2} v(x) w(r) \frac{w'(r)|w'(r)|^{p-2}}{r} v(\sigma_\alpha x) d\mu_k(x)\\
&= \int_{\mathbb{R}^N} \Big(\frac{|w'(r)|^{p-2} w'(r) w(r)}{r}\Big)^{\frac{p-1}{p}} v(x)|v|^{p-2} \Big(\frac{|w'(r)|^{p-2} w'(r) w(r)}{r}\Big)^{\frac{1}{p}} v(\sigma_\alpha x) d\mu_k(x)\\
&\leq \bigg(\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1}}{r} |v(x)|^p d\mu_k(x)\bigg)^{\frac{p-1}{p}}\\
&\quad \times \bigg(\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1} w(r)}{r} |v(\sigma_\alpha x)|^p d\mu_k(x)\bigg)^{\frac{1}{p}}.
\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^{p-2} v(x) v(\sigma_\alpha x) \frac{w(r)w'(r)}{r} |\nabla_0 w|^{p-2} d\mu_k(x)\\
&\qquad \leq \int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1} w(r)}{r} |v(x)|^p d\mu_k(x).
\end{aligned} \tag{3.14}$$

Using this we can rewrite equation (3.12) as

$$\begin{aligned}
p \int_{\mathbb{R}^N} & |v|^{p-2} |\nabla_k w|^{p-2} v w \nabla_k v \nabla_k w d\mu_k(x)\\
&\geq p \int_{\mathbb{R}^N} |v|^{p-2} |\nabla_k w|^{p-2} v w \nabla_0 v . \nabla_0 w d\mu_k(x)\\
&\quad + p\gamma_k \int_{\mathbb{R}^N} |v|^{p-2} v^2(x) w(x) \frac{w'(r)}{r} |\nabla_k w|^{p-2} d\mu_k(x)\\
&\quad - p\gamma_k \int_{\mathbb{R}^N} |v|^p \frac{|w'(r)|}{r} w(x) |\nabla_k w|^{p-2} d\mu_k(x)\\
&= p \int_{\mathbb{R}^N} |v|^{p-2} |\nabla_k w|^{p-2} v w \nabla_0 v . \nabla_0 w d\mu_k(x).
\end{aligned} \tag{3.15}$$

Now by repeating exactly same steps of the proof for Theorem 3.1 we get the generalized Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geq \int_{\mathbb{R}^N} V |u|^p d\mu_k(x)$$

with some function V and w satisfies (3.5).

(2) Let $w(x) = |x|^{-\frac{d_k - p}{p}}$ with $d_k < p$. Then $w'(r) \geq 0$ and, by using Remark 3.1(1) we get the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) \geq \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} |u|^p d\mu_k(x).$$

The above inequality is optimal and it is true for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$.

(3) If $w'(r) < 0$ equation (3.15) will be of the form

$$\begin{aligned} p\int_{\mathbb{R}^N} &|v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v \nabla_k w d\mu_k(x) \\ \geq & p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x) r \\ &+ p\gamma_k \int_{\mathbb{R}^N} |v|^{p-2} v^2(x) w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2} d\mu_k(x) \\ &- p\gamma_k \int_{\mathbb{R}^N} |v|^p \frac{|w'(r)|}{r} w(x)|\nabla_k w|^{p-2} d\mu_k(x) \\ = & p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x) \\ &+ 2p\gamma_k \int_{\mathbb{R}^N} |v|^p(x) w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2} d\mu_k(x). \end{aligned} \tag{3.16}$$

Now using (3.11) and (3.16) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} &|\nabla_k(vw)|^p d\mu_k(x) \\ &\geq -\int_{\mathbb{R}^N} w|v|^p \operatorname{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) d\mu_k(x) \\ &\quad + 2\gamma_k(p-1)\int_{\mathbb{R}^N} |v|^p(x) w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2} d\mu_k(x) \\ &= -\int_{\mathbb{R}^N} w|v|^p \Big(\operatorname{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) - 2\gamma_k(p-1)\frac{|w'(r)|^{p-2}w'(r)}{r}\Big) d\mu_k(x). \end{aligned}$$

If w is a weak solution of the equation $L_p w + V w^{p-1} = 0$ where

$$\begin{aligned} L_p w :=& \operatorname{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) - 2\gamma_k(p-1)\frac{|w'(r)|^{p-2}w'(r)}{r} \\ =& \operatorname{div}_k(|\nabla_0 w|^{p-2}\nabla_0 w) - 2\gamma_k p\frac{|w'(r)|^{p-2}w'(r)}{r}, \end{aligned}$$

we have the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) \geq \int_{\mathbb{R}^N} V|u|^p d\mu_k(x).$$

(4) In $C_0^\infty(\mathbb{R}^N)$, let $w := |x|^{-\frac{d_k-p}{p}}$ with $d_k > p$ and $v = |x|^{\frac{d_k-p}{p}}u$. Now using the calculation carried out in (3.1) we can write

$$\begin{aligned}\mathrm{div}_0(|\nabla_0 w|^{p-2}\nabla_0 w) &= (p-1)|w'(r)|^{p-2}w''(r) + \frac{(N-1)}{r}|w'(r)|^{p-2}w'(r)\\ &= -\Big(\frac{d_k-p}{p}\Big)^{p-1}\Big(\frac{d_k-p}{p} - 2\gamma_k\Big)r^{-((\frac{(d_k-p)}{p})(p-1)+p)}.\end{aligned}$$

Using this and the expression for L_p we have

$$L_p(w) = -\Big(\frac{d_k-p}{p}\Big)^{p-1}\Big(\frac{d_k-p}{p} - 2\gamma_k(p-1)\Big)r^{-((\frac{(d_k-p)}{p})(p-1)+p)}.$$

Now for

$$V(x) = -\Big(\frac{d_k-p}{p}\Big)^{p-1}\Big(\frac{d_k-p}{p} - 2\gamma_k(p-1)\Big)|x|^{-p}$$

we have the Hardy inequality

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla_k(u)|^p d\mu_k(x)\\ &\qquad\geq\Big(\frac{d_k-p}{p}\Big)^{p-1}\Big(\frac{d_k-p}{p} - 2\gamma_k(p-1)\Big)\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}d\mu_k(x).\end{aligned}\tag{3.17}$$

(5) We don't know about the sharpness of the constant appearing in (3.17). However, for $p = 2$ it has been shown in [17] that the optimal constant for the Hardy inequality is $\frac{(d_k-2)^2}{4}$ without the restriction that the function is G-invariant.

Recall the algebraic inequality given in [3, Equation 2.13]: for $p \geq 2$

$$|a+b|^p \geq |a|^p + p|a|^{p-2}a.b + c_p|b|^p,$$

where a and b are real numbers, and constant c_p is given by

$$c_p := \min_{0<\tau<1/2}((1-\tau)^p - \tau^p + p\tau^{p-1})\tag{3.18}$$

and is sharp for this inequality. Using this, inequality (3.3) can be written as

$$\begin{aligned}|\nabla_k u|^p =&|\nabla_k(vw)|^p\\ \geq&|v|^p|\nabla_k w|^p + p|v|^{p-2}|\nabla_k w|^{p-2}vw\nabla_k v.\nabla_k w + c_p|w|^p|\nabla_k v|^p.\end{aligned}\tag{3.19}$$

For radial function w and reflection invariant function v such that

$$u = vw \in C_0^\infty(\mathbb{R}^N),$$

if we use inequality (3.19) instead of (3.3), then inequality (3.13) turns out to be

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_k(vw)|^p d\mu_k(x) \\ \geq - \int_{\mathbb{R}^N} w(x)|v(x)|^p div(|\nabla_k w|^{p-2}\nabla_k w) d\mu_k(x) + c_p \int_{\mathbb{R}^N} |w|^p |\nabla_k v|^p d\mu_k(x).\end{aligned}$$

This improves the following Hardy inequality with a remainder term for $p \geq 2$.

Corollary 3.2: *Let $2 \leq p < \infty$. Let u be a real-valued G-invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$, and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$, then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \Big|\frac{d_k - p}{p}\Big|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \geq c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x), \tag{3.20}$$

where c_p is given by (3.18)*. When $p = 2$ the equality holds with $c_2 = 1$.*

Remark 3.2: By observing Remark 3.1 we can make another remark on Corollary 3.2. If $w(x) = |x|^{-\frac{d_k - p}{p}}$ with $d_k < p$, we obtain the following improved Hardy inequality for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$:

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \Big|\frac{d_k - p}{p}\Big|^p \int_{\mathbb{R}^N} |u|^p d\mu_k(x) \geq c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x).$$

Also, if $u \in C_0^\infty(\mathbb{R}^N)$ and if $w := |x|^{-\frac{d_k - p}{p}}$ with $d_k > p$ and $v = |x|^{\frac{d_k - p}{p}} u$, then again by Remark 3.1 we obtain the following improved Hardy inequality:

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) - \Big(\frac{d_k - p}{p}\Big)^{p-1} \Big(\frac{d_k - p}{p} - 2\gamma_k(p-1)\Big) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \\ \geq c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x).\end{aligned}$$

Now we prove a generalized Hardy inequality which generalizes Theorem 3.1. Fix $1 \leq l \leq N$; we write $x \in \mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Let R_1 be a root system on $\mathbb{R}^l$, and k_1 be a multiplicity function on R_1. The Dunkl weight function associated with R_1 and k_1 is given by

$$h_{k_1}^2(x) = \prod_{\alpha \in R_{1,+}} |\langle x, \alpha \rangle|^{2k_1(\alpha)}.$$

Since k_1 is G-invariant, we have $k_1(\alpha) = k_1(-\alpha)$ and thus the choice of any arbitrary positive subsystem $\mathbb{R}_{1,+}$ does not make any impact on the weight function. Now similarly for a root system R_2 and a multiplicity function k_2

on $\mathbb{R}^{N-l}$, we have the weight function $h_{k_2}^2(x) = \prod_{\alpha\in R_{2,+}} |\langle x,\alpha\rangle|^{2k_2(\alpha)}$. Define a root system on $\mathbb{R}^N$ as

$$R := (R_1 \times (0)_{N-l}) \cup ((0)_l \times R_2).$$

Also define the multiplicity function k on R as $k(y,0)=k_1(y)$ and $k(0,z)=k_2(z)$, where y and z belong to R_1 and R_2 respectively. It is straightforward to check that R is a root system on $\mathbb{R}^N$ and k is a multiplicity function from R to positive reals. Corresponding to this R and k, one can see that the Dunkl weighted measure on $\mathbb{R}^N$, denoted by $d\mu_k(x)$, is nothing but the product of the Dunkl weighted measures on $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$. That is,

$$d\mu_k(x) = d\mu_{k_1}(y)d\mu_{k_2}(z) = h_k^2(x)dx = h_{k_1}^2(y)h_{k_2}^2(z)dydz.$$

With this preparation we state the following theorem.

Theorem 3.3: *Let $1 \le p < \infty$ and let $1 \le l \le l \le N$. Let u be a real-valued G-invariant function. Assume that $u \in C_0^\infty(\mathbb{R}^N)$ if $d_{k_1} > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_{k_1} < p$. Then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \ge \left|\frac{d_{k_1}-p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} d\mu_k(x). \tag{3.21}$$

The constant $|\frac{d_{k_1}-p}{p}|^p$ given in the inequality is optimal.

Proof. The root system R with which we started allows us to write

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} d\mu_k(x) = \int_{\mathbb{R}^{N-l}} d\mu_{k_1}(z) \int_{\mathbb{R}^l} \frac{|u(x)|^p}{|y|^p} d\mu_{k_2}(y). \tag{3.22}$$

Let $\nabla_{k_1,y}$ and $\nabla_{k_2,z}$ be the Dunkl gradient on $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$ respectively. It is easy to see that

$$|\nabla_{k_1,y} u(y,z)| \le |\nabla_k u(x)|.$$

By applying Theorem 3.1 to (3.22) we obtain the inequality (3.21). Now by using Lemma 3.1 and following the arguments from [11] we can prove that $|\frac{d_{k_1}-p}{p}|^p$ is optimal. ∎

Remark 3.3: Remark 3.1 can be extended to Theorem 3.3 similarly.

4. Fractional Hardy inequality for $L^p(\mathbb{R}^N, d\mu_k(x))$

For the classical Laplacian $\Delta = -\sum_{j=1}^N \partial_j^2$ the L^2 Hardy inequality can be written as

$$\langle \Delta u, u\rangle \geq \Big(\frac{N-2}{2}\Big)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx.$$

For $0 < s < 1$ the fractional power of a Laplacian is defined as

$$\Delta^s u(x) := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta} u(x) - u(x)) \frac{dt}{t^{s+1}},$$

where $e^{-t\Delta} u = u * q_t$ with q_t denoting the Euclidean heat kernel. A straightforward calculation using the definition of $e^{-t\Delta} u$ yields that

$$\Delta^s u(x) = C \ P.V. \int_{\mathbb{R}^N} \frac{(u(x)-u(y))}{|x-y|^{N+2s}} dy,$$

for some constant C. Using the symmetry of the kernel $|x-y|^{-(N+2s)}$ with a constant $\tilde{C}$,

$$\|(-\Delta^{s/2})u\|_2^2 = \langle \Delta^s u, u\rangle = \tilde{C} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy, \tag{4.1}$$

and thus the fractional L^2 Hardy inequality takes the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \geq C(N,s) \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx,$$

where the constant depends on N and s. One of the references to see the explicit calculation of this L^2 fractional Hardy inequality is [9, Appendix A]. However, when $p \neq 2$ one cannot have the equivalence of

$$\|(-\Delta^{s/2})u\|_p^p \quad \text{and} \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy$$

which we stated for $p = 2$ in (4.1). There are many studies in the literature regarding the fractional Hardy inequality of the form

$$\|(-\Delta^{s/2})u\|_p^p \geq C(N,s,p) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx;$$

for instance, Herbst in [8] calculated the sharp constant in the above inequality. But in this paper we are interested in the fractional Hardy inequalities of the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq C'(N,s,p) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \tag{4.2}$$

in the Dunkl setting.

The basic study of fractional power of the Dunkl Laplacian can be conducted in a similar fashion to the Euclidean case. The kernel $|x-y|^{-(N+ps)}$ in (4.2) is actually the translation of the function $|x|^{-(N+ps)}$. We are motivated to consider the kernel which is the Dunkl translation of $|x|^{-(d_k+ps)}$. Motivated by [6, Lemma 2.3] we define the kernel $\Phi(x,y)$ as

$$\Phi(x,y) := \frac{1}{\Gamma((d_k+ps)/2)} \int_0^\infty s^{\frac{d_k+ps}{2}-1} \tau_y^k(e^{-s|.|^2})(x)ds. \tag{4.3}$$

Theorem 4.1: *Let $d_k \geq 1$ and $0 < s < 1$. If $u \in \dot{W}_p^s(\mathbb{R}^N)$ when $2 \leq p < d_k/s$ or $u \in \dot{W}_p^s(\mathbb{R}^{\mathbb{N}} \setminus \{0\})$ when $p > d_k/s$, the following inequality holds:*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y) \geq C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x), \tag{4.4}$$

where $\Phi(x,y)$ is given in (4.3) and

$$C_{d_k,s,p} := 2\int_0^1 r^{ps-1}|1-r^{(d_k-ps)/p}|^p \Phi_{N,s,p}(r)dr, \tag{4.5}$$

with

$$\begin{aligned} \Phi_{N,s,p}(r) :=& \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})} \int_0^\pi \frac{\sin^{d_k-2}\theta}{(1-2r\cos\theta+r^2)^{\frac{d_k+ps}{2}}} d\theta, \quad && N \geq 2, \\ \Phi_{1,s,p}(r) :=& (\tau_r^k(|.|^{d_k+ps}) + \tau_{-r}^k(|.|^{d_k+ps}))(1), && N=1. \end{aligned} \tag{4.6}$$

The constant $C_{d_k,s,p}$ is sharp. If $p=1$, equality holds iff u is proportional to a symmetric decreasing function. If $p>1$, the inequality is strict for any function $0 \not\equiv u \in \dot{W}_p^s(\mathbb{R}^N)$ or $\dot{W}_p^s(\mathbb{R}^N \setminus \{0\})$, respectively. Further, for $p \geq 2$ the following inequality holds:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y) \\ &\quad \geq C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x) \\ &\qquad + c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi(x,y) \frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}} \frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}}, \end{aligned} \tag{4.7}$$

where $v := |x|^{(d_k-ps)/p}u$, Φ is as in equation (4.3), $C_{d_k,s,p}$ is given by (4.5) and c_p is given in (3.18). If $c_2 = 1$ the equality holds in $p=2$ case.

Remark 4.1: The case when we choose the multiplicity function $k \equiv 0$, the Dunkl case will reduce to the classical case. So in that case we get the main results in [3] as a corollary of the above theorems. That is [3, Theorem 1.1] and [3, Theorem 1.2] are obtained as corollaries to Theorem 4.1.

Here is an auxiliary lemma which is proven in [3].

LEMMA 4.2 (R. Frank, R. Seiringer): *Let $p \geq 1$. Then for all $0 \leq t \leq 1$ and $a \in \mathbb{C}$ one has*

$$|a - t|^p \geq (1-t)^{p-1}(|a|^p - 1). \tag{4.8}$$

For $p > 1$ this inequality is strict, unless $a = 1$ or $t = 0$. Moreover, if $p \geq 2$ then, for all $0 \leq t \leq 1$ and all $a \in \mathbb{C}$, one has

$$|a - t|^p \geq (1-t)^{p-1}(|a|^p - t) + c_p t^{p/2}|a-1|^p, \tag{4.9}$$

with $0 < c_p \leq 1$ and c_p given in (3.18). *For $p = 2$,* (4.9) *is an equality with $c_2 = 1$. For $p > 2$,* (4.9) *is a strict equality unless $a = 1$ or $t = 0$.*

For $N, p \geq 1$, let $\Phi_\epsilon(x, y)$ be symmetric positive real-valued functions defined on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\Phi_\epsilon \to \Phi$ as $\epsilon \to 0$ with $\Phi_\epsilon \leq \Phi$. Let us define the energy functional $E[u]$ as

$$E[u] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^p \Phi(x, y) d\mu_k(x) d\mu_k(y),$$

where $\Phi(x, y)$ is the kernel given in (4.3). Let us define the functions V_ϵ and V as

$$V_\epsilon(x) := 2w(x)^{-p+1} \int_{\mathbb{R}^N} (w(x) - w(y))|w(x) - w(y)|^{p-2} \Phi_\epsilon(x, y) d\mu_k(y) \tag{4.10}$$

and

$$\int_{\mathbb{R}^N} V f d\mu_k(x) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} V_\epsilon f d\mu_k(x)$$

for every $f \in C_0^\infty(\mathbb{R}^N)$. Following a similar argument as in the proof of [3, Proposition 2.2, Proposition 2.3] gives us the following two lemmas.

LEMMA 4.3: *Let $u \in C_0^\infty(\mathbb{R}^N)$. If $E[u]$ and $\int V|u|^p$ are finite we have*

$$E[u] \geq \int_{\mathbb{R}^N} V(x)|u(x)|^p d\mu_k(x). \tag{4.11}$$

Lemma 4.4: *Let $p \geq 2$ and $u \in C_0^\infty(\mathbb{R}^N)$. If $E[u]$, $\int V|u|^p$ are finite and*

$$\int_{\mathbb{R}^N} |v(x)-v(y)|^p w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} \Phi(x,y) d\mu_k(x) d\mu_k(y) < \infty, \tag{4.12}$$

then we have

$$\begin{aligned} E[u] - &\int_{\mathbb{R}^N} V(x)|u(x)|^p d\mu_k(x) \\ &\geq c_p \int_{\mathbb{R}^N} |v(x)-v(y)|^p w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} \Phi(x,y) d\mu_k(x) d\mu_k(y), \end{aligned} \tag{4.13}$$

where c_p is as in (3.18)*. If $p=2$,* (4.11) *becomes an equality with $c_2 = 1$.*

We will prove the following lemma which states that $w(x) = |x|^{-\frac{d_k - ps}{p}}$ solves the Euler–Lagrange equation related to equation (4.4). For convenience in calculations we write $\alpha := (d_k - ps)/p$. Let $\Phi_\epsilon := \Phi \chi_{||x|-|y||>\epsilon}$; then the Φ_ϵ's are positive symmetric real-valued functions which converge to Φ, with $0 < \Phi_\epsilon \leq \Phi$.

Lemma 4.5: *Let $w(x) = |x|^{-\frac{d_k - ps}{p}}$. The following limit converges uniformly for any compact subsets of $\mathbb{R}^N$:*

$$\begin{aligned} 2\lim_{\epsilon\to 0} \int_{||x|-|y||>\epsilon} &(w(x)-w(y))|w(x)-w(y)|^{p-2} \Phi_\epsilon(x,y) d\mu_k(y) \\ &= \frac{C_{d_k,s,p}}{|x|^{ps}} w(x)^{p-1}. \end{aligned} \tag{4.14}$$

Proof. Let $|x| = r$ and $|y| = \rho$ and write $x = rx'$ and $y = \rho y'$. Using polar coordinates we obtain

$$\begin{aligned} &\int_{||x|-|y||>\epsilon} (w(x)-w(y))|w(x)-w(y)|^{p-2} \Phi(x,y) d\mu_k(y) \\ &= \int_{|\rho - r|>\epsilon} \int_{\mathbb{S}^{N-1}} (r^{-\alpha} - \rho^{-\alpha})|r^{-\alpha} - \rho^{-\alpha}|^{p-2} \Phi(rx', \rho y') \rho^{2\lambda_k+1} d\rho d\sigma_k(y'), \end{aligned} \tag{4.15}$$

where

$$d\sigma_k(y') = h_k^2(y') d\sigma(y')$$

with $d\sigma(y')$ the (Euclidean) surface measure on the sphere $\mathbb{S}^{N-1}$. If $\rho < r$ we use the fact from [6, Lemma 2.3] that $\Phi(rx', \rho y') = r^{-d_k - ps} \Phi(x', \frac{\rho}{r} y')$ to get

$$\begin{aligned} &\int_{||x|-|y||>\epsilon} (w(x)-w(y))|w(x)-w(y)|^{p-2} \Phi(x,y) d\mu_k(y) \\ &= \int_{|\rho - r|>\epsilon} \int_{\mathbb{S}^{N-1}} \frac{\operatorname{sgn}(\rho^\alpha - r^\alpha)|\rho^{-\alpha} - r^{-\alpha}|^{p-1}}{r^{d_k+ps}} \Phi\left(x', \frac{\rho}{r} y'\right) \rho^{2\lambda_k+1} d\sigma_k(y') d\rho. \end{aligned} \tag{4.16}$$

Similarly, if $r < \rho$, from [6, Lemma 2.3] it follows that

$$\begin{aligned}(4.17)\quad &\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)\\ &=\int_{|\rho-r|>\epsilon}\int_{\mathbb{S}^{N-1}}\frac{\operatorname{sgn}(\rho^{\alpha}-r^{\alpha})|\rho^{-\alpha}-r^{-\alpha}|^{p-1}}{\rho^{1+ps}}\Phi(\frac{r}{\rho}x',y')d\sigma_k(y')d\rho.\end{aligned}$$

It follows from [6, Lemma 2.3] that

$$(4.18)\quad \int_{\mathbb{S}^{N-1}}\Phi(rx',\rho y')d\sigma_k(y')=\frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}\int_0^{\pi}\frac{sin^{d_k-2}\theta}{(r^2-2r\rho\cos\theta+\rho^2)^{\frac{d_k+ps}{2}}}d\theta.$$

Using (4.16), (4.17) and (4.18) we can write (4.15) as

$$\begin{aligned}(4.19)\quad &\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)\\ &=\frac{1}{r^{d_k-1}}\int_{|\rho-r|>\epsilon}\frac{\operatorname{sgn}(\rho^{\alpha}-r^{\alpha})}{|\rho-r|^{2-p(1-s)}}\varphi(\rho,r)d\rho,\end{aligned}$$

where $\varphi(\rho,r)$ is given by

$$(4.20)\quad \varphi(\rho,r)=\Big|\frac{\rho^{-\alpha}-r^{-\alpha}}{r-\rho}\Big|^{p-1}\cdot\begin{cases}\rho^{d_k-1}(1-\frac{\rho}{r})^{1+ps}\Phi_{N,s,p}(\frac{\rho}{r}), & \text{if } \rho<r,\\ r^{d_k-1}(1-\frac{r}{\rho})^{1+ps}\Phi_{N,s,p}(\frac{r}{\rho}), & \text{if } \rho>r,\end{cases}$$

with $\Phi_{N,s,p}$ given in (4.6).

We need to show the convergence of the integral

$$(4.21)\quad \int_{|\rho-r|>\epsilon}\frac{\operatorname{sgn}(\rho^{\alpha}-r^{\alpha})}{|\rho-r|^{2-p(1-s)}}\varphi(\rho,r)d\rho.$$

It is enough to show that the function $\phi(\rho,r)$ is Lipschitz continuous as a function of ρ at $\rho=r$. Writing $t=\rho/r$ it is sufficient to show the function $(1-t)^{1+ps}\Phi_{N,s,p}(t)$ and its t-derivative are bounded at $t\to 1-$. As $N=1$ it is trivial; we do it for $N\geq 2$. The identity in [7, 3.665] states that

$$(4.22)\quad \int_{\mathbb{R}^N}\frac{\sin^{2\mu-1}x dx}{(1+2a\cos x+a^2)^{\nu}}=B\Big(\mu,\frac{1}{2}\Big)F\Big(\nu,\nu-\mu+\frac{1}{2},\mu+\frac{1}{2};a^2\Big),$$

where F is a hypergeometric function with $Re\ \mu>0$ and $|a|<1$. Using (4.22) we can write

$$(4.23)\quad \Phi_{N,s,p}(t)=\frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}B\Big(\frac{d_k-1}{2},\frac{1}{2}\Big)F\Big(\frac{d_k+ps}{2},\frac{ps+2}{2};\frac{d_k}{2};t^2\Big).$$

Using the property that both $(1-z)^{a+b-c}F(a,b,c;z)$ and its derivative has a limit at $z\to 1-$ if $a+b-c>1$, we conclude that $(1-t)^{1+ps}\Phi_{N,s,p}(t)$ and its t-derivative are bounded at $t\to 1-$.

Continuing the same argument from [3] we get (4.14) with

$$C_{d_k,s,p}=2\lim_{\epsilon\to 0}\int_{|\rho-1|>\epsilon}\frac{\operatorname{sgn}(\rho^\alpha-1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho.$$

Now we will prove that this constant coincides with the constant given in (4.5).

$$\begin{aligned}
&2\lim_{\epsilon\to 0}\int_{|\rho-1|>\epsilon}\frac{\operatorname{sgn}(\rho^\alpha-1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho\\
&\quad=2\lim_{\epsilon\to 0}\Big[\int_0^{1-\epsilon}\frac{\operatorname{sgn}(\rho^\alpha-1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho+\int_{1+\epsilon}^{\infty}\frac{\operatorname{sgn}(\rho^\alpha-1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho\Big]\\
&\quad=2\Big[\int_0^{1}\frac{\operatorname{sgn}(\rho^\alpha-1)}{(1-\rho)^{2-p(1-s)}}\varphi(\rho,1)d\rho+\int_0^{1}\frac{\operatorname{sgn}(1-\rho^\alpha)\rho^{-p(1-s)}}{(1-\rho)^{2-p(1-s)}}\varphi(\rho^{-1},1)d\rho\Big]\\
&\quad=2\operatorname{sgn}(\alpha)\int_0^1\frac{(\rho^{-p(1-s)}\varphi(\rho^{-1},1)-\varphi(\rho,1))}{(1-\rho)^{2-p(1-s)}}d\rho.
\end{aligned}$$

A straightforward calculation gives

$$(\rho^{-p(1-s)}\varphi(\rho^{-1},1)-\varphi(\rho,1))=|1-\rho^\alpha|^{p-1}(1-\rho^\alpha)\Phi_{N,s,p}(\rho)(1-\rho)^{2-p(1-s)}$$

and it follows that

$$C_{d_k,s,p}=2\int_0^1\rho^{ps-1}|1-\rho^\alpha|^p\Phi_{N,s,p}(\rho)d\rho. \qquad ∎$$

4.1. Proof of Theorem 4.1. Now the Hardy inequalities (4.4) and (4.7) follow by repeating the arguments of [3]. In case of the strictness $p\geq 2$ due to the positive remainder term in (4.7), it is immediate that the inequality in (4.4) is strict. With similar arguments used to obtain [3, (2.18)], in our case we obtain

$$E[u]=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_u(x,y)\Phi(x,y)d\mu_k(x)d\mu_k(y)+\int_{\mathbb{R}^N}V|u|^p d\mu_k(x), \tag{4.24}$$

for all $u\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$ with

$$\begin{aligned}
\phi_u(x,y)=&|w(x)v(x)-w(y)v(y)|^p\\
&-(w(x)|v(x)|^p-w(y)|v(y)|^p)(w(x)-w(y))|w(x)-w(y)|^{p-2}.
\end{aligned}$$

It can be proven easily that $\phi_u\geq 0$ (see [3]). This can be extended to $\dot{W}_p^s(\mathbb{R}^N\setminus\{0\})$ when $d_k<ps$ and to $\dot{W}_p^s(\mathbb{R}^N)$ when $d_k>ps$ by approximation.

Suppose $E[u] = \int_{\mathbb{R}^N} V|u|^p d\mu_k(x)$ for some $u \in \dot{W}_p^s(\mathbb{R}^N \setminus \{0\})$. Then it is true for $|u|$. Observing that $\Phi_{|u|} \geq 0$ and $\Phi(x,y)$ is positive in (4.24) we can see that $\Phi_{|u|} \equiv 0$. From Lemma 4.2 we obtain that v is a constant function and since $v = w^{-1}u$ we conclude that $u \equiv 0$. This gives that for any non-zero $u \in \dot{W}_p^s(\mathbb{R}^N \setminus \{0\})$ in case $d_k < ps$ or $u \in \dot{W}_p^s(\mathbb{R}^N)$ in case $d_k > ps$, inequality (4.11) is strict.

Now for $p = 1$, we shall prove that the equality of (4.4) holds if and only if u is proportional to a symmetric decreasing function. Let χ_t be the characteristic function of a ball centered at the origin with radius $R(t)$. Define $u = \int_0^\infty \chi_t dt$. Then for $p = 1$, we can write the right-hand side of the inequality (4.4) as

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{|x|^s} = \frac{\|\mathbb{S}^{N-1}\|_k}{d_k - s} \int_0^\infty R(t)^{d_k - s} dt,$$

where $\|\mathbb{S}^{N-1}\|_k$ is the surface measure of $\mathbb{S}^{N-1}$ with Dunkl weighted measure; one can calculate $\|\mathbb{S}^{N-1}\|_k = c_k^{-1}/(2^{(\frac{d_k}{2}-1)}\Gamma(d_k/2))$. Now the left-hand side of the same inequality (4.4) can be written as

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y)\\
=&2\iint_{\{|x|<|y|\}} \left| \int (\chi_t(x) - \chi_t(y)) dt \right| \Phi(x,y) d\mu_k(x) d\mu_k(y)\\
=&2\iiint_{\{|x|<R(t)<|y|\}} \Phi(x,y) d\mu_k(x) d\mu_k(y) dt\\
=&2\iint_{\{|x|<1<|y|\}} \Phi(x,y) d\mu_k(x) d\mu_k(y) \int_0^\infty R(t)^{d_k-s} dt.
\end{aligned}$$

It gives the equality of (4.4) for the function u and $p = 1$.

The sharpness of the constant $C_{d_k,s,p}$ can be proved by the same arguments in [3]. But for the completion we give the proof here. To prove this, we will use the trial functions u_n and show that, as $n \to \infty$,

$$\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^p k(x,y) d\mu_k(x) d\mu_k(y)}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)} \leq C_{d_k,s,p}(1 + \mathcal{O}(1)).$$

Let us first define the functions u_n for $d_k > ps$. Let

$$\begin{aligned}
I &:= \{x \in \mathbb{R}^N : 0 \leq |x| < 1\},\\
M_n &:= \{x \in \mathbb{R}^N : 1 \leq |x| \leq n\},\\
O_n &:= \{x \in \mathbb{R}^N : |x| \geq n\}.
\end{aligned}$$

Define

$$u_n(x) := \begin{cases} 1 - n^{-\alpha}, & \text{if } x \in I, \\ |x|^{-\alpha} - n^{-\alpha}, & \text{if } x \in M_n, \\ 0, & \text{if } x \in O_n, \end{cases} \tag{4.25}$$

where $\alpha = \frac{(d_k - ps)}{p}$. Multiply the integrand of (4.14) by $u_n(x)$ and integrate with respect to x. Using the symmetry of $\Phi(x,y)$ we obtain, as $\epsilon \to 0$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u_n(x)-u_n(y))(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(x)d\mu_k(y) \\ &\qquad = C_{d_k,s,p}\int_{\mathbb{R}^N} \frac{u_n(x)w(x)^{p-1}}{|x|^{ps}}d\mu_k(x). \end{aligned} \tag{4.26}$$

Write

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u_n(x)-u_n(y))(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(x)d\mu_k(y) \\ &\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_n(x)-u_n(y)|^p\Phi(x,y)d\mu_k(x)d\mu_k(y) + 2\mathcal{R}_0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_0 := &\int_{x\in I}\int_{y\in M_n} (1-w(y))((w(x)-w(y)^{p-1}) - (1-w(y))^{p-1}) \\ &\qquad \times \Phi(x,y)d\mu_k(x)d\mu_k(y) \\ &+ \int_{x\in M_n}\int_{y\in O_n} (w(x)-n^{-\alpha})((w(x)-w(y))^{p-1} - (w(x)-n^{-\alpha})^{p-1}) \\ &\qquad \times \Phi(x,y)d\mu_k(x)d\mu_k(y) \\ &+ \int_{x\in I}\int_{y\in O_n} (1-n^{-\alpha})((w(x)-w(y))^{p-1} - (1-N^{-\alpha})^{p-1}) \\ &\qquad \times \Phi(x,y)d\mu_k(x)d\mu_k(y). \end{aligned}$$

Since all the terms within all the three integrals are non-negative, we have $\mathcal{R} \geq 0$. Divide the right-hand side of (4.26) by $C_{d_k,s,p}$ and add and subtract $\frac{u_n^p}{|x|^{ps}}$ to the integrand: we obtain

$$\int_{\mathbb{R}^N} \frac{u_n^p}{|x|^{ps}}d\mu_k(x) + \mathcal{R}_1 + \mathcal{R}_2, \tag{4.27}$$

where

$$\mathcal{R}_1 := \int_I (1-n^{-\alpha})(w(x)^{p-1} - (1-n^{-\alpha})^{p-1} \frac{d\mu_k(x)}{|x|^{ps}},$$
$$\mathcal{R}_2 := \int_{M_n} (w(x)-n^{-\alpha})(w(x)^{p-1} - (w(x)-n^{-\alpha})^{p-1}) \frac{d\mu_k(x)}{|x|^{ps}}.$$

Observe that the integrands on both of the integrals are non-negative and we will show that $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{O}(1)$ as $n \to \infty$:

$$\begin{aligned} &\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_n(x)-u_n(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y)}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)} \\ &= C_{d_k,s,p}\Big(1 + \frac{\mathcal{R}_1\mathcal{R}_2}{\int_{\mathbb{R}^N} |u_n(x)|^p|x|^{-ps} d\mu_k(x)}\Big) - \frac{2\mathcal{R}_0}{\int_{\mathbb{R}^N} |u_n(x)|^p|x|^{-ps} d\mu_k(x)} \\ &\le C_{d_k,s,p}(1+o(1)). \end{aligned} \tag{4.28}$$

Now we need to prove that $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{O}(1)$ as $n \to \infty$. Note that the integrand of $\mathcal{R}_1$ is bounded by $|x|^{\alpha - d_k}$ and it allows us to write

$$\mathcal{R}_1 \le \int_{|x|<1} |x|^{\alpha-d_k} d\mu_k(x) < \infty.$$

Observe that $1-(1-t)^{p-1} \le t$ for $1 \le p \le 2$ and $1-(1-t)^{p-1} \le (p-1)t$ for $p > 2$, where $0 \le t \le 1$. Using this we can write

$$(w(x)-n^{-\alpha})(w(x)^{p-1} - (w(x)-n^{-\alpha})^{p-1}) \le C_p n^{-\alpha} w(x)^{p-1}, \tag{4.29}$$

where $C_p = 1$ for $1 \le p \le 2$ and $C_p = p-1$ for $p > 2$. Now it is not hard to see that

$$\mathcal{R}_2 \le C_p \int_{|x|<1} |x|^{\alpha-d_k} d\mu_k(x) < \infty.$$

The case $d_k < ps$ can be treated similarly using the sequence of trial functions described in [3] taking $\alpha = (d_k - ps)/p$.

5. Fractional Hardy inequality for the half-space

Let R_1 be a root system on $\mathbb{R}^{N-1}$ and k_1 be a multiplicity function on R_1. Extend R_1 to a root system R of $\mathbb{R}^N$ as $R = R_1 \times \{0\} = \{(x,0) : x \in R_1\}$. Clearly it is a root system on $\mathbb{R}^N$ and the multiplicity function k_1 can be extended to k which acts on R by $k(x_1, x_2, \dots, x_{N-1}, x_N) = k_1(x_1, x_2, \dots, x_{N-1})$. Let $R_{1,+}$ be a positive subsystem of R_1 with $R_1 = R_{1,+} \cup (-R_{1,+})$. Then we can write $R = R_+ \cup (-R_+)$, where the positive subsystem R_+ of R is given by $R_+ = \{(x,0) : x \in R_{1,+}\}$; γ_k remains the same as

$$\gamma_k = \sum_{\nu \in R_+} k(\nu) = \sum_{\nu \in R_{1,+}} k_1(\nu) = \gamma_{k_1}.$$

The Dunkl measure corresponding to the root system R and the multiplicity function k will be

$$\begin{aligned} d\mu_k(x) =& d\mu_k(x) = \prod_{\nu \in R_+} |\langle x, \nu\rangle|^{2k(\nu)} dx \\ &= \prod_{\nu \in R_{1,+}} |\langle x', \nu\rangle|^{2k_1(\nu)} dx'.dx_N = d\mu_{k_1}(x')dx_N, \end{aligned}$$

where $x = (x', x_N) \in \mathbb{R}^N$.

Theorem 5.1: *Let $N \geq 1$, $1 \leq p < \infty$, and $0 < s < 1$ with $ps \neq 1$. Then for all $u \in \dot{W}^p_s(\mathbb{R}^N_+)$*

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |u(x) - u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y) \geq D_{N,\gamma_k,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x), \tag{5.1}$$

where

$$D_{N,\gamma_k,p,s} := c_{k_1}^{-1} 2^{-\lambda_{k_1}} \frac{\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \int_0^1 |1 - r^{(ps-1)/p}|^p \frac{dr}{(1-r)^{1+ps}} \tag{5.2}$$

and the constant $D_{N,\gamma_k,p,s}$ is optimal. If $p = 1$ and $N = 1$, equality holds iff u is proportional to a non-increasing function. If $p = 1$ or if $p = 1$ and $N \geq 2$, the inequality is strict for any non-zero function in $\dot{W}^s_p(\mathbb{R}^N_+)$. Further, for $p \geq 2$ we also have

$$\begin{aligned} &\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |u(x) - u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y) \\ &\quad \geq D_{N,\gamma_k,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x) \\ &\qquad + c_p \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |v(x) - v(y)|^p \Phi(x,y) \frac{d\mu_k(x)}{|x_N|^{(1-ps)/2}} \frac{d\mu_k(y)}{|y_N|^{(1-ps)/2}}, \end{aligned} \tag{5.3}$$

where $v := x_N^{(1-ps)/p} u$, Φ is as in (4.3), $D_{N,\gamma_k,p,s}$ is given in (5.2) and c_p is given in (3.18); $c_2 = 1$ and the equality holds in the $p = 2$ case.

Proof. Let $x = (x', x_N)$ and $y = (y', y_N)$ be elements of $\mathbb{R}^N$. Choose $w(x) = |x_N|^{(1-ps)/p}$ and $V(x) = D_{N,\gamma_k.p,s}|x_N|^{-ps}$. Since for the fixed root system R

$$\tau_y^k(e^{-s|.|^2})(x) = e^{-s|x_N-y_N|^2}\tau_{y'}^{k_1}(e^{-s|.|^2})(x'),$$

the definition of $\Phi(x,y)$ in (4.3) takes the form

$$\begin{aligned}\Phi(x,y) :=&\frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}\tau_y^k(e^{-s|.|^2})(x)ds\\ =&\frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}e^{-s|x_N-y_N|^2}\tau_{y'}^{k_1}(e^{-s|.|^2})(x')ds.\end{aligned}$$

We start with the Euler–Lagrange equation corresponding to (5.1) and let us verify that $w(x) = |x_N|^{-\frac{1-ps}{p}}$ solves it:

$$\begin{aligned}&\int_{\substack{y\in\mathbb{R}^N_+\\ |x_N-y_N|>\epsilon}}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)\\ &=\frac{1}{\Gamma((d_k+ps)/2)}\int_{\substack{y\in\mathbb{R}^N_+\\ |x_N-y_N|>\epsilon}}(w(x)-w(y))|w(x)-w(y)|^{p-2}\\ &\quad\times\int_0^\infty s^{\frac{d_k+ps}{2}-1}\tau_y(e^{-s|.|^2})(x)dsd\mu_k(y)\\ &=\frac{1}{\Gamma((d_k+ps)/2)}\int_{\mathbb{R}^{N-1}}\int_{|x_N-y_N|>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\\ &\quad\times\int_0^\infty s^{\frac{d_k+ps}{2}-1}e^{-s|x_N-y_N|^2}\tau_{y'}^{k_1}(e^{-s|.|^2})(x')dsdy_Nd\mu_{k_1}(y').\end{aligned}\tag{5.4}$$

The property of translation of a radial function [15, Theorem 3.8] gives that

$$\int_{\mathbb{R}^{N-1}}\tau_{y'}^{k_1}(e^{-s|.|^2})(x')d\mu_{k_1}(y') = \int_{\mathbb{R}^{N-1}}e^{-s|y'|^2}d\mu_{k_1}(y').\tag{5.5}$$

From the definition of the Gamma function we get

$$\begin{aligned}&\frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}e^{-s(|x_N-y_N|^2+|y'|^2)}ds\\ &\qquad=\frac{1}{(|x_N-y_N|^2+|y'|^2)^{\frac{d_k+ps}{2}}}.\end{aligned}\tag{5.6}$$

Applying (5.5) and (5.6) to (5.4) we find that

$$\int_{\substack{y\in\mathbb{R}^N_+,\\ |x_N-y_N|>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y) = \int_{\substack{y\in\mathbb{R}^N_+,\\ |x_N-y_N|>\epsilon}} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x_N-y_N|^2+|y'|^2)^{\frac{d_k+ps}{2}}} d\mu_k(y).$$

Let us calculate the following integral separately for convenience and set $m=|x_N-y_N|^2$, and keep in mind that $d_{k_1}=d_k-1$:

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \frac{1}{(m^2+|y'|^2)^{\frac{d_k+ps}{2}}} d\mu_k(y') &= \|\mathbb{S}^{N-2}\|_{k_1} \int_0^\infty \frac{1}{(m^2+r^2)^{\frac{d_k+ps}{2}}} r^{d_k-2} dr\\ &= \|\mathbb{S}^{N-2}\|_{k_1} \frac{1}{m^{1+ps}} \int_0^\infty \frac{t^{d_k-2}}{(1+t^2)^{\frac{d_k+ps}{2}}} dt\\ &= \|\mathbb{S}^{N-2}\|_{k_1} \frac{1}{2m^{1+ps}} \frac{\Gamma((d_k-1)/2)\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)}.\end{aligned}$$

We now return to the equation and use [3, Theorem 1.1] for $N=1$ to conclude. Also substitute the value of $\|\mathbb{S}^{N-2}\|_{k_1} = (c_{k_1}^{-1}2^{-\lambda_{k_1}})/\Gamma(d_{k_1}/2)$. We use the same notation w for the function $w(x_N)=|x_N|^{-(1-ps)/p}$:

$$\begin{aligned}&\int_{y\in\mathbb{R}^N_+,|x_N-y_N|>\epsilon} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x_N-y_N|^2+|y'|^2)^{\alpha/2}} d\mu_k(y)\\ &= \frac{c_{k_1}^{-1}2^{-\lambda_{k_1}}\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \int_{|x_N-y_N|>\epsilon} \frac{(w(x_N)-w(y_N))|w(x_N)-w(y_N)|^{p-2}}{|x_N-y_N|^{1+ps}} dy_N.\end{aligned} \tag{5.7}$$

From [3, Lemma 3.1], considering $x_N, y_N \in \mathbb{R}$, we can write

$$\begin{aligned}&\frac{C_{1,p,s}}{|x_N|^{ps}} w(x_N)^{p-1}\\ &\quad= 2\lim_{\epsilon\to 0} \int_{\substack{\mathbb{R},\\ ||x_N|-|y_N||>\epsilon}} \frac{(w(x_N)-w(y_N)||w(x_N)-w(y_N)|^{p-2})}{|x_N-y_N|^{1+ps}} dy_N\\ &\quad= 2\int_0^\infty (w(x_N)-w(y_N))|w(x_N)-w(y_N)|^{p-2}\\ &\qquad \times\Big(\frac{1}{|x_N-y_N|^{1+ps}+|x_N+y_N|^{1+ps}}\Big) dy_N.\end{aligned} \tag{5.8}$$

This gives the constant in [3, Theorem 1.1] as

$$C_{1,p,s} = 2\int_0^1 |1-r^{(1-ps)/p}|^p\Big(\frac{1}{(1-r)^{1+ps}}+\frac{1}{(1+r)^{1+ps}}\Big)dr. \tag{5.9}$$

But in our case we are only interested in the case $y_N > 0$, so (5.8) and (5.9) imply that

$$\begin{aligned} 2\lim_{\epsilon\to 0}\int^{\infty}_{\substack{0,\\ |x_N-y_N|>\epsilon}} &(w(x_N)-w(y_N))\frac{|w(x_N)-w(y_N)|^{p-2}}{|x_N-y_N|^{1+ps}}dy_N \\ &=\frac{\tilde{C}_{1,p,s}}{|x_N|^{ps}}w(x)^{p-1}, \end{aligned} \tag{5.10}$$

where

$$\tilde{C}_{1,p,s} := 2\int_0^1 \frac{|1-r^{(1-ps)/p}|^p}{(1-r)^{1+ps}}dr. \tag{5.11}$$

Now by using (5.10) and (5.7) we can conclude that

$$\begin{aligned} 2\lim_{\epsilon\to 0}\int_{y\in\mathbb{R}^N_+,|x_N-y_N|>\epsilon} &\frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x_N-y_N|^2+|y'|^2)^{\alpha/2}}d\mu_k(y) \\ &=\frac{c_{k_1}^{-1}2^{-\lambda_{k_1}-1}\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)}\frac{\tilde{C}_{1,p,s}}{|x_N|^{ps}}w(x)^{p-1}. \end{aligned} \tag{5.12}$$

We can see that the constant appearing in (5.2) and

$$\frac{c_{k_1}^{-1}2^{-\lambda_{k_1}-1}\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)}\tilde{C}_{1,p,s}$$

are same.

The Hardy inequalities (5.1) and (5.3), the strictness for $p > 1$ and the equality in the case of $p = 1$ follow from the proof of [4, Theorem 1.1]. Optimality comes from the optimality of Theorem 4.1. ∎

6. Fractional Hardy inequality for the cone

For $0 \le l \le N$, a cone $\mathbb{R}^N_{l+}$ is defined as a subset of $\mathbb{R}^N$ which is precisely the set

$$\{x=(x_1,\dots,x_N)\in\mathbb{R}^N : x_{N-l+1}>0,\dots,x_N>0\}.$$

In the case of a half-space we extended a root system of $\mathbb{R}^{N-1}$ to a root system of $\mathbb{R}^N$ and found a corresponding multiplicity function and Dunkl weighted measure on $\mathbb{R}^N_+$. In the case of a cone we write $\mathbb{R}^N = \mathbb{R}^{N-l}\times\mathbb{R}^l$ and extend a root system of $\mathbb{R}^{N-l}$ to $\mathbb{R}^N$. For an element $x \in \mathbb{R}^N$ we write $x = (x', x_{N-l+1}, x_{N-l+2}, \dots, x_N)$ where $x' \in \mathbb{R}^{N-l}$. Let R_1 be a root system on $\mathbb{R}^{N-l}$ and k_1, $d\mu_{k_1} := h_k^2(x')$ be the corresponding multiplicity function and Dunkl weighted measure. Define $R := \{(x,0) \in \mathbb{R}^N : x \in R_1\}$. It is easy to verify that R is a root system on $\mathbb{R}^N$. Now as in the case of the upper half-space, extend the multiplicity function to k of $\mathbb{R}^N$ as $k(x',0) = k_1(x)$ and the corresponding Dunkl weighted measure $d\mu_k(x) = d\mu_{k_1}(x')dx_{N-l+1}\cdots dx_N$. For the convenience of the calculations we write $x \in \mathbb{R}^N$ as $x = (x', x'')$ with $x' \in \mathbb{R}^{N-l}$ and $x'' \in \mathbb{R}^l$.

Theorem 6.1: *Let $N \in \mathbb{N}$, $1 \le p < \infty$. Further, $0 < s < 1$ with a condition $ps \neq 1$. Then for all $u \in \dot{W}^p_s(\mathbb{R}^N_{l_+})$ the following inequality holds:*

$$\begin{aligned}(6.1)\qquad &\int_{\mathbb{R}^N_{l_+}}\int_{\mathbb{R}^N_{l_+}} |u(x)-u(y)|^p \Phi(x,y)d\mu_k(x)d\mu_k(y)\\ &\quad\ge D_{N_l,\gamma_k,p,s}\int_{\mathbb{R}^N_{l_+}} \frac{|u(x)|^2}{x^2_{N-l+1}+\cdots+x_N^2}d\mu_k(x),\end{aligned}$$

where

$$(6.2)\qquad D_{N_l,\gamma_k,p,s} = \frac{c_{k_1}^{-1}2^{-\lambda_k}\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\int_0^1 r^{ps-1}|1-r^{(l-ps)/p}|^p\tilde{\Phi}_{l_+,s,p}(r)dr,$$

with

$$\tilde{\Phi}_{l_+,s,p}(r) = \int_{\mathbb{S}^{l-1}_{l_+}} \frac{1}{|\tilde{x}-r\tilde{y}|^{l+ps}}d\sigma(\tilde{y}),$$

where $\tilde{x} \in \mathbb{S}^{l-1}_{l_+}$ and $\mathbb{S}^{l-1}_{l_+} = \mathbb{S}^{l-1}\cap\mathbb{R}^l_{l_+}$. The constant $D_{N_l,\gamma_k,p,s}$ is optimal. If $p=1$ and $N=l$, equality holds iff u is proportional to a non-increasing function. Also, for $p \ge 2$ the following inequality holds:

$$\begin{aligned}(6.3)\qquad &\int_{\mathbb{R}^N_{l_+}}\int_{\mathbb{R}^N_{l_+}} |u(x)-u(y)|^p \Phi(x,y)d\mu_k(x)d\mu_k(y)\\ &\quad\ge D_{N_l,\gamma_k,p,s}\int_{\mathbb{R}^N_{l_+}} \frac{|u(x)|^p}{|x''|^{ps}}d\mu_k(x)\\ &\qquad + c_p\int_{\mathbb{R}^N_{l_+}}\int_{\mathbb{R}^N_{l_+}} |v(x)-v(y)|^p\Phi(x,y)\frac{d\mu_k(x)}{|x''|^{(1-ps)/2}}\frac{d\mu_k(y)}{|y''|^{(1-ps)/2}}\end{aligned}$$

where $v := |x''|^{(l-ps)/p}u$, Φ is as in (4.3), $D_{N,\gamma_k,p,s}$ is given in (6.2) and c_p is given in (3.18). Moreover, $c_2 = 1$ and the equality holds in the $p=2$ case.

Proof. The proof is very similar to that of Hardy inequality of the half-space. Similar steps will lead to the desired conclusion very easily. In order to find a positive solution of the Euler–Lagrange equation corresponding to (6.1), we set $w(x) = |x''|^{-(l-ps)/2}$ and $V(x) = D_{N_l,\gamma_k,p,s}|x''|^{-ps}$. The $\Phi(x,y)$ given in (4.3) will take the form

$$\begin{aligned}\Phi(x,y) :=& \frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}\tau_y^k(e^{-s|.|^2})(x)ds\\ =& \frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}e^{-s\sum_{j=N-l+1}^N |x_j-y_j|^2}\tau_{y'}(e^{-s|.|^2})(x')ds,\end{aligned}$$

since

$$\tau_y^k(e^{-s|.|^2})(x) = e^{-s\sum_{j=N-l+1}^N |x_j-y_j|^2}\tau_{y'}(e^{-s|.|^2})(x')$$

with our root system R on $\mathbb{R}^N$.

Repeating the same arguments as in the proof of Theorem 5.1 we obtain

$$\begin{aligned}&\int_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)\\ &\qquad = \int_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x''-y''|^2+|y'|^2)^{\frac{d_k+ps}{2}}}d\mu_k(y).\end{aligned}$$

We evaluate $\int_{\mathbb{R}^{N-l}} \frac{1}{(m^2+|y'|^2)^{\alpha/2}}d\mu_k(y')$ as in the previous proof with $m=|x''-y''|$ and find that

$$\int_{\mathbb{R}^{N-l}} \frac{1}{(|x''-y''|^2+|y'|^2)^{\frac{d_k+ps}{2}}}d\mu_k(y') = \pi^{\frac{d_{k_1}}{2}}\frac{\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\frac{1}{|x''-y''|^{l+ps}},$$

where $d_{k_1} = N-l+2\gamma_{k_1}$. Now the Euler–Lagrange equation corresponding to (6.1) is of the form

$$\begin{aligned}&2\lim_{\epsilon\to 0}\int_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)\\ &\qquad = \frac{c_{k_1}^{-1}2^{-\lambda_{k_1}}\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\\ &\qquad\quad \times\lim_{\epsilon\to 0}\int_{\substack{y\in\mathbb{R}^l_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{(|x''-y''|)^{l+ps}}dy'',\end{aligned}\tag{6.4}$$

with $w(x'') = |x''|^{-(l-ps)/p}$.

If $\mathbb{S}^{l-1}_{l_+} = \mathbb{S}^{l-1} \cap \mathbb{R}^l_{l_+}$, the polar decomposition of the right-hand-side integral of (6.4) can be written as

$$\lim_{\epsilon\to 0} \int_{\substack{y\in\mathbb{R}^l_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{(|x''-y''|)^{l+ps}} dy''$$
$$= \int_{|\rho-r|>\epsilon} \int_{\mathbb{S}^{l-1}_{l_+}} \frac{(r^{-\alpha}-\rho^{-\alpha})|r^{-\alpha}-\rho^{-\alpha}|^{p-2}}{|r\tilde{x}-\rho\tilde{y}|^{l+ps}} d\sigma(\tilde{y})d\rho,$$

where $x'' = r\tilde{x}$, $y'' = \rho\tilde{y}$ and $\alpha = (l-ps)/p$. Using similar steps in the proof of [3, Lemma 3.1] we can prove that

$$(6.5)\quad 2\lim_{\epsilon\to 0} \int_{\mathbb{R}^l_{l_+}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{|x''-y''|^{l+ps}} dy'' = \frac{\tilde{C}_{l_+,s,p}}{|x''|^{p}s} w(x'')^{p-1},$$

where for $l \geq 2$

$$\tilde{C}_{l_+,s,p} = 2\int_0^1 r^{ps-1}|1-r^{(l-ps)/p}|^p \tilde{\Phi}_{l_+,s,p}(r)dr$$

with

$$\tilde{\Phi}_{l_+,s,p}(r) = \int_{\mathbb{S}^{l-1}_{l_+}} \frac{1}{|\tilde{x}-r\tilde{y}|^{l+ps}} d\sigma(\tilde{y}),$$

and when $l = 1$ then $\tilde{C}_{1_+,s,p} = \tilde{C}_{1,p,s}$ given in equation (5.11). The constant $\tilde{C}_{l_+,s,p}$ is different from the constant $C_{l,s,p}$ given in [3, Theorem 1.1], since instead of integrating over the whole sphere $\mathbb{S}^{l-1}$ we are only integrating over the points on the sphere which intersect with the cone, that is only on $\mathbb{S}^{l-1}_{l_+}$.

Define

$$D_{N_l,\gamma_k,p,s} := \frac{c_{k_1}^{-1} 2^{-\lambda_{k_1}} \Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)} \tilde{C}_{l_+,s,p}.$$

From (6.4) and (6.5), we get w as the positive solution of the Euler–Lagrange equation corresponding to (6.1)

$$2\lim_{\epsilon\to 0} \int_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi(x,y)d\mu_k(y)$$
$$= \frac{D_{N_l,\gamma_k,p,s}}{|x''|^{ps}} w(x)^{p-1}.$$

Proof of the Hardy inequalities (6.1) and (6.3) and the proof of optimality of the constant $D_{N_l,\gamma_k,p,s}$(it follows from the optimality of $\tilde{C}_{1_+,s,p}$) can be obtained by the same techniques used in the proof of [3, Theorem 1.1, Theorem 1.2]. ∎

Remark 6.1: Since we could not calculate the integral $\int_{\mathbb{S}^{l-1}_{l_+}} \frac{1}{|\tilde{x}-r\tilde{y}|^{l+ps}} d\sigma(\tilde{y})$ explicitly, the expression of the constant $D_{N_l,\gamma_k,p,s}$ in Theorem 6.1 is not explicit compared to the constants given in Theorem 4.1 and Theorem 5.1 .

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