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THE SEPARABLE QUOTIENT PROBLEM FOR TOPOLOGICAL GROUPS

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ABSTRACT

The famous Banach–Mazur problem, which asks if every infinite-dimensional Banach space has an infinite-dimensional separable quotient Banach space, has remained unsolved for 85 years, though it has been answered in the affirmative for reflexive Banach spaces and even Banach spaces which are duals. The analogous problem for locally convex spaces has been answered in the negative, but has been shown to be true for large classes of locally convex spaces including all non-normable Fréchet spaces.

For a topological group G there are four natural analogous problems: Does G have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable. Positive answers to all four questions are proved for groups G which belong to the important classes of (a) all compact groups; (b) all locally compact abelian groups; (c) all σ-compact locally compact groups; (d) all abelian pro-Lie groups; (e) all σ-compact pro-Lie groups; (f) all pseudocompact groups.

However, a surprising example of an uncountable precompact group G is produced which has no non-trivial separable quotient group other than the trivial group. Indeed G^τ has the same property, for every cardinal number $\tau \geq 1$.

1. Introduction

It is natural to attempt to describe all objects of a certain kind in terms of basic building blocks of that kind. For example, one may try to describe general Banach spaces in terms of separable Banach spaces. Recall that a topological space is said to be **separable** if it has a countable dense subset.

To put our investigation into context, we begin with a famous unsolved problem in Banach space theory. The Separable Quotient Problem for Banach Spaces has its roots in the 1930s and, according to a private communication from Władysław Orlicz to Jerzy Kąkol, is due to Stefan Banach and Stanisław Mazur.

Problem 1.1 (Separable quotient problem for Banach spaces): Does every infinite-dimensional Banach space have a quotient Banach space which is separable and infinite-dimensional?

The related Quotient Schauder Basis Problem for Banach Spaces is due to Aleksander (Olek) Pełczyński [33].

Problem 1.2 (Quotient Schauder basis problem for Banach spaces): Does every infinite-dimensional Banach space have a quotient Banach space which is infinite-dimensional and has a Schauder basis?

Of course any Banach space with a Schauder basis is separable. Mazur's Problem 153 in The Scottish Book [29], for which the prize was a live goose, was answered by Per Enflo [12] in 1973. Enflo proved that there exist separable Banach spaces which do not have a Schauder basis (and hence lack the approximation property).

However William Johnson and Haskell Rosenthal proved the following result:

Theorem 1.3 ([21]): *Every separable infinite-dimensional Banach space has a quotient infinite-dimensional Banach space with a Schauder basis.*

Corollary 1.4: *The Quotient Schauder Basis Problem for Banach Spaces 1.2 and the Separable Quotient Problem for Banach Spaces 1.1 are equivalent.*

Steve Saxon and Albert Wilansky proved some equivalent versions of the Separable Quotient Problem for Banach Spaces.

Theorem 1.5 ([39]): *The following are equivalent for an infinite-dimensional Banach space B:*

(i) *B has a quotient Banach space which is separable and infinite-dimensional;*
(ii) *B has a dense subspace which is not barrelled;*
(iii) *B has a dense subspace E which is the union of a strictly increasing sequence of closed linear subspaces.*

Some extensions of this result to topological vector spaces are obtained in [24]. We have then that Problem 1.1 is equivalent to each of Problem 1.6 and Problem 1.7:

Problem 1.6: Does every infinite-dimensional Banach space have a dense subspace E which is the union of a strictly increasing sequence of closed linear subspaces?

Problem 1.7: Does every infinite-dimensional Banach space have a dense infinite-dimensional subspace which is not barrelled?

As a corollary of Theorem 1.5 (see Corollary 3.5 of [32]) one obtains the result first proved by Dan Amir and Joram Lindenstrauss.

Corollary 1.8 ([1]): *Every infinite-dimensional weakly compactly generated (WCG) Banach space has a separable infinite-dimensional quotient Banach space.*

As reflexive Banach spaces (and separable Banach spaces) are WCG, one obtains:

Corollary 1.9 ([33]): *Every infinite-dimensional reflexive Banach space has a separable infinite-dimensional quotient Banach space.*

From Corollary 1.9 one easily obtains:

Corollary 1.10: *Let B be a Banach space such that the dual Banach space B^* has an infinite-dimensional reflexive subspace E. Then B has a quotient Banach space isomorphic to E^*. So B has an infinite-dimensional separable quotient Banach space.*

Spiros Argyros, Pandelis Dodos and Vassilis Kanellopoulos in 2008 generalized Corollary 1.9.

Theorem 1.11 ([2]): *If B is the Banach dual of any infinite-dimensional Banach space, then B has a separable infinite-dimensional quotient Banach space.*

In the literature many special cases of the Separable Quotient Problem for Banach Spaces have been proved, however the general problem remains unsolved.

Turning to locally convex spaces one can state the analogous problem. Throughout this paper all locally convex spaces will be assumed to be Hausdorff.

Problem 1.12 (Separable quotient problem for locally convex spaces): Does every infinite-dimensional locally convex space have a quotient locally convex space which is separable and infinite-dimensional?

This question was answered in the positive for a wide class of locally convex spaces by M. Eidelheit [11]. (See also Chapter 6, Section 31.4 of [25].)

Theorem 1.13 ([11]): *Every infinite-dimensional Fréchet space (= locally convex space with its topology determined by a complete translation invariant metric) which is non-normable has the separable metrizable topological vector space $\mathbb{R}^\omega$ as a quotient space.*

Definition 1.14 ([35]): *A topological vector space is said to be* **properly separable** *if it has a proper dense vector subspace of countably infinite (Hamel) dimension.*

Wendy Robertson [35] observed that a Fréchet space is properly separable if and only if it is separable and that a metrizable barrelled locally convex space is properly separable if and only if it is separable. The following result generalizes Theorem 1.13. For further generalizations along this line, see [38].

Theorem 1.15 ([35]): *Every strict inductive limit of a strictly increasing sequence (E_m) of Fréchet spaces with at least one E_m non-normable has a properly separable quotient locally convex space.*

Jerzy Kąkol, Steve Saxon and Aaron Todd [23] answered Problem 1.12 in the negative.

Theorem 1.16 ([23]): *There exist infinite-dimensional barrelled locally convex spaces which do not have any infinite-dimensional separable quotient locally convex spaces.*

However the following results go in the positive direction.

Theorem 1.17 ([23]): *Let X be any infinite Tychonoff space and $C_c(X)$ the linear space of all real-valued continuous functions on X endowed with the compact-open topology. If $C_c(X)$ is barrelled, then it has a quotient locally convex space which is infinite-dimensional and separable.*

Theorem 1.18 ([23]): *Let X be any infinite Tychonoff space. Both the strong and weak duals of $C_c(X)$ have a quotient locally convex space which is infinite-dimensional and separable.*

Once again we note that there are many partial positive solutions in the literature to Problem 1.12 (see also [22]) which are out of the scope of these introductory remarks.

Now we turn to the specific topic of this paper, the Separable Quotient Problem(s) for Topological Groups. We shall in fact state nine quite natural problems.

Terminology and basic facts. A topological space X is said to be **hereditarily separable** if X and every subspace of X is separable. A topological space is said to be **second countable** if its topology has a countable base. A topological space X is said to have a **countable network** if there exists a countable family $\mathcal{B}$ of (not necessarily open) subsets such that each open set of X is a union of members of $\mathcal{B}$.

(i) Any space with a countable network is hereditarily separable;
(ii) a metrizable space is separable if and only if it is second countable;
(iii) any continuous image of a separable space is separable;
(iv) countable networks are preserved by continuous images;
(v) a quotient image of a second countable space need not be second countable;
(vi) a closed subgroup of a separable topological group is not necessarily separable. However, the class of topological groups with the property that every closed subgroup is separable is quite large and natural, and in particular it includes all separable locally compact groups. This class of topological groups has been thoroughly investigated in [27], [28] (see also [26]).

An abstract group is called **simple** if it has no proper non-trivial normal subgroup, where a group is called **non-trivial** if it has at least two elements. A topological group G is said to be **topologically simple** if it has no proper non-trivial closed normal subgroup. As the connected component of the identity of any topological group is a closed normal subgroup, every topologically simple group is either totally disconnected or connected.

Throughout this paper all topological groups are assumed to be Hausdorff. The circle group with the usual multiplication and compact topology inherited from the complex plane is denoted by $\mathbb{T}$. The product of elements $x, y \in \mathbb{T}$ will be denoted by $x \cdot y$, while we will use additive notation for all abelian groups distinct from $\mathbb{T}$. A **character** of a group H is a homomorphism of H to $\mathbb{T}$. The additive topological group of all real numbers with the euclidean topology is denoted by $\mathbb{R}$. The cardinality of the continuum is denoted by $\mathfrak{c}$, so $\mathfrak{c} = 2^\omega$. By Lie group we mean a real finite-dimensional Lie group; the 0-dimensional Lie groups are discrete. ∎

We begin with an example which says that we must exclude some discrete groups and indeed some totally disconnected groups.

Example 1.19: Let $\aleph$ be any uncountable cardinal number and S a set of cardinality $\aleph$. Let $A_\aleph$ be the **finitary alternating group** on the set S; that is, $A_\aleph$ is the group of all even permutations of the set S which fix all but a finite number of members of S. That $A_\aleph$ is an uncountable simple group follows easily from [36, 3.2.4]. So if $A_\aleph$ is given the discrete topology, it is an infinite discrete topologically simple group and so does not have a (proper) quotient group which is a non-trivial topological group.

It is proved in [31, Corollary 7] that for each positive integer n, there exists a non-discrete Hausdorff totally disconnected group G_n of cardinality $\aleph_n$ such that each proper subgroup is discrete (that is, G_n is strongly minimal) and each proper subgroup has cardinality strictly less than $\aleph_n$ (that is G_n a Johnson group). Assuming the Continuum Hypothesis this implies that for $n > 1$ the group G_n has no separable quotient group, since all non-trivial quotient groups have cardinality strictly greater than $\aleph_1 = 2^{\aleph_0}$.

Without the assumption of the Continuum Hypothesis, George A. Willis [43, §3] (see also [7]) extended the examples of discrete topologically simple groups to produce, for each infinite cardinal number $\aleph$, a non-discrete totally disconnected locally compact topologically simple group of cardinality (and weight) $\aleph$.

By contrast, Theorem 2.13 shows that every infinite discrete abelian group G does indeed have a quotient group which is a countably infinite (discrete) group. (Indeed G has a quotient group of cardinality κ, for each infinite κ less than the cardinality of G.) ∎

Remark 1.20: Comprehensive surveys about the decomposition theory of totally disconnected locally compact groups can be found in the recent book [9].

Problem 1.21 (Separable quotient problem for topological groups): Does every non-totally disconnected topological group have a quotient group which is a non-trivial separable topological group?

Problem 1.22 (Separable infinite quotient problem for topological groups): Does every non-totally disconnected topological group have a quotient group which is an infinite separable topological group?

Problem 1.23 (Separable metrizable quotient problem for topological groups): Does every non-totally disconnected topological group have a quotient group which is a non-trivial separable metrizable topological group?

Problem 1.24 (Separable infinite metrizable quotient problem for topological groups): Does every non-totally disconnected topological group have a quotient group which is an infinite separable metrizable topological group?

Regarding Problem 1.24, one might reasonably ask: If the topological group G has a quotient group which is infinite and separable, does G necessarily have a quotient group which is infinite, separable and metrizable? This question is answered negatively in Proposition 4.4.

Corollary 1.9 suggests the following problem for **reflexive** topological groups; that is, topological groups for which the natural map of the topological group into the Pontryagin dual of its Pontryagin dual is an isomorphism of topological groups.

Problem 1.25 (Separable quotient problem for reflexive topological groups): Does every infinite reflexive abelian topological group, G, have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

A special case of Problem 1.25 is:

Problem 1.26 (Separable quotient problem for locally compact abelian groups): Does every infinite locally compact abelian group, G, have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

As another special case of Problem 1.25 one might be tempted to ask whether every topological group, G, which is the underlying topological group of an infinite-dimensional Banach space has a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable? But a positive answer to all these questions follows immediately from the fact that $\mathbb{R}^n$, for every $n \in \mathbb{N}$, is a quotient locally convex space of every infinite-dimensional locally convex space. Noting that according to the Anderson–Kadec theorem each infinite-dimensional separable Banach space (indeed each infinite-dimensional separable Fréchet space) is homeomorphic to $\mathbb{R}^\omega$ ([6, Chapter VI, Theorem 5.2]), the next question is pertinent. Of course a positive answer to Problem 1.1 would yield a positive answer to Problem 1.27.

Problem 1.27 (Separable quotient problem for Banach topological groups): Does every topological group, which is the underlying topological group of an infinite-dimensional Banach space have a separable quotient group which is homeomorphic to $\mathbb{R}^\omega$?

Related to this problem we mention the following recent result in [14].

Theorem 1.28 ([14]): *Every infinite-dimensional Fréchet space, and in particular every infinite-dimensional Banach space, has the infinite separable metrizable tubby torus group $\mathbb{T}^\omega$ as a quotient group.*

The non-abelian version of Problem 1.26 is:

Problem 1.29 (Separable quotient problem for locally compact groups): Does every non-totally disconnected locally compact group, G, have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

As a special case of Problem 1.29 we have:

Problem 1.30 (Separable quotient problem for compact groups): Does every infinite compact group, G, have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

We shall address these problems in subsequent sections.

2. Locally compact groups and pro-Lie groups

In this section we provide a positive answer to each of Problem 1.30 (i), (ii), (iii), and (iv) and Problem 1.26 (i), (ii), (iii), and (iv), and a partial answer to Problem 1.29. We prove satisfying results for pro-Lie groups. We also prove stronger structural results for compact abelian groups, connected compact groups and totally disconnected compact groups.

Theorem 2.1: *Every non-metrizable compact abelian group G has a quotient group Q which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group, Q, is therefore an infinite separable metrizable group.*

Proof. As G is non-metrizable, Theorem 29 of [30] implies that the dual group $\widehat{G}$ is uncountable. By Kulikov's Theorem, Corollary 18.4 of [13], every abelian group A is the union of an ascending chain

$$A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots$$

of subgroups, where each A_n is a direct sum of (finite or infinite) cyclic groups. Putting $A = \widehat{G}$ implies that $\widehat{G}$ has a subgroup A_n which is a direct sum of uncountably many cyclic subgroups, each of which is a finite cyclic or infinite cyclic group. By the Pontryagin van-Kampen Duality Theorem, then, G has a quotient group $\widehat{H}$ which is an infinite product of finite-dimensional Lie groups, each of which is either a discrete finite cyclic group or the (compact) circle group $\mathbb{T}$. So G has a quotient group Q which is a countably infinite product of finite-dimensional abelian Lie groups. ∎

We shall use the following lemma which follows immediately from the Sandwich Theorem for compact connected groups, Corollary 9.25, and Theorems 9.19 and 9.24 of [19]. We have previously defined what we mean by a simple group and a topologically simple group. We now record that a **simple** Lie group is a finite-dimensional Lie group which has no non-trivial connected normal subgroup. The center of a group S will be denoted by $Z(S)$.

Lemma 2.2: *Let G be a connected compact group and let G' be its closed commutator subgroup. Then*

(i) *G' is a quotient group of $\prod_{j\in J} S_j/Z(S_j)$, where J is some index set, S_j is a simple simply connected compact finite-dimensional Lie group, for each $j \in J$; and*
(ii) *$G/G' \times \prod_{j\in J} S_j/Z(S_j)$ is a quotient group of G.*

Theorem 2.3: *Every non-metrizable connected compact group, G, has a quotient group, Q, which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group, Q, is therefore an infinite separable metrizable group.*

Proof. We apply Lemma 2.2(ii) and consider three cases.

Case 1. J is infinite. Then G has $\prod_{j\in J} S_j/Z(S_j)$ as a quotient group. Consequently G has a quotient group which is a countably infinite product of non-trivial compact finite-dimensional Lie groups.

Case 2. J is finite and $\operatorname{rank}\widehat{G/G'}$ is finite. By Theorem 9.52 of [19], G' is a compact Lie group and G/G' is a connected finite-dimensional compact abelian group. Thus G' is metrizable, and G/G' also is metrizable by Theorem 8.49 of [19]. As metrizability is a Three Space Property by Theorem 12.13 of [37], this implies that G is metrizable, which contradicts our assumption that G is non-metrizable. So Case 2 does not occur.

Case 3. J is finite and $\operatorname{rank}\widehat{G/G'}$ is infinite. By the remark immediately following Proposition 8.15 of [19], G/G', and hence also G, have a quotient group which is a torus group of dimension equal to $\operatorname{rank}\widehat{G/G'}$, which completes our proof. ∎

Remark 2.4: No discrete group has a quotient group which is a countably infinite product of non-trivial topological groups since every quotient of a discrete group is evidently discrete. In particular then, a locally compact abelian group need not have a quotient group which is a countably infinite product of non-trivial topological groups.

Theorem 2.5: *Every non-metrizable connected locally compact abelian group, G, has a quotient group, Q, which is a countably infinite product of non-trivial compact finite-dimensional Lie groups. The quotient group, Q, is therefore an infinite separable metrizable group.*

Proof. By Theorem 26 of [30], G is isomorphic as a topological group to $\mathbb{R}^n \times K$, where n is a non-negative integer and K is a non-metrizable compact group. The required result then follows from Theorem 2.3. ∎

Theorem 2.6: *Every infinite totally disconnected compact group G has a quotient group, Q, which is homeomorphic to a countably infinite product of finite discrete topological groups. The quotient group, Q, is thus homeomorphic to the Cantor set and therefore is an infinite separable metrizable group.*

Proof. As G is a profinite group it is isomorphic as a topological group to a closed subgroup of a product $\prod_{i\in I} F_i$, where I is an index set, and each F_i is a discrete finite group. Let p_i be the projection mapping of G to F_i, for $i \in I$. We can assume that $p_i(G) = F_i$, for each $i \in I$. As G is an infinite group and each F_i is a finite group, for each $y \in G$ and $i \in I$ there is an infinite number of members z of G such that $p_i(y) = p_i(z)$. Fixing $g_1 \in G$ and $i_1 \in I$, let $t_1 = p_{i_1}(g_1) \in F_{i_1}$. There exists an index i_2 and an element g_2

in G such that $p_{i_1}(g_1) = p_{i_1}(g_2)$ and $p_{i_2}(g_1) \neq p_{i_2}(g_2) = t_2 \in F_{i_2}$. There are infinitely many y in G such that $p_{i_1}(y) = t_1$ and $p_{i_2}(y) = t_2$. So there exists an index i_3 and an element $g_3 \in G$ such that $p_{i_1}(g_2) = p_{i_1}(g_3)$, $p_{i_2}(g_2) = p_{i_2}(g_3)$ but $p_{i_3}(g_2) \neq p_{i_3}(g_3) = t_3 \in F_{i_3}$. So by induction we obtain infinite sequences of indices $i_1, i_2, i_3, \ldots, i_n, \ldots$ and elements $g_1, g_2, g_3, \ldots, g_n, \ldots$ in G such that $p_{i_j}(g_{n+1}) = p_{i_j}(g_n)$, $j = 1, 2, \ldots, n$ but $p_{i_{n+1}}(g_{n+1}) \neq p_{i_{n+1}}(g_n)$. Therefore the image $p(G)$ of G in the product $\prod_{j\in\mathbb{N}} F_{i_j}$ is infinite, where p is the projection mapping.

So $p(G)$ is an infinite separable metrizable totally disconnected compact group. By Theorem 10.40 of [19], $p(G)$ is a Cantor set, and so is homeomorphic to a countably infinite product of finite discrete topological groups. As G is compact, p is a quotient mapping, which proves the result. ∎

We now give a positive answer to Problem 1.30 (i), (ii), (iii), and (iv).

Theorem 2.7 (Separable quotient theorem for compact groups): *Let G be an infinite compact group. Then G has a quotient group which is an infinite separable metrizable (compact) group.*

Proof. By Corollary 2.43 of [19], G is a strict projective limit of compact Lie groups. So G is isomorphic as a topological group to a subgroup of a product $\prod_{i\in I} L_i$, where I is an index set and each L_i, $i \in I$, is a Lie group. Let p_i be the projection map of G into L_i, $i \in I$.

Assume first that each group $p_i(G)$ is finite. Then G is a compact group which is isomorphic as a topological group to a subgroup of a product of finite Lie groups; that is, G is a totally disconnected compact group. The required result then follows from Theorem 2.6.

So, assume that for some $n \in I$, $p_n(G)$ is infinite. As G is compact, p_n is a quotient mapping of G onto the infinite compact metrizable (separable) group $p_n(G)$, as required. ∎

Remark 2.8: Note that the Kakutani–Kodaira–Montgomery–Zippin Theorem, in the particular case of compact groups, immediately shows that G has a separable metrizable (compact) quotient group. However, this quotient might be finite, and we have to spend additional effort in order to prove there is an infinite one.

Theorem 2.7 for compact groups is used in the proof of the more general Theorem 2.17 dealing with σ-compact locally compact groups.

Having settled the compact group case, we mention three related problems.

Problem 2.9 (Separable quotient problem for **σ**-compact groups): Does every non-discrete σ-compact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

Problem 2.10 (Separable quotient problem for precompact groups): Does every infinite precompact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) non-trivial metrizable; (iv) infinite metrizable?

Problem 2.11 (Separable quotient problem for pseudocompact groups): Does every infinite pseudocompact group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) non-trivial metrizable; (iv) infinite metrizable?

Problem 2.9 is answered in Corollary 4.3 and Proposition 4.4. Problem 2.11 is answered in Proposition 4.5, while Problem 2.10 is answered in Theorem 3.5.

We now give a positive answer to Problem 1.26 (i), (ii), (iii), and (iv). First we state a proposition due to W. R. Scott [40]. (See EA1.12 of [19] and 16.13(c) of [17].)

Proposition 2.12: *Let A be an uncountable abelian group and $\aleph$ a cardinal number satisfying $\aleph_0 \leq \aleph < |A|$, where $|A|$ denotes the cardinality of the group A. Then A has a subgroup B such that $|A/B|$, the cardinality of the quotient group A/B, equals $\aleph$. In particular, every uncountable abelian group has a quotient group which is countably infinite.*

Theorem 2.13 (Separable quotient theorem for locally compact abelian groups): *Let G be an infinite locally compact abelian group. Then G has a quotient group which is an infinite separable metrizable group.*

Proof. By the principal structure theorem for locally compact abelian groups, Theorem 26 of [30], G has an open subgroup H isomorphic as a topological group to $\mathbb{R}^n \times K$, where n is a non-negative integer and K is a compact abelian group.

Case 1. $G = H$. Then the theorem follows from Theorem 2.7.

Case 2. G/H is an infinite discrete abelian group. Then by Proposition 2.12 G/H, and hence also G, has a quotient group which is countably infinite and discrete.

Case 3. G/H is finite. Then G is a compactly-generated locally compact abelian group. By Exercise Set 14 #3 of [30], G is isomorphic as a topological group to $\mathbb{R}^n \times \mathbb{Z}^m \times C$, where n and m are non-negative integers and C is a compact abelian group. The required result then follows from Theorem 2.7. ∎

Recall that a **proto-Lie group** is defined in [18, Definition 3.25] to be a topological group G for which every neighborhood of the identity contains a closed normal subgroup N such that the quotient group G/N is a Lie group. If G is also a complete topological group, then it is said to be a **pro-Lie group**. If G is a proto-Lie group (respectively, pro-Lie group) with all the quotient Lie groups G/N discrete then G is said to be **protodiscrete** (respectively, **prodiscrete**). It is immediately clear that if G is a proto-Lie group which is not a Lie group, then it is not topologically simple.

Theorem 2.14 (Separable quotient theorem for proto-Lie groups): *Let G be an infinite proto-Lie group which is not protodiscrete; that is, G is not totally disconnected. Then G has a quotient group which is an infinite separable metrizable (Lie) group.*

Proof. By definition, G has a quotient group G/N which is a Lie group L.

Case 1. All such L are discrete. Then G is a subgroup of a product of discrete groups; that is G is a protodiscrete group. This contradicts the assumption in the theorem.

Case 2. At least one such L is not discrete. Then G has a quotient group which is a non-discrete Lie group and so is infinite separable and metrizable. ∎

Theorem 2.15 (Separable quotient theorem for $\boldsymbol{\sigma}$-compact pro-Lie groups): *Let G be an infinite σ-compact pro-Lie group. Then G has a quotient group which is an infinite separable metrizable group.*

Proof. If G is not prodiscrete, the theorem follows immediately from Theorem 2.14.

So consider the case that G is prodiscrete. Whenever N is a closed normal subgroup such that G/N is discrete, the group G/N must also be σ-compact and therefore countable. If it is countably infinite, then we have immediately that G has an infinite separable metrizable quotient group. If each such G/N is finite, then G is compact (and totally disconnected) and the theorem follows from Theorem 2.7. ∎

Another significant generalization of Theorem 2.13 is Theorem 2.16.

Theorem 2.16 (Separable quotient theorem for abelian pro-Lie groups): *Let G be an infinite abelian pro-Lie group. Then G has a quotient group which is an infinite separable metrizable group.*

Proof. We modify Case 1 in the proof of Theorem 2.14. If all such L are discrete then we have one of the two cases:

Case (i) All such L_i are finite. Then G is isomorphic as a topological group to a closed subgroup of a product of finite discrete groups. So G is compact. Thus by Theorem 2.7, G has a quotient group which is an infinite separable metrizable group.

Case (ii) At least one such L is infinite and discrete. So G has a quotient group which is an infinite discrete group. Thus by Proposition 2.12, G has a quotient group which is an infinite separable metrizable group, which completes the proof of the theorem. ∎

The next theorem, which generalizes Theorem 2.7, provides a partial but significant answer to Problem 1.29.

Theorem 2.17 (Separable quotient theorem for $\boldsymbol{\sigma}$-compact locally compact groups): *Every infinite σ-compact locally compact group has a quotient group which is an infinite separable metrizable group.*

Proof. By the Kakutani–Kodaira–Montgomery–Zippin Theorem, every σ-compact locally compact group, G, has a compact normal subgroup, K, such that G/K is a separable metrizable group. (See Theorem 2 of [20] for a more general statement and elegant proof.)

Case 1. G/K is finite. Then G must be compact as compactness is a Three Space Property by Theorem 5.25 of [17]. The required result then follows from Theorem 2.7.

Case 2. G/K is infinite. Then G has a quotient group which is infinite separable and metrizable, as required. ∎

Pierre-Emmanuel Caprace studied the class $\mathscr{S}$ of all non-discrete compactly-generated locally compact groups that are topologically simple. He writes: *"Simple Lie groups and simple algebraic groups over local fields are the most* *prominent members of the class $\mathscr{S}$*" [7]. (See also [8].) Recall that, in contradistinction with the notions of simple abstract group and topologically simple topological group, a Lie group is said to be simple if it has no non-trivial connected normal subgroup.

As an immediate corollary of Theorem 2.17, we observe the following apparently new result which reveals the topological structure of members of the class $\mathscr{S}$.

PROPOSITION 2.18: *Every topological group in the class $\mathscr{S}$ is a separable metrizable group.*

Proof. A quotient group of any $G \in \mathscr{S}$ is trivial or it is G itself. But every compactly-generated topological group is σ-compact, and therefore by Theorem 2.17, every $G \in \mathscr{S}$ has a quotient group which is an infinite separable metrizable group. So G itself is an infinite separable metrizable group. ∎

Recall that a topological group G is said to be **almost connected** if the quotient group G/G_0 is compact, where G_0 denotes the connected component of the identity of G.

THEOREM 2.19 (Separable quotient theorem for almost connected locally compact groups): *Every infinite almost connected locally compact group has a quotient group which is an infinite separable metrizable group.*

Proof. Let G be an almost connected locally compact group and G_0 the connected component of the identity, so that G/G_0 is a compact group. If G/G_0 is infinite, then by Theorem 2.7 it has an infinite separable metrizable quotient group. Thus G also has an infinite separable metrizable quotient group.

So assume G/G_0 is finite. Noting that every connected locally compact group is σ-compact, it follows in this case that G is σ-compact. So by Theorem 2.17, G has an infinite separable metrizable quotient group. ∎

The class of $\mathbb{R}$-**factorizable** groups (see [4, Chapter 8]) contains all pseudocompact groups as well as σ-compact groups. We note that by Theorem 8.1.9 of [4], a locally compact group is $\mathbb{R}$-factorizable if and only if it is σ-compact. This suggests the following problem which will be answered in Section 3.

Problem 2.20 (Separable quotient problem for $\mathbb{R}$-factorizable groups): Does every $\mathbb{R}$-factorizable group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable?

3. A precompact counterexample

In this section, firstly we address Problem 2.10 and Problem 2.20 and answer both in the negative by producing a counterexample.

We present an example of a "weird" precompact topological group G with no non-trivial separable quotient groups. Indeed we proceed to prove that the group G^τ has no non-trivial separable quotient groups, for τ any cardinal number.

Recall that a subset X of an abelian group G with identity 0_G is called **independent** if the equality

$$n_1 x_1 + \cdots + n_k x_k = 0_G,$$

where $n_1, \ldots, n_k \in \mathbb{Z}$ and $x_1, \ldots, x_k$ are pairwise distinct elements of X, implies that

$$n_1 x_1 = \cdots = n_k x_x = 0_G.$$

The conclusion of the following lemma is well-known; it is a form of saying that the dual $\Pi^\wedge$ of the product $\Pi = \prod_{i \in I} K_i$ of compact abelian groups K_i is isomorphic to the discrete group

$$\bigoplus_{i \in I} K_i^\wedge,$$

the direct sum of the discrete dual groups $K_i^\wedge$ (see [30, Theorem 17] or [4, Proposition 9.6.25]; the latter result is dual to the lemma).

Lemma 3.1: *Let A be a non-empty set and χ a continuous character on the group $\mathbb{T}^A$. Then one can find pairwise distinct indices $\alpha_1, \ldots, \alpha_k \in A$ and integers $n_1, \ldots, n_k$ such that*

$$\chi(x) = \prod_{i=1}^{k} x(\alpha_i)^{n_i},$$

for each $x \in \mathbb{T}^A$.

Lemma 3.2: *Let G be an uncountable precompact abelian group such that every countable subgroup of G is closed. Then the following are equivalent:*

(a) *every uncountable subgroup of G is dense in G;*

(b) *the kernel of every non-trivial continuous character of G is countable.*

Furthermore, each of the items (a), (b) implies

(c) *every quotient group of G is either trivial or non-separable.*

Proof. (a)⇒(b). Assume that every uncountable subgroup of G is dense in G. Let χ be a continuous non-trivial character on G. Then the kernel K of χ must be countable—otherwise $K = G$ and χ is trivial.

(b)⇒(a). Let D be an uncountable subgroup of G. Then $K = \mathrm{cl}_G(D)$ is a closed subgroup of G. If $K \neq G$, then the quotient group G/K is non-trivial and precompact, so there exists a non-trivial continuous character ψ on G/K. Let $p\colon G \to G/K$ be the quotient homomorphism. Then $\chi = \psi \circ p$ is a non-trivial continuous character on G and $K \subset \ker \chi$. This contradicts (b) of the lemma and proves that D is dense in G.

We have thus proved that (a) and (b) of the lemma are equivalent.

(b)⇒(c). Let $h\colon G \to H$ be a continuous open homomorphism of G onto a topological group H with $|H| > 1$. Then H is precompact and abelian. Since $|H| > 1$, there exists a non-trivial continuous character χ_H on H. Hence $\chi_G = \chi_H \circ h$ is a non-trivial continuous character on G. By (b) of the lemma, the kernel of χ_G is countable. Therefore the kernel of h is countable as well. This implies that

$$|H| = |G| > \omega.$$

Let C be a countable subgroup of H. Then $h^{-1}(C)$ is a countable subgroup of G, so $h^{-1}(C)$ is closed in G by (a) of the lemma. Since the homomorphism h is continuous and open, C is closed in H. We have thus proved that all countable subgroups of H are closed. If S is a countable subset of H, then the subgroup C of H generated by S is also countable and, hence, C is closed in H. Since the group H is uncountable, we conclude that it cannot be separable. ∎

The proof of the following fact is quite simple and left to the reader.

Lemma 3.3: *Let A, B, C be subgroups of an abelian group K.*

(a) *The equality $(A+B)\cap C = (A^*+B)\cap C$ holds, where $A^* = A\cap(B+C)$.*
(b) *If A and $B + C$ have trivial intersection, then $(A + B) \cap C = B \cap C$.*

Notice that item (b) of the above lemma is immediate from (a).

Following [42, Section 4] we say that a subgroup S of an abelian topological group G is ***h*-embedded** in G if every homomorphism f of S to the circle group $\mathbb{T}$ extends to a continuous character of G. In particular, every homomorphism of an h-embedded subgroup of G to $\mathbb{T}$ is continuous. If the group G is precompact, then a subgroup S of G is h-embedded if and only if S inherits from G the maximal precompact group topology, i.e., the Bohr topology.

The following fact is a weaker version of [5, Proposition 2.1]. It also follows from [15, Lemma 2.3].

Lemma 3.4: *If every countable subgroup of an abelian topological group G is h-embedded in G, then the countable subgroups of G are closed.*

In the next theorem we present an uncountable precompact group without non-trivial separable quotients.

Theorem 3.5: *There exists an uncountable dense subgroup G of the compact group $\mathbb{T}^{\mathfrak{c}}$ satisfying $\dim G = 0$ such that every countable subgroup of G is h-embedded in G and closed and every uncountable subgroup of G is dense in G. Hence every quotient group of G is either trivial or non-separable.*

Proof. Let X be a set of cardinality $\mathfrak{c}$ and $A(X)$ the free abelian group on X. We will define a monomorphism f of the group $A(X)$ to $\mathbb{T}^{\mathfrak{c}}$ and then take G to be $f(A(X))$, considered as a topological subgroup of $\mathbb{T}^{\mathfrak{c}}$.

Since $|A(X)| = |X| = \mathfrak{c}$, the group $A(X)$ contains exactly $\mathfrak{c}^{\omega} = \mathfrak{c}$ countable subgroups. Let $\mathcal{C}$ be the family of all countable subgroups of $A(X)$. For every $C \in \mathcal{C}$, the family $H(C)$ of all homomorphisms of C to $\mathbb{T}$ has cardinality at most $\mathfrak{c}$, so the family $\mathcal{H} = \bigcup\{H(C) : C \in \mathcal{C}\}$ satisfies $|\mathcal{H}| = \mathfrak{c}$.

Let $\{(C_\alpha, g_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of the family

$$\{(C, g) : C \in \mathcal{C},\ g \in H(C)\}.$$

The subgroup C_α of $A(X)$ being countable, there exists a countable subset Y_α of X such that $C_\alpha \subset \langle Y_\alpha \rangle$, where $\alpha < \mathfrak{c}$. Let

$$Z_\alpha = X \setminus Y_\alpha.$$

Further, let E be an independent subset of $\mathbb{T}$ such that $|E| = \mathfrak{c}$ and each element of E has infinite order (see [4, Lemma 7.1.6]). Let also $\{E_\alpha : \alpha < \mathfrak{c}\}$ be a partition of E into $\mathfrak{c}$ pairwise disjoint subsets E_α, each of cardinality $\mathfrak{c}$.

For every $\alpha < \mathfrak{c}$, we will define a homomorphism $f_\alpha \colon A(X) \to \mathbb{T}$ satisfying the following conditions:

(i) f_α and g_α coincide on the subgroup C_α of $A(X)$;

(ii) $b_\alpha = f_\alpha \restriction Z_\alpha$ is a bijection of Z_α onto a subset of E_α, so the restriction of f_α to $\langle Z_\alpha \rangle$ is a monomorphism;

(iii) the subgroups of $\mathbb{T}$ generated by the sets $f_\alpha(Y_\alpha) \cup (\bigcup_{\nu<\alpha} f_\nu(X))$ and $f_\alpha(Z_\alpha)$ have trivial intersection;

(iv) $f_\alpha(A(X)) \subset \langle f_\alpha(Y_\alpha) \rangle \cdot \langle E_\alpha \rangle$.

Notice that (ii) and (iii) together imply that $\ker f_\alpha \subset \langle Y_\alpha \rangle$, so the kernel of f_α is countable for each $\alpha < \mathfrak{c}$. Indeed, it follows from the equality $X = Y_\alpha \cup Z_\alpha$ that every element $g \in \ker f_\alpha$ has the form $g = y \cdot z$ with $y \in \langle Y_\alpha \rangle$ and $z \in \langle Z_\alpha \rangle$. Then $1 = f_\alpha(g) = f_\alpha(y) \cdot f_\alpha(z)$, where $f_\alpha(y) \in \langle f_\alpha(Y_\alpha) \rangle$ and $f_\alpha(z) \in \langle f_\alpha(Z_\alpha) \rangle$. So we conclude that $f_\alpha(z) = f_\alpha(y)^{-1} \in \langle f_\alpha(Y_\alpha) \rangle \cap \langle f_\alpha(Z_\alpha) \rangle$ and, by (iii), $f_\alpha(z) = 1$. According to (ii) the latter implies that z is the identity element of $A(X)$, whence $g = y \in \langle Y_\alpha \rangle$. This shows that $\ker f_\alpha \subset \langle Y_\alpha \rangle$.

Also, since the equality $X = Y_\alpha \cup Z_\alpha$ holds for each $\alpha < \mathfrak{c}$, (iv) follows from (ii). However, we isolate (iv) for further applications. To construct the family $\{f_\alpha : \alpha < \mathfrak{c}\}$ satisfying (i)–(iv) we argue as follows.

For every $\alpha < \mathfrak{c}$, we extend g_α to a homomorphism h_α of $\langle Y_\alpha \rangle$ to $\mathbb{T}$. There exists a countable set $D_0 \subset E$ such that

$$\langle h_0(Y_0) \rangle \cap \langle E \rangle \subset \langle D_0 \rangle.$$

Let $F_0 = E_0 \setminus D_0$. Then $|F_0| = \mathfrak{c}$, so we choose a bijection b_0 of $Z_0 = X \setminus Y_0$ onto F_0. Since

$$A(X) = \langle Y_0 \rangle \oplus \langle Z_0 \rangle,$$

there exists a homomorphism $f_0 \colon A(X) \to \mathbb{T}$ which extends h_0 and coincides with b_0 on Z_0. Clearly the restriction of f_0 to $\langle Z_0 \rangle$ is a monomorphism. One can easily verify that the groups $\langle f_0(Y_0) \rangle$ and $\langle F_0 \rangle = \langle f_0(Z_0) \rangle$ have trivial intersection. Therefore f_0 satisfies (i)–(iv).

Assume that for some $\alpha < \mathfrak{c}$, we have defined a family $\{f_\nu : \nu < \alpha\}$ of homomorphisms of $A(X)$ to $\mathbb{T}$ satisfying (i)–(iv). The subgroup H_α of $\mathbb{T}$ generated by $h_\alpha(Y_\alpha) \cup \bigcup_{\nu<\alpha} f_\nu(Y_\nu)$ has cardinality at most $|\alpha + 1| \cdot \omega < \mathfrak{c}$. Let

$$H_\alpha^* = H_\alpha \cap \Big\langle \bigcup_{\nu \le \alpha} E_\nu \Big\rangle.$$

Then $|H_\alpha^*| \le |H_\alpha| < \mathfrak{c}$, so we can find a subset D_α of $\bigcup_{\nu \le \alpha} E_\nu$ with $|D_\alpha| < \mathfrak{c}$ such that $H_\alpha^* \subset \langle D_\alpha \rangle$. Let $F_\alpha = E_\alpha \setminus D_\alpha$. Notice that $|F_\alpha| = \mathfrak{c}$.

Claim 1: *The groups $\langle F_\alpha \rangle$ and $H_\alpha \cdot \langle \bigcup_{\nu<\alpha} E_\nu \rangle$ have trivial intersection.*

Indeed, we apply (a) of Lemma 3.3 with $A = H_\alpha$, $B = \langle \bigcup_{\nu<\alpha} E_\nu \rangle$, and $C = \langle F_\alpha \rangle$ to deduce that

$$(A \cdot B) \cap C = (A^* \cdot B) \cap C \subset \Big(H_\alpha^* \cdot \Big\langle \bigcup_{\nu<\alpha} E_\nu \Big\rangle \Big) \cap \langle F_\alpha \rangle, \tag{1}$$

where $A^* = A \cap (B \cdot C) \subset H^*_\alpha$. Further, F_α and $D_\alpha \cup \bigcup_{\nu<\alpha} E_\nu$ are pairwise disjoint subsets of the independent set $E \subset \mathbb{T}$, so the groups $\langle D_\alpha\rangle \cdot \langle\bigcup_{\nu<\alpha} E_\nu\rangle$ and $\langle F_\alpha\rangle$ have trivial intersection. Since $H^*_\alpha \subset \langle D_\alpha\rangle$, it follows from (1) that $(A \cdot B) \cap C = \{1\}$. This proves Claim 1.

Consider an arbitrary bijection $b_\alpha\colon Z_\alpha \to F_\alpha$. There exists a homomorphism $f_\alpha\colon A(X) \to \mathbb{T}$ which extends h_α and coincides with b_α on Z_α. It is clear that f_α extends g_α and the restriction of f_α to $\langle Z_\alpha\rangle$ is a monomorphism. Hence conditions (i) and (ii) hold true at the step α. Condition (iv) is also fulfilled. Indeed, it follows from the definition of f_α that $f_\alpha(Z_\alpha) = F_\alpha \subset E_\alpha$. Hence, applying the equality $X = Y_\alpha \cup Z_\alpha$ we deduce that

$$f_\alpha(A(X)) = \langle f_\alpha(X)\rangle \subset \langle f_\alpha(Y_\alpha)\rangle \cdot \langle E_\alpha\rangle.$$

It remains to verify that (iii) is also valid. Indeed, according to (iv) we have

$$f_\nu(X) \subset \langle f_\nu(Y_\nu)\rangle \cdot \langle E_\nu\rangle = \langle f_\nu(Y_\nu) \cup E_\nu\rangle,$$

for each $\nu < \alpha$. Hence the group generated by $f_\alpha(Y_\alpha) \cup \bigcup_{\nu<\alpha} f_\nu(X)$ (see condition (iii)) is contained in

$$\begin{aligned}\Big\langle f_\alpha(Y_\alpha) \cup \bigcup_{\nu<\alpha} (f_\nu(Y_\nu) \cup E_\nu)\Big\rangle &= \Big\langle \bigcup_{\nu\le\alpha} f_\nu(Y_\nu) \cup \bigcup_{\nu<\alpha} E_\nu\Big\rangle \\ &= \Big\langle \bigcup_{\nu\le\alpha} f_\nu(Y_\nu)\Big\rangle \cdot \Big\langle \bigcup_{\nu<\alpha} E_\nu\Big\rangle = H_\alpha \cdot \Big\langle \bigcup_{\nu<\alpha} E_\alpha\Big\rangle.\end{aligned}$$

By Claim 1, the groups $\langle F_\alpha\rangle = \langle f_\alpha(Z_\alpha)\rangle$ and $H_\alpha \cdot \langle\bigcup_{\nu<\alpha} E_\alpha\rangle$ have trivial intersection. Clearly this implies (iii) and finishes our recursive construction.

Let f be the diagonal product of the family $\{f_\alpha : \alpha < \mathfrak{c}\}$. We claim that f is a monomorphism of $A(X)$ to $\mathbb{T}^{\mathfrak{c}}$. Indeed, take an arbitrary element $x \in A(X)$ distinct from the identity and let C_x be the cyclic subgroup of $A(X)$ generated by x. Take a homomorphism g of C_x to $\mathbb{T}$ such that $g(x) \neq 1$. There exists $\alpha < \mathfrak{c}$ such that $(C_\alpha, g_\alpha) = (C_x, g)$. Then (i) implies that $f_\alpha(x) = g_\alpha(x) = g(x) \neq 1$, so the element $f(x)$ is distinct from the identity of the group $\mathbb{T}^{\mathfrak{c}}$. This proves our claim.

We consider the group $G = f(A(X))$ with the topology inherited from the compact group $\mathbb{T}^{\mathfrak{c}}$. Let p_α be the projection of $\mathbb{T}^{\mathfrak{c}}$ to the αth factor $\mathbb{T}_{(\alpha)}$, where $\alpha < \mathfrak{c}$. Our definition of f implies that the equality $p_\alpha \circ f = f_\alpha$ holds for each $\alpha < \mathfrak{c}$. Since f is a monomorphism and $\ker f_\alpha$ is countable, the restriction of each projection p_α to G has countable kernel as well.

To show that every countable subgroup, say, D of G is h-embedded, we consider a homomorphism $h\colon D \to \mathbb{T}$. Take a countable subgroup C of G such that $f(C) = D$ and let $g = h \circ f$. There exists $\alpha < \mathfrak{c}$ such that $(C, g) = (C_\alpha, g_\alpha)$. Then, for every $x \in C$, we have the equalities

$$(h \circ f)(x) = g(x) = g_\alpha(x) = f_\alpha(x) = (p_\alpha \circ f)(x).$$

Since f is a monomorphism, we conclude that h and p_α coincide on $D = f(C)$. Hence $p_\alpha \restriction G$ is a continuous character on G extending h. It follows from Lemma 3.4 that every countable subgroup of G is closed.

Our next step is to show that G is dense in $\mathbb{T}^\mathfrak{c}$. This is equivalent to the density of the projection of G to every subproduct $\mathbb{T}^A$, where A is a finite subset of the index set $\mathfrak{c}$. So let $A \subset \mathfrak{c}$ be a finite non-empty set. Then $Y = \bigcup_{\alpha \in A} Y_\alpha$ is a countable subset of X, so we can take an element $x \in X \setminus Y$. It follows from condition (ii) of our construction that $f_\alpha(x) \in E_\alpha$, for each $\alpha \in A$. Since the sets $E_\alpha \subset E$ with $\alpha \in A$ are pairwise disjoint, we see that the coordinates $\{f_\alpha(x) : \alpha \in A\}$ of the element $p_A(f(x)) \in \mathbb{T}^A$ are pairwise distinct and form a subset of E; here $p_A\colon \mathbb{T}^\mathfrak{c} \to \mathbb{T}^A$ is the projection. Notice that the set E is independent and consists of elements of infinite order. Hence the element $p_A(f(x))$ generates a dense subgroup of $\mathbb{T}^A$ (this follows, e.g., from [34, Example 65]). Therefore $p_A(G)$ is dense in $\mathbb{T}^A$, as claimed.

Let us verify that $\dim G = 0$. It follows from (iv) and our choice of the partition $E = \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ that for every $\alpha < \mathfrak{c}$, $G_\alpha = f_\alpha(A(X))$ is a proper subgroup of $\mathbb{T}$. Hence the space G_α is zero-dimensional (all classic definitions of dimension coincide for separable metrizable spaces). Since G is a topological subgroup of the product $\prod_{\alpha<\mathfrak{c}} G_\alpha$ of zero-dimensional groups, we see that $\operatorname{ind} G = 0$ or, equivalently, G has a base of clopen sets. The group G being precompact, it follows from [41, Corollary 3.4] that $\dim G = 0$.

To show that every non-trivial quotient group of G is not separable it suffices, by Lemma 3.2, to verify that the kernel of every non-trivial continuous character of G is countable.

Let $\varphi\colon G \to \mathbb{T}$ be a continuous homomorphism with $|\varphi(G)| > 1$. Then φ extends to a continuous homomorphism $\psi\colon \mathbb{T}^\mathfrak{c} \to \mathbb{T}$ (see [16, Lemma 2.2]). By Lemma 3.1, there exist pairwise distinct indices $\alpha_1, \ldots, \alpha_k \in \mathfrak{c}$ and non-zero integers $n_1, \ldots, n_k$ such that

$$\psi(x) = \prod_{i=1}^{k} x(\alpha_i)^{n_i},$$

for each $x \in \mathbb{T}^{\mathfrak{c}}$. We can assume that $\alpha_1 < \cdots < \alpha_k$. Since $p_\alpha \circ f = f_\alpha$ for each $\alpha \in \mathfrak{c}$, the expression for $\psi(x)$ can be rewritten as follows:

$$\varphi(f(b)) = \psi(f(b)) = \prod_{i=1}^{k} f_{\alpha_i}(b)^{n_i}, \quad \text{for each } b \in A(X). \tag{2}$$

Clearly the set $Y = \bigcup_{i=1}^{k} Y_{\alpha_i} \subset X$ is countable.

Claim 2: *The kernel of the homomorphism $\varphi \circ f$ is contained in $\langle Y \rangle$.*

Indeed, take an arbitrary element $b \in A(X) \setminus \langle Y \rangle$. Let $Z = X \setminus Y$. Then $b = y + z$, where $y \in \langle Y \rangle$ and $z \in \langle Z \rangle$. Clearly z is distinct from the identity of $A(X)$. For simplicity, let $\alpha = \alpha_k$. It follows from (2) that

$$\varphi(f(b)) = f_\alpha(z)^{n_k} \cdot f_\alpha(y)^{n_k} \cdot \prod_{i=1}^{k-1} f_{\alpha_i}(y)^{n_i} \cdot \prod_{i=1}^{k-1} f_{\alpha_i}(z)^{n_i}. \tag{3}$$

Since $Z \subset Z_\alpha$ and $f_\alpha \restriction Z_\alpha$ is a monomorphism, we see that $f_\alpha(z)^{n_k} \neq 1$.

Let us put

$$A = \left\langle f_\alpha(Y_\alpha) \cup \bigcup_{i=1}^{k-1} f_{\alpha_i}(X) \right\rangle, \quad B = \langle f_\alpha(Y \setminus Y_\alpha) \rangle, \quad \text{and} \quad C = \langle f_\alpha(Z) \rangle.$$

Since $Y = Y_\alpha \cup (Y \setminus Y_\alpha)$ and $y \in \langle Y \rangle$, the element on the right hand side of equality (3) is in $C \cdot B \cdot A$. Suppose for a contradiction that $\varphi(f(b)) = 1$. Then, by (3), we have that $1 \neq f_\alpha(z)^{n_k} \in C \cap (B \cdot A)$. Further, the group $B \cdot C$ is algebraically generated by the set

$$f_\alpha(Y \setminus Y_\alpha) \cup f_\alpha(X \setminus Y) = f_\alpha(X \setminus Y_\alpha) = f_\alpha(Z_\alpha),$$

so condition (iii) of our construction implies that $A \cap (B \cdot C) = \{1\}$. Hence, by (b) of Lemma 3.3, the equality $(A \cdot B) \cap C = B \cap C$ holds. Since $Y \setminus Y_\alpha$ and $X \setminus Y$ are disjoint subsets of the independent set Z_α, condition (ii) implies that the groups B and C have trivial intersection. Hence the intersection $(B \cdot A) \cap C$ is trivial. This contradiction shows that kernel of $\varphi \circ f$ is a subgroup of $\langle Y \rangle$, which proves Claim 2.

Since the set Y is countable, Claim 2 implies that $|\ker(\varphi \circ f)| \leq \omega$. We already know that f is a monomorphism, so the kernel of φ is also countable. Hence every non-trivial quotient of the group $G = f(A(X))$ is not separable, according to the implication (b) $\Rightarrow$ (c) in Lemma 3.2. ∎

We now give an answer to Problem 2.20.

Corollary 3.6: *There exist infinite $\mathbb{R}$-factorizable groups without non-trivial separable or metrizable quotients.*

Proof. Since the group G in Theorem 3.5 is precompact, it is $\mathbb{R}$-factorizable according to [4, Corollary 8.1.17]. Every quotient of G is also precompact. Notice that every precompact metrizable group is separable (this follows, e.g., from [4, Proposition 3.4.5]). Therefore the precompact group G does not have non-trivial separable or metrizable quotients. ∎

Our next aim is to show that every power of the group G in Theorem 3.5 does not have non-trivial separable quotients. The proof of this fact requires several preliminary results.

Lemma 3.7: *Let H be an abelian topological group, $k \in \mathbb{N}$, and $n_1, \dots, n_k$ be integers, not all equal to zero, such that the highest common divisor of $n_1, \dots, n_k$ equals 1. Let also φ be a homomorphism of H^k to H defined by $\varphi(x_1, \dots, x_k) = n_1 x_1 + \cdots + n_k x_k$, for each $(x_1, \dots, x_k) \in H^k$. Then the homomorphism φ is open and surjective. In particular, the quotient group H^k/N is isomorphic as a topological group to H, where $N = \ker \varphi$.*

Proof. Clearly the homomorphism φ is continuous. By our assumption about $n_1, \dots, n_k$, there exist integers $m_1, \dots, m_k$ such that $m_1 n_1 + \cdots + m_k n_k = 1$. Consider the homomorphism λ of H to H^k defined by $\lambda(x) = (m_1 x, \dots, m_k x)$. The continuity of λ is evident. Notice that $\varphi \circ \lambda = Id_H$, so φ is surjective. Therefore it suffices to verify that φ is open.

Take an arbitrary open neighborhood U of the identity element in H^k. We can assume that $U = V \times \cdots \times V$, for some open neighborhood V of the identity element in H (V is taken k times as a factor). Let $M = \max\{|m_1|, \dots, |m_k|\}$ and choose an open symmetric neighborhood W of the identity in H such that

$$\underbrace{W + \cdots + W}_{M \text{ times}} \subset V.$$

It follows from our choice of M and W that $\lambda(W) \subset U$. Further, the equality $\varphi \circ \lambda = Id_H$ implies that

$$W = \varphi(\lambda(W)) \subset \varphi(U),$$

so the set $\varphi(U)$ contains a non-empty open neighborhood of the identity. Hence the homomorphism φ is open. ∎

The conclusion of the next lemma is only formally stronger than that of Theorem 3.5.

Lemma 3.8: *Let $G \subset \mathbb{T}^{\mathfrak{c}}$ be the group constructed in Theorem 3.5 and let H be a quotient group of G. Then every countable subgroup of H is h-embedded and closed in H, while every quotient group of H is either trivial or non-separable.*

Proof. Since every quotient group of H is also a quotient of G, the second part of the conclusion is immediate from Theorem 3.5. Therefore, according to Lemma 3.4, it suffices to verify that every countable subgroup of H is h-embedded.

If H is trivial, there is nothing to prove. Assume therefore that $|H| > 1$. Denote by p an open continuous homomorphism of G onto H and let K be the kernel of p. Clearly the group H is precompact and, hence, there is a non-trivial continuous character χ on H. Since the kernel of p is contained in the kernel of the continuous character $\chi \circ p$ of G and each non-trivial character of G has countable kernel, we conclude that the kernel of p is countable.

Let C be a countable subgroup of H and f a homomorphism of C to $\mathbb{T}$. Then $D = p^{-1}(C)$ is a countable subgroup of G and $f_D = f \circ p \restriction D$ is a homomorphism of D to $\mathbb{T}$. Since D is h-embedded in G, f_D admits an extension to a continuous character f^* on G. It follows from $D = p^{-1}(C)$ and the definition of f_D and f^* that $\ker p \subset \ker f_D \subset \ker f^*$. Therefore, by [4, Corollary 1.5.11], there exists a continuous character g of H satisfying $f^* = g \circ p$. Then

$$g \circ p \restriction D = f^* \restriction D = f_D = f \circ p \restriction D,$$

whence it follows that g extends f. Hence the group $C = p(D)$ is h-embedded in H. ∎

In the following lemma we establish that one of the properties of the group G constructed in Theorem 3.5 is finitely productive.

Lemma 3.9: *Let $H_1, \ldots, H_n$ be abelian topological groups such that for each $i \le n$, all countable subgroups of H_i are h-embedded. Then all countable subgroups of the group $H = H_1 \times \cdots \times H_n$ are also h-embedded.*

Proof. Let C be a countable subgroup of H and $f\colon C \to \mathbb{T}$ a homomorphism. For every $i \le n$, let $C_i = p_i(C)$, where $p_i\colon H \to H_i$ is the projection. Then C is a subgroup of the countable group $D = C_1 \times \cdots \times C_n$. Since the group $\mathbb{T}$ is divisible, f extends to a homomorphism $f_D \colon D \to \mathbb{T}$. Taking the restrictions of f_D to the factors, we can find homomorphisms $f_i \colon C_i \to \mathbb{T}$, where $i = 1, \ldots, n$ such that

$$f_D(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$$

for each $(x_1, \ldots, x_n) \in C_1 \times \cdots \times C_n$. For every $i \le n$, there exists a continuous character g_i on H_i extending f_i. We define a continuous character g on H by

$$g(y_1, \ldots, y_n) = g_1(y_1) \cdots g_n(y_n),$$

for each $(y_1, \ldots, y_n) \in H_1 \times \cdots \times H_n$. Then g extends both f_D and f. So the group C is h-embedded in H. ∎

Let $p \colon G \to H$ be a continuous surjective homomorphism of topological groups and τ_H be the topology of H. Denote by τ_H^p the finest topology on H such that the mapping p of G to $G^* = (H, \tau_H^p)$ is continuous. It is clear that G^* is a topological group, $\tau_H \subset \tau_H^p$, and the homomorphism $p \colon G \to G^*$ is open. We will say that τ_H^p is the **p-quotient** topology on H. It follows that the groups G^* and G/K are isomorphic as topological groups, where K is the kernel of p.

The proof of the following lemma is elementary, so we omit it.

Lemma 3.10: *Let $p \colon G \to H$, $q \colon H \to K$ and $r \colon G \to K$ be continuous surjective homomorphisms of topological groups satisfying $r = q \circ p$. Then the r-quotient topology on K is finer than the q-quotient topology of K. If the homomorphism p is open, then two topologies on K coincide.*

The next lemma extends the conclusion of Lemma 3.8 to finite powers of the group G.

Theorem 3.11: *Let $G \subset \mathbb{T}^{\mathfrak{c}}$ be the group constructed in Theorem 3.5 and $k \ge 1$ be an integer. Then every quotient group of G^k is either trivial or non-separable.*

Proof. Let $\varphi \colon G^k \to H$ be an open continuous homomorphism onto a topological group H with $|H| > 1$. It is clear that the group H is precompact and abelian. Hence there exists a non-trivial continuous character χ on H. Then $\chi^* = \chi \circ \varphi$ is a non-trivial continuous character on the dense subgroup G^k of $(\mathbb{T}^{\mathfrak{c}})^k$. Let the group $H^* = \chi^*(G^k)$ carry the χ^*-quotient topology. By Lemma 3.10, H^* is a continuous homomorphic image of H, so it suffices to prove that H^* is not separable.

Denote by ψ an extension of χ^* to a continuous character of $(\mathbb{T}^{\mathfrak{c}})^k$. Then there exist continuous characters $\psi_1, \dots, \psi_k$ on $\mathbb{T}^{\mathfrak{c}}$ such that

$$\psi(x_1, \dots, x_k) = \prod_{i=1}^{k} \psi_i(x_i),$$

for all $x_1, \dots, x_k \in \mathbb{T}^{\mathfrak{c}}$.

It follows from Lemma 3.1 that for every $i \in \{1, \dots, k\}$, one can find ordinals $\alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,n_i}$ in $\mathfrak{c}$ and integers $m_{i,1}, m_{i,2}, \dots, m_{i,n_i}$ such that

$$\psi_i(x) = \prod_{j=1}^{n_i} x(\alpha_{i,j})^{m_{i,j}}, \quad \text{for each } x \in \mathbb{T}^{\mathfrak{c}}.$$

Some of the integers $m_{i,1}, m_{i,2}, \dots, m_{i,n_i}$ can be zeros. Therefore we can assume that the sets $A_i = \{\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n_i}\}$ coincide for all $i \le k$. Let $A_1 = A_2 = \cdots = A_k = A$. In particular, $n_i = n_{i'} = n$ for all $i, i' \le k$ and, for each $j \le n$, there exists an ordinal $\alpha_j \in \mathfrak{c}$ such that $\alpha_{1,j} = \alpha_{2,j} = \cdots = \alpha_{k,j} = \alpha_j$; hence $A = \{\alpha_1, \dots, \alpha_n\}$. In other words, the columns of the rectangular $k \times n$ matrix $(\alpha_{i,j})$ are constant. Hence, combining the expressions for the characters ψ and ψ_i, $1 \le i \le k$, and changing the order of multiplication, we get the following equality:

$$\psi(x_1, \dots, x_k) = \prod_{j=1}^{n} \prod_{i=1}^{k} x_i(\alpha_j)^{m_{i,j}}. \tag{4}$$

Furthermore, we can additionally assume that the set $A \subset \mathfrak{c}$ is minimal by inclusion among those that admit a representation of ψ in the form (4). Therefore

$$\sum_{i=1}^{k} |m_{i,j}| > 0,$$

for each $j \in \{1, \dots, n\}$.

For every $j \le n$, denote by λ_j the continuous character on $(\mathbb{T}^{\mathfrak{c}})^k$ defined by

$$\lambda_j(x_1, \dots, x_k) = \prod_{i=1}^{k} x_i(\alpha_j)^{m_{i,j}}, \tag{5}$$

where $x_1, \dots, x_k \in \mathbb{T}^{\mathfrak{c}}$. It follows from (4) and (5) that $\psi = \lambda_1 \cdots \lambda_n$. For every $j \le n$, the character λ_j can be represented in the form $\lambda_j^* \circ (p_{\alpha_j})^k$, where p_{α_j} is the projection of $\mathbb{T}^{\mathfrak{c}}$ to the factor $\mathbb{T}_{(\alpha_j)}$ and λ_j^* is the continuous character on $\mathbb{T}^k_{(\alpha_j)}$ defined by

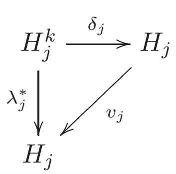
$$\lambda_j^*(t_1, \dots, t_k) = t_1^{m_{1,j}} \cdots t_k^{m_{k,j}}.$$

We supply the subgroup $H_j = p_{\alpha_j}(G)$ of $\mathbb{T}$ with the q_j-quotient topology, where $q_j = p_{\alpha_j} \restriction G$ and $1 \le j \le n$. Since the kernel of q_j is countable, the group H_j is uncountable. By Lemma 3.8, every countable subgroup of H_j is h-embedded. Let M_j be the maximal common divisor of the integers $m_{1,j}, \ldots, m_{k,j}$ (we recall that at least one of these integers is distinct from zero). For each $i \le k$, we put $n_{i,j} = m_{i,j}/M_j$ and consider the character δ_j on $\mathbb{T}^k$ defined by

$$\delta_j(t_1, \ldots, t_k) = t_1^{n_{1,j}} \cdots t_k^{n_{k,j}}.$$

It is clear from the definition that $\lambda_j^* = \delta_j^{M_j}$. Since the maximal common divisor of $n_{1,j}, \ldots, n_{k,j}$ is equal to 1, it follows from Lemma 3.7 that the homomorphism $\delta_j \colon H_j^k \to H_j$ is open and surjective. Notice that the group H_j is algebraically a subgroup of $\mathbb{T}$.

Let $v_j \colon \mathbb{T} \to \mathbb{T}$ be the homomorphism defined by $v_j(t) = t^{M_j}$, for each $t \in \mathbb{T}$. It follows from $\lambda_j^* = \delta_j^{M_j}$ that $\lambda_j^* = v_j \circ \delta_j$. Clearly the kernel of v_j is finite and the restriction $v_j \restriction H_j$ is continuous when considered as a homomorphism of H_j to itself. Hence the group $\lambda_j^*(H_j^k) = v_j(H_j)$ is uncountable.



Strictly speaking, one should replace δ_j, λ_j^* and v_j in the above diagram by $\delta_j \restriction H_j^k$, $\lambda_j^* \restriction H_j^k$ and $v_j \restriction H_j$, respectively.

We define a homomorphism

$$\lambda^* \colon (\mathbb{T}_{(\alpha_1)})^k \times \cdots \times (\mathbb{T}_{(\alpha_n)})^k \to \mathbb{T}_{(\alpha_1)} \times \cdots \times \mathbb{T}_{(\alpha_n)}$$

by

$$\lambda^*(y_1, \ldots, y_n) = (\lambda_1^*(y_1), \ldots, \lambda_n^*(y_n)).$$

In other words, $\lambda^* = \lambda_1^* \times \cdots \times \lambda_n^*$. Since each group H_j is algebraically identified with the subgroup $p_{\alpha_j}(G)$ of $\mathbb{T}$ we have that $\lambda^*(H_1^k \times \cdots \times H_n^k) \subset H_1 \times \cdots \times H_n$. Further, the multiplication mapping P of $\mathbb{T}^n$ to $\mathbb{T}$ defined by

$$P(t_1, \ldots, t_n) = t_1 \cdots t_n$$

is a continuous character on $\mathbb{T}^n$. It follows from the equality

$$\psi = \lambda_1 \cdots \lambda_n = P \circ (\lambda_1^* \times \cdots \times \lambda_n^*) \circ (p_{\alpha_1}^k \times \cdots \times p_{\alpha_n}^k)$$

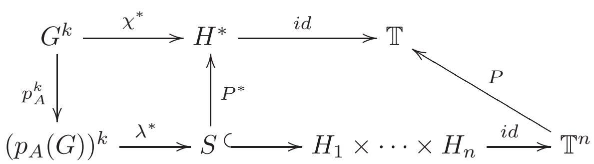
and our definition of λ^* that

$$\chi^* = \psi \restriction G^k = P \circ \lambda^* \circ (p_A)^k \restriction G^k,$$

where p_A is the projection of $\mathbb{T}^{\mathfrak{c}}$ to $\mathbb{T}^A$. Let P^* be the restriction of P to the (abstract) subgroup

$$S = \lambda^*(p_A(G)^k)$$

of $\mathbb{T}^n$. We consider S with the topology inherited from $H_1 \times \cdots \times H_n$.



The natural isomorphic embeddings of the abstract groups H^* to $\mathbb{T}$ and $H_1 \times \cdots \times H_n$ to $\mathbb{T}^n$ in the above diagram are denoted by the same symbol *id*.

Our next step is to verify the following:

Claim 3: *The kernel of the homomorphism $P\colon H_1 \times \cdots \times H_n \to \mathbb{T}$ is countable, where each group H_j is identified algebraically with the corresponding subgroup $p_{\alpha_j}(G)$ of $\mathbb{T}$.*

Indeed, take an arbitrary element $(y_1, \ldots, y_n) \in H_1 \times \cdots \times H_n$ and assume that $y_1 \cdots y_n = 1$. It follows from condition (iv) in the proof of Theorem 3.5 that for every $j \le n$, the element $y_j \in H_j = p_{\alpha_j}(G) = f_{\alpha_j}(A(X))$ can be written in the form $y_j = t_j \cdot z_j$, where $t_j \in \langle f_{\alpha_j}(Y_{\alpha_j})\rangle$ and $z_j \in \langle E_{\alpha_j}\rangle$. Therefore we have the equality $z_1 \cdots z_n = t_1^{-1} \cdots t_n^{-1} \in \langle Y\rangle$, where

$$Y = \bigcup_{j=1}^{n} f_{\alpha_j}(Y_{\alpha_j}).$$

Since each set Y_{α_j} is countable, the group $\langle Y\rangle$ is countable as well. Further, since the set $E \subset \mathbb{T}$ is independent and the subsets $E_{\alpha_1}, \ldots, E_{\alpha_n}$ of E are pairwise disjoint, the products $z_1 \cdots z_n$ and $z_1' \cdots z_n'$ are distinct provided $(z_1, \ldots, z_n)$ and $(z_1', \ldots, z_n')$ are distinct elements of $\langle E_1\rangle \times \cdots \times \langle E_n\rangle$. Therefore it follows from $|\langle Y\rangle| \le \omega$ that there exist at most countably many n-tuples $(z_1, \ldots, z_n) \in \langle E_1\rangle \times \cdots \times \langle E_n\rangle$ with $z_1 \cdots z_n \in \langle Y\rangle$. Finally, since the group $\langle Y\rangle$ is countable, we conclude that the kernel of P is countable. This proves Claim 3.

To finish the proof of the theorem we argue as follows. We know that for every $j \leq n$, all countable subgroups of H_j are h-embedded. By Lemma 3.9, the countable subgroups of $H_1 \times \cdots \times H_n$ are also h-embedded. Hence every countable subgroup of $H_1 \times \cdots \times H_n$ is closed (see Lemma 3.4), and the same holds for its subgroup S. For every $j \leq n$, let π_j be the projection of $\mathbb{T}_{(\alpha_1)} \times \cdots \times \mathbb{T}_{(\alpha_n)}$ to the factor $\mathbb{T}_{(\alpha_j)}$. It follows from the definition of λ^* that $\pi_j \circ \lambda^* = \lambda_j^* \circ \pi_j^k$. Therefore

$$\pi_j(S) = \pi_j(\lambda^*(p_A(G)^k)) = \lambda_j^*(H_j^k) = v_j(H_j)$$

and the latter group is uncountable. Hence $|S| > \omega$. By Claim 3, the kernel of the homomorphism P restricted to $H_1 \times \cdots \times H_n$ is countable. Therefore the group $H^* = P^*(S)$ is uncountable as well.

By Lemma 3.10, the χ^*-quotient topology of H^*, i.e., the original topology of H^*, is finer than the P^*-quotient topology of H^* denoted by τ^*. Since all countable subgroups of S are closed, it follows that the group (H^*, τ^*) has the same property. Hence all countable subgroups of H^* are closed as well. Since the group H^* is uncountable it cannot be separable. ∎

We have established in the proof of Theorem 3.11 that all countable subgroups of the group $H^* = \chi^*(G^k)$ are closed provided that H^* carries the χ^*-quotient topology. Since χ^* is a continuous character on G^k, this prompts the following question:

Problem 3.12: Let G be the group constructed in Theorem 3.5 and H a quotient group of G^k, for some integer $k \geq 1$. Are all countable subgroups of H h-embedded or closed?

Finally we extend Theorem 3.11 to arbitrary powers of the group G.

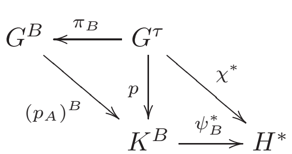
THEOREM 3.13: *Let $G \subset \mathbb{T}^{\mathfrak{c}}$ be the group constructed in Theorem 3.5 and $\tau \geq 1$ be a cardinal. Then every quotient group of G^τ is either trivial or non-separable.*

Proof. We start as in the proof of Theorem 3.11. Let $\varphi \colon G^\tau \to H$ be an open continuous homomorphism onto a non-trivial Hausdorff topological group H. Then the group H is precompact and abelian. Hence there exists a continuous character χ on H distinct from the trivial one, so $\chi^* = \chi \circ \varphi$ is a non-trivial continuous character on G^τ. We consider the group $H^* = \chi^*(G^\tau)$ endowed with the χ^*-quotient topology. Since H^* is a continuous homomorphic image of H, it suffices to verify that H^* is not separable.

The group G^τ is a dense subgroup of the compact group $(\mathbb{T}^{\mathfrak{c}})^\tau \cong \mathbb{T}^{\mathfrak{c}\times\tau}$. Let ψ be a continuous character on $\mathbb{T}^{\mathfrak{c}\times\tau}$ extending χ^*. By Lemma 3.1, the character ψ depends on at most finitely many coordinates. In other words, we can find finite non-empty sets $A \subset \mathfrak{c}$ and $B \subset \tau$ and a continuous character ψ^* on the group $\mathbb{T}^{A\times B} \cong (\mathbb{T}^A)^B$ such that $\psi = \psi^* \circ p_{A,B}$, where $p_{A,B}$ is the projection of $\mathbb{T}^{\mathfrak{c}\times\tau}$ onto $\mathbb{T}^{A\times B}$. Let the subgroup $K = p_A(G)$ of $\mathbb{T}^A$ carry the quotient topology with respect to the homomorphism $p_A \restriction G$, where $p_A \colon \mathbb{T}^{\mathfrak{c}} \to \mathbb{T}^A$ is the projection. Let also ψ_B^* be the restriction of ψ^* to $K^B = p_{A,B}(G^\tau)$. Then

$$\chi^* = \psi_B^* \circ p_{A,B} \restriction G^\tau.$$

Further, we can represent the restriction $p = p_{A,B} \restriction G^\tau$ as the composition of the projection $\pi_B \colon G^\tau \to G^B$ and the homomorphism $(p_A)^B \colon G^B \to K^B$, which makes the following diagram commute:



Since the mapping p is open, all homomorphisms in the diagram are continuous. Clearly $f = \psi_B^* \circ (p_A)^B$ is a continuous homomorphism of G^B onto H^*. Since $\chi^* = f \circ \pi_B$, it follows from Lemma 3.10 that the χ^*-quotient topology on H^* is finer than the f-quotient topology on H^*, say, τ_f. Further, as the set B is finite, we can apply Theorem 3.11 to the continuous homomorphism $f \colon G \to H^*$ and deduce that the group (H^*, τ_f) is not separable. Therefore H^* with the χ^*-quotient topology is not separable either. This completes the proof. ∎

Problem 3.14: Does there exist a precompact abelian group G as in Theorem 3.5 which has one of the following additional properties:

(a) G is connected;
(b) G is Baire;
(c) G is reflexive?

Remark 3.15: We observe that Theorem 1.5 for Banach spaces suggests questions for topological groups. Let us define a topological group to be a $\boldsymbol{G_\sigma}$**-group** if it has a dense subgroup H which is the union of a strictly increasing sequence of closed subgroups. We might ask: Does every non-discrete G_σ-group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable? We answer each of these questions in the negative in Theorem 3.16.

THEOREM 3.16: *For every cardinal $\tau \geq \mathfrak{c}$, there exists a precompact topological abelian group H with the following properties:*

(a) $w(H) = \tau$;
(b) $H = \bigcup_{n\in\omega} H_n$, *where* $H_0 \subset H_1 \subset H_2 \subset \cdots$ *are proper closed subgroups of* H;
(c) *every quotient group of H is either trivial or non-separable.*

Proof. Let τ be a cardinal with $\tau \geq \mathfrak{c}$ and G the precompact abelian group as in Theorem 3.5. Then G is a dense subgroup of $\mathbb{T}^\mathfrak{c}$, so $w(G) = \mathfrak{c}$. Clearly the group $K = G^\tau$ is precompact, abelian, has weight τ, and every non-trivial quotient of K is not separable (see Theorem 3.13).

We denote by Π the group K^ω endowed with the Tychonoff product topology. For every $n \in \omega$, let p_n be the projection of Π to the nth factor $K_{(n)}$. Given an element $x \in \Pi$, we denote by $\mathrm{supp}(x)$ the set $\{n \in \omega : p_n(x) \neq e\}$, where e is the identity element of K. We also put

$$H = \{x \in \Pi : |\,\mathrm{supp}(x)| < \omega\}.$$

It is clear that H is a dense subgroup of Π and

$$w(H) = w(\Pi) = w(K) = \tau.$$

For every $n \in \omega$, denote by H_n the subgroup of H which consists of all $x \in \Pi$ with $\mathrm{supp}(x) \subset \{0, 1, \ldots, n\}$. It is easy to see that $H_n \cong K^{n+1}$ is closed in H, $H_n \subset H_{n+1}$ for each $n \in \omega$, and $H = \bigcup_{n\in\omega} H_n$.

Let us show that every non-trivial quotient of H is not separable. Consider an open continuous homomorphism $f : H \to L$ onto a non-trivial topological group L. Let N be the kernel of f and $\overline{N}$ the closure of N in Π. Denote by π the quotient homomorphism of Π onto $\Pi/\overline{N}$. Since N is dense in $\overline{N}$, the dense subgroup $\pi(H)$ of $\Pi/\overline{N}$ is isomorphic as a topological group to $L \cong H/N$ (see [4, Theorem 1.5.16]). The group Π is isomorphic as a topological group to $(G^\tau)^\omega \cong G^\tau$, so Theorem 3.13 implies that the non-trivial group $\Pi/\overline{N}$ is not separable. Thus neither is the group $L \cong H/N$ which is isomorphic as a topological group to the dense subgroup $\pi(H)$ of $\Pi/\overline{N}$. ∎

4. σ-compact groups, Lindelöf Σ-groups and pseudocompact groups

Here we give a positive answer to (i) and (ii) of Problem 2.9 for the wider class of Lindelöf Σ-groups and then answer (iii) and (iv) of Problem 2.9 in the negative. Recall that the class of **Lindelöf Σ-groups** contains all σ-compact and all separable metrizable topological groups, and is closed with respect to countable products, closed subgroups and continuous homomorphic images (see [4, Section 5.3]).

Proposition 4.2 answers (i) and (ii) of Problem 2.9 in a stronger form. First we present an important fact about Lindelöf Σ-groups (see [4, Lemma 5.3.24]).

Lemma 4.1: *Let G be a Lindelöf Σ-group. Then*

(a) *for every closed normal subgroup N of type G_δ in G, the quotient group G/N has a countable network;*

(b) *the family*

$$\{\pi^{-1}(V) : \pi \text{ is a continuous homomorphism of } G \text{ to a topological group } K \text{ with a countable network, } V \text{ is open in } K\}$$

constitutes a base for G.

Proposition 4.2 (Separable quotient theorem for Lindelöf Σ-groups): *Let G be an infinite Lindelöf Σ-group. Then G has a quotient group which is infinite and separable. Indeed, the topology of G is initial with respect to the family of quotient homomorphisms onto infinite groups with a countable network.*

Proof. It follows from (b) of Lemma 4.1 that the topology of G is initial with respect to the family of quotient homomorphisms onto topological groups with a countable network. Note that the identity of a topological group with a countable network is a G_δ-set (in fact, the singletons in every Hausdorff space with a countable network are G_δ-sets). Hence (b) of Lemma 4.1 implies that every neighborhood of the identity in G contains a closed normal subgroup of type G_δ in G. Therefore, as G is infinite, there exists a closed normal subgroup N_0 of G of type G_δ such that G/N_0 is infinite.

Finally, let $\mathcal{N}$ be the family of closed normal subgroups N of type G_δ in G with $N \subset N_0$. Then the family of quotient homomorphisms $\pi_N : G \to G/N$, with $N \in \mathcal{N}$, generates the topology of G and each group G/N is infinite. According to (a) of Lemma 4.1, the group G/N has a countable network for each $N \in \mathcal{N}$. ∎

Since every σ-compact topological group is evidently a Lindelöf Σ-group, the next result is immediate from Proposition 4.2.

Corollary 4.3 (Separable quotient theorem for σ-compact groups): *Let G be an infinite σ-compact topological group. Then G has a quotient group which is infinite and separable. Indeed, the topology of G is initial with respect to the family of quotient homomorphisms of the group onto infinite groups with a countable network.*

In Proposition 4.4 we give a negative answer to (iii) and (iv) of Problem 2.9 by producing a countably infinite (hence, totally disconnected) σ-compact group which has no non-trivial separable metrizable quotient group.

Proposition 4.4: *There exists a countably infinite precompact abelian group H such that every quotient group of H is either trivial or non-metrizable.*

Proof. For a given prime number p, denote by $\mathbb{C}_p$ the quasicyclic p-group

$$\{z \in \mathbb{T} : z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

considered as a subgroup of the group $\mathbb{T}$. Clearly $\mathbb{C}_p$ is a countable infinite abelian group. Let τ be the Bohr topology of $\mathbb{C}_p$, i.e., the maximal precompact topological group topology of $\mathbb{C}_p$ [4, Section 9.9]. We claim that the group

$$H = (\mathbb{C}_p, \tau)$$

is as required. Indeed, let N be a proper subgroup of H. Then there exists an integer $n \geq 0$ such that

$$N = \{z \in \mathbb{C}_p : z^{p^n} = 1\}$$

(see [36, Chapter 4]), so N is finite and, hence, closed in H. Therefore the quotient group H/N is infinite and, in fact, algebraically isomorphic to $\mathbb{C}_p$. It follows from [4, Proposition 9.9.9 c)] that the quotient topology of H/N and the Bohr topology of the abstract group H/N coincide. We conclude, therefore, that the groups H and H/N are isomorphic as topological groups. The group H/N being infinite and precompact is not discrete. Further, according to [4, Theorem 9.9.30], all compact subsets of the groups H and H/N are finite, so the space H/N does not contain non-trivial convergent sequences. This implies that H/N is not metrizable. ∎

We now answer Problem 2.11 in the affirmative. In fact, the argument in Theorem 4.5 provides an alternative proof of Theorem 2.7 for compact groups and complements it implying that the topology of an infinite compact group is initial with respect to the family of continuous open homomorphisms onto infinite compact metrizable groups.

Theorem 4.5 (Separable quotient theorem for pseudocompact groups): *The topology of every infinite pseudocompact topological group, G, is initial with respect to the family of quotient homomorphisms onto infinite compact metrizable groups. In particular, G has a quotient group which is infinite separable compact and metrizable.*

Proof. The completion, ϱG, of the infinite pseudocompact topological group G is a compact group that contains G as a dense topological subgroup. Hence ϱG is a Lindelöf Σ-group and we can apply Proposition 4.2 to conclude that the topology of ϱG is initial with respect to the family of quotient homomorphisms onto infinite topological groups with a countable network or, equivalently, countable base (every compact Hausdorff space with a countable network has a countable base [3]).

Let $p\colon \varrho G \to K$ be an open continuous homomorphism of ϱG onto an infinite topological group K with a countable base. In particular K is metrizable. Denote by π the restriction of p to G. Since the group G is pseudocompact it meets every non-empty G_δ-set in ϱG [10]. The points in K are G_δ-sets, and so are the fibers $p^{-1}(y)$ in ϱG, for all $y \in K$. Therefore

$$G \cap p^{-1}(y) \neq \emptyset$$

for each $y \in K$, which implies the equality

$$p(G) = K.$$

It is also clear that $G \cap \ker p$ is dense in $\ker p$, for $\ker p$ is a G_δ-set in ϱG. Hence the restriction of p to G is an open continuous homomorphism of G onto K, by [4, Theorem 1.5.16]. Since the group K is infinite, compact and metrizable, this completes the proof. ∎

Finally, we mention that Theorem 4.5 cannot be extended to precompact topological groups, even in the weak form of the existence of non-trivial separable quotients, as Theorem 3.5 shows.

5. Conclusion

Throughout this paper many problems for Banach spaces, locally convex spaces and topological groups have been listed. We have addressed the questions for topological groups some of which are answered here completely, and the others partially. For clarity we state the status of each problem in a table and for easy reference purposes we include a table listing properties. All topological groups in the left column of the first table are assumed to be infinite.

Groups	**Non-trivial Separable Quotient**	**Infinite Separable Quotient**	**Metrizable Separable Quotient**	**Infinite Metrizable Separable Quotient**
non-totally disconnected	?	?	?	?
non-totally disconnected locally compact	?	?	?	?
reflexive	?	?	?	?
locally compact abelian	Yes	Yes	Yes	Yes
compact	Yes	Yes	Yes	Yes
non-discrete σ-compact	Yes	Yes	No	No
pseudocompact	Yes	Yes	Yes	Yes
non-totally disconnected proto-Lie	Yes	Yes	Yes	Yes
σ-compact pro-Lie	Yes	Yes	Yes	Yes
abelian pro-Lie	Yes	Yes	Yes	Yes
σ-compact locally compact	Yes	Yes	Yes	Yes
almost connected locally compact	Yes	Yes	Yes	Yes
precompact	No	No	No	No
$\mathbb{R}$-factorizable	No	No	No	No

Problem	**Answer**	**Theorems**
1.1	Partial	1.4, 1.5, 1.8, 1.9, 1.10, 1.11
1.2	Partial	1.3, 1.4, 1.5, 1.8, 1.9, 1.10, 1.11
1.6	Partial	1.5, 1.8, 1.9, 1.10, 1.11
1.7	Partial	1.5, 1.8, 1.9, 1.10, 1.11
1.12	No	1.13, 1.15, 1.16, 1.17, 1.18
1.21	Partial	1.28, 2.14, 2.15, 2.16, 2.17, 4.2, 4.5
1.22	Partial	1.28, 2.14, 2.16, 2.15, 2.17, 4.2, 4.5,
1.23	Partial	1.28, 2.14, 2.15, 2.16, 2.17, 4.5
1.24	Partial	1.28, 2.16, 2.15, 2.17, 4.5
1.25	Partial	1.5, 1.8, 1.9, 1.10, 1.11, 2.13
1.26	Yes	2.13
1.27	Partial	1.5, 1.8, 1.9, 1.10, 1.11
1.29	Partial	2.7, 2.13, 2.17, 2.19, 4.3
1.30	Yes	2.7, 4.3, 4.5
2.9(i)&(ii)	Yes	4.3
2.9(iii)&(iv)	No	4.4
2.10	No	3.5
2.11	Yes	4.5
2.20	No	3.6
3.12	?	—
3.14	?	—

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