PROFINITE RIGIDITY OF GRAPH MANIFOLDS, II: KNOTS AND MAPPING CLASSES

BY

Gareth Wilkes

Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
e-mail: G.Wilkes@dpmms.cam.ac.uk

ABSTRACT

In this paper we study some consequences of the author's classification of graph manifolds by their profinite fundamental groups. In particular we study commensurability, the behaviour of knots, and relation to mapping classes. We prove that the exteriors of graph knots are distinguished among all 3-manifold groups by their profinite fundamental groups. We also prove a strong conjugacy separability result for certain mapping classes of surfaces.

Introduction

There has been a recent slew of papers dealing with the properties of 3-manifold groups which may be detected via the finite quotients of the group. Examples of major advances in this field include the detection of fibring for compact 3-manifolds by Jaikin-Zapirain [9]; the detection of the geometry of a closed 3-manifold by Wilton and Zalesskii [16]; the proof that each once-punctured torus bundle may be distinguished from all other 3-manifolds by the set of finite quotients of its fundamental group [3]; and the classification of those Seifert fibre spaces whose fundamental groups have the same finite quotients by the author [14].

Instead of talking of sets of finite quotients of a group G, one generally packs this information into the **profinite completion** $\widehat{G}$ of G, defined as the inverse limit of the system of finite groups

$$\{G/N \mid N \lhd G, G/N \text{ finite}\},$$

which is naturally a compact Hausdorff topological group. See [11, Section 3.2] for basic facts about profinite completions. We say that a 3-manifold M is **profinitely rigid** if, for any 3-manifold N, a (topological) isomorphism $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ implies that N is homeomorphic to M. We may abuse this terminology by saying instead that $\pi_1 M$ is profinitely rigid—that is, a 3-manifold group $\pi_1 M$ is 'profinitely rigid' if, whenever any other 3-manifold group $\pi_1 N$ has the same profinite completion as $\pi_1 M$, the manifolds M and N must be homeomorphic. Note that other authors use the term 'profinitely rigid' in a broader sense to say that the profinite completion of a finitely generated residually finite group distinguishes it from all other finitely generated residually finite groups. In this paper however we only speak of profinite rigidity in the more restricted sense of rigidity within the class of 3-manifold groups.

In the paper [15] the author proved a result classifying graph manifolds by the profinite completions of their fundamental groups. This classification (see Theorem 1.6 of the present paper) takes the form of a finite list of numerical conditions which are easy to check for a given pair of graph manifolds when presented in a certain standard form. This present paper represents a continuation of [15]. A certain familiarity with at least Section 10 of [15] will be required. Initially we seek to shed light on the classification theorem by means of examples. In Section 3 we prove the following corollary (Proposition 3.1).

THEOREM A: *Every closed orientable graph manifold has a finite-sheeted cover which is profinitely rigid. Hence if two graph manifold groups have isomorphic profinite completions, then they are commensurable.*

Following this we will use the techniques from [15] to investigate two other entities of great interest to low dimensional topologists—knots and mapping classes. In Section 4 we will study those knot exteriors in $\mathbb{S}^3$ which are graph manifolds and prove that they are all determined by the profinite completions of their fundamental groups. Strikingly this result does not assume any condition on the behaviour with respect to the peripheral structures of the groups of the isomorphisms of profinite completions. The following appears as Theorem 4.1.

THEOREM B: *Let M_K be the exterior of a graph knot K. Let N be another compact orientable 3-manifold and assume that $\widehat{\pi_1 M_K} \cong \widehat{\pi_1 N}$. Then*

$$\pi_1 M_K \cong \pi_1 N.$$

In particular, if K is prime and N is also a knot exterior then N is homeomorphic to M_K.

Finally in Section 5 we use the behaviour of profinite completions of fibred graph manifolds to deduce the following result (Theorem 5.5) concerning mapping classes. A 'piecewise periodic but not periodic' mapping class is one for which the corresponding mapping torus is a graph manifold (but not a Seifert fibre space).

THEOREM C: *If ϕ_1 and ϕ_2 are piecewise periodic but not periodic automorphisms of a closed surface group $\pi_1 S$ which are not conjugate in $\mathrm{Out}(\pi_1 S)$, then ϕ_1 is not conjugate to ϕ_2^κ in $\mathrm{Out}(\widehat{\pi_1 S})$ for any $\kappa \in \widehat{\mathbb{Z}}^\times$.*

Conventions: In this paper, we will adopt the following conventions.
- Generally speaking, discrete groups will generally be given Roman letters $A, G, H, \ldots$.
- A finite graph of spaces will be denoted $(X, M_\bullet)$ where X is a finite graph and $M_\bullet$ will be an edge or vertex space. Similarly for graphs of groups.
- For us, a graph manifold will be required to be non-geometric, i.e., not a single Seifert fibre space or a Sol-manifold, hence not a torus bundle. All 3-manifolds will be orientable.
- For two elements g, h of a group, g^h will denote $h^{-1}gh$. That is, conjugation will be a right action.
- We will sometimes shorten the phrase 'profinite completion of the fundamental group' to 'profinite fundamental group'. In such cases we may use the term 'discrete fundamental group' to mean the fundamental group itself.

1. Results from previous papers

In this section we shall recall those results and notions from [15] and other papers which will be required for our discussions. We begin with some information and conventions concerning 2-orbifolds and Seifert fibre spaces.

Definition 1.1 (Definition 2.6 of [15]): Let O be an orientable 2-orbifold of genus g with $s + 1$ boundary components and r cone points, with fundamental group

$$B = \pi_1^{\mathrm{orb}} O = \langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g \mid a_i^{p_i} \rangle$$

where the boundary components of O are represented by the conjugacy classes of the elements $e_1, \ldots, e_s$ together with

$$e_0 = (a_1 \cdots a_r e_1 \cdots e_s [u_1, v_1] \cdots [u_g, v_g])^{-1}.$$

Let $\mu \in \widehat{\mathbb{Z}}^\times$. An **exotic automorphism of O of type** μ is an automorphism $\psi \colon \widehat{B} \to \widehat{B}$ such that

$$\psi(a_i) \sim a_i^\mu \quad \text{and} \quad \psi(e_i) \sim e_i^\mu$$

for all i, where $\sim$ denotes conjugacy in $\widehat{B}$.

There is a corresponding notion for non-orientable orbifolds, but this will not be relevant to this paper.

PROPOSITION 1.2 (Proposition 10.5 of [15]): *Let O be an orientable 2-orbifold with boundary. Then O admits an exotic automorphism of type μ for any $\mu \in \widehat{\mathbb{Z}}$. Moreover, this automorphism may be induced by an automorphism of the orbifold $\mathring{O}$ obtained from O by removing a small disc about each cone point.*

Remark: In this proposition the word 'induced' is supposed to have the following meaning. By excising a small disc around each cone point one obtains a surface with boundary whose fundamental group has a free basis consisting of the a_i, e_i, u_i and v_i. There is an exotic automorphism of this orbifold (i.e., an automorphism of the free profinite group on these letters) sending each a_i to a conjugate of a_i^μ and so on. On passing to the quotient by the relations $a_i^{p_i} = 1$ this gives an exotic automorphism of the original orbifold. It is occasionally necessary in applications to know that the automorphism is specified in this way (that is, it lifts to a certain automorphism of a free profinite group).

THEOREM 1.3 (Theorem 2.7 of [15]; adaption of Theorems 5.8 and 5.9 of [14]): *Let M and N be Seifert fibre spaces whose boundary components are $\partial M_1, \ldots, \partial M_n$ and $\partial N_1, \ldots, \partial N_n$ respectively. Suppose Φ is an isomorphism of group systems*

$$\Phi \colon (\widehat{\pi_1 M}; \widehat{\pi_1 \partial M_1}, \ldots, \widehat{\pi_1 \partial M_n}) \to (\widehat{\pi_1 N}; \widehat{\pi_1 \partial N_1}, \ldots, \widehat{\pi_1 \partial N_n})$$

(where we allow the replacement of peripheral subgroups by conjugates). Then:

- *If M is a twisted $\mathbb{I}$-bundle over the Klein bottle or a copy of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$, then $M = N$.*
- *Otherwise, the base orbifolds of M and N may be identified with the same orbifold O such that Φ splits as an isomorphism of short exact sequences*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & \widehat{\pi_1 M} & \longrightarrow & \widehat{\pi_1^{orb} O} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \cdot \lambda} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & \widehat{\pi_1 N} & \longrightarrow & \widehat{\pi_1^{orb} O} & \longrightarrow & 1
\end{array}
$$

where λ is some invertible element of $\widehat{\mathbb{Z}}$ and ϕ is an exotic automorphism of O of type μ.

   Hence if N is a surface bundle over the circle with fibre a hyperbolic surface Σ with periodic monodromy ψ, then M is also such a surface bundle with monodromy ψ^k where k is congruent to $\kappa = \mu^{-1}\lambda$ modulo the order of ψ.

In this theorem, as elsewhere, we follow the convention that the short exact sequence is the profinite completion of a short exact sequence for the discrete fundamental group, so that the generator $1 \in \widehat{\mathbb{Z}}$ is the homotopy class of a regular fibre rather than an arbitrary generator of $\widehat{\mathbb{Z}}$.

Definition 1.4: If M, N are as in the latter case of the above theorem, we say that (M, N) is a **Hempel pair of scale factor** κ, where $\kappa = \mu^{-1}\lambda$. Note that κ is only well-defined modulo the order of ψ, which may be taken to be the lowest common multiple of the orders of the cone points of M. Note that a Hempel pair of scale factor ± 1 is a pair of homeomorphic Seifert fibre spaces.

Finally we state the results from [15] concerning graph manifolds. The reader is warned that at certain points in this paper we will be needing details from the proofs of these theorems as well as just the statements.

THEOREM 1.5 (Theorem 6.2 of [15]): *Let M and N be closed graph mani-folds with respective JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$. Assume that there is an isomorphism $\Phi \colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$. Then the isomorphism Φ induces an isomorphism of JSJ decompositions in the following sense:*

- *there is a graph isomorphism $\varphi \colon X \to Y$;*
- *Φ restricts to an isomorphism from $\widehat{\pi_1 M_x}$ to a conjugate of $\widehat{\pi_1 N_{\varphi(x)}}$ for every $x \in V(X) \cup E(X)$.*

After performing an automorphism of $\widehat{\pi_1 N}$, one may eliminate the conjugacy in the last point.

We will henceforth restrict attention to those graph manifolds whose vertex spaces have orientable base orbifolds. Let us recall how one obtains numerical invariants of a graph manifold M. Let the JSJ decomposition be $(X, M_\bullet)$. In this section we shall adopt the convention (from Serre) that each 'geometric edge' of a finite graph is a pair $\{e, \bar{e}\}$ of oriented edges, with $\bar{e}$ being the 'reverse' of e. Fix presentations in the standard form [10, Section 5.3]

$$\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g, h \mid a_i^{p_i} h^{q_i}, h \text{ central} \rangle$$

for each M_x $(x \in V(X))$. This determines an ordered basis (h, e_i) for the fundamental group of each boundary torus of M_x, where the final boundary component is described by

$$e_0 = (a_1 \cdots a_r e_1 \cdots e_s [u_1, v_1] \cdots [u_g, v_g])^{-1}.$$

Then with these bases the gluing map along an edge e (from the boundary component of $M_{d_0(e)}$ to that of $M_{d_1(e)}$) takes the form of a matrix (acting on the left of a column vector)

$$\begin{pmatrix} \alpha(e) & \beta(e) \\ \gamma(e) & \delta(e) \end{pmatrix}$$

where $\gamma(e)$, the intersection number of the fibre of $d_0(e)$ with that of $d_1(e)$, is non-zero by the definition of a graph manifold. The number $\gamma(e)$ is well-defined up to a choice of orientation of the fibres of the two vertex groups. This matrix has determinant -1 from the requirement that the graph manifold be orientable. Once an orientation of the fibre and base are fixed, the number $\delta(e)$ becomes independent of the choice of presentation, modulo $\gamma(e)$. Changing these orientations multiplies the matrix by -1. The precise values of the $\delta(e)$

and q_i may be changed by arbitrary integer multiples of $\gamma(e)$ and p_i using 'Dehn twists' of the form

$$e_i \mapsto e_i h,\ e_j \mapsto e_j h^{-1} \quad \text{or} \quad e_i \mapsto e_i h^{\pm 1},\ a_j \mapsto a_j h^{\mp 1}.$$

These operations however leave the **total slope**

$$\tau(x) = \sum_{d_0(e)=x} \frac{\delta(e)}{\gamma(e)} - \sum \frac{q_i}{p_i}$$

of the vertex space invariant. Note also that these quantities are all invariant under the conjugation action of the group on itself, i.e., it does not matter which conjugate of each edge or vertex group we consider.

THEOREM 1.6 (Theorem 10.9 of [15]): *Let M and N be closed orientable graph manifolds with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively, in which all Seifert fibred pieces have orientable base orbifold. Suppose M and N are not homeomorphic.*

(1) *If X is not bipartite, then $\pi_1 M$ is profinitely rigid.*

(2) *If X is bipartite, on two sets R and B, then $\pi_1 M$ and $\pi_1 N$ have isomorphic profinite completions if and only if, for some choices of orientations on fibre subgroups, there is a graph isomorphism $\phi \colon X \to Y$ and some $\kappa \in \widehat{\mathbb{Z}}^\times$ such that:*

  (a) *For each edge e of X, $\gamma(\phi(e)) = \gamma(e)$.*

  (b) *The total slope of every vertex space of M or N vanishes.*

  (c1) *If $d_0(e) = r \in R$ then $\delta(\phi(e)) = \kappa \delta(e)$ modulo $\gamma(e)$, and $(M_r, N_{\phi(r)})$ is a Hempel pair of scale factor κ.*

  (c2) *If $d_0(e) = b \in B$ then $\delta(\phi(e)) = \kappa^{-1} \delta(e)$ modulo $\gamma(e)$, so that $(M_b, N_{\phi(b)})$ is a Hempel pair of scale factor κ^{-1}.*

Remark: In the present paper we shall only need to deal with manifolds whose Seifert fibred pieces have orientable base orbifold, so we have cleaned up this statement slightly from Theorem 10.9 of [15] which deals with the general case.

2. Examples

We now give some simple illustrative examples to demonstrate some of the phenomena that may occur in consequence of Theorem 1.6. For some of the examples we will also explicitly describe the isomorphisms of profinite completions.

Example 2.1 (Changing a vertex space): Fix a positive integer p and let $0 < q < p/2$ be coprime to p. Consider the following family of graph manifold groups, whose vertex groups for different values of q are not isomorphic. The manifolds themselves are schematically illustrated in Figure 1.

$$G_q = \langle a_1, a_2, u, v, h, u', v', h' \mid a_1^p h^q, a_2^p h^{-q}, e' = h, h' = e,$$
$$[h, a_1], [h, a_2], [h, u], [h, v], [h', u'], [h', v']\rangle$$
$$= \langle a_1, a_2, u, v, h\rangle *_{\mathbb{Z}^2} \langle u', v', h'\rangle$$

where $e = (a_1 a_2 [u, v])^{-1}$ and $e' = [u', v']^{-1}$. One may readily check that the conditions of Theorem 1.6 are satisfied so that all these groups as q varies have isomorphic profinite completions. The isomorphism from $\widehat{G}_1$ to $\widehat{G}_q$ defined in the theorem may be described as follows. Let $\kappa = q + p\rho$ be an element of $\widehat{\mathbb{Z}}^\times$ congruent to q modulo p. Let ϕ be an automorphism of the free profinite group on the generators u' and v' such that

$$\phi([u', v']) = [u', v']^\kappa,$$

i.e., an exotic automorphism of type κ of a once-punctured torus, which exists by Proposition 1.2. Then define $\Phi \colon \widehat{G}_1 \to \widehat{G}_q$ by sending u' and v' to their images under ϕ, by mapping

$$h \mapsto h^\kappa, \quad a_1 \mapsto a_1 h^{-\rho}, \quad a_2 \mapsto a_2 h^\rho$$

and by the 'identity' on all other generators. The reader may readily verify that this gives a well-defined surjection of profinite groups. As argued in the proof of the theorem it is in fact an isomorphism. This may also be seen from the fact that the map so given is an isomorphism of graphs of profinite groups.

Example 2.2 (Changing a gluing map): Consider the two graph manifolds depicted schematically in Figure 2. Each is composed of two product Seifert fibre spaces $S \times \mathbb{S}^1$ and $S' \times \mathbb{S}^1$ glued together, where S and S' are copies of a torus with two open discs removed. One readily verifies that the conditions of Theorem 1.6 hold, so the fundamental groups have isomorphic profinite completions. As in the previous example, we will write down the isomorphism explicitly. The fundamental group of the first manifold has presentation

$$G_1 = \langle e_1, u, v, h, e_1', u', v', h', t \mid [h, e_1], [h, u], [h, v], [h', e_1'], [h', u'], [h', v']$$
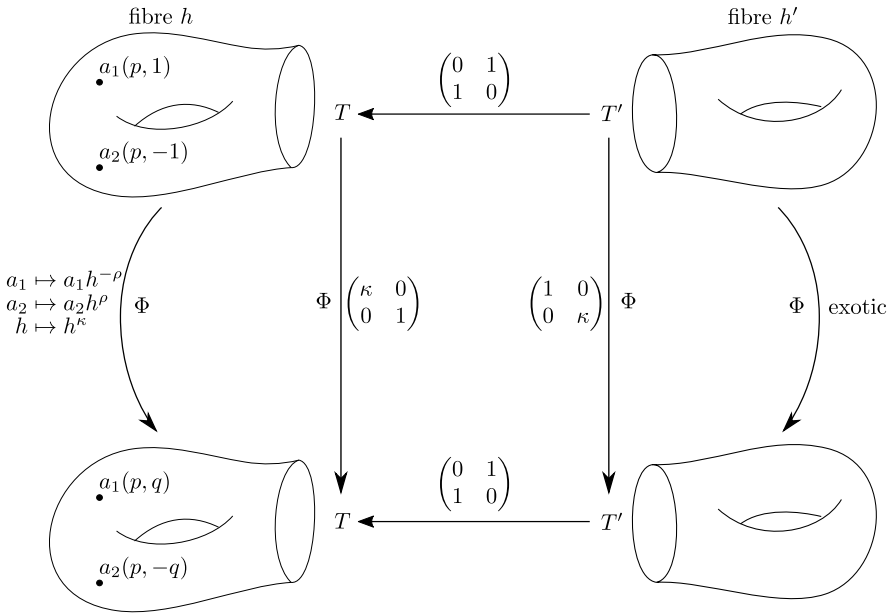$$e_1' = e_1, h' = h^{-1} e_1^5, (e_0')^t = e_0^{-1}, (h')^t = h e_0^5\rangle$$

Figure 1. A schematic depiction of the graph manifolds considered in Example 2.1. Here we represent each Seifert fibred piece as a surface with some marked points, where the marked points represent exceptional fibres and are labelled with the Seifert invariants of that fibre. Each boundary subgroup $T^{(\prime)}$ is given the ordered basis $(h^{(\prime)}, e^{(\prime)})$. The top pair of Seifert fibre spaces, glued together by the map on the horizontal arrow, gives the manifold whose fundamental group is G_1. The bottom pair yields G_q.

where $e_0 = (e_1[u, v])^{-1}$ and similarly for e_0'. The second group has presentation

$$G_2 = \langle e_1, u, v, h, e_1', u', v', h', t \mid [h, e_1], [h, u], [h, v], [h', e_1'], [h', u'], [h', v']$$
$$e_1' = he_1^{-2}, h' = h^{-2}e_1^5, (e_0')^t = he_0^2, (h')^t = h^2e_0^5\rangle.$$

Let $\kappa \in \widehat{\mathbb{Z}}^\times$ be congruent to 2 modulo 5, so that -2κ is congruent to 1 modulo 5. Take $\lambda, \mu \in \widehat{\mathbb{Z}}$ such that
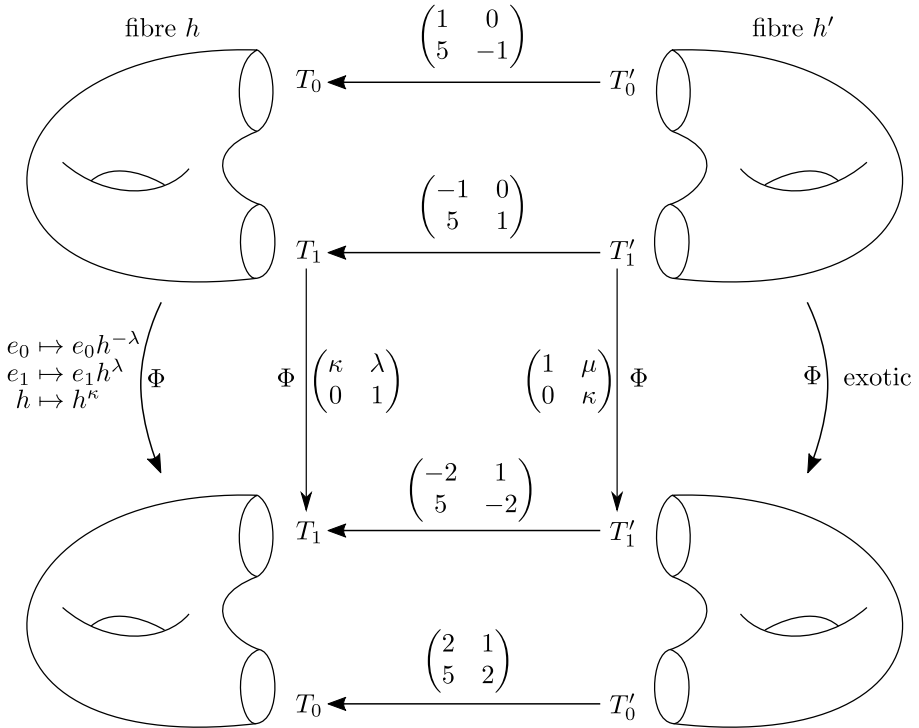
$$\kappa = 2 + 5\lambda, \quad 1 = -2\kappa + 5\mu.$$

fibre h $\begin{pmatrix} 1 & 0 \\ 5 & -1 \end{pmatrix}$ fibre h'

$T_0 \longleftarrow T_0'$

$\begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}$

$T_1 \longleftarrow T_1'$

$\begin{aligned} e_0 &\mapsto e_0 h^{-\lambda} \\ e_1 &\mapsto e_1 h^{\lambda} \\ h &\mapsto h^{\kappa} \end{aligned}$ Φ $\Phi \begin{pmatrix} \kappa & \lambda \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & \mu \\ 0 & \kappa \end{pmatrix} \Phi$ Φ exotic

$\begin{pmatrix} -2 & 1 \\ 5 & -2 \end{pmatrix}$

$T_1 \longleftarrow T_1'$

$\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$

$T_0 \longleftarrow T_0'$

Figure 2. The graph manifolds considered in Example 2.2. Here each Seifert fibred piece is a product of a surface with $\mathbb{S}^1$. Each boundary subgroup $T_i^{(\prime)}$ is given an ordered basis $(h^{(\prime)}, e_i^{(\prime)})$. Please note that for clarity we have swapped the roles of T_0 and T_1 in the lower diagram and omitted the maps induced by Φ on T_0 and T_0'.

Let ϕ be an exotic automorphism of S' of type κ, such that

$$\phi(e_1') = (e_1')^{\kappa}, \quad \phi(e_0') = [(e_0')^{\kappa}]^g$$

for some g in the subgroup of $\widehat{G}_2$ generated by u', v' and e_1'. Now define $\Phi \colon \widehat{G}_1 \to \widehat{G}_2$ as follows:

$$\begin{aligned} h &\mapsto h^{\kappa}, & e_1 &\mapsto e_1 h^{\lambda}, & u &\mapsto u, & v &\mapsto v, & t &\mapsto g^{-1}t, \\ h' &\mapsto h', & e_1' &\mapsto \phi(e_1')(h')^{\mu}, & u' &\mapsto \phi(u'), & v' &\mapsto \phi(v'). \end{aligned}$$

The reader is left to verify that this map is well-defined. The only real issue is whether the maps on the edge tori match up correctly. As indicated in Figure 2, this amounts to checking a matrix equation

$$\begin{pmatrix} -2 & 1 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} \kappa & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & \kappa \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\kappa + 5\lambda & \mu - 5\lambda\kappa^{-1}\mu + \lambda\kappa^{-1} \\ 5 & \kappa^{-1}(1 - 5\mu) \end{pmatrix}$$

on the 'e_1 edge' (and a similar one on the 'e_0 edge'). These equations hold by the definitions of λ and μ.

Example 2.3 (Fibred examples): Consider the surface S formed from a sphere by removing 10 small discs spaced equidistantly along the equator. This surface has an order 5 self-homeomorphism φ given by a rotation. The surface bundle

$$M_q = S \rtimes_{\varphi^q} \mathbb{S}^1$$

with monodromy φ^q (for q coprime to 5) is a Seifert fibre space whose base orbifold has genus 0, two boundary components and two exceptional fibres, each of order 5 and with Seifert invariants $(5, q)$ and $(5, -q)$ in some order. The surfaces S describe parallel circles on each boundary torus; choose one such curve to give the second basis element of the fundamental group of the boundary. Take two such Seifert fibre spaces M_{q_1} and M_{q_2}. Glue the 'e_0 boundary' of M_{q_2} to the 'e_0 boundary' of M_{q_1} by a map

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

(interpreted as a map from the boundary of M_{q_2} to the boundary of M_{q_1}) and glue the 'e_1 boundary' of M_{q_2} to the 'e_1 boundary' of M_{q_1} by a map

$$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

These gluings give a graph manifold L_{q_1,q_2} (see Figure 3). The choice of the second column guarantees that the glued-up manifold is still fibred, since in each case the fibre surfaces, each running exactly five times over the boundary components in curves isotopic to e_0 or e_1, may be matched by this gluing homeomorphism. By construction the total slope at each vertex space M_{q_i} is

zero. We may now apply Theorem 1.6 to conclude that the distinct fibred graph manifolds $L_{1,1}$ and $L_{2,-2}$ have isomorphic profinite completions of fundamental groups.
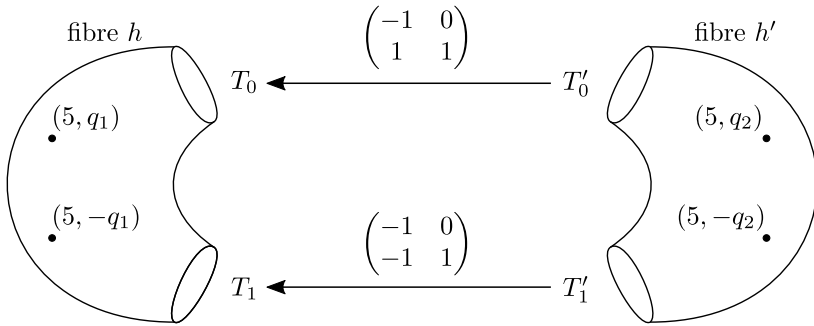

Figure 3. Schematic picture of the graph manifold L_{q_1,q_2} considered in Example 2.3

Remark: The study of fibred manifolds is closely connected with the study of mapping class groups. This will be discussed in Section 5.

3. Commensurability of graph manifolds

The following proposition answers a question asked of me by Michel Boileau at a conference in Marseille. All other known pairs of 3-manifold groups with the same profinite completions are commensurable—both Seifert fibred examples and examples with Sol geometry. The graph manifolds given by Theorem 1.6 also fit in this pattern.

PROPOSITION 3.1: *Every closed orientable graph manifold has a finite-sheeted cover with profinitely rigid fundamental group. Hence if two graph manifold groups have isomorphic profinite completions, then they are commensurable.*

Proof. The second statement follows easily from the first. For the first, we will consider the following class of graph manifolds. We will say that a graph manifold M with JSJ decomposition $(X, M_\bullet)$ is **right-angled** if the following conditions hold:

(RA1) the total slope at every vertex space of M is zero,

(RA2) for every edge space the intersection number of the fibres of the adjacent vertex spaces is 1, and

(RA3) every vertex space is of the form $S \times \mathbb{S}^1$ for S an orientable surface with genus at least 2 (that is, if all boundary components are filled in with discs, the result has genus at least 2).

Conditions (RA1) and (RA2) together imply that, after performing suitable Dehn twists, the gluing maps on every torus are simply $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. That is, the fibres of adjacent pieces 'are at right angles to each other'. A right-angled graph manifold is thus determined completely by its underlying graph and the first Betti numbers of the vertex groups. This graph, and the first Betti numbers of the vertex spaces, are profinite invariants by Theorem 1.5. Therefore to show that right-angled graph manifolds are indeed profinitely rigid it only remains to show that the property of being right-angled is a profinite invariant.

Theorem 1.6.2(b) immediately shows that having all total slopes zero is a profinite invariant. Theorem 1.6.2(a) shows that property (RA2) is preserved by profinite completions. Finally (RA3) is a profinite invariant by Theorems 1.3 and 1.5. Hence right-angled graph manifolds are indeed profinitely rigid.

Now consider a closed orientable graph manifold M. If the total slope of some vertex is non-zero then we have rigidity by Theorem 1.6, so we will ignore this case for the rest of the proof. Also notice that the vanishing of total slopes is essentially equivalent to the vanishing of Euler numbers of the Seifert fibre spaces obtained by filling each boundary torus of a Seifert fibred piece by a solid torus with meridian running along the fibre of the adjacent piece; so this condition is preserved by taking finite-sheeted covers of M. We will show that every graph manifold with zero total slope at each vertex has a finite-sheeted cover which is right-angled.

First pass to a suitable index 1 or 2 cover to eliminate all pieces which are homeomorphic to a twisted $\mathbb{I}$-bundle over a Klein bottle (in the terminology of [15], the 'minor' pieces). The base orbifold of every other Seifert fibre space appearing in the graph manifold has a finite cover which is a surface S of genus at least two; hence every Seifert fibred piece has a finite cover of the form $S \times \mathbb{S}^1$. The JSJ decomposition is efficient in the profinite topology (Theorem A of [16]) so some finite-sheeted cover of the graph manifold induces such a cover on every Seifert-fibred piece (it may be necessary to pass to a deeper cover on each vertex space than the one specified—but this is still of the required form $S \times \mathbb{S}^1$).

We must now aim to satisfy the condition on intersection numbers. Consider some edge group $T_e = \mathbb{Z}^2$. The fibres of the adjacent vertex groups are primitive elements generating an index γ subgroup of T_e, so there is a quotient map

$$T_e \twoheadrightarrow \mathbb{Z}/\gamma\mathbb{Z}$$

sending both fibres to zero. The fibres therefore lift to the corresponding degree γ cover of the torus, and the intersection number of any choice of such lifts is 1. So if we can find a finite-sheeted cover of the graph manifold inducing this precise cover of each boundary torus, we are done.

For each vertex space M_v we may, as previously noted, fill in each boundary torus T_e by gluing in a solid torus whose meridian is the fibre of the adjacent piece. This gives a Seifert fibre space whose base orbifold is the base orbifold of M_v with each boundary component collapsed and replaced by a cone point of order $\gamma(e)$, the intersection number of the two fibres. The fundamental group of this orbifold is residually finite, so there is a finite quotient into which all the isotropy groups of cone points inject. That is, we have a quotient $G_v \twoheadrightarrow Q_v$ such that the map on each boundary subgroup T_e is precisely the map $T_e \twoheadrightarrow \mathbb{Z}/\gamma(e)\mathbb{Z}$ discussed above.

We may now piece all these quotient groups together into a quotient graph of groups

$$(X, G_\bullet) \twoheadrightarrow (X, Q_\bullet)$$

where each Q_v is as above and each edge group Q_e is a copy of $\mathbb{Z}/\gamma(e)\mathbb{Z}$. This is a graph of finite groups, hence is residually finite (see, for example, [12], Section II.2.6, Proposition 11). So there is some finite quotient into which all the finite groups Q_x inject. The corresponding finite-sheeted cover of the graph manifold is the required rigid cover. ∎

4. Behaviour of knot complements

A **graph knot** is a knot in $\mathbb{S}^3$ whose exterior is a graph manifold. In this section we will prove the following theorem.

THEOREM 4.1: *Let M_K be the exterior of a graph knot K. Let N be another compact orientable 3-manifold and assume that $\widehat{\pi_1 M_K} \cong \widehat{\pi_1 N}$. Then $\pi_1 M_K \cong \pi_1 N$. In particular, if K is prime and N is also a knot exterior then N is homeomorphic to M_K.*

While the theorems of [15] were stated for closed manifolds, for most of the paper the assumption that the boundary was empty was not relevant. The arguments prior to Section 10 of [15] were based on the existence and properties of a JSJ decomposition along tori. This structure, and its properties, holds just as well for manifolds with toroidal boundary. In particular for graph knot exteriors we find that an isomorphism of profinite fundamental groups still induces an isomorphism of JSJ graphs and isomorphisms of the profinite fundamental groups of each vertex space of the JSJ decomposition. It is not necessarily true that any such isomorphism preserves the fundamental group of the boundary component (even up to conjugacy). Indeed this is not even true for the discrete fundamental group. However, we shall see that the only ambiguities in determining the manifold from its profinite fundamental group come from ambiguities in the discrete fundamental group.

Before proving the theorem we will need to discuss in more detail the Seifert fibre spaces arising in the JSJ decomposition of a graph knot exterior. Statements made in this discussion will be used without explicit reference in the following proofs.

Since a loop in the JSJ graph would imply the existence of a non-trivial element of $H^1(M_K; \mathbb{Z})$ vanishing on the boundary component of M_K, which is impossible by standard properties of knots, the JSJ graph is a rooted tree with root given by the single boundary component. It follows from Section 7 and Corollary 9.3 of [5] that the only possible vertex spaces are those described in the list below. A paraphrase of these results would be that all graph knots are built up from torus knots by the operations of cabling and connected-summation. See also Proposition 3.2 of [4]. Note that there are additional possibilities when considering exteriors of graph links. We do not consider this issue here.

Torus knot exteriors. The exterior $E_{p,q}$ of a (p, q)-torus knot, for $|p|, |q| \geq 2$. This is a Seifert fibre space with two exceptional fibres of orders p and q. Since p and q are coprime there exist $\bar{p}$ and $\bar{q}$ such that $\bar{p}p + \bar{q}q = 1$. Then a presentation for the fundamental group is

$$\langle a, b, h \mid h \text{ central}, a^p h^{\bar{q}}, b^q h^{\bar{p}} \rangle$$

where h is the homotopy class of a regular fibre and ab is a meridian curve of the knot. We remark that replacing $\bar{q}$ and $\bar{p}$ by any integers coprime to p and q respectively does not change this group up to isomorphism.

The profinite completion of this group has centre $\widehat{\mathbb{Z}}$ generated by h, and the quotient by this centre is $\mathbb{Z}/p \amalg \mathbb{Z}/q$, where $\amalg$ denotes the free profinite product. By the techniques of [14] any Seifert fibre space group with the same profinite completion has the same base orbifold (a disc with two cone points of orders p and q), and will therefore have the same discrete fundamental group by the above remark. It will also have only one boundary component. See also [7, Section 4].

Theorem 1.3 implies that the Seifert fibre spaces with the same profinite fundamental group as $E_{p,q}$ preserving peripheral structures are precisely those with the Seifert invariants $\bar{q}$ and $\bar{p}$ replaced by $k\bar{q}$ and $k\bar{p}$ for any k coprime to pq. Note that the requirement $\bar{p}p + \bar{q}q = 1$ additionally shows that, for k not congruent to 1 modulo pq, this Seifert fibre space is not a knot exterior.

If a torus knot exterior arises as a JSJ piece of a graph knot exterior then this piece must of course be a leaf of the JSJ tree. As the other possibilities in the list will show, the converse also holds: any leaf (except the root) is a torus knot exterior.

Products. Pieces of the form $S \times \mathbb{S}^1$, where S is a sphere with at least $k + 1 \geq 3$ open discs removed. The only Seifert fibre spaces with the same profinite fundamental group as $S \times \mathbb{S}^1$ also have the same discrete fundamental group—see Theorem 5.5 of [14]—and are therefore also of the form $F \times \mathbb{S}^1$ for some surface F with

$$\pi_1 F \cong \pi_1 S.$$

Note that if F is orientable then either $F = S$ or F has at most two fewer boundary components than S.

The presence of such a piece in the JSJ decomposition of a graph knot exterior represents the procedure of taking the connected sum of several graph knots. Note that under this operation, the meridian of each summand becomes a fibre of the product piece. Such a piece is either of valence $k + 1$ in the JSJ graph, or k if it happens to be the root piece.

Cable spaces. A cable space $C_{s,t}$ of type (s,t), for $|s| \geq 2$, consists of the space formed from a fibred solid torus T with Seifert invariants (s,t) by removing a neighbourhood of a regular fibre. Equivalently this is the orientable Seifert fibre space with base orbifold an annulus with a single cone point of Seifert invariants (s,t) where s is coprime to t. A presentation for the fundamental

group is

$$\pi_1 C_{s,t} = \langle c, e, j \mid j \text{ central}, c^s j^t \rangle$$

where the regular fibre is j and the boundary components are given by the conjugacy classes of the subgroups $\langle j, e \rangle$ and $\langle j, (ce)^{-1} \rangle$. Note also that, if the boundary component of T is represented in $\pi_1 C_{s,t}$ by $\langle j, e \rangle$, then the boundary of a meridian disc of T is given by $m = j^t e^{-s}$.

Notice that these groups $\pi_1 C_{s,t}$ as t varies are abstractly isomorphic to each other, and that such isomorphisms can be chosen to fix one (but not both) boundary components. Theorem 1.3 shows that the profinite fundamental groups of $C_{s,t}$ as t varies are isomorphic while preserving all of the given peripheral structure.

Furthermore (the proof of) Theorem 5.5 of [14] shows that the only orientable Seifert fibre spaces with the same profinite fundamental group (ignoring the peripheral structure) as a cable space have orientable base orbifolds with fundamental group $\mathbb{Z} * \mathbb{Z}/s$. There is only one such orbifold, so the Seifert fibre space in question is again a cable space (with the same invariant s).

The presence of a cable space piece in the JSJ decomposition of a graph knot exterior represents the cabling operation on knots. The corresponding vertex of the JSJ graph has valence two, or one if it happens to be the root. Note that in this case the meridian curve of the initial knot becomes the element m given above (i.e., the meridian curve of T). A longitude of the initial knot will therefore be given by $l = j^{\bar{s}} e^{\bar{t}}$ for some integers $\bar{s}, \bar{t}$ such that $s\bar{s} + t\bar{t} = 1$. The ambiguity in the choice of these integers reflects the ambiguity in the choice of longitude.

LEMMA 4.2: *Let K be a graph knot with exterior M_K. Let $(X, M_\bullet)$ be the JSJ decomposition of M_K, where X is viewed as a rooted tree. Let x be a leaf of X that is not the root. Then the total slope at x is non-zero.*

Proof. Since x is a leaf, the corresponding manifold M_x is the exterior of a (p, q) torus knot for some coprime integers p and q. As above choose a presentation

$$\pi_1 M_x = \langle a, b, h \mid h \text{ central}, a^p h^{\bar{q}}, b^q h^{\bar{p}} \rangle$$

where $\bar{p} p + \bar{q} q = 1$ and the meridian of the torus knot is ab. Let y be the unique vertex of X adjacent to x. There are two cases to consider: whether M_y is a cable space or a product.

Suppose first that M_y is a product. Then by the discussions above the meridian ab of the torus knot is isotopic to the fibre j of the product. Thus the gluing matrix along the edge joining x and y, oriented from x to y, has the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence the total slope at x is

$$\tau(x) = 0 - \frac{\bar{q}}{p} - \frac{\bar{p}}{q} = -\frac{1}{pq} \neq 0.$$

Next we consider the case when M_y is an (s,t)-cable space whose fundamental group has presentation

$$\pi_1 C_{s,t} = \langle c, e, j \mid j \text{ central}, c^s j^t \rangle$$

where the regular fibre is j and the boundary components are given by the conjugacy classes of the subgroups $\langle j, e \rangle$ and $\langle j, (ce)^{-1} \rangle$. Without loss of generality let the boundary component glued to M_x be $\langle j, e \rangle$. As discussed above the meridian ab of the torus knot is given by $ab = j^t e^{-s}$ and the fibre h—which is a longitude of the torus knot—is given by $h = j^{\bar{s}} e^{\bar{t}}$ for some integers $\bar{s}, \bar{t}$ such that $s\bar{s} + t\bar{t} = 1$.

Thus the gluing matrix along the edge joining x and y, again oriented from x to y, has the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \bar{s} & t \\ \bar{t} & -s \end{pmatrix}.$$

And the vanishing of the total slope would imply

$$0 = pq\bar{t}\tau(x) = pq(-s) - \bar{t}(\bar{q}q + \bar{p}p) = -pqs - \bar{t}$$

and hence s would divide $\bar{t}$, so $s = 1$ giving a contradiction. So the total slope is non-zero as required. ∎

Proof of Theorem 4.1. Let $(X, M_\bullet)$ and $(Y, N_\bullet)$ be the JSJ decompositions of M_K and N respectively. Let $G = \pi_1 M_K$, $G_\bullet = \pi_1 M_\bullet$, $H = \pi_1 N$ and $H_\bullet = \pi_1 N_\bullet$. Let $\Phi \colon \widehat{G} \to \widehat{H}$ be an isomorphism. By Theorem 1.5 and Theorem 7.1 of [15] (whose proofs do not rely on the manifolds being closed) we find that N is a graph manifold and, possibly after post-composing Φ with some automorphism of $\widehat{H}$, there is a graph isomorphism $\phi \colon X \to Y$ such that Φ restricts to an isomorphism $\widehat{G}_x \cong \widehat{H}_{\phi(x)}$ for each $x \in X$. By the discussions above, this implies that G_x and $H_{\phi(x)}$ are abstractly isomorphic.

Now X is a rooted tree with root r distinguished by the single boundary component of M_K. For standard cohomological reasons (for example, Corollary 4.2 of [2]) the manifold N cannot be closed. We claim that N has exactly one boundary component, located in the piece $N_{\phi(r)}$. We do not claim—indeed it may not be true even for isomorphisms of discrete fundamental groups—that this boundary component ∂N satisfies

$$\widehat{\partial N} = \Phi(\pi_1(\widehat{\partial M_K})),$$

even up to conjugacy. However, its position in the JSJ decomposition is fixed because of the following argument. Here a 'free boundary component' of a JSJ piece of a 3-manifold will mean a boundary torus not glued to any JSJ piece—that is, those boundary components which survive in the boundary of the ambient manifold.

Every Seifert fibre space with fundamental group isomorphic to that of a torus knot exterior has exactly one boundary component—and every such piece of N has valence 1 in Y because of the isomorphism Φ, hence has no free boundary components. Similarly every Seifert fibre space with fundamental group isomorphic to that of a cable space has two boundary components, and all such pieces N_y have valence 2 in Y unless $y = \phi(r)$, when there is one free boundary component ∂N. Comparing X and Y, every piece N_y with fundamental group isomorphic to $F_k \times \mathbb{Z}$ has valence $k + 1$ in Y unless $y = \phi(r)$, when it has valence r. Now any orientable Seifert fibre space with fundamental group $F_k \times \mathbb{Z}$ has either $k + 1$ boundary components or has strictly fewer than k—hence all these pieces have no free boundary components except if $y = \phi(r)$ when it has exactly one. So we see that N may have at most one boundary component, and it is located in the piece $N_{\phi(r)}$.

Now consider the isomorphisms $\Phi_x \colon \widehat{G}_x \to \widehat{H}_{\phi(x)}$. For $x \neq r$, the fact that Φ preserves the JSJ decomposition implies that the peripheral structure is preserved by Φ_x. So by Theorem 1.3 the isomorphism Φ_x determines constants $\lambda_x, \mu_x \in \widehat{\mathbb{Z}}^\times$ such that λ_x gives the map from the fibre subgroup of M_x to the fibre subgroup of $N_{\phi(x)}$ and the map on base orbifolds is an exotic isomorphism of type μ_x. Analysing the gluing maps in M and N as in the proof of Theorem 1.6 we find that if x' is adjacent to x (and neither is the root) we have $\lambda_x = \mu_{x'}$ and $\mu_x = \lambda_{x'}$, perhaps up to choosing appropriate orientations on fibre subgroups to eliminate minus signs. Furthermore the total slopes of M_x and $N_{\phi(x)}$ are related by multiplication by λ_x/μ_x. (Strictly speaking we have

not defined multiplication of an element of $\mathbb{Q}$ by an element of $\widehat{\mathbb{Z}}$; we really mean that after clearing denominators by an integer $n \in \mathbb{Z}$ we have

$$n\tau(\phi(x)) = \lambda_x \mu_x^{-1} \tau(x).$$

Alternatively one can consider the ring obtained from $\widehat{\mathbb{Z}}$ by inverting all the elements of $\mathbb{Z}$.)

However the total slope $\tau(x)$ of any piece M_x which is a torus knot exterior is non-zero by Lemma 4.2. So the total slope of $N_{\phi(x)}$ is a rational number equal to λ_x/μ_x times a non-zero rational number. This implies that $\lambda_x = \mu_x$ (up to changes in orientations) using Lemma 2.2 of [15]. Every connected component of $X \smallsetminus \{r\}$ contains a leaf of X, hence has a vertex space which is a torus knot exterior. It follows that $\lambda_\bullet$ and $\mu_\bullet$ are constant on connected components of $X \smallsetminus \{r\}$.

Now on the root pieces of M and N, the fibre subgroup is the unique maximal central subgroup of the relevant profinite fundamental group by Theorem 5.5 of [14] and so is preserved by Φ_r. Let the map on fibre subgroups be multiplication by λ_r (as usual, identifying this fibre subgroup with $\widehat{\mathbb{Z}}$ via a generator in the discrete fundamental group). Again, examining the gluing maps along the tori gluing r to other vertices and using the same analysis as in Theorem 1.6 we find that if x is adjacent to r then $\pm\mu_x = \lambda_r$. Therefore we find that $\lambda_\bullet$ and $\mu_\bullet$ are constants over all of X (up to reversing all the orientations on components of $X \smallsetminus \{r\}$ to fix the signs).

By Theorem 1.3, the fact that $\lambda_x/\mu_x = 1$ (together with the fact that away from the root all boundary tori of JSJ pieces are edge groups in the JSJ decomposition, hence are preserved by Φ) implies that M_x and $N_{\phi(x)}$ are homeomorphic for $x \neq r$.

Now, the fundamental group of the graph manifold N is determined up to isomorphism by the following data: the JSJ graph; the isomorphism types of group pairs given by each vertex group and its adjacent edge groups; the intersection numbers $\gamma(e)$ of adjacent fibres, for each edge e of the JSJ graph; the invariants $\delta(e)$ modulo $\gamma(e)$; and the total slope of each vertex space that is not the root. These allow us to reconstruct the total group uniquely—as explained in the preliminaries, the indeterminacy of $\delta(e)$ modulo $\gamma(e)$ may be resolved using Dehn twists given that the total slope is fixed. For the root one may use the free boundary component for Dehn twists, hence no sort of total slope condition is required.

Now the analysis in Theorem 10.1 of [15] shows that the fact that λ_x/μ_x is always equal to 1 fixes all of these data for N to be equal to those for M. Hence the manifolds have the same fundamental group.

In the case when K is prime (that is, the root piece is a cable space) and N is also a knot exterior, then the fundamental group determines the homeomorphism type and final statement of the theorem also follows. See Section 7 and Corollary 9.3 of [5] for the proof of this for graph knots. The more general result is [6, Corollary 2.1], which states the stronger result that the fundamental group determines the knot in $\mathbb{S}^3$. ∎

Remark: We comment that the properties deriving from the fact that M_K was a knot exterior were crucial to the rigidity in this theorem. In particular, the fact that every leaf of the JSJ tree had non-zero total slope provides strong rigidity to all complements of the root piece, leaving little flexibility in what remained. If one had some complement of the root piece which was not profinitely rigid (relative to the boundary component joining it to the root) one can easily extend this to further non-rigid examples. The free boundary component in the root means that results such as Theorem 1.3 which forces all the boundary components of the root to behave in roughly the same way simply do not apply. Possibly one could impose extra constraints on the boundary to extend the analysis of Theorem 1.6. However, a large part of the interest of Theorem 4.1 is that no such boundary condition is needed.

5. Relation to mapping class groups

In this section, we will only discuss closed orientable manifolds, to avoid worries on the boundary. We will view a fibred 3-manifold (M, ζ) as a 3-manifold M equipped with a choice of homomorphism $\zeta \colon \pi_1 M \twoheadrightarrow \mathbb{Z}$ with finitely generated kernel $\pi_1 S$, where S is a closed orientable surface. By Stallings' theorem on fibred 3-manifolds [13, Theorem 2], this is equivalent to the topological definition. For such a fibred manifold, $\pi_1 M$ has many expressions as a semidirect product $\pi_1 S \rtimes_\phi \mathbb{Z}$ each given by a section of ζ. The different maps ϕ differ by composition with an inner automorphism of $\pi_1 S$, hence give a well-defined element of $\mathrm{Out}(\pi_1 S)$. If we have two automorphisms ϕ_1 and ϕ_2 of $\pi_1 S$, these are conjugate in $\mathrm{Aut}(\pi_1 S)$ by some automorphism ψ if and only if there is an

isomorphism of semidirect products

$$(\psi, \mathrm{id})\colon \pi_1 S \rtimes_{\phi_1} \mathbb{Z} \to \pi_1 S \rtimes_{\phi_2} \mathbb{Z}.$$

Allowing for a change in section, we find that ϕ_1 and ϕ_2 are conjugate in $\mathrm{Out}(\pi_1 S)$ if and only if there is a commuting diagram

$$
\begin{array}{ccc}
\pi_1 M_1 & \xrightarrow{\ \zeta_1\ } & \mathbb{Z} \\
\Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
\pi_1 M_2 & \xrightarrow{\ \zeta_2\ } & \mathbb{Z}
\end{array}
$$

where (M_1, ζ_1) and (M_2, ζ_2) are the fibred manifolds corresponding to ϕ_1 and ϕ_2.

All the above equivalences still hold when one replaces all manifold groups with their profinite completions and $\mathrm{Aut}(\pi_1 S)$ with $\mathrm{Aut}(\widehat{\pi_1 S})$. There is a canonical injection

$$\mathrm{Aut}(\pi_1 S) \hookrightarrow \mathrm{Aut}(\widehat{\pi_1 S})$$

and we will abuse notation by identifying an automorphism of $\pi_1 S$ with the induced automorphism of the profinite completion. As was noted by Boileau and Friedl [1, Corollary 3.6], Theorem 5.2 of [14] implies that the canonical map

$$\mathrm{Out}(\pi_1 S) \to \mathrm{Out}(\widehat{\pi_1 S})$$

is also an injection.

Thus related to the question of whether two fibred manifolds can have isomorphic profinite fundamental groups we have a question concerning automorphisms of surface groups.

Question 5.1: Do there exist automorphisms ϕ_1, ϕ_2 of a surface group $\pi_1 S$ which are conjugate in $\mathrm{Out}(\widehat{\pi_1 S})$ but not in $\mathrm{Out}(\pi_1 S)$?

When S is a once-punctured torus, this question has been proven to have a negative answer by Bridson, Reid and Wilton [3, Theorem A].

While a positive answer to Question 5.1 would give examples of fibred manifold groups with isomorphic profinite completions, the converse does not always hold; one could conceivably have profinite isomorphisms of the manifold groups which do not in any sense preserve the fibrations. For instance, Example 2.3 above does not give a positive solution to Question 5.1; as we will see below, no other graph manifold does either. Let us make this precise.

Definition 5.2: Let (M, ζ_M) and (N, ζ_N) be fibred graph manifolds. Suppose

$$\Psi \colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$$

is an isomorphism. We say that Ψ **weakly preserves the fibration** if there is a commuting diagram

$$
\begin{array}{ccc}
\widehat{\pi_1 M} & \xrightarrow{\ \hat{\zeta}_M\ } & \widehat{\mathbb{Z}} \\
\Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\cdot\kappa} \\
\widehat{\pi_1 N} & \xrightarrow{\ \hat{\zeta}_N\ } & \widehat{\mathbb{Z}}
\end{array}
$$

for some $\kappa \in \widehat{\mathbb{Z}}^\times$. We say that Ψ **strongly preserves the fibration** if there exists such a diagram with, additionally,

$$\kappa = +1.$$

Relating this definition to the discussion above, we see that strong fibre preservation yields conjugacy in $\mathrm{Out}(\widehat{\pi_1 S})$. Weak fibre preservation says that ϕ_1 is conjugate to ϕ_2^κ in $\mathrm{Out}(\widehat{\pi_1 S})$.

Finite order automorphisms give rise to Seifert fibre spaces of geometry $\mathbb{H}^2 \times \mathbb{R}$. Hempel's original paper giving examples of Seifert fibre spaces which are not profinitely rigid do not give an explicit isomorphism of profinite groups, so do not say anything about mapping classes. The isomorphisms Ψ constructed in [14] preserve the fibration weakly. For suitable choices of sections, the mapping classes involved are ϕ^k and ϕ for some k which is coprime to the order of ϕ. In terms of mapping classes, the weak fibre preservation in this case then says that ϕ^k is conjugate to ϕ^κ in $\mathrm{Out}(\pi_1 S)$, for $\kappa \in \widehat{\mathbb{Z}}^\times$ such that κ is congruent to k modulo the order of ϕ. This is not exactly surprising, since in fact these automorphisms are equal. Indeed this gives yet another way to see that the corresponding Seifert fibre space groups must have isomorphic profinite completions.

We can however say more. The mapping classes ϕ^k and ϕ are genuinely conjugate in $\mathrm{Out}(\widehat{\pi_1 S})$—that is, there exists an isomorphism which strongly preserves the fibration. The isomorphisms in [14] do not do this, but armed with the exotic automorphisms of surface groups from Proposition 1.2 we can build new isomorphisms. The case with non-empty boundary was covered by Theorem 10.7 of [15] as the focus was then on constructing graph manifolds. We now deal with the case of closed Seifert fibre spaces.

THEOREM 5.3: *Let S be a closed hyperbolic surface and let φ be a periodic self-homeomorphism of S. Let k be coprime to the order of φ. Let (M, ζ) be the surface bundle with fibre S and monodromy φ and let (M', ζ') be the surface bundle with fibre S and monodromy φ^k. Then there is an isomorphism*

$$\Psi \colon \widehat{\pi_1 M} \to \widehat{\pi_1 M'}$$

which strongly preserves the fibration.

Proof. We may choose presentations for these two Seifert fibre spaces in the standard form. Note that since M is fibred over the circle, the base orbifold is orientable and the geometry is $\mathbb{H}^2 \times \mathbb{R}$. Given ζ, there is a presentation of the form

$$\pi_1 M = \langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \mid$$
$$h \in Z(\pi_1 M),\ a_i^{p_i} h^{q_i},\ a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^b \rangle$$

where $b = -\sum q_i / p_i$ since the Euler number vanishes, and where the map ζ is given by

$$h \mapsto \prod_j p_j, \quad a_i \mapsto -q_i \prod_{j \neq i} p_j$$

and by sending u_i, v_i to zero. Similarly we have

$$\pi_1 M = \langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \mid$$
$$h \in Z(\pi_1 M),\ a_i^{p_i} h^{q'_i},\ a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^{b'} \rangle$$

where

$$b' = -\sum \frac{q'_i}{p_i}.$$

Again the map ζ' is given by

$$h \mapsto \prod_j p_j, \quad a_i \mapsto -q'_i \prod_{j \neq i} p_j.$$

Note that by construction we have $q_i \equiv \kappa q'_i$ modulo p_i for every i, for $\kappa \in \widehat{\mathbb{Z}}^\times$ congruent to k modulo the order of φ. See Proposition 5.1 of [8] for the translation of data from surface bundles to Seifert invariants. Let $\rho_i \in \widehat{\mathbb{Z}}$ be such that $q_i = \kappa q'_i + \rho_i p_i$. Now by Proposition 1.2 there is an automorphism ψ of the free group on the generators $\{a_i, u_i, v_i\}$ which sends each a_i to a conjugate of a_i^κ, sends every other generator to a conjugate of a power of itself, and sends

$$a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] \mapsto (a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g])^\kappa.$$

Now define $\Psi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 M}$ by

$$h \mapsto h, \quad a_i \mapsto \psi(a_i)h^{-\rho_i}, \quad u_i \mapsto \psi(u_i), \quad v_i \mapsto \psi(v_i).$$

The reader may readily check that this map Ψ is a well-defined isomorphism of profinite groups and that $\hat{\zeta} = \hat{\zeta}'\Psi$ as required. ∎

Definition 5.4: A **piecewise periodic** self-diffeomorphism of a surface is one for which the corresponding fibred manifold 3-manifold is a graph manifold or Seifert fibre space. We abuse this terminology by also using it for the induced automorphism of the fundamental group.

Remark: Those monodromies giving rise to Seifert fibre spaces are of course those which are genuinely periodic. The terminology 'piecewise periodic' is supposed to reflect the idea that there is a decomposition of the surface along simple closed curves, fixed by the monodromy, so that the restriction of the self-diffeomorphism to each complementary subsurface is isotopic to a periodic one. These isotopies will not in general extend across all these simple closed curves; if one can choose isotopies which do, then the self-diffeomorphism is periodic.

THEOREM 5.5: *If M and N are non-homeomorphic closed fibred graph manifolds, and $\Psi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$ is any isomorphism, then Ψ does not preserve any fibrations of M and N, even weakly. Hence if ϕ_1 and ϕ_2 are automorphisms of a closed surface group $\pi_1 S$ which are piecewise periodic but not periodic, and which are not conjugate in $\mathrm{Out}(\pi_1 S)$, then ϕ_1 is not conjugate to any power of ϕ_2 in $\mathrm{Out}(\widehat{\pi_1 S})$.*

Proof. For suppose (M, ζ_M) and (N, ζ_N) are fibred graph manifolds and Ψ is an isomorphism of the profinite completions of their fundamental groups weakly preserving the fibration. Suppose M and N are not homeomorphic. Let the JSJ decompositions be $(X, M_\bullet)$ and $(Y, N_\bullet)$ and denote the regular fibre of a vertex group $\pi_1 M_x$ or $\pi_1 N_x$ by h_x. It follows from [5, Theorem 4.2] that any fibre surface in a graph manifold must cut the Seifert fibres of all vertex spaces transversely, so every Seifert fibre survives as a non-trivial element of $\mathbb{Z}$ under the map ζ_M (or ζ_N). Also, for an orientable fibred graph manifold all the base orbifolds of major pieces must be orientable. We may therefore apply the analysis in the proof of Theorem 10.1 of [15] to conclude that there is a graph isomorphism $\psi\colon X \to Y$ and numbers $\lambda, \mu \in \widehat{\mathbb{Z}}^\times$ such that for adjacent

vertices x and y of X, we have (where $\sim$ denotes conjugacy in $\widehat{\pi_1 N}$)

$$\Psi(h_x) \sim h^{\lambda}_{\psi(x)}, \quad \Psi(h_y) \sim h^{\mu}_{\psi(y)}$$

(or vice versa). Since M and N are not homeomorphic the ratio λ/μ is not equal to ± 1. Now if Ψ weakly preserves the fibre, then we have equations

$$\kappa \zeta_M(h_x) = \zeta_N(\Psi(h_x)) = \zeta_N(h^{\lambda}_{\psi(x)}) = \lambda \zeta_N(h_{\psi(x)})$$

so that λ/κ, when multiplied by a non-zero element of $\mathbb{Z}$, remains in $\mathbb{Z}$. Thus by standard theory (for example Lemma 2.2 of [15]), $\kappa = \pm \lambda$. Applying the same argument to y gives $\kappa = \pm \mu$, so that $\lambda/\mu = \pm 1$, giving a contradiction. $\blacksquare$

COROLLARY 5.6: *A closed fibred graph manifold of first Betti number one is profinitely rigid.*

Proof. By Theorem 7.1 of [15] and Theorem 1.1 of [9] any other manifold with the same profinite fundamental group as the given manifold is also a closed fibred graph manifold. If the first Betti number is 1, then there is a unique map to $\widehat{\mathbb{Z}}$ so that any isomorphism weakly preserves the fibration. Theorem 5.5 now gives the result. $\blacksquare$

Remark: There is another proof of this corollary, which we will now sketch, which is more closely related to Theorem 1.6. For consider a closed fibred graph manifold M of first Betti number one. Then there is an essentially unique homomorphism $\zeta\colon \pi_1 M \to \mathbb{Q}$ which, as noted above, does not vanish on the regular fibre of any Seifert-fibred piece of M. This in turn implies that the JSJ graph of M is a tree and that the base orbifolds of all pieces are spheres with cone points with discs removed.

Consider piece M_l corresponding to a leaf l in the JSJ graph. We will show that the total slope at M_l is non-zero, so that M is profinitely rigid by Theorem 1.6. Let e be the edge emanating from l, let h be the homotopy class of a regular fibre of M_l, and let h' be the regular fibre of $d_1(e)$. If $\pi_1 M_l$ has a standard form presentation

$$\langle a_1, \ldots, a_r, h \mid a_i^{p_i} h^{q_i}, h \text{ central} \rangle$$

then, if $e_0 = (a_1 \cdots a_r)^{-1}$, we have

$$h' = e_0^{-\gamma(e)} h^{\delta(e)} = (a_1 \cdots a_r)^{\gamma} h^{\delta}$$

where $\gamma \neq 0$. Without loss of generality suppose $\zeta(h) = 1$. Then $\zeta(a_i) = -q_i/p_i$ and

$$\zeta(h') = -\gamma \sum \frac{q_i}{p_i} + \delta = \gamma\tau(l).$$

Since $\zeta(h') \neq 0$ we find that the total slope at l is non-zero as claimed.

Theorem 5.5 shows that Theorem 1.6, while finding examples of non-rigid fibred manifolds, leaves open the possibility that Question 5.1 could have a negative answer for all infinite order mapping classes. It also raises the possibility that even if profinite rigidity for hyperbolic 3-manifolds fails, the weaker statement about mapping class groups could still hold.

Remark: It is curious to compare the directions of the proofs in this section and in [3]. In the latter paper, strong properties of $\mathrm{Out}(F_2)$ ('congruence omnipotence for elements of infinite order') were used to deduce that independent mapping classes $\phi_1, \phi_2 \in \mathrm{Out}(F_2)$ were not conjugate to any power of each other in $\mathrm{Out}(\widehat{F_2})$, so that no isomorphism of the profinite completions of once-punctured torus bundles could (in our terminology) weakly preserve the fibration. Meanwhile the assumption that the first Betti number is 1 showed that any such isomorphism must weakly preserve the fibration. In this way [3] obtained a profinite rigidity theorem for once-punctured torus bundles.

In our situation, the direction is quite different; we investigated profinite completions of graph manifolds, and in doing so learnt about conjugacy of certain elements in $\mathrm{Out}(\widehat{\pi_1 S})$. One rather suspects that the results about the mapping class group should come first, but seem to be lacking except in the case described above.

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