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STRONGLY DEPENDENT ORDERED ABELIAN GROUPS AND HENSELIAN FIELDS

BY

Yatir Halevi*

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: yatir.halevi@mail.huji.ac.il
URL: http://ma.huji.ac.il/˜yatirh/

AND

Assaf Hasson**

Department of mathematics, Ben Gurion University of the Negev
Be'er Sheva 84105, Israel
e-mail: hassonas@math.bgu.ac.il
URL: http://www.math.bgu.ac.il/˜hasson/

ABSTRACT

Strongly dependent ordered abelian groups have finite dp-rank. They are precisely those groups with finite spines and $|\{p \text{ prime} : [G:pG]=\infty\}|<\infty$. We apply this to show that if K is a strongly dependent field, then (K,v) is strongly dependent for any Henselian valuation v.

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1. Introduction

Ordered abelian groups were classified up to elementary equivalence (and beyond) by Gurevich [12] and Schmitt [29] (and references therein). One significant application was the proof in [11] that ordered abelian groups are dependent (i.e., do not have the independence property). This result, when combined with transfer principles (such as [5] and [2], and most recently [18]), reduced—under fairly general conditions—the task of checking whether a (pure) Henselian valued field is dependent to checking whether its residue field is.

The finer classification of Henselian dependent fields, motivated mainly by Shelah's conjecture ([32]) that all infinite (strongly) dependent fields are separably closed, real closed or admit a definable Henselian valuation, called for a finer classification of ordered abelian groups. The immediate motivation for the investigation carried out in the present paper was the lack of worked out examples of strongly dependent ordered abelian groups (and Henselian fields) that are not dp-minimal. We prove, generalising the classification of dp-minimal ordered abelian groups of [21]:

Theorem 1: *Let G be an ordered abelian group. The following are equivalent:*

(1) *G is strongly dependent;*
(2) $\text{dp-rk}(G) < \aleph_0$*;*
(3) *G has finite spines and $|\{p \text{ prime} : [G : pG] = \infty\}| < \infty$;*
(4) *G is elementary equivalent to a lexicographic sum $\bigoplus_{i\in I} G_i$, where*
    (b) *for every prime p, $|\{i \in I : pG \neq G\}| < \infty$ and*
    (b) *$[G_i : pG_i] = \infty$ for only finitely many primes p.*

The spines of an ordered abelian group, in the terminology of [30], are (interpretable) coloured linear orders determining the first order theory of the group. To the best of our knowledge, no systematic study of ordered abelian groups with finite spines has been carried out before. In Section 2, we collect a few useful facts about ordered abelian groups. In Section 3 we apply Schmitt's characterization of lexicographic sums of ordered archimedian groups to characterize groups with finite spines.

Theorem 1 is proved in Section 4. The proof proceeds by showing that strongly dependent ordered abelian groups have finite spines and explicitly calculating the dp-rank of the latter. This is done by first calculating the dp-rank of a certain 1-based reduct of the group, and then studying the effect of reintroducing the order into that structure.

We have recently learned that Rafel Farré [9], Alfred Dolich and John Goodrick [7] have obtained, independently and using different methods, some of the results concerning ordered abelian groups obtained in this paper.

In Section 5 we apply our classification of strongly dependent ordered abelian groups to the study of strongly dependent Henselian fields. Our main result is:

Theorem 2: *Let K be a strongly dependent field and v any Henselian valuation on K. Then (K, v) is strongly dependent. The value group, vK, is stably embedded in (K, v) as a pure ordered abelian group (up to one constant), and the residue field, Kv, is stably embedded as a pure field.*

As a corollary we deduce (using results of Johnson, [22]) that strongly dependent fields are defectless (and therefore also algebraically maximal) with respect to any Henselian valuation. Our study of strongly dependent valued fields builds on ideas of Jahnke and Simon ([17], [18]).

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2. Preliminaries and notation

Throughout the text G will denote a group, usually abelian and often ordered, $\mathfrak{C}$ will denote a sufficiently saturated model of $\mathrm{Th}(G)$. By definable we will mean definable with parameters. We will need a few results from [29]. Since this text is not readily available, we try to keep the present work as self-contained as possible, referring to more accessible sources whenever we are aware of such. In particular, for the study of ordered abelian groups we chose the language of [4], rather than the language used by Schmitt. The next sub-section is dedicated to a quick overview of (parts) of the language we are using, and to the basic properties of definable sets.

2.1. Ordered abelian groups. Recall that an abelian group $(G;+)$ is ordered if it is equipped with a linear ordering $<$ such that $a<b$ implies $a+g<b+g$ for all $a,b,g\in G$. An ordered abelian group is **discrete** if it has a minimal positive element, and **dense** otherwise. It is **archimedean** if for all $a,b\in G$ there exists $n\in\mathbb{Z}$ such that $na>b$. In particular, archimedean ordered abelian groups do not have non-trivial convex subgroups.

Schmitt and Gurevich [12, 29] were the first to provide quantifier elimination for ordered abelian groups. For most of our needs in the present paper a slightly different language introduced by Cluckers and Halupczok in [4] will be more convenient. We recall some of the notation and conventions from [4]:

For any $n\in\mathbb{N}$ and $a\in G\setminus nG$ let $H_n(a)$ be the largest convex subgroup of G such that $a\notin H_n(a)+nG$ (equivalently, it is the largest convex subgroup not meeting $a+nG$), and $H_n(a)=0$ if $a\in nG$. By [4, Lemma 2.1] the groups $H_n(a)$ are definable (uniformly in a) in the language of ordered abelian groups. We set $\mathcal{S}_n:=G/{\sim}$, with $a\sim a'$ if and only if $H_n(a)=H_n(a')$, and let $\mathfrak{s}_n:G\to\mathcal{S}_n$ be the canonical map; we denote $H_n(a)$ by G_α for $\mathfrak{s}_n(a)=\alpha$.

Since the system of convex subgroups of an ordered abelian group are linearly ordered, $\mathcal{S}_n$ is an interpretable set linearly ordered by $\alpha\le\alpha'$ if $G_\alpha\subseteq G_{\alpha'}$.

For any $\alpha\in\mathcal{S}_n$ and $m\in\mathbb{N}$ define

$$G_\alpha^{[m]}:=\bigcap\{H+mG : G_\alpha\subsetneq H\subseteq G,\ H \text{ a convex subgroup}\}.$$

Other than the sorts $\mathcal{S}_p$, Cluckers–Halupczok define two more auxiliary sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ parametrizing more definable convex subgroups of G. It suffices, for our needs, to know that they are intersections and unions of convex subgroups G_α for α ranging in $\mathcal{S}_p$.

Remark: As we will need results from [29] we note that the groups denoted $H_n(a)$ in [4] (and in the present text) are denoted $F_n(a)$ by Schmitt.

We conclude this section with some basic results.

Fact 2.1 ([29, Lemmas 2.8, 2.9, 2.10]):

(1) *$H_n(a)=H_n(a+ng)$, for any $g\in G$.*
(2) *If $H_n(a)\subsetneq H_n(b)$ then $(a+nG)\cap H_n(b)\neq\emptyset$,*
(3) *as a result, if $H_n(a)\subsetneq H_n(b)$ then $H_n(a+b)=H_n(b)$*
(4) *and if $H_n(a)=H_n(b)$ then $H_n(a+b)\subseteq H_n(a)$.*
(5) *For every prime p, $H_{p^m}(a)=H_{p^{m+k}}(p^k a)$.*

2.2. Examples. Some important examples of ordered abelian groups:

Example 2.2 ([29, Lemma 1.19]): Let $\chi : \{\text{primes}\} \to \mathbb{N} \cup \{\aleph_0\}$ be a function and $B = \bigcup_p \{B_p : p \text{ prime}\}$ be a linearly independent subset of $\mathbb{R}$ as a $\mathbb{Q}$-vector space such that the B_p are disjoint and $|B_p| = \chi(p)$. Let

$$G = \sum_p \mathbb{Z}_{(p)} \otimes \langle B_p \rangle,$$

where $\mathbb{Z}_{(p)} = \{n/m \in \mathbb{Q} : \gcd(m,p) = 1\}$ and $\langle B_p \rangle$ is the $\mathbb{Z}$ module generated by B_p. Due to the linear independence of B,

$$G = \bigoplus \{\mathbb{Z}_{(p)} \cdot b : p \text{ prime}, b \in B_p\},$$

and thus

$$[G : pG] = p^{\chi(p)}$$

for every prime p. Letting G inherit the order from $\mathbb{R}$ we get a dense archimedean group with the same property.

Example 2.3: Any discrete archimedean group is isomorphic (as an ordered abelian group) to $\mathbb{Z}$.

Example 2.4: Let $(I, <)$ be an ordered set and for each $i \in I$ let G_i be an ordered abelian group. Let $\prod_{i \in I} G_i$ be the direct product of the groups, as abelian groups. For $f \in \prod_{i \in I} G_i$ we define

$$\operatorname{supp}(f) = \{i \in I : f(i) \neq 0\}.$$

The **Hahn-product** of the G_i is the subgroup

$$H := \{f \in \prod_{i \in I} G_i : \operatorname{supp}(f) \text{ is a well ordered subset of } I\}$$

endowed with an order defined by

$$f < g \Leftrightarrow f(i) < g(i) \text{ where } i = \min \operatorname{supp}(g - f).$$

The subgroup

$$\bigoplus_{i \in I} G_i = \{f \in H : \operatorname{supp}(f) \text{ is finite}\}$$

is called the **lexicographic product\sum.**

2.3. Strong dependence, burden and dp-rank. We recall the basic model-theoretic definitions with which this paper is concerned:

Definition 2.5: Let T be a complete theory and $\mathfrak{C}$ a sufficiently saturated model. All elements and sequences below are taken from $\mathfrak{C}$.

(1) T has an **inp-pattern of depth κ over** A if there are $(b_i^\alpha)_{i<\omega}$, where $\alpha < \kappa$, integers $k^\alpha < \omega$ and formulas $\varphi^\alpha(x, y^\alpha)$ such that each system $\{\varphi^\alpha(x, b_i^\alpha) : i < \omega\}$ is k^α-inconsistent, but for any function $\eta \in \omega^\kappa$ the partial type $\{\varphi^\alpha(x, b_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent.
(2) The **burden** (over A) of T is the supremum over all κ such that there is an inp-pattern of depth κ (over A).
(3) The **dp-rank** (over A) of T is the supremum over all κ such that there is a b and a system of κ sequences mutually indiscernible over A such that none of them is indiscernible over Ab.
(4) For a structure M, define burden(M) and dp-rk(M) over A to be burden(Th(M)) and dp-rk(Th(M)) over A, respectively.
(5) T is **strongly dependent** if there are no $\aleph_0$ mutually indiscernible sequences and b such that none of them is indiscernible over b.

Remark: (1) In the compuation of the dp-rank of a theory T the parameter set A appearing in the definition does not make a difference.
(2) In the definition of an inp-pattern, we may assume the $(b_i^\alpha)_{i<\omega}$ are mutually indiscernible in which case we may require only that $\{\varphi^\alpha(x, b_i^\alpha) : i < \omega\}$ be inconsistent.

The above definitions are tied together by:

Fact 2.6 ([1]): *If T is dependent then burden$(T) =$ dp-rk(T).*

Fact 2.7 ([32, Observation 2.1]): *T is strongly dependent if and only if for any infinite indiscernible sequence $\langle \bar{a}_t : t \in I\rangle$ (the $\bar{a}_t$ may be infinite sequences themselves) and c a singleton there exists a finite convex equivalence relation E on I such that if $s \in I$ then $\langle \bar{a}_t : t \in (s/E)\rangle$ is indiscernible over c.*

In Section 5 Shelah's expansion, $\mathcal{M}^{sh}$ of a structure $\mathcal{M}$, will play an important role. We recall that $\mathcal{M}^{sh}$ is obtained by expanding $\mathcal{M}$ with all externally definable sets. Shelah, [31], shows that if $\mathcal{M}$ is dependent, $\mathcal{M}^{sh}$ has quantifier elimination, and is therefore dependent. It follows immediately from the above definitions (and is well known) that if $\mathcal{M}$ is dp-minimal (resp., strongly dependent) then $\mathcal{M}^{sh}$ is dp-minimal (resp., strongly dependent).

3. Ordered abelian groups with finite spines

We start by defining our main object of interest for the present and the following sections:

Definition 3.1: A pure ordered abelian group G has **finite spines** if $\mathcal{S}_p$ is finite for every prime p.

Remark: If $\mathcal{S}_p$ is finite for all p then $\mathcal{S}_n$ is finite for all n [4, Lemma 2.2].

We will see in Proposition 4.14 that every strongly dependent ordered abelian group has finite spines. We collect a few easy or known facts about groups with finite spines.

Lemma 3.2: *Let G be an ordered abelian group with finite spines. For $n \in \mathbb{N}$ denote*

$$H_n^-(g) := \bigcup \{H_n(h) : g \notin H_n(h), h \in G\}.$$

Then

$$X = \{H_n^-(g) : g \in G\} = \{H_n(g) : g \in G\} = Y$$

for all n.

Proof. Because $\mathcal{S}_n$ is finite and convex subgroups are linearly ordered by inclusion, $X \subseteq Y$. In the other direction, if $H_n(h)$ is maximal within the set X then $H_n(h) = \bigcup \{H_n(g) : h \notin H_n(g)\} = H_n^-(h)$. Otherwise let $x \in H_n(h') \setminus H_n(h)$ where $H_n(h')$ is the immediate successor of $H_n(h)$ in Y. It is easy to see that $H_n^-(x) = H_n(h)$. ∎

Proposition 3.3: *Let G be an ordered abelian group with finite spines. Then $\{G_\alpha : \alpha \in \mathcal{S}_n, n \in \mathbb{N}\}$ are all the definable convex subgroups of G. In particular, there are only countably many definable convex subgroups.*

Proof. By [6, Theorem 4.1],[1] for every definable convex subgroup of (any) ordered abelian group, there exists $n \in \mathbb{N}$ such that

$$H = \bigcap_{g \notin H} H_n^-(g).$$

If G has finite spines, then by Lemma 3.2, $H = H_n(g)$ for some $n \in \mathbb{N}$ and $g \in G$. ∎

[1] By [4, Section 1.5], what Schmitt and Delon–Farré denote by $A_n(g)$ is equal to $H_n^-(g)$.

Quantifier elimination for G with finite spines is considerably simpler than in the case of arbitrary ordered abelian groups:

Proposition 3.4: *Let G be an ordered abelian group with finite spines and let $\{H_i\}_{i<\alpha}$ be its definable convex subgroups (including $\{0\}$) for some $0 < \alpha \leq \omega$. Then G has quantifier elimination in the the following language:*

$$L = L_{oag} \cup \{(x =_{H_i} y + k_{G/H_i})_{k\in\mathbb{Z}, i<\alpha}, (x \equiv_{m,H_i} y + k_{G/H_i})_{k\in\mathbb{Z}, m\in\mathbb{N}, i<\alpha}\},$$

where

- *for each $k \in \mathbb{Z}$, "$x =_H y + k_{G/H}$" is defined by*

  $$\pi(x) = \pi(y) + k_{G/H}$$

  for $\pi : G \to G/H$ and $k_{G/H}$ denotes k times the minimal positive element of G/H, if it exists, and 0 otherwise;
- *for each $k \in \mathbb{Z}$ and each $m \in \mathbb{N}$, "$x \equiv_{m,H} y + k_{G/H}$" is defined by*

  $$\pi(x) \equiv_m \pi(y) + k_{G/H}.$$

Proof. This is a direct consequence of the main theorem of [4]. The auxiliary sorts $\mathcal{T}_n$ and $\mathcal{T}_n^+$ do not add any new convex subgroups because they are unions or intersection of convex subgroups coming from $\mathcal{S}_n$, and $\mathcal{S}_n$ is finite. Also the ternary relation given by $x \equiv_{m,\alpha}^{[m']} y$ if and only if $x - y \in G_\alpha^{[m']} + mG$ is not needed, since by [4, Lemma 2.4], and the finiteness of $\mathcal{S}_n$, $G_\alpha^{[n]} = G_{\alpha'} + nG$ for some $\alpha' \in \mathcal{S}_n$. ∎

Remark: We do not need predicates for $\pi(x) > \pi(y) + k_{G/H}$ since, for example,

$$\pi(x) > \pi(y) + 1_{G/H} \Leftrightarrow x > y \wedge x \neq_H y \wedge x \neq_H y + 1_{G/H}.$$

We will need the following result, due to Schmitt:

Fact 3.5 ([29, Theorem 4.13]): *An ordered abelian group G is elementary equivalent to a lexicographic sum of archimedean groups if and only if for all $n, m \in \mathbb{N}$ and $0 \neq x \in G$ there exists $y \in G$ such that*

$$H_n(x) = H_{n\cdot m}^-(y).$$

The application of the above fact to groups with finite spines is summed up in the next two results:

Corollary 3.6: *Every ordered abelian group with finite spines is elementary equivalent to a lexicographic sum of non-zero archimedean groups.*

Proof. Let $n, m \in \mathbb{N}$ and $0 \neq x \in G$. Since, by [4, Lemma 2.2], $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n\cdot m}$, there exists $z \in G$ such that $H_n(x) = H_{n\cdot m}(z)$ and by Lemma 3.2 there exists $y \in G$ such that

$$H_n(x) = H_{n\cdot m}(z) = H^-_{n\cdot m}(y). \qquad \blacksquare$$

Recall the notation and definitions from Example 2.4.

Lemma 3.7: *Let*

$$G = \bigoplus_{i \in I} G_i$$

be a lexicographic sum of non-zero archimedean groups.

(1) *For $g \notin nG$,*

$$H_n(g) = \{h \in G : \text{for all } k \leq j,\ h(k) = 0\},$$

where j is the smallest index in $\operatorname{supp}(g)$ such that $g(j) \notin nG_j$.

(2) *$\mathcal{S}_p$ is finite if and only if $|\{i \in I : G_i \text{ not } p\text{-divisible}\}| < \infty$.*

Proof. (1) Straightforward calculation.

(2) Let $e_i(j) = \delta_{i,j}$. It follows from (1) above that every $i \in I$ such that G_i is not p-divisible gives a different group $H_p(e_i) \in \mathcal{S}_p$. $\blacksquare$

Example 3.8: A group G with finite spines may be strongly dependent, even dp-minimal even if it has infinitely many definable convex subgroups. For instance,

$$G = \bigoplus_{p \text{ prime}} \mathbb{Z}_{(p)}$$

where $\mathbb{Z}_{(p)}$ is as in Example 2.2. Indeed, since $[G : pG] < \infty$ for every prime p, by [21, Proposition 5.1] G is dp-minimal. By an easy direct calculation G has finite spines (see Proposition 4.14 for an abstract proof).

By Lemma 3.7(1) the definable convex subgroups are all of the form

$$\bigoplus_{p \leq p_0} 0 \oplus \bigoplus_{p > p_0} \mathbb{Z}_{(p)},$$

for prime p_0.

4. Calculating the dp-rank

In the present section we combine the results and observations collected in the previous sections to calculate the dp-rank of ordered abelian groups with finite spines. Let G be an ordered abelian group with finite spines. We consider G as a structure in the language L of Proposition 3.4. The **reduct** of G to the group language is the restriction of G obtained by dropping the order symbol. Namely, it is G considered as a structure in the language:

$$\mathcal{L}_{\mathrm{reduct}} = \mathcal{L}_{\mathrm{grp}} \cup \{(x =_{H_i} y + k_{G/H_i})_{k\in\mathbb{Z},i<\alpha}, (x \equiv_{m,H_i} y + k_{G/H_i})_{k\in\mathbb{Z},m\in\mathbb{N},i<\alpha}\}.$$

Recall that a group $(G, +, 0, \dots)$ is **1-based** if every definable set (of G^n) is a boolean combination of cosets of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups (of G^n). In the following, by **abelian structure** we mean an abelian group A with some predicates for subgroups of powers of A. The key fact about abelian structures is:

Fact 4.1 ([34, Theorem 4.2.8]): *Every abelian structure is 1-based.*

This will allow us to compute the dp-rank of strongly dependent ordered abelian groups by, first, computing the dp-rank of their reduct to the group language (using [14]), and then compute the effect of re-introducing the order on the dp-rank. Of course, quantifier elimination will play a crucial role in this computation. In the next proposition we will prove that G in the language $\mathcal{L}_{\mathrm{reduct}}$ is 1-based. Since we do not, a priori, know what are, in the present setting, all $\emptyset$-definable subgroups (allowing imaginary elements) we will essentially reprove below (Proposition 4.7) a variant of this better suited for our needs.

Proposition 4.2: *The reduct of G to the language $\mathcal{L}_{\mathrm{reduct}}$ is 1-based.*

Proof. Consider G as an abelian group with predicates for $\{H_i\}_{i<\alpha}$; it is 1-based. Adding constants, it is still 1-based (see [27, Remark 4.1.8]). The group G in the language $\mathcal{L}_{\mathrm{reduct}}$ is a reduct of this structure (in fact, they are bi-interpretable), hence it is also 1-based (see [27, Proposition 4.6.4]). ∎

In what follows we will be using the following fact.

Fact 4.3 ([14, Proposition 3.3]): *Let G be a 1-based group. Then there is an inp-pattern of depth κ over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ if and only if there exist $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups $(H_\alpha)_{\alpha<\kappa}$ such that for any $i_0 < \kappa$*

$$\left[\bigcap_{i_0\neq\alpha<\kappa} H_\alpha : \bigcap_{\alpha<\kappa} H_\alpha\right] = \infty.$$

Furthermore, if such subgroups exist, they witness an inp-pattern of depth κ, i.e., there exists an indiscernible array $(b_i^\alpha)_{\alpha<\kappa,i<\omega}$, such that $\{x\in b_i^\alpha H_\alpha\}_{\alpha<\kappa,i<\omega}$ forms an inp-pattern of depth κ.

Remark: The proof of the above actually shows:

(1) If every definable set is a boolean combination of cosets of some family $\mathcal{F}$ of definable groups, then the inp-pattern may be witnessed by intersections of definable groups from $\mathcal{F}$ (see [14, Remark 3.3]).
(2) The collection of subgroups witnessing such an inp-pattern of subgroups has the property that their intersection has unbounded index in any proper subintersection.

Thus, in order to compute the dp-rank we must first study the definable subgroups. We start by collecting some useful well-known observations:

Lemma 4.4: *Let G be an ordered abelian group.*

(1) *Let $A\subseteq B$ and $C\subseteq D$ be subgroups of G; then*
$$(A+D)\cap(B+C)=A+(D\cap B)+C.$$
(2) *Let H be a convex subgroup; then $nG\cap H=nH$.*
(3) *Let $H_1\subseteq\cdots\subseteq H_k$ be convex subgroups and $n_1|n_2|\cdots|n_k$ be integers; then*
$$(n_1H_1+n_2H_k)\cap(n_1H_2\cap n_3H_k)\cap\ldots\cap(n_1H_{k-1}+n_kH_k)\\ =n_1H_1+n_2H_2+\cdots+n_kH_k.$$
(4) *Let H be a subgroup and $n=p_1^{e_1}\cdots p_k^{e_k}$ be the prime decomposition of an integer n; then*
$$H+nG=(H+p_1^{e_1}G)\cap\cdots\cap(H+p_k^{e_k}G).$$

Proof. (1) By an old (and easy) fact due to Dedekind, the lattice of subgroups of an abelian group is modular (i.e., if $x\leq z$ then $x\vee(y\wedge z)=(x\vee y)\wedge z$), so
$$\begin{aligned}(A+D)\cap(B+C)&=C+((A+D)\cap B)\\&=C+(A+(D\cap B))\\&=A+(D\cap B)+C.\end{aligned}$$

(2) Let $h \in nG \cap H$, and write $ng = h$ for $g \in G$. Replacing h with $-h$ if needed, we may assume that $0 < g$. Since $0 < g < ng$ and $ng = h \in H$, convexity of H, $g \in H$.

(3) By induction on k: The case $k = 1$ is clear, so we proceed to the induction step.

$$\begin{aligned}&(n_1H_1 + n_2H_k) \cap \cdots \cap (n_1H_{k-2} + n_{k-1}H_k) \cap (n_1H_{k-1} + n_kH_k)\\ &=(n_1H_1 + n_2H_2 + \cdots + n_{k-2}H_{k-2} + n_{k-1}H_k) \cap (n_1H_{k-1} + n_kH_k).\end{aligned}$$

Since $(n_1H_1 + n_2H_2 + \cdots + n_{k-2}H_{k-2}) \subseteq n_1H_{k-1}$, we may use (1) and thus it is equal to

$$(n_1H_1 + n_2H_2 + \cdots + n_{k-2}H_{k-2}) + (n_{k-1}H_k \cap n_1H_{k-1}) + n_kH_k).$$

Finally, using (2), we get our result.

(4) This is just the Chinese remainder theorem for $\mathbb{Z}$-modules (i.e., abelian groups) in G/H. ∎

The next lemma follows directly from the definition of $H_n(g)$:

Lemma 4.5: *Let G be an ordered abelian group and $H_1 \subsetneq H_2$ be convex subgroups. Then H_2/H_1 is not p-divisible if and only if there exists $H' \in \mathcal{S}_p$ with*

$$H_1 \subseteq H' \subsetneq H_2.$$

The following is a special case of [15, Lemma A.2.1]:

Fact 4.6 ([15, Lemma A.2.1]): *The theory of torsion free abelian groups proves that for every $n_\alpha, \lambda_{\alpha,j} \in \mathbb{Z}$, the formula*

$$\exists \bar{y} \bigwedge_{\alpha \in J} \left(n_\alpha x + \sum_{j=1}^{|\bar{y}|} \lambda_{\alpha,j} y_j = 0 \right)$$

is equivalent to $n|x$ for some integer n, where $n|x$ is shorthand for $\exists\, y(ny{=}x)$.

Proposition 4.7: *Let G be an ordered abelian group with finite spines and $\{H_i\}_{i<\kappa}$ be all the definable convex subgroups of G, where $H_0 = \{0\}$. Then every formula in one variable in the reduct language $\mathcal{L}_{\mathrm{reduct}}$ is a boolean combination of cosets of subgroups of the form*

$$H_i \text{ or } H_i + p^nG, \quad \text{for } n \geq 0 \text{ and } H_i \in \mathcal{S}_p.$$

Proof. For simplicity of notation we assume that the H_i are enumerated by inclusion, i.e., if $\alpha < \beta < \kappa$ then $H_\alpha < H_\beta$.

As in the proof of Proposition 4.2, we may expand the reduct language to

$$\{G, +, 0, \{H_i\}_{i<\kappa}\}$$

(possibly with some constants). In that language, by [15, Theorem A.1.1], every formula $\varphi(x, \bar{b})$ is equivalent to a boolean combination of formulas of the form

$$(*) \qquad \exists \bar{y} \bigwedge_{\alpha \in J} (n_\alpha x + t_\alpha(\bar{b}) + \sum_{j=1}^{|\bar{y}|} \lambda_{\alpha,j} y_j \in H_\alpha),$$

where $n_\alpha, \lambda_{\alpha,j}$ are integers, and $t_\alpha(\bar{x})$ is a term.

Note that if $g_1, g_2 \models \varphi(x, \bar{b})$, then $g_1 - g_2 \models \psi(x)$ where

$$\psi(x) := \exists \bar{y} \bigwedge_{\alpha \in J} \left(n_\alpha x + \sum_{j=1}^{|\bar{y}|} \lambda_{\alpha,j} y_j \in H_\alpha \right).$$

So $\varphi(x, \bar{b})$ defines a coset of the subgroup defined by $\psi(x)$. Thus it will suffice to show that any definable subgroup of G of the form $\psi(x)$ is the intersection of subgroups of the desired form. Since $\psi(x)$ is $\emptyset$-definable, we may apply Corollary 3.6, and assume that

$$G = \bigoplus_{i \in I} G_i,$$

where all the G_i are non-zero archimedean ordered abelian groups.

By Lemma 3.7(1) and Proposition 3.3 all definable convex subgroups of G are of the form

$$H_\alpha = \bigoplus_{j \leq \alpha^-} 0 \oplus \bigoplus_{j > \alpha^-} G_j$$

for some $\alpha^- \in I$.

Because the H_α are enumerated by inclusion, we get—considering each $a_i \in G_i$ separately—that $(a_i)_{i \in I} \models \psi(x)$ if and only if for every $\beta \in J$ and $i \leq \beta^-$ (for β^- as appearing in the above representation of H_β)

$$(**) \qquad a_i \models \exists \bar{y}_i \bigwedge_{\beta \geq \alpha \in J} \left(n_\alpha x_i + \sum_{j=1}^{|\bar{y}_i|} \lambda_{\alpha,j} y_{i,j} = 0 \right).$$

Note that, by Fact 4.6, for a fixed $\beta \in J$ there exists $m_\beta \in \mathbb{N}$ such that the formula $(**)$ is equivalent to $m_\beta \mid x$ (with m_β independent of $i \leq \beta^-$). Assume that $J = \{\beta_1, \ldots, \beta_k\}$, $\beta_1 < \cdots < \beta_k < \kappa$.

Claim: $\psi(G) = H_{\beta_1} + m_{\beta_1} H_{\beta_2} + \cdots + m_{\beta_{k-1}} H_{\beta_k} + m_{\beta_k} G$.

Proof. Let $g = g_{i_1} + \cdots + g_{i_m} \models \psi(x)$, where $\mathrm{supp}(g) = \{i_1, \ldots, i_m\}$ and $i_1 < \cdots < i_m$. Since, clearly, $\psi(G) \supseteq H_{\beta_1}$, we may assume that $i_m \leq \beta_1^-$. So for each $1 \leq j \leq m$ there exists $1 \leq \ell \leq k$ such that $i_j \leq \beta_\ell^-$ so i_j satisfies the corresponding formula $(**)$, implying that, considered in G_{i_j}, $m_{\beta_\ell} \mid g_{i_j}$. So $g_{i_j} \in m_{\beta_\ell} H_{\beta_{\ell-1}}$.

The other inclusion follows in a similar way from the characterization of those elements realizing $\psi(G)$ in $(**)$, and the fact that if $\beta > \beta'$ then

$$\exists \bar{y} \bigwedge_{\beta \geq \alpha \in J} \left(n_\alpha x + \sum_{j=1}^{|\bar{y}|} \lambda_{\alpha,j} y_j = 0 \right)$$

defines a subgroup of

$$\exists \bar{y} \bigwedge_{\beta' \geq \alpha \in J} \left(n_\alpha x + \sum_{j=1}^{|\bar{y}|} \lambda_{\alpha,j} y_j = 0 \right). \qquad \blacksquare \text{(claim)}$$

Since for any natural numbers n, m and $H \subseteq H'$ convex subgroups,

$$nH + mH' = nH + mH + mH' = \gcd(n,m)H + mH',$$

we may assume that $m_{\beta_1} | m_{\beta_2} | \cdots | m_{\beta_k}$ and that all the m_{β_i} are distinct. By Lemma 4.4(3) this implies that $\psi(G)$ is the intersection of subgroups of the form $n_i H_i \cap n_j H_j$. Finally, we finish by applying Lemma 4.4(4) with the observation that for every $n \mid m$ and convex subgroups $H \subseteq H'$,

$$nH + mH' = (\{0\} + nH') \cap (H + mH')$$

and

$$H + mH' = (H + mG) \cap (H' + \{0\}).$$

To show that H_i can be taken in $\mathcal{S}_p$, consider the subgroup $H_i + p^n G$. If $H_i \notin \mathcal{S}_p$, then let $H \in \mathcal{S}_p \cup \{G\}$ be such that there is no $H' \in \mathcal{S}_p$ with $H_i \subsetneq H' \subsetneq H$. Since $\mathcal{S}_p$ is finite such a subgroup H exists. By Lemma 4.5, H/H_i must be p-divisible. Thus $H_i = H + p^n H$ so $H_i + p^n G = H + p^n G$. $\blacksquare$

Remark: For future reference we note that the proof of the previous proposition shows that any p.p. formula $\varphi(x, \bar{b})$ as in $(*)$ defines a coset of a $\emptyset$-definable group $A \leq G$ not depending on the constant $\bar{b}$ (or indeed, on the terms t_α as in $(*)$).

We will first compute the dp-rank of G in the reduct language.

Notation: Let G be an ordered abelian group with finite spines and p a prime. Denote by k_p the maximal n for which there exist definable convex subgroups $H_0 \subsetneq \cdots \subsetneq H_{n-1} \subsetneq H_n = G$ such that for all $i < n$,

$$[H_{i+1}/H_i : p(H_{i+1}/H_i)] = \infty.$$

LEMMA 4.8: *If G has finite spines and $k_p = n$ there are $H_0 \subsetneq \cdots \subsetneq H_{n-1}$ witnessing it such that $H_i \in \mathcal{S}_p$ for all i.*

Proof. Take any sequence $H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{k_p-1}$ of definable convex subgroups with

$$[H_{i+1}/H_i : p(H_{i+1}/H_i)] = \infty$$

for every $0 \leq i \leq k_p - 1$. Choose the H_i so that a maximal number among them is in $\mathcal{S}_p$. Assume towards a contradiction that there is some $H_i \notin \mathcal{S}_p$. By Lemma 4.5 there exists $H \in \mathcal{S}_p$ with $H_i \subseteq H \subsetneq H_{i+1}$. Take such a subgroup H which is minimal possible (such a minimal subgroup exists because $\mathcal{S}_p$ is finite). Then (excatly) one of $[H_{i+1}/H : p(H_{i+1}/H)] = \infty$ or $[H/H_i : p(H/H_i)] = \infty$. By minimality of H it must be that H_i/H is p-divisible (otherwise, apply Lemma 4.5 with H_i and H), so

$$[H_{i+1}/H : p(H_{i+1}/H)] = \infty.$$

Replacing H_i with H we get a contradiction to the choice of the sequence $H_0, \ldots, H_{k_p-1}$. ∎

Remark: The above lemma gives a simple way of computing k_p. Writing

$$\mathcal{S}_p = \{H_0, \ldots, H_{n-1}\}$$

and denoting

$$\mathcal{S}_p^\infty = \{H_i \in \mathcal{S}_p : [H_{i+1}/H_i : p(H_{i+1}/H_i)] = \infty\},$$

with $H_n = G$, the previous lemma shows that $|\mathcal{S}_p^\infty| = k_p$.

Since $\mathcal{S}_p^\infty \subseteq \mathcal{S}_p$ we immediately have:

LEMMA 4.9: *If G is an ordered abelian group with finite spines and p a prime, then k_p is finite.*

In the following proposition we study subgroups of the form $H_i + p^{e_i}G$ and of the form H_i. In order to avoid dividing into cases, we will allow $e_i = \infty$ with the convention that $H_i + p_i^\infty G = H_i$.

Lemma 4.10: *Let G be an ordered abelian group with finite spines, p a prime number, $H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{n-1}$ definable convex subgroups and $e_0 < \cdots < e_{n-1}$, where e_{n-1} may be ∞. If $e_{n-1} \neq \infty$, then for every $r < n$,*

$$\Big[\bigcap_{r\neq i<n}(H_i + p^{e_i}G) : \bigcap_{i<n}(H_i + p^{e_i}G)\Big] = \infty$$
$$\iff [H_{r+1}/H_r : p(H_{r+1}/H_r)] = \infty.$$

If $e_{n-1} = \infty$ then it is always true that

$$\Big[\bigcap_{i<n-1}(H_i + p^{e_i}G) : \bigcap_{i<n}(H_i + p^{e_i}G)\Big] = \infty,$$

and for every $r < n-1$,

$$\Big[\bigcap_{r\neq i<n}(H_i + p^{e_i}G) : \bigcap_{i<n}(H_i + p^{e_i}G)\Big] = \infty$$
$$\iff [H_{r+1}/H_r : p(H_{r+1}/H_r)] = \infty.$$

Proof. As the lemma is elementary and involves no parameters, we may assume that

$$G = \bigoplus_{i\in I} G_i,$$

where all the G_i are non-zero archimedean ordered abelian groups. For every $i < n$ there exists $i^- \in I$ such that

$$H_i = \bigoplus_{j\leq i^-} 0 \oplus \bigoplus_{j>i^-} G_j.$$

By Lemma 4.4(3), $\bigcap_{i<n}(H_i + p^{e_i}G)$ is equal to

$$\begin{aligned} &H_0 + p^{e_0}H_1 + p^{e_1}H_2 + \cdots + p^{e_{n-2}}H_{n-1} + p^{e_{n-1}}G \\ &\quad = p^{e_{n-1}} \cdot \Big(\bigoplus_{j\leq(n-1)^-} G_j\Big) \oplus \cdots \oplus p^{e_0} \cdot \Big(\bigoplus_{1^-<j\leq 0^-} G_j\Big) \oplus \bigoplus_{j>0^-} G_j. \end{aligned}$$

Likewise, $\bigcap_{r\neq i<n}(H_i + p^{e_i}G)$ is equal to

$$H_0 + p^{e_0}H_1 + \cdots + p^{e_{r-1}}H_{r+1} + p^{e_{r+1}}H_{r+2} + \cdots + p^{e_{n-2}}H_{n-1} + p^{e_{n-1}}G.$$

Thus, $[\bigcap_{r\neq i<n}(H_i + p^{e_i}G) : \bigcap_{i<n}(H_i + p^{e_i}G)] = \infty$ is equivalent to

$$\Big[p^{e_{r-1}}\Big(\bigoplus_{(r+1)^-<j\leq r^-} G_j\Big) : p^{e_r}\Big(\bigoplus_{(r+1)^-<j\leq r^-} G_j\Big)\Big] = \infty,$$

which is equivalent (if $e_r \neq \infty$), since G is torsion-free, to

$$[H_{r+1}/H_r : p(H_{r+1}/H_r)] = \infty.$$

The case where $e_{n-1} = \infty$ is obvious since for all i we have that

$$[H_{i+1} : H_i] = \infty. \qquad \blacksquare$$

Before computing the dp-rank we combine the above results to obtain:

Proposition 4.11: *Let G be an ordered abelian group with finite spines, $\{H_i\}_{i<\kappa\leq\omega}$ a collection of definable convex subgroups with $H_i \subsetneq H_j$ if $i < j$ and $\{e_i\}_{i<\kappa} \subseteq \mathbb{N} \cup \{0, \infty\}$. Assume that for every $i_0 < \kappa$*

$$\left[\bigcap_{i_0\neq i<\kappa} (H_i + p^{e_i}G) : \bigcap_{i<\kappa}(H_i + p^{e_i}G)\right] = \infty.$$

Then

(1) *$e_i \neq 0$ for every $i < \kappa$, and if $e_{i_0} = \infty$ then i_0 is maximal in κ;*
(2) *$i < j < \kappa$ if and only if $e_i < e_j$;*
(3) *$\kappa \leq k_p + 1$, and is, therefore, finite.*

Proof. (1) If $e_{i_0} = 0$, then $H_{i_0} + p^{e_{i_0}}G = G$ and thus

$$\left[\bigcap_{i_0\neq i<\kappa} (H_i + p^{e_i}G) : \bigcap_{i<\kappa}(H_i + p^{e_i}G)\right] = 1.$$

If $e_{i_0} = \infty$, then $H_{i_0} + p^{e_{i_0}}G = H_i$ and note that i_0 must be maximal in κ. For otherwise, if $i_0 < i_1$, then $H_{i_0} \subsetneq H_{i_1} + p^{e_{i_1}}G$ and thus

$$\left[\bigcap_{i_1\neq i<\kappa} (H_i + p^{e_i}G) : \bigcap_{i<\kappa}(H_i + p^{e_i}G)\right] = 1.$$

(2) Assume $i < j < \kappa$, and by the above we may assume that $e_j \neq \infty$. If $e_j \leq e_i$ then, since $H_i \subsetneq H_j$, $H_i + p^{e_i}G \subsetneq H_j + p^{e_j}G$ leading to that same contradiction as above. Since both the index set and the set of e_i are ordered, this also proves the other implication.

(3) For any group, G, if $H, K \leq G$ are subgroups then $[K : K \cap H] \leq [G : H]$. So any (finite) sub-family of the H_i will also satisfy the assumptions of the lemma (with the associated e_i). So assume towards a contradiction that $\kappa > k_p + 1$ and fix a sub-family of size $k_p + 2$ of the H_i. Applying Lemma 4.10 to this sub-family, we see that this sub-family witnesses $k_p > k_p + 1$, which is absurd. $\blacksquare$

Notation: (1) For an abelian group G, let

$$\mathbb{P}_\infty(G) = \{p \text{ prime} : [G : pG] = \infty\}.$$

(2) For an ordered abelian group G with finite spines and prime $p \in \mathbb{P}_\infty(G)$ let H_p denote the maximal element in $\mathcal{S}_p^\infty$. If there exists a definable convex subgroup strictly containing all the $\{H_p\}_{p\in\mathbb{P}_\infty(G)}$ then set $c_G = 1$, otherwise set $c_G = 0$.

The need for introducing the error-term c_G is illustrated in the following example:

Example 4.12: Let A_1, A_2 be archimedean ordered abelian groups such that A_1 is p-divisible for every prime $p \neq 2$, $[A_1 : 2A_1] = \infty$, and A_2 p-divisible for every prime $p \neq 2, 3$ and $[A_2 : 2A_2] = [A_2 : 3A_2] = \infty$. Consider $G_1 = \mathbb{Q} \oplus A_1 \oplus A_2$ and $G_2 = \mathbb{Z} \oplus A_1 \oplus A_2$.

The convex subgroups of G_1, G_2 are 0, A_2, $A_1 \oplus A_2$ (as direct summands) and G_i. Among those 0 and A_2 are definable in both (as $H_2(g)$ for $g \in 0\oplus H_1 \oplus 0$ with $g \notin 2G$). For similar reasons $H_1 \oplus H_2$ is in $\mathcal{S}_2(G_2)$ but not in $\mathcal{S}_2(G_1)$. It follows that

$$\mathcal{S}_2^\infty(G_i) = \{0, H_2\}$$

(because in G_2 we have that $[G_2/(H_1 \oplus H_2) : 2(G/(H_1 \oplus H_2))] = 2$).

Similar arguments show that $S_3 = \{0\}$ in both groups. Thus, $k_2 = 2$ and $k_3 = 1$ in both groups. It follows from Proposition 3.3 that $H_1 \oplus H_2$ is not definable in G_1. So $c_{G_2} = 1$ whereas $c_{G_1} = 0$. We will see in the next proposition that, despite the fact that k_p is equal in both groups for all p, in the reduct language, dp-rk$(G_1) = 3$ whereas dp-rk$(G_2) = 4$.

Proposition 4.13: *Let G be an ordered abelian group with finite spines, considered in the reduct language. Then* dp-rk(G) *is equal to*

$$(\star) \qquad \begin{cases} c_G + \sum_{p\in\mathbb{P}_\infty(G)} k_p & \text{if } \mathbb{P}_\infty(G) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $\{H_\alpha\}_{\alpha<\gamma\le\omega}$ be the definable convex subgroups of G. If dp-rk$(G) = \kappa$, then by Fact 4.3 we may find definable subgroups $\{N_\beta\}_{\beta<\kappa}$, and an indiscernible array $(b_s^\beta)_{s<\omega,\beta<\kappa}$ such that

$$\{x \in b_s^\beta + N_\beta\}_{s<\omega,\beta<\kappa}$$

is an inp-pattern of depth κ. Furthermore, by the remark following Fact 4.3 and Proposition 4.7, we may assume that the N_β appearing in such an inp-pattern are of the form $H_i + p^{e_i}G$, where possibly $e_i = \infty$ (recall the convention that $H_i + p^\infty G = H_i$). So we fix once and for all such an inp-pattern of maximal depth. Call a prime p meaningful for H if some $H + p^{e_i}G$ appears in our fixed inp-pattern (with $e_i < \infty$). Call p meaningful if it is meaningful for some H.

Fix a meaningful p and let

$$\{H_\alpha + p^{e_\alpha}G\}_{\alpha<m_p}$$

be the family of all occurrences of p in our fixed inp-pattern. By Fact 4.3, for every $i_0 < m_p$

$$\Big[\bigcap_{i_0\neq\alpha<m_p}(H_\alpha + p^{e_\alpha}G) : \bigcap_{\alpha<m_p}(H_\alpha + p^{e_\alpha}G)\Big] = \infty.$$

This implies that if $i \neq j$ then $H_i \neq H_j$. Otherwise, assuming without loss of generality that $e_i < e_j$, we would get

$$\Big[\bigcap_{i\neq\alpha<m_p}(H_\alpha + p^{e_\alpha}G) : \bigcap_{\alpha<m_p}(H_\alpha + p^{e_\alpha}G)\Big] = 1.$$

This allows us to apply Proposition 4.11, with the implication that $m_p \leq k_p + 1$ (in particular m_p is finite), if $H_\alpha \subsetneq H_\beta$ then $e_\alpha < e_\beta$ and $e_\alpha \neq 0$ for every $\alpha < m_p$. Also, note that necessarily $[G : pG] = \infty$, for, otherwise, this would entail $[G : H_\alpha + p^{e_\alpha}G] < \infty$ which, as noted in the proof of Proposition 4.11(3), is impossible.

Summing up the above observations, we may assume the inp-pattern is witnessed by a family of subgroups

$$\{H_{\alpha_p} + p^{e_{\alpha_p}}G\}_{\alpha_p<m_p, p\in\mathbb{P}_\infty(G)}.$$

There can be only one prime p for which $e_{\alpha_p} = \infty$, for some α_p. For otherwise, we would have $H_\alpha \subsetneq H_\beta$ both arising as subgroups in the inp-pattern but this can not be (as already mentioned above). Hence for all but (maybe) one prime p, $m_p \leq k_p$. This proves that $(\star)$ is an upper bound on dp-rk(G) and that $\kappa \leq \omega$ with equality possible only if $\mathbb{P}_\infty(G)$ is infinite. We will now show that this bound is attained.

If $\mathbb{P}_\infty(G)$ is empty, then any sequence of pairwise distinct elements gives an inp-pattern of depth 1 (with the formula $x = y$) so assume that $\mathbb{P}_\infty(G) \neq \emptyset$. Let $p \in \mathbb{P}_\infty(G)$ and let $H_{i_{p,1}} \subsetneq \cdots \subsetneq H_{i_{p,k_p}}$ be $\mathcal{S}_p^\infty \cup \{G\}$ (so $H_{i_{p,k_p+1}} = G$).

Claim: *The subgroups*

$$\{H_{i_{p,1}} + pG, \ldots, H_{i_{p,k_p}} + p^{k_p}G\}$$

witness an inp-pattern of depth k_p

Proof. This follows from the paragraph concluding the statement of Fact 4.3 using Lemma 4.10. To apply this last lemma note that the groups $H_{i_{p,j}}$ were chosen specifically so that they satisfy the assumptions of the lemma. ∎(claim)

As a result, for every $p \in \mathbb{P}_\infty(G)$ we have an inp-pattern of depth k_p. The next claim shows that we can combine these inp-patterns into one large pattern:

Claim: *The subgroups*

$$\bigcup_{p \in \mathbb{P}_\infty(G)} \{H_{i_{p,1}} + pG, \ldots, H_{i_{p,k_p}} + p^{k_p}G\}$$

witness an inp-pattern of depth $\sum_{p \in \mathbb{P}_\infty(G)} k_p$.

Proof. Since cosets are always 2-inconsistent, we only need to check the consistency part of the definition. For each $p \in \mathbb{P}_\infty(G)$, consider a collection

$$\{H_{i_{p,1}} + pG + b_{i_{p,1}}, \ldots, H_{i_{p,k_p}} + p^{k_p}G + b_{i_{p,k_p}}\}$$

of cosets with non-empty intersection. Note that if b_p is any element witnessing this, then

$$\{H_{i_{p,1}} + pG + b_p, \ldots, H_{i_{p,k_p}} + p^{k_p}G + b_p\}$$

defines the exact same set. So our task is to show that the (partial) type

$$(\dagger) \qquad \bigcup_{p \in \mathbb{P}_\infty(G)} \{x \in H_{i_{p,1}} + pG + b_p, \ldots, x \in H_{i_{p,k_p}} + p^{k_p}G + b_p\}$$

is consistent. Note that for every p,

$$p^{k_p}G + b_p \subseteq (H_{i_{p,1}} + pG + b_p) \cap \cdots \cap (H_{i_{p,k_p}} + p^{k_p}G + b_p).$$

By the Chinese remainder theorem for abelian groups, there is an element $b \in G$ such that $b \equiv_{p^{k_p}G} b_p$ for all $p \in \mathbb{P}_\infty(G)$, proving the consistency of the type (†) and finishing the proof of the claim. ∎(claim)

The last claim finishes the proof of the proposition in case $c_G = 0$. If $c_G = 1$ let H be a definable convex subgroup witnessing it. Consider the following collection of definable subgroups from above

$$\bigcup_{p\in\mathbb{P}_\infty(G)} \{H_{i_p,j}+p^jG\}_{1\le j\le k_p}.$$

Enumerate these subgroups by $\{A_\alpha\}_{\alpha<\lambda}$, where $\lambda=\sum_{p\in\mathbb{P}_\infty(G)} k_p$. Let $A_\lambda=H$. We will show that for every $i_0<\lambda+1$

$$\left[\bigcap_{i_0\neq\alpha<\lambda+1} A_\alpha : \bigcap_{\alpha<\lambda+1} A_\alpha\right]=\infty.$$

If $i_0=\lambda$ then

$$\left[\bigcap_{\alpha<\lambda} A_\alpha : \bigcap_{\alpha<\lambda+1} A_\alpha\right]=\infty,$$

as in Lemma 4.10. If $i_0<\lambda$ then we need to show that

$$\left[\bigcap_{i_0\neq\alpha<\lambda} A_\alpha\cap H : \bigcap_{\alpha<\lambda} A_\alpha\cap H\right]=\infty.$$

But since $(H_{i_p,j}+p^jG)\cap H=H_{i_p,j}+p^jH$, this boils down to showing that the index is ∞ when we do the calculation inside H. By quantifier elimination H is a stably embedded convex subgroup of G. As a result, this follows from the same calculation we conducted when $c_G=0$. ∎

The following argument is similar to the one given by Farré in [9, Theorem 6.2].

Proposition 4.14: *Let G be an ordered abelian group, possibly with additional structure. If G is strongly dependent then G has finite spines, i.e., $\mathcal{S}_p$ is finite for all p, and $\mathbb{P}_\infty(G)$ is finite.*

Proof. If $\mathbb{P}_\infty(G)$ is infinite then G is already not strongly dependent in the group language (see, for example, [14]).

Since $\mathcal{S}_p$ is an interpretable linear order, in order to show that it is finite it is enough, by compactness, to show that it has no infinite ascending chain. By Lemma 4.5, if $H_1\subsetneq H_2$ are in $\mathcal{S}_p$ then H_2/H_1 is not p-divisible. Therefore, if G is sufficiently saturated and $\mathcal{S}_p$ is infinite we can find for all n an increasing sequence $\langle\alpha_i\in\mathcal{S}_p : i<\omega\rangle$ with

$$[G_{\alpha_{i+1}}/G_{\alpha_i} : p(G_{\alpha_{i+1}}/G_{\alpha_i})]>n,$$

for every $i < \omega$. By compactness and saturation we may find such an increasing sequence $\langle \beta_i \in \mathcal{S}_p : i < \omega \rangle$ such that for every $i < \omega$

$$[G_{\beta_{i+1}}/G_{\beta_i} : p(G_{\beta_{i+1}}/G_{\beta_i})] = \infty.$$

As in the proof of Proposition 4.13, by Lemma 4.10 and Fact 4.3, the definable subgroups

$$\{G_{\beta_i} + p^{i+1}G : i < \omega\}$$

witness an inp-pattern of depth ω, contradicting strong dependence. ∎

We now proceed to reintroducing the order:

Lemma 4.15: *Let $(G; +, -, 0, <, \dots)$ be an ordered abelian group, possibly with some more relational symbols and constants, admitting quantifier elimination. Let $c \in G$ and $I_1 = \langle a_i : i < \omega \rangle$, $I_2 = \langle b_i : i < \omega \rangle$ be mutually indiscernible sequences which are also indiscernible over c in the language without the order. Then at least one of I_1, I_2 is indiscernible over c in the full language.*

Proof. Every term $t(x_1, \dots, x_n)$ is equivalent to a term of the form

$$\sum_{i=1}^{n} z_i \cdot x_i + d,$$

where $z_i \in \mathbb{Z}$ and d is a $\mathbb{Z}$-linear combination of constants. Thus every quantifier free formula in the ordered group language, not using equality, is equivalent to

$$\sum_{i=1}^{n} a_i \cdot x_i + d > 0.$$

Assume towards a contradiction that there are terms $t_1(\bar{x})$ and $t_2(\bar{y})$, of the above form, such that $t_1(\bar{a}_I) < zc < t_1(\bar{a}_{I'})$ and $t_2(\bar{b}_J) < wc < t_2(\bar{b}_{J'})$ where $w, z \in \mathbb{N}$ and $I, I', J, J' \subseteq \omega$ are some index sets of the appropriate lengths. By replacing t_1 with wt_1, t_2 with zt_2 and c with wzc we may assume that

$$t_1(\bar{a}_I) < c < t_1(\bar{a}_{I'})$$

and that

$$t_2(\bar{b}_J) < c < t_2(\bar{b}_{J'}).$$

Without loss of generality $t_2(\bar{b}_J) \leq t_1(\bar{a}_I) < c$ so $t_1(\bar{a}_I) < t_2(\bar{b}'_J)$, contradicting mutual indiscernibility. ∎

PROPOSITION 4.16: *Let $(G;+,-,0,<,\{H_i\}_{i<\omega})$ be an ordered abelian group with finite spines, possibly with some more constants, admitting quantifier elimination. Then, in the above notation,*

$$\text{dp-rk}(G) \leq \sum_{p\in\mathbb{P}_\infty(G)} k_p + 1.$$

In particular, if $c_G = 1$ then

$$\text{dp-rk}_{\text{reduct}}(G) = \text{dp-rk}(G).$$

Proof. Because G has finite spines dp-rk$(G) \leq \aleph_0$, and in case dp-rk$(G) = \aleph_0$ we get from Lemma 4.9 combined with Proposition 4.13 and the previous lemma that $|\mathbb{P}_\infty(G)| = \infty$, and the result follows. So we assume that G is of finite dp-rank. Let $\kappa = \sum_{p\in\mathbb{P}_\infty(G)} k_p$ and $\langle I_i : i < \kappa+2\rangle$ be a sequence of mutually indiscernible sequences. Fix some $c \in G$. We will show that at least one of the I_i is indiscernible over c.

If there are two sequences I_{i_1} and I_{i_2}, both indiscernible over c in the reduct language then by Lemma 4.15 at least one of them is indiscernible over c in the full language. We may thus assume that there is at most one of the I_i which is indiscernible over c in the reduct language and as dp-rk$_{\text{reduct}}(G) \leq \kappa+1$, such I_i does exist.

Assume, without loss of generality, that I_0 is indiscernible over c in the reduct language, but not indiscernible over c in the full language; furthermore, assume that for $i > 0$, I_i is not indiscernible over c in the reduct language. Consequently, for each such $i > 0$, there is a formula $\varphi_i(\bar{x}_i, c)$ in the reduct language witnessing this. Namely, if $I_i = \langle a_{i,j} : j < \omega\rangle$ then $\varphi_i(a_{J_{i,1}}, c)$ and $\neg\varphi_i(a_{J_{i,2}}, c)$ for some $J_{i,1}, J_{i,2} \subseteq \{i\}\times\omega$ of the same order type.

Note that if $\varphi_i(\bar{x}, c)$ is a boolean combination of some formulas, then already one of the formulas in this combination witnesses non-indiscernibility over c. By Proposition 4.7 and the remark following it we may assume that for every tuple $\bar{b}$ and each $\varphi_i(\bar{x}_i, x)$, the formula $\varphi_i(\bar{b}, x)$ defines a coset of A_i, where A_i is a definable subgroup of the form H_i or $H_i + p_i^{e_i}G$ with $H_i \in \mathcal{S}_{p_i}$.

CLAIM: (1) *For every distinct $i, j > 0$, $A_i \not\subseteq A_j$.*

(2) *For every distinct $i, j > 0$, if A_i, A_j are of the form $H_i + p^{e_i}G$, $H_j + p^{e_j}G$ then $p \in \mathbb{P}_\infty(G)$, and either $[A_i : A_j \cap A_i] = \infty$ or $[A_i : A_j \cap A_i] = \infty$.*

(3) *For every $i > 0$, A_i is not of the form H_i.*

Proof. Let $i,j>0$ be distinct, and assume that $A_i \subseteq A_j$. Thus $\varphi_i(a_{J_{i,1}},x)$ also defines a coset of A_j, and since c satisfies $\varphi_i(a_{J_{i,1}},x)\wedge\varphi_j(a_{J_{j,1}},x)$, necessarily

$$\varphi_i(a_{J_{i,1}},x)\rightarrow\varphi_j(a_{J_{j,1}},x).$$

Therefore, by mutual indiscernibility,

$$\varphi_i(a_{J_{i,1}},x)\rightarrow\varphi_j(a_{J_{i,2}},x)$$

but this contradicts the fact that $\neg\varphi_j(a_{J_{j,2}},c)$. This gives (1).

As for (2), let I_i' and I_j' be sequences of tuples of order type $J_{i,1}$ (resp. $J_{j,2}$) tuples in I_i (resp. I_j) such that the convex hulls of any two tuples are disjoint. By mutual indiscernibility of I_i and I_j, I_i' and I_j' are also mutual indiscernible sequences, and together with $\varphi_i(\bar{x}_i,x)$ and $\varphi_j(\bar{x}_j,x)$ they form an inp-pattern of depth 2. Indeed, inconsistency is clear, as for consistency,

$$c\models\varphi_i(a_{J_{i,1}},x)\wedge\varphi_j(a_{J_{j,1}},x),$$

but since I_i' and I_j' are mutually indiscernible any path in the array is consistent. By Fact 4.3 the desired conclusion follows. Note that this also proves that $p\in\mathbb{P}_\infty(G)$, for otherwise $[G:A_i]<\infty$.

Finally, for (3), if A_i is of the form H_i then, since I_0 is not indiscernible over c, we can find $J_0,J_0'\subseteq\{0\}\times\omega$ and $J_{i,1},J_{i,2}\subseteq\{i\}\times\omega$ such that, after replacing c with mc for some $m\in\mathbb{Z}$, we get

$$t_0(a_{J_0})>c \text{ but } t_0(a_{J_0'})<c,$$

and

$$\varphi_i(a_{J_{i,1}},c) \text{ but } \neg\varphi_i(a_{J_{i,2}},c).$$

We may assume that $c>0$. Note that by indiscernibility of I_i necessarily $c\notin H_i$, indeed otherwise

$$\varphi_i(a_{J_{i,1}},x)\leftrightarrow x\in H_i.$$

Since H_i is convex, necessarily

$$\neg\varphi_i(a_{J_{i,2}},t_0(a_{J_0}))$$

and

$$\varphi_i(a_{J_{i,1}},t_0(a_{J_0'})),$$

which contradicts mutual indiscernibility. ∎(claim)

We can now finish the proof. If $c_G = 0$, then $\text{dp-rk}(G) = \kappa$ and so we must have two sequences which are indiscernible over c in the reduct language, so we finish by Lemma 4.15. Otherwise, $c_G = 1$ and $\text{dp-rk}_{\text{reduct}}(G) = \kappa + 1$. By (3) of the above claim for all $i > 0$, if A_i is a definable group appearing above, then A_i is of the form $H_i + p_i^{e_i} G$ with $e_i < \infty$. By (2) of the claim and Lemma 4.8, if $\mathcal{A}_p$ is the collection of all groups A_i above associated with the same prime p, then $|\mathcal{A}_p| \leq k_p$. By (2) again all primes p appearing above belong to $\mathbb{P}_\infty(G)$ and by (1) of the claim they are, in particular, distinct. So, all in all, there are at most

$$\kappa = \sum_{p \in \mathbb{P}_\infty} k_p$$

groups A_i appearing in the above. By assumption, I_i is associated with some definable group A_i for all $0 \leq i \leq k+1$. This is a contradiction. ∎

The following example shows that quantifier elimination is essential for the proposition.

Example 4.17: In the notation of Example 2.2 consider

$$G = \bigoplus_{i<\omega} \mathbb{Z}_{(2)}$$

in the language of ordered abelian groups. It has infinitely many definable convex subgroups. Indeed, fixing

$$e_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

we get that the groups $H_2(e_i)$ (in the sense of Section 2.1) are all definable and distinct for $i < \omega$. But by Proposition 4.14, G is not strongly dependent. On the other hand, as an abelian group G is dp-minimal (see, for example, [14]).

Summing up all of the above we can finally conclude our computation of the dp-rank:

PROPOSITION 4.18: *Let G be an ordered abelian groups with finite spines. Then*

$$\text{dp-rk}(G) = 1 + \sum_{p \in \mathbb{P}(G)} k_p.$$

Proof. If $[G : pG] < \infty$ for every prime p, then G is dp-minimal by [21, Proposition 5.1] in which case the proposition holds. So we may assume this is not the case.

Case 1: Assume $c_G = 0$, and hence

$$\text{dp-rk}_{\text{reduct}}(G) = \sum_{p\in\mathbb{P}_\infty(G)} k_p.$$

Since G is not dp-minimal, there exists a prime q with $[G : qG] = \infty$. By Corollary 3.6, we may assume that

$$G = \bigoplus_{i\in I} G_i,$$

where the G_i are non zero archimedean groups. Since every discrete archimedean ordered abelian group is isomorphic to $\mathbb{Z}$, the existence of a prime q such that $[G : qG] = \infty$, and the fact that G is with finite spines, guarantee the existence of a dense archimedean G_j.

Let $(b_i)_{i<\omega}$ be an ascending indiscernible sequence of elements of the ordered set G_j and C_i be the definable convex subset defined by

$$x \in ((\dots, 0, b_i, 0, \dots), (\dots, 0, b_{i+1}, 0, \dots)).$$

The proof of Proposition 4.13 provides an inp-pattern witnessing the fact that

$$\text{dp-rk}_{\text{reduct}}(G) = \sum_{p\in\mathbb{P}(G)} k_p.$$

Our goal is to augment this inp-pattern by adjoining the formulas $\{x \in C_i\}_{i<\omega}$. By Proposition 4.16, it will suffice to show that this augmented pattern is an inp-pattern. Since inconsistency is automatic, we only have to check consistency of paths. As before, since $c_G = 0$ and

$$p^{e_k}G \subseteq (H_{i_1} + p^{e_1}G) \cap \dots \cap (H_{i_k} + p^{e_k}G),$$

we only need to show that $nG \cap C_i$ is consistent for every $n \in \mathbb{N}$ and $i < \omega$. This is an easy exercise (see, e.g., [3, Lemma 1.1]).

Case 2: If $c_G = 1$, the result is given by Proposition 4.16. ∎

Corollary 4.19: *Let G_1 and G_2 be ordered abelian groups with finite spines; then*

$$\text{dp-rk}(G_1 \oplus G_2) = \text{dp-rk}(G_1) + \text{dp-rk}(G_2) - 1.$$

Finally we obtain as a direct corollary of Propositions 4.14 and 4.18:

Theorem 4.20: *Let G be an ordered abelian group. The following are equivalent:*

1. *G is strongly dependent;*
2. dp-rk$(G) < \aleph_0$*;*
3. *G is with finite spines and $|\mathbb{P}_\infty(G)| < \infty$;*
4. *G is elementary equivalent to a lexicographic sum $\bigoplus_{i\in I} G_i$, where*
   (a) *for every prime p, $|\{i \in I : pG \neq G\}| < \infty$ and*
   (b) *$[G_i : pG_i] = \infty$ for only finitely many primes p.*

Proof. (1)⇒(2): If G is strongly dependent, then by Proposition 4.14, G has finite spines and $\mathbb{P}_\infty(G)$ is finite and thus dp-rk(G) is finite by Proposition 4.18.

(2)⇒(3): Since every structure of finite dp-rank is strongly dependent (see Definition 2.5), the result follows from Proposition 4.14.

(3)⇒(4): An ordered abelian group with finite spines is elementary equivalent to a lexicographic sum of non zero archimedean groups by Corollary 3.6. The rest follows from the analysis in Lemma 3.7.

(4)⇒(1): Again, by Lemma 3.7 it is easily seen that G has finite spines and that $\mathbb{P}_\infty(G)$ is finite. Thus by Proposition 4.18, G has finite dp-rank and thus strongly dependent. ∎

The following is now easy:

Corollary 4.21: *Let G be an ordered abelian group, $H \leq G$ a convex subgroup. If G/H and H are strongly dependent as pure ordered abelian groups, then so is G.*

Proof. We readily get that $\mathbb{P}_\infty(G) = \mathbb{P}_\infty(H)+\mathbb{P}_\infty(G/H)$. Similarly, the p-spine of G is naturally isomorphic to the ordered union of the p-spine of H and the p-spine of G/H. ∎

5. Strongly dependent Henselian fields

As an application of our results on strongly dependent ordered abelian groups we show that if (K, v) is Henselian, with K strongly dependent (as a pure field), then (K, v) is strongly dependent. The heart of the proof, and the main new ingredient, will be showing that the value group vK is strongly dependent. To conclude we adapt a transfer theorem (due, essentially, to Jahnke, [17], after Johnson, [22]) to the strongly dependent setting. For a valued field (K, v) we denote by vK its value group, Kv its residue field and $\mathcal{O}_v$ its valuation ring. All other standard valuation-theoretic terminology used in this section can be found in any textbook on the subject, e.g, [25] or [8].

The following fact will be used repeatedly

Fact 5.1 ([32, Proof of Claim 5.40]): *Every strongly dependent field is perfect.*

First we show that the residue field must be strongly dependent, hence perfect.

Proposition 5.2: *Let K be a strongly dependent field and let v be a Henselian valuation on K. Then Kv is strongly dependent.*

Proof. If Kv is not separably closed, then v is definable in K^{sh}, the Shelah expansion of K ([17, Theorem A]), and as K^{sh} is strongly dependent so is Kv.

If Kv is separably closed and perfect it is algebraically closed and hence strongly dependent. If it is not perfect then, by an argument of Scanlon's [17, Proposition 3.7], v is definable in K and hence (K, v) is strongly dependent, so that Kv is perfect, a contradiction. ∎

Dealing with the value group is more complicated. The valuation itself may not be definable but under mild assumptions Theorem 4.20 allows us to find a definable (non-trivial) coarsening of it. We need the following:

Definition 5.3 ([20]): Let G be an ordered abelian group and p a prime. Then, G is p-**antiregular** if no non-trivial quotient of G is p-divisible and G has no rank one quotient.

Remark: p-antiregularity is an elementary property of G; see [20, Section 3].

Proposition 5.4: *Let G be a non-divisible ordered abelian group with finite spines. Then there exists a prime p such that G is not p-divisible and not p-antiregular.*

Proof. By the above remark and by Corollary 3.6 we may assume that $G = \bigoplus_{i \in I} G_i$, where all the G_i are non-zero archimedean groups. Let p be a prime with G not p-divisible. Since G has finite spines, $\mathcal{S}_p$ is finite and hence there is a maximal element $\alpha \in \mathcal{S}_p$. Let $g \in G$ be such that $\mathfrak{s}_p(g) = \alpha$ (i.e., $G_\alpha = H_p(g)$). By Lemma 3.7(1) we may assume that $|\operatorname{supp}(g)| = 1$, so if $\operatorname{supp}(g) = \{i_0\}$ then $g(i_0) \notin pG$ and

$$H_p(g) = \bigoplus_{j \le i_0} 0 \oplus \bigoplus_{j > i_0} G_j.$$

Define the following convex subgroup:

$$H = \bigoplus_{j<i_0} 0 \oplus \bigoplus_{j\geq i_0} G_j.$$

Aiming for a contradiction, assume that G is p-antiregular. If $G = H$ then $G/H_p(g) = G_{i_0}$, which is rank one, contradiction. Otherwise, by maximality of $H_p(g)$ and Lemma 4.5, G/H is p-divisible, contradiction. ∎

Recall that if (K, v) is a valued field and u is a coarsening of v, then there exists a convex subgroup $\Delta \leq vK$ such that

$$uK \cong vK/\Delta.$$

In this situation v induces a valuation $\bar{v}$ on Ku with valuation ring $\{xu : x\in\mathcal{O}_v\}$, where xu is the residue of x in the valued field (K, u), and there exists an isomorphism

$$\bar{v}(Ku) \cong \Delta.$$

For a field K and a prime p, let $K(p)$ be the compositum of all Galois extensions of K of p-power degree. A field K is p-closed if $K = K(p)$. A valued field (K, v) is called p-Henselian if v extends uniquely to $K(p)$. If there exists a p-Henselian valuation with p-closed residue field, then there exists a unique coarsest p-Henselian valuation whose residue field is p-closed. It is denoted by v_K^p and called the **canonical p-Henselian valuation**. For more, and the definition of the canonical p-Henselian valuation, see, e.g., [19]. We can now show:

Proposition 5.5: *Let K be a strongly dependent field. Assume that K admits some Henselian valuation v with vK non-divisible. Then K admits a non-trivial $\emptyset$-definable Henselian coarsening u of v. Moreover, if Kv is separably closed and q is such that vK is not q-divisible, then u may be chosen so that the convex subgroup corresponding to u is q-divisible.*

Proof. K is necessarily not separably closed, otherwise, together with Fact 5.1, we would get that vK is divisible.

Case 1: If the residue field is separably closed, and hence algebraically closed by Fact 5.1 and Proposition 5.2, then K admits a $\emptyset$-definable non-trivial Henselian valuation by [19, Theorem 3.10]. As the result we care about (i.e., that we actually get a coarsening) appears only in the proof of that theorem (not in its statement) we give the details:

Let q be such that vK is not q-divisible (so q is different from the characteristic of K). As Kv is algebraically closed, by definition, the canonical q-Henselian valuation has a q-closed residue field. As $K \neq K(q)$ (since vK is not q-divisible) it is also non-trivial (see [19, Section 2.2]). Denote it by v_K^q. It is coarser than v, and—by definition—also coarser than the canonical Henselian valuation on K.

If K contains a primitive q^{th} root of unity, then v_K^q is a $\emptyset$-definable coarsening of v ([19, Theorem 2.7]). If K does not contain a primitive q^{th} root of unity, we repeat the same argument with $L := K(\zeta_q)$ to obtain a $\emptyset$-definable u coarsening the unique extension of v to L. Since L is a $\emptyset$-definable extension, $u|K$ is a $\emptyset$-definable coarsening of v.

Finally, if v_L^q is the canonical q-Henselian valuation on L, then by definition

$$Lv_L^q = Lv_L^q(q).$$

Since $[L : K] < \infty$ and v_L^q extends v_K^q we get that

$$[Lv_L^q : Kv_K^q] < \infty.$$

Note that Kv_K^q is not real closed. Indeed, since (K, v) is Henselian, so is $(Kv_K^q, \bar{v})$. Hence if it were real closed, by [8, Lemma 4.3.6] we would get that $(Kv_K^q)\bar{v} = Kv$ is orderable, contradiction. We conclude that

$$Kv_K^q = Kv_K^q(q).$$

So any valuation on the residue field has q-divisible valuation group. In particular $\bar{v}(Kv_K^q)$ is q-divisible, as required.

Case 2: If Kv is not separably closed, as in Proposition 5.2, (K, v) is strongly dependent and hence so is vK. So by Proposition 5.4, vK is not p-divisible and not p-antiregular for some p. Thus, by [20, Corollary 3.7] K admits some $\emptyset$-definable non-trivial Henselian coarsening of v.

As any coarsening of a Henselian valuation is Henselian, the proposition is proved. ∎

The following observation will not be used for the proof of our main result, but may be interesting in its own right:

Corollary 5.6: *Let K be a strongly dependent field, $(\mathcal{K}, v)$ a Henselian field with $v\mathcal{K}$ not divisible, and $\mathcal{K}$ elementarily equivalent to K (as pure fields). Then K is Henselian (i.e., admits a non-trivial Henselian valuation).*

Proof. By the last proposition $\mathcal{K}$ admits a $\emptyset$-definable non-trivial Henselian valuation. Since $K \equiv \mathcal{K}$ the same is true of K. ∎

Remark: Recall ([28]) that a field is t-Henselian if it is elementarily equivalent (in the language of rings) to a Henselian field. The assumptions of the last corollary are equivalent to K being t-Henselian, admitting some valuation v with vK non-divisible.

Using the above results we can finally conclude the following:

Proposition 5.7: *Let K be a strongly dependent field, v a Henselian valuation on K. Then the value group vK is strongly dependent as a pure group.*

Proof. If K is separably closed, and hence algebraically closed, the result follows from the strong dependence of ACVF. So we assume this not to be the case.

If $\mathbb{P}_\infty(vK) = \emptyset$ we get by [21, Proposition 5.1] that vK is dp-minimal, and we are done. So we may assume that $|\mathbb{P}_\infty(vK)| > 0$ and fix some prime $p \in \mathbb{P}_\infty(vK)$. We may assume that Kv is algebraically closed, otherwise, v is K^{sh}-definable by [17, Theorem A]), and we are done (as in the proof of Proposition 5.2).

Proposition 5.5 supplies us with a non-trivial $\emptyset$-definable Henselian coarsening u of v. Consider Ku, equipped with the valuation $\bar{v}$. By Proposition 5.5, $\bar{v}(Ku)$, the corresponding convex subgroup of vK, may be chosen to be p-divisible. So

$$\mathbb{P}_\infty(\bar{v}(Ku)) \subsetneq \mathbb{P}_\infty(vK).$$

Claim: $|\mathbb{P}_\infty(vK)| < \infty$.

Proof. Either by [24, Corollary 3.12] or by [14], since $K^\times$ is a strongly dependent abelian group $|\mathbb{P}_\infty(K^\times)| < \infty$. Now notice that

$$|\mathbb{P}_\infty(vK)| \leq |\mathbb{P}_\infty(K^\times)|. \qquad \blacksquare\text{(claim)}$$

We conclude by induction on $|\mathbb{P}_\infty(vK)|$: by the induction hypothesis $\bar{v}(Ku)$ is strongly dependent (because $\bar{v}$ is Henselian). It follows from Corollary 4.21 that vK is strongly dependent since $vK/\bar{v}(Ku)$ and $\bar{v}(Ku)$ are strongly dependent. ∎

Before proceeding to the proof of our main result, we need to sort out some technicalities:

Fact 5.8 ([18, Proposition2.5]): *Let T be dependent in a relational language L, let $\mathcal{M} \models T$ and let D be a definable set. Assume that D is stably embedded. Let $\mathcal{D}_{ind}$ be the structure with universe $D(M)$ and the induced L-language. Consider an expansion $\mathcal{D}'$ of $\mathcal{D}_{ind}$ in a relational language $\mathcal{L}_p$ and let $\mathcal{M}'$ be the corresponding expansion of $\mathcal{M}$ in the language $\mathcal{L}' = \mathcal{L} \cup \mathcal{L}_p$. Then the definable set D is stably embedded in $\mathcal{M}'$. Furthermore, if $\mathcal{D}'$ is dependent, then so is $\mathcal{M}'$.*

Proposition 5.9: *With the same assumptions and definitions as in Fact 5.8, if we assume that T and $\mathcal{D}'$ are strongly dependent then so is $\mathcal{M}'$.*

Proof. The proof is similar to that of [18, Proposition 2.5] (and uses it). By Fact 5.8, D is stably embedded in M'. As the conclusion of the proposition does not depend on the choice of language, we may assume that $\mathcal{D}'$ admits quantifier elimination in the relational language $\mathcal{L}_p$ and that $\mathcal{M}$ admits quantifier elimination in $\mathcal{L}$.

Let $\langle \bar{a}_t : t \in I \rangle$ be an infinite indiscernible sequence in $\mathcal{M}'$ and c a singleton. By [32, Observation 2.1], we may assume that each $\bar{a}_t = \langle a_{t,\alpha} : \alpha < \alpha^* \rangle$ enumerates a model M_t. By [32, Observation 2.1], in order to show that $\mathcal{M}'$ is strongly dependent, we need to find a convex equivalence relation E on I with finitely many equivalence classes such that $sEt \Rightarrow \mathrm{tp}(\bar{a}_s/c) = \mathrm{tp}(\bar{a}_t/c)$.

Since $\mathcal{D}'$ and $\mathcal{M}$ are strongly dependent there exists a finite convex equivalence relation E on I such that if sEt, then $\mathrm{tp}(\bar{a}_t/c)$ and $\mathrm{tp}(\bar{a}_s/c)$ agree on formulas of the form

$$\varphi(\bar{x}, y) \wedge \chi(\bar{x}, y),$$

where $\varphi(\bar{x}, y)$ is a quantifier-free $\mathcal{L}$-formula and $\chi(\bar{x}, y)$ is a quantifier-free $\mathcal{L}_p$-formula (with all variables restricted to D). In particular, if $c \notin D$ the variable y does not appear in $\chi(\bar{x}, y)$.

Let $s, t \in I$ be such that sEt. As in [18, Proposition 2.5], in order to show that the types $\mathrm{tp}(\bar{a}_t/c)$ and $\mathrm{tp}(\bar{a}_s/c)$ are equal, we must show that they also agree on D-bounded formulas, i.e., formulas of the sort

$$(Q_1 z_1 \in D) \dots (Q_n z_n \in D) \bigvee_i (\varphi_i(\bar{x}\bar{z}, y) \wedge \chi_i(\bar{x}\bar{z}, y)),$$

where φ_i and χ_i are as before. We proceed by induction on the number of quantifiers $Qz \in D$ appearing in formulas. If there are no quantifiers, this follows from the previous paragraph (namely, from the assumption on E). Now consider

$$(\exists z \in D)\psi(\bar{x}z, y),$$

where $\psi(\bar{x}z, y)$ is a D-bounded formula for which the inductive hypothesis holds. If the types do not agree on this formula, there are $\alpha_1 < \cdots < \alpha_k < \alpha^*$, where $k = |\bar{x}|$, such that

$$(\exists z \in D)\psi(a_{t,\alpha_1}, \ldots, a_{t,\alpha_k}, z, c), \text{ but}\neg(\exists z \in D)\psi(a_{s,\alpha_1}, \ldots, a_{s,\alpha_k}, z, c).$$

Since M_t is a model, there exists $a \in D(M_t)$ with

$$\psi(a_{t,\alpha_1}, \ldots, a_{t,\alpha_k}, a, c).$$

Without loss of generality, assume that $a = a_{t,\alpha}$ for some $\alpha_1 \leq \alpha \leq \alpha_2$. But by the second formula,

$$\neg\psi(a_{t,\beta_1}, \ldots, a_{t,\beta_k}, b, c)$$

for every $b = a_{t,\beta}$ with $\beta_1 \leq \beta \leq \beta_2$. This is a contradiction to the assumption that the inductive hypothesis holds on $\psi(\bar{x}z, y)$. ∎

Recall the following definition:

Definition 5.10: A valued field (K, v) of residue characteristic $p > 0$ is a **Kaplansky field** if the value group is p-divisible, the residue field is perfect and the residue field does not admit any finite separable extensions of degree divisible by p.

In [18, Theorem 3.3], Jahnke–Simon show that any theory of separably algebraically maximal Kaplansky fields of a fixed finite degree of imperfection is dependent if and only if the residue field and value group are.

Proposition 5.11: *Any theory of an algebraically maximal Kaplansky field is strongly dependent if and only if the residue field and value group are.*

Proof. Passing to an elementary extension we may assume that such a field has an angular component map (see [33, Corollary 5.18]). In [26, Section 3], Kuhlmann proves that if F and L are any such valued fields with F, $|L|^+$-saturated and K a common substructure, then any embedding $RV_L \hookrightarrow RV_F$ (over RV_K) may be lifted to an embedding $L \hookrightarrow F$ (over K), where RV is the rv-structure (see, for instance, [10] for the connection to the amc-structures defined by Kuhlmann). In [2, Lemma 4.3] Belaír deduces elimination of field quantifiers in the Denef-Pas language (the 3-sorted language with an angular component map) from precisely this data. The result now follows from [32, Claim 1.17(2)].[2] We may finally drop the ac-map; the valued field remains strongly dependent. For a direct proof of this fact see also a subsequent paper [13]. ∎

Remark: By elimination of field quantifiers we only need to check that the residue field and value group are strongly dependent as pure structures.

Lemma 5.12: *Let K be a strongly dependent field of characteristic $p > 0$. Then (K, v) is an algebraically maximal Kaplansky field with respect to any Henselian valuation v. Furthermore, (K, v) is strongly dependent.*

Proof. Since $\mathrm{char}(K) = p$ it is perfect and so vK is p-divisible. Moreover, as K is dependent it follows from the proof of [23, Proposition 5.3] that Kv is Artin–Schreier closed, and therefore infinite.

Recall that by [23, Corollary 4.4] infinite dependent fields of characteristic p have no separable extensions of degree divisible by p, the characteristic of the field. Thus, strongly dependent fields, which are perfect, have no finite extensions of degree dividing p. The residue field Kv is strongly dependent by Proposition 5.2 and hence (K, v) is Kaplansky.

Since the degree of every finite extension of K is prime to p, K is defectless and thus, by Henselianity, algebraically maximal. By Propositions 5.2, 5.7 and 5.11 (K, v) is strongly dependent. ∎

Proposition 5.13 ([22, The proof of Theorem 4.3.1]): *Let K be a strongly dependent field and (K, v) Henselian of mixed characteristic $(0, p)$. Then*

1. *either $[0, v(p)]$ is finite or there exists a non-trivial p-divisible convex subgroup of vK,*
2. *if $[0, v(p)]$ is infinite then Kv is infinite.*

Proof. (1) Assume $[0, v(p)]$ is infinite. Let Δ_p be the maximal p-divisible convex subgroup of vK.

Claim: *There is a formula defining, in any ordered abelian group, the maximal p-divisible convex subgroup.*

[2] The model-theoretic implications of elimination of field quantifiers used in the proof appear in [32, Claim 1.16] for which no reference is given. With the exception of cell decomposition (which is not used in the proof), they are an immediate consequence of [5, Theorem 5].

Proof. Consider

$$X = \{g \in vK : \text{for all } 0 \leq |x| \leq |g|,\ x \text{ is } p\text{-divisible}\}.$$

All elements of X are obviously p-divisible and it is closed under inverses. Let $g, h \in X$ and assume for simplicity that $0 < g+h$. We may assume that $g, h > 0$ and let $0 < c < g+h$. If $c \leq g$ or $c \leq h$, then c is p-divisible so assume without loss of generality that $h < c$, but then $0 < c-h < g$, hence $c-h$ is p-divisible and thus so is c. So X is a subgroup. By definition we must have that

$$X = \Delta_p. \qquad \blacksquare \text{(claim)}$$

As a result, what we want to prove is first order expressible so we may assume that (K, v) is sufficiently saturated and specifically that

$$|[0, v(p)]| > |\mathbb{R}|.$$

Let Δ be the minimal convex subgroup of vK containing $v(p)$ and Δ_0 the maximal convex subgroup not containing $v(p)$. Since Δ/Δ_0 is archimedean it embeds into $\mathbb{R}$. If Δ_0 were trivial then, since $[0, v(p)] \subseteq \Delta$, necessarily $|\Delta| > |\mathbb{R}|$, which is impossible.

The following claim will finish (1).

Claim: *Δ_0 is p-divisible and thus*

$$\Delta_0 \subseteq \Delta_p.$$

Proof. The coarsening $v_0 : K \to vK/\Delta$ of v is Henselian of equi-characteristic 0. In particular, $K_1 := Kv_0$ is strongly dependent by Proposition 5.2. Also, the valuation $v_1 : K_1 \to \Delta/\Delta_0$ of mixed characteristic $(0, p)$ is Henselian.

Finally, consider the valuation $v_2 : K_2 \to \Delta_0$, where $K_2 := K_1 v_1$. Note that K_2 is of characteristic $p > 0$ and that K_2 is strongly dependent by Proposition 5.2. By Fact 5.1, K_2 is perfect and hence Δ_0 is p-divisible. $\blacksquare$ (claim)

(2) Keeping the same notation, assume that $[0, v(p)]$ is infinite. As before, Δ_0 is non-trivial. The proof of [23, Proposition 5.3] shows that if K is dependent and (K, v) is a valued field of characteristic $p > 0$, then Kv is infinite. Applying this fact to the valuation $v_2 : K_2 \to \Delta_0$, whose residue field is Kv, finishes the proof. $\blacksquare$

We can now prove the main part of Theorem 2:

Theorem 5.14: *Let K be a strongly dependent field. Assume that v is a Henselian valuation on K. Then (K, v) is strongly dependent.*

Proof. We may move to a sufficiently saturated extension of (K, v), keeping the base field strongly dependent. By Propositions 5.2 and 5.7, vK and Kv are strongly dependent. The proof now splits into three cases:

Case 1: If $\mathrm{char}(Kv) = 0$ then, since vK and Kv are strongly dependent, by [32, Claim 1.17], (K, v) is also strongly dependent. Moreover, we note that in this case vK and Kv are stably embedded as pure structures by [33, Corollary 5.25].

Case 2: Assume that $\mathrm{char}(K) = \mathrm{char}(Kv) = p > 0$. This case follows by Lemma 5.12. Moreover, we note that vK and Kv are stably embedded as pure structures by [18, Lemma 3.1].

Case 3: Assume that (K, v) is of mixed characteristic $(0, p)$. Let Δ_0 be the largest convex subgroup of

$$\Gamma := vK$$

not containing $v(p)$ and Δ the smallest convex subgroup containing $v(p)$.

The coarsening $v_0 : K \to \Gamma/\Delta$ of v is Henselian of equi-characteristic 0. So by Case 1, (K, v_0) is strongly dependent. In particular, $K_1 := Kv_0$ is strongly dependent. Also, the valuation $v_1 : K_1 \to \Delta/\Delta_0$ of mixed characteristic $(0, p)$ is Henselian.

Finally, consider the valuation $v_2 : K_2 \to \Delta_0$, where $K_2 := K_1v_1$. It is of equi-characteristic (p, p) and thus (K_2, v_2) is strongly dependent by Case 2.

Case 3.1: If $K_1v_1 = K_2$ is finite then so is Kv, and hence it is not separably closed, by [17, Theorem A], v is definable in K^{sh} so (K, v) is strongly dependent.

Case 3.2: Assume K_2 is infinite.

Claim 1: (K_1, v_1) *is unboundedly ramified, i.e.,* $[0, v_1(p)]$ *is infinite, and* Δ/Δ_0 *is p-divisible.*

Proof. If $[0, v_1(p)]$ is finite then the valuation $v_1 : K_1 \to \Delta/\Delta_0$ is discrete, hence by [16, Theorem 4] v_1 is definable in K_1, so (K_1, v_1) is strongly dependent. Now, by [22, Lemma 4.2.1], $K_1v_1 = K_2$ is finite, a contradiction.

We may now apply Proposition 5.13 to (K_1, v_1). Since Δ/Δ_0 is archimedean, if it contains a non-trivial p-divisible convex subgroup, Δ/Δ_0 itself must be p-divisible. ∎ (claim)

We can now show:

Claim 2: (K_1, v_1) *is strongly dependent, Kaplansky and algebraically maximal. Moreover, the value group* v_1K_1 *and residue field* K_1v_1 *are stably embedded as pure structures.*

Proof. The following argument is taken from [22, Theorem 4.3.1]. Since (K, v) is sufficiently saturated, any countable chain of balls in (K, v) has non-empty intersection. Therefore, the same is true for (K_1, v_1). On the other hand, Δ/Δ_0 embeds into $\mathbb{R}$ and thus every cut has countable cofinality, consequently (K_1, v_1) is spherically complete and thus algebraically maximal. It is obviously Kaplansky and hence, by Proposition 5.11, (K_1, v_1) is strongly dependent. The second part of the claim is due, again, to [18, Lemma 3.1]. ∎(claim)

It will be enough to show that the structure $(K, v_0, K_1, v_1, K_2, v_2)$ is strongly dependent, since v is definable there. We apply Proposition 5.9 twice. Since (K, v_0) is strongly dependent, and K_1 is stably embedded as a pure structure and (K_1, v_1) is strongly dependent, (K, v_0, K_1, v_1) is strongly dependent. Doing this again, we get our result. ∎

Corollary 5.15: *Let* K *be a strongly dependent field. Then for every Henselian valuation* v *on* K*, the valued field* (K, v) *is defectless, and therefore algebraically maximal.*

Proof. By [22, Theorem 4.3.2] every strongly dependent (K, v) is defectless. As defectless Henselian fields are algebraically maximal [25, Theorem 11.31], the corollary follows. ∎

To finish the proof of Theorem 2 we need to show that in every strongly dependent Henselian field the value group is stably embedded as an ordered abelian group and the residue field is stably embedded as a pure field. In a different paper, we show that every strongly dependent Henselian field admits elimination of field quantifiers; the result follows (see [13]).

Remark: Theorem 5.14 can also be deduced from elimination of field quantifiers and [32, Claim 1.17(2)]; see [13].

We end with the following consequence of some of the results discussed in this paper. Note that it answers [22, Question 9.9.3] in the affirmative.

Proposition 5.16: *Let K be a dp-minimal field and v a Henselian valuation on K. Then (K, v) is also dp-minimal.*

Proof. We use Johnson's classification of dp-minimal valued fields [22, Theorem 9.8.1]. If Kv is not algebraically closed then, by [17, Theorem A], v is definable in K^{sh} and, as K^{sh} is dp-minimal, so is (K, v). We may thus assume that Kv is algebraically closed and hence dp-minimal. Since

$$|\mathbb{P}_\infty(vK)| \leq |\mathbb{P}_\infty(K^\times)|$$

and $K^\times$ is a dp-minimal abelian group, vK is also dp-minimal (see [21, Proposition 5.1]). By Corollary 5.15, (K, v) is defectless.

If $\text{char}(K) = p$ then Kv is p-divisible by Lemma 5.12.

Finally, assume that $\text{char}(K) = 0$ and $\text{char}(Kv) = p$ and let Δ, Δ_0, K_1, K_2, v_0, v_1 and v_2 be as in the proof of Theorem 5.14. Since Kv is infinite, so is K_1v_1 and, by Claim 1 of the proof of Theorem 4.20, Δ/Δ_0 is p-divisible. Since Δ_0 is p-divisible by Lemma 5.12, then Δ, and hence $[0, v(p)]$, is p-divisible. Now we may apply [22, Theorem 9.8.1]. ∎

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