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SPECTRUM AND COMBINATORICS OF TWO-DIMENSIONAL RAMANUJAN COMPLEXES

BY

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ABSTRACT

Ramanujan graphs have extremal spectral properties, which imply a remarkable combinatorial behavior. In this paper we compute the high dimensional Hodge–Laplace spectrum of Ramanujan triangle complexes, and show that it implies a combinatorial expansion property, and a pseudorandomness result. For this purpose we prove a Cheeger-type inequality and a mixing lemma of independent interest.

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1. Introduction

Expanders are graphs whose nontrivial adjacency spectrum is concentrated in a narrow strip. This implies remarkable combinatorial properties, such as isoperimetric expansion [AM85], pseudo-randomness [AC88, FP87], rapid convergence of random walk and a large chromatic number [Hof70]. We refer to the surveys [HLW06, Lub12] for the applications of expanders in mathematics and computer science.

A k-regular graph is called **Ramanujan** if its nontrivial spectrum is contained within the L^2-spectrum of its universal cover, which is the k-regular tree T_k:

$$\mathrm{Spec}(\mathrm{Adj}_{T_k}) = [-2\sqrt{k-1}, 2\sqrt{k-1}]$$

(the **trivial** spectrum is, by definition, the eigenvalues $\pm k$). By the Alon–Boppana theorem (cf. [HLW06, Thm. 2.7]), this is asymptotically the best one can hope for, so that Ramanujan graphs are optimal expanders. Such graphs were first constructed in [LPS88, Mar88], as quotients of the Bruhat–Tits tree associated with $PGL_2(\mathbb{Q}_p)$ by arithmetic lattices. It was suggested by several authors [CSŻ03, Li04, LSV05a, Sar07] that Ramanujan complexes should be defined as quotients of Bruhat–Tits buildings whose spectral properties agree with those of the building. The cited papers show that such complexes do exist, and the papers [FGL$^+$12, EGL15] use these spectral properties to obtain some types of combinatorial expansion.

However, all of the definitions and applications in the cited papers only refer to the spectrum of operators acting on the vertices of the complex in question. The spectrum of these operators is encoded in the spherical representations of the group PGL_d, and these correspond to representations of the Hecke algebra of the group, which is commutative [Mac79]. In this paper, we investigate the spectrum of the high-dimensional Hodge–Laplace operators, which encode the homology of the complex in all dimensions [Eck44]. To achieve this, we interpret (see Proposition 3.1) the simplicial boundary and coboundary maps on Ramanujan complexes as intertwining maps between different representations of the (non-commutative) Iwahori–Hecke algebra of PGL_d.

For Ramanujan complexes of dimension two, we compute the Hodge–Laplace spectra in all dimensions, and show that unlike the situation in graphs, the nontrivial spectrum in dimension one is concentrated in two narrow strips. We give here a loose version, and a tight one appears in Theorem 2.3.

Theorem 1.1: *Let X be a Ramanujan complex of type $\widetilde{A}_2$, and $\Delta_i = \delta^*\delta + \delta\delta^*$ the simplicial Hodge–Laplace operator in dimension $i \in \{0, 1, 2\}$ (see definitions in §2). Then the nontrivial spectrum of Δ_i is contained in*

$$\Delta_0 : \mathfrak{S}_0, \quad \Delta_1 : \mathfrak{S}_1 \cup \mathfrak{S}_0, \quad \Delta_2 : \{0\} \cup \mathfrak{S}_1,$$

where

$$\mathfrak{S}_0 = [k_0 - 6\sqrt{k_0 - 1}, k_0 + 3\sqrt{k_0 - 1}],$$
$$\mathfrak{S}_1 = [k_1 - 2\sqrt{k_1 - 1}, k_1 + 2\sqrt{k_1 - 1}] \bigcup [2k_1 - 1, 2k_1 + 8],$$

and k_0 (resp. k_1) is the vertex degree (resp. edge degree) in X.

In the second part of the paper (Section 4) we explore the combinatorial information which is encoded in the Hodge–Laplace spectrum. The results apply to any complex, and not only to quotients of Bruhat–Tits buildings, so that Section 4 can be read independently. In §4.1 we prove the following theorem, which generalizes the isoperimetric inequalities from [AM85, PRT16]:

Theorem 1.2: *Let X be a d-dimensional complex on n vertices, and $Z_i = Z_i(X)$ the space of i-dimensional cycles. If* $\operatorname{Spec} \Delta_i|_{Z_i} \subseteq [k_i - \mu_i, k_i + \mu_i]$ *for* $0 \le i \le d-2$ *and* $\operatorname{Spec} \Delta_{d-1}|_{Z_{d-1}} \subseteq [\lambda_{d-1}, \infty)$, *then for any partition* $\operatorname{Verts}(X) = \coprod_{i=0}^{d} A_i$

$$\frac{|X(A_0, \ldots, A_d)| n^d}{|A_0| \cdots |A_d|}$$
$$\geq k_0 \cdots k_{d-2} \cdot \lambda_{d-1} \Big(1 - \frac{\mu_{d-2}}{k_{d-2}} - C_d \Big(\frac{\mu_0}{k_0} + \cdots + \frac{\mu_{d-2}}{k_{d-2}}\Big) \frac{n^{d+1}}{\prod_{i=0}^{d} |A_i|}\Big),$$

where $X(A_0, \ldots, A_d)$ are the d-cells of X in $A_0 \times \cdots \times A_d$, and C_d depends only on d.

For a complex with a complete skeleton, one has $k_i = n$ and $\mu_i = 0$ for $0 \le i \le d-2$, hence the r.h.s. above reads as $n^{d-1}\lambda_{d-1}$, recovering Theorem 1.2 in [PRT16].

In Theorem 4.1 we prove a generalization of the Expander Mixing Lemma (cf. [HLW06, §2.4]), showing that concentration of the Hodge–Laplace spectrum implies a pseudo-random behavior. Combining these combinatorial theorems with Theorem 2.3, we obtain the following results on Ramanujan complexes of type $\widetilde{A}_2$, which show that they enjoy isoperimetric expansion, pseudo-randomness and large chromatic number.

Theorem 1.3: *Let X be a Ramanujan complex of type $\widetilde{A}_2$ with n vertices, vertex degree $k_0 = 2(q^2+q+1)$ and edge degree $k_1 = q+1$. Fix a constant $\vartheta > 0$.*

(1) (Isoperimetry) *If X is not tripartite, then for any partition of the vertices of X into sets A_0, A_1, A_2 of sizes at least ϑn,*

$$\frac{|X(A_0,A_1,A_2)|n^2}{|A_0||A_1||A_2|} \geq 2q^3 - 4q^{2.5} - C\cdot \frac{q^2}{\vartheta^3}.$$

(2) (Pseudo-randomness) *If X is tripartite, let A, B, C, D be disjoint sets of vertices such that each of $A \cup D$, B and C is contained in a different block of the tripartition of X. If A, B, C and D are of sizes at most ϑn, then*

$$\left||X^2(A,B,C,D)| - \frac{27q^4|A||B||C||D|}{n^3}\right| \leq (65q^{3.5}\vartheta + 244q^{2.5})\vartheta n$$

where $X^2(A,B,C,D)$ are the pairs of triangles $t_1 \in A\times B\times C$, $t_2 \in B \times C \times D$ which share an edge ($|t_1 \cap t_2| = 2$).

(3) *If X is not tripartite, the chromatic number of X is at least $\frac{\sqrt[3]{q}}{5}$.*

Here the chromatic number is the minimal number of colors needed to color the vertices of the complex with no monochromatic triangle. It is interesting to compare the last result with that of [EGL15], which studies mixing in Ramanujan complexes using the spherical representations alone (which correspond to operators on the vertices of the complex). They show that the chromatic number of such a complex is at least $\frac{\sqrt[6]{q}}{2}$, and we expect that in higher dimensions our new methods should lead to an even greater advantage over the spherical analysis.

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2. Complexes and buildings

We recall the basic elements of simplicial Hodge theory, and of Bruhat–Tits buildings of type $\widetilde{A}_d$. For a more relaxed exposition of the former we refer to [PRT16, §2], and for the latter to [Li04, LSV05a, Lub14].

2.1. Simplicial Hodge Theory. For a finite simplicial complex X of dimension d, denote by X^i the set of cells of dimension i in X (**i-cells**). The **degree** of an i-cell is the number of $(i+1)$-cells which contain it. Denote by $\Omega^i = \Omega^i(X)$ the space of i-forms, namely skew-symmetric complex functions on the oriented i-cells, equipped with the inner product

$$\langle f, g\rangle = \sum_{\sigma\in X^i} f(\sigma)\overline{g(\sigma)}.$$

The i-th boundary map $\partial_i : \Omega^i \to \Omega^{i-1}$ is defined by

$$(\partial_i f)(\sigma) = \sum_{v: v\sigma\in X^i} f(v\sigma),$$

its dual is the i-th coboundary map $\delta_i = \partial_i^* : \Omega^{i-1} \to \Omega^i$, given by

$$(\delta_i f)(\sigma) = \sum_{j=0}^{i} (-1)^j f(\sigma\backslash\sigma_j),$$

and

$$Z_i = \ker\partial_i, \quad Z^i = \ker\delta_{i+1}, \quad B_i = \operatorname{im}\partial_{i+1} \quad \text{and} \quad B^i = \operatorname{im}\delta_i$$

are the cycles, cocycles, boundaries and coboundaries, respectively.

The **upper**, **lower** and **full i-Laplacians** are

$$\Delta_i^+ = \partial_{i+1}\delta_{i+1}, \quad \Delta_i^- = \delta_i\partial_i \quad \text{and} \quad \Delta_i = \Delta_i^+ + \Delta_i^-,$$

respectively. Their spectra are closely related: $\mathrm{Spec}\Delta_i^+$ coincides with $\mathrm{Spec}\Delta_{i+1}^-$, up to a difference in the multiplicity of zero (which is determined by the number of i-cells and $i+1$-cells), and the spectrum of Δ_i is, up to zeros, the union of $\mathrm{Spec}\,\Delta_i^+$ and $\mathrm{Spec}\,\Delta_i^-$. It is most convenient for our purposes to work with the upper Laplacian, and Δ_0^+ is the classical graph Laplacian:

$$(\Delta_0^+ f)(v) = \deg(v)f(v) - \sum_{w\sim v} f(w). \tag{2.1}$$

2.2. Bruhat–Tits buildings. Let F be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformizer π, and residue field ${}^{\mathcal{O}}/_{\pi\mathcal{O}}$ of size q, which we identify with $\mathbb{F}_q$. We denote by $\mathcal{B} = \mathcal{B}_d(F)$ the **building of type** $\widetilde{A}_{d-1}$ associated with F, which is defined as follows. The vertices of $\mathcal{B}$ are the left K-cosets in G, where $G = PGL_d(F)$ and $K = PGL_d(\mathcal{O})$. Each vertex gK is associated with the homothety class of the $\mathcal{O}$-lattice $g\mathcal{O}^d$. A collection of vertices $\{g_iK\}_{i=0..r}$ forms an r-cell if, possibly after reordering, there exist representatives $g_i' \in GL_d(F)$ for g_i, such that

$$\pi g_0'\mathcal{O}^d < g_r'\mathcal{O}^d < g_{r-1}'\mathcal{O}^d < \cdots < g_1'\mathcal{O}^d < g_0'\mathcal{O}^d. \tag{2.2}$$

The group G acts on $\mathcal{B}$ by left translation, and if Γ is a torsion-free lattice in G then the quotient $X = \Gamma\backslash\mathcal{B}$ is a finite complex. In the case $d = 2$, the building $\mathcal{B}_2(\mathbb{Q}_p)$ is a $(p+1)$-regular tree, and its quotients by lattices in $G = PGL_2(\mathbb{Q}_p)$ are $(p+1)$-regular graphs. Certain lattices give rise to Ramanujan quotients:

Theorem 2.1 ([LPS88, Mar88], cf. [Sar90, Lub94]): *If Γ is a congruence subgroup of a torsion-free arithmetic lattice in G, then $\Gamma\backslash\mathcal{B}_2$ is a Ramanujan graph.*

Namely, its spectrum is contained within $\{-p-1\} \cup [-2\sqrt{p}, 2\sqrt{p}] \cup \{p+1\}$, where the eigenvalues $\pm(p+1)$ are the trivial ones: $p+1$ corresponds to the constant function on the vertices, and if the graph $\Gamma\backslash\mathcal{B}_2$ is bipartite, $-(p+1)$ appears as an eigenvalue of the function which takes one value on one side and the opposite value on the other.

2.3. Trivial spectrum of complexes. The function

$$\tau = \mathrm{ord}_\pi \det : PGL_d(F) \to {}^{\mathbb{Z}}/_{d\mathbb{Z}}$$

induces a d-partition on the vertices of $\mathcal{B}_d$. We define the type of a cell σ in $\mathcal{B}_d$ to be the subset $\{\tau(v) \,|\, v \in \sigma\}$ of ${}^{\mathbb{Z}}/_{d\mathbb{Z}}$, and say that a function on $X^i = \Gamma\backslash\mathcal{B}_d^i$ is trivial if its lift to $\mathcal{B}_d^i$ is constant on each type. An eigenvalue of Δ_i (or of $\Delta_i^{\pm}$) is called **trivial** if it is obtained from a trivial eigenfunction, and thus the nontrivial spectrum of these operators is obtained from their restriction to the functions which sum to zero on each type.

In dimension two, the Bruhat–Tits building $\mathcal{B}_3$ has constant vertex and edge degrees

$$k_0 = 2(q^2+q+1) \quad \text{and} \quad k_1 = q+1$$

respectively, and τ partites the vertices of $\mathcal{B}_3$ into three parts. We say that $X = \Gamma\backslash\mathcal{B}_3$ is tripartite when Γ preserves this partition. In this case the trivial eigenfunctions on vertices are

Eigenfunction	Eigenvalue of Δ_0^+	Eigenvalue of Δ_0^-
$\mathbb{1}$	0	$\lvert X^0\rvert$
$v \mapsto \omega^{\tau(v)}$, $\omega = e^{\frac{\pm 2\pi i}{3}}$	$\frac{3k_0}{2}$	0

When X is not tripartite, only the constant function is in the trivial spectrum. The edges of $\mathcal{B}_3$ have a canonical orientation by setting

$$\tau(\mathrm{term}(e)) \equiv \tau(\mathrm{orig}(e)) + 1 \pmod 3,$$

and we can thus define a form in $f \in \Omega^1(\mathcal{B}_3)$ by assigning a value to the positively oriented edges. Furthermore, the action of Γ always preserves this orientation, so that the same holds for $X = \Gamma\backslash\mathcal{B}_3$.[1] The trivial eigenforms in $\Omega^1(X)$ for a tripartite X are

Eigenform on positive direction	Eigenvalue of Δ_1^+	Eigenvalue of Δ_1^-
$\mathbb{1}$	$3k_1$	0
$e \mapsto \omega^{\tau(\mathrm{orig}(e))}$, $\omega = e^{\frac{\pm 2\pi i}{3}}$	0	$\frac{3k_0}{2}$

and again, in the non-tripartite case only the constant one appears.

2.4. Ramanujan complexes. There are several plausible ways to define what are Ramanujan complexes, and these are discussed in [CSŻ03, Li04, LSV05a, KLW10, Fir16]. However, it can be shown that they all agree for complexes of type $\widetilde{A}_2$, and amount to the following.

Definition 2.2: The complex $X = \Gamma\backslash\mathcal{B}_3$ is **Ramanujan** if the nontrivial spectrum of the Laplace operators in every dimension is contained within that of the corresponding Laplace operators on $L^2(\mathcal{B}_3)$.

The papers [Li04, LSV05b, Sar07] give several constructions of Ramanujan complexes, some of which are the clique complexes of Cayley graphs.

[1] Namely, X is always **disorientable**, see [PR17, Def. 2.6].

The following theorem determines the spectrum of the upper Laplacians on two-dimensional Ramanujan complexes. The spectra of the full and lower Laplacians can be inferred from it (see §2.1). By definition, this is the same as determining the L^2-spectrum of the Laplacians on $\mathcal{B}_3$ itself, but in addition we determine the multiplicity of eigenvalues on the finite quotients.

THEOREM 2.3: *Let X be a Ramanujan quotient of $\mathcal{B}_3$ with n vertices, and vertex and edge degrees $k_0 = 2(q^2+q+1)$ and $k_1 = q+1$. If X is non-tripartite, then*

(1) Δ_0^+ *has the trivial eigenvalue* 0, *and* $n-1$ *nontrivial eigenvalues in* $[k_0 - 6q, k_0 + 3q]$.
(2) Δ_1^+ *has:*
  (a) *The trivial eigenvalue* $3k_1$.
  (b) $n-1$ *zeros, corresponding to* $B^1(X)$ *(coboundaries).*
  (c) *For every nontrivial* $\lambda \in \operatorname{Spec} \Delta_0^+$, *the eigenvalues* $\frac{3k_1}{2} \pm \sqrt{(\frac{3k_1}{2})^2 - \lambda}$. *This amounts to* $n-1$ *eigenvalues in each of the strips*

$$\mathcal{I}_- = \left[\tfrac{3k_1}{2} - \sqrt{(\tfrac{k_1}{2})^2 + 8q}, k_1 + 1\right], \qquad \mathcal{I}_+ = \left[2k_1 - 1, \tfrac{3k_1}{2} + \sqrt{(\tfrac{k_1}{2})^2 + 8q}\right]. \tag{2.3}$$

  (d) $n(q^2+q-2)+2$ *eigenvalues in the strip*

$$\mathcal{I} = [k_1 - 2\sqrt{q}, k_1 + 2\sqrt{q}]. \tag{2.4}$$

If X is tripartite, then:

(1) Δ_0^+ *has trivial spectrum* $\{0, \frac{3k_0}{2}, \frac{3k_0}{2}\}$ *(see §2.3), and* $n-3$ *eigenvalues in* $[k_0 - 6q, k_0 + 3q]$.
(2) Δ_1^+ *has:*
  (a) *The trivial eigenvalue* $3k_1$, *and two trivial zeros (both coming from* $B^1(X)$*).*
  (b) $n-3$ *nontrivial zeros, all coming from* $B^1(X)$.
  (c) $n-3$ *eigenvalues in each of* $\mathcal{I}_\pm$, *corresponding to* $\frac{3k_1}{2} \pm \sqrt{(\frac{3k_1}{2})^2 - \lambda}$ *for* λ *a nontrivial eigenvalue of* Δ_0^+.
  (d) $n(q^2+q-2)+6$ *eigenvalues in* $\mathcal{I}$.

Let us make a few remarks:

(1) The spectrum of Δ_0^+ is well-known [Mac79, Li04, LSV05a], but our methods are different, and give the spectrum in all dimensions in a unified manner.
(2) These bounds are sharp: a sequence of Ramanujan complexes with injectivity radius growing to infinity (as constructed in [LM07]) has Laplace spectra which accumulate to any point in these intervals. This follows from [Li04] for dimension zero and from [PR17, §3.5] for general dimension.
(3) All the zeros in the spectra of Δ_0^+ and Δ_1^+ come from $B^1(X)$, so that the zeroth and first Betti numbers of X vanish, in accordance with [Gar73, Cas74].

It is interesting to compare Theorem 2.3 with Garland's spectral bounds:

Theorem (Garland's bound, [Gar73, Pap08, GW13]): *If X is a finite complex such that* $\operatorname{Spec}\Delta_0^+(\operatorname{link}(\sigma))|_{Z_0(\operatorname{link}(\sigma))} \subseteq [\lambda, \Lambda]$ *for every* $\sigma \in X^{j-2}$ *and* $k \le \deg\sigma \le K$ *for every* $\sigma \in X^{j-1}$, *then*

$$\operatorname{Spec}\Delta_j^+(X)|_{Z_j(X)} \subseteq [(j+1)\lambda - jK, (j+1)\Lambda - jk].$$

The links of vertices in $\mathcal{B}_3$ are incidence graphs of projective planes over $\mathbb{F}_q$, which have $\lambda = k_1 - \sqrt{q}$ and $\Lambda = 2k_1$, so that Garland's bound implies that any quotient of $\mathcal{B}_3$ satisfies

$$\operatorname{Spec}\Delta_1^+|_{Z_1} \subseteq [k_1 - 2\sqrt{q}, 3k_1].$$

Theorem 2.3 shows that both ends are tight! The drawback of Garland's method is that it misses the sparse picture within this interval, which is crucial for our combinatorial purposes, namely, the results in §4.1 and §4.2. The proof of Theorem 2.3 occupies Section 3.

3. Computation of the Laplace spectrum

3.1. Boundary maps as Iwahori–Hecke operators. In this section we translate the simplicial boundary and coboundary maps into intertwining operators between different representations arising from the group PGL_3. Keeping the notations of §2.2, we fix the "fundamental" vertex $v_0 = K$ in $\mathcal{B}^0 = {}^{G}\!/\!_{K}$. It follows from the fact that Γ is torsion-free that it acts freely on vertices, and thus if we normalize the Haar measure on G so that $\mu(K) = 1$, we have $\mu(\Gamma\backslash G) = n$. Furthermore, this implies a linear isometry

$$\Omega^0(X) \cong L^2(\Gamma\backslash G/K),$$

given explicitly by $f(gv_0) = f(\Gamma gK)$. We identify $L^2(\Gamma\backslash G/K)$ with $L^2(\Gamma\backslash G)^K$, the space of K-fixed vectors in the G-representation $L^2(\Gamma\backslash G)$.

The element

$$\sigma = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{smallmatrix}\right) \in G$$

acts on $\mathcal{B}$ by rotation on the triangle consisting of the vertices v_0, σv_0, and $\sigma^2 v_0$. We fix the oriented edge $e_0 = [v_0, \sigma v_0]$, and define

$$E = \mathrm{stab}_G\, e_0 = K \cap \sigma K \sigma^{-1} = \left\{ \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ x & y & * \end{smallmatrix}\right) \in K \,\middle|\, x, y \in \pi\mathcal{O} \right\}. \tag{3.1}$$

Since G acts transitively on the non-oriented edges of $\mathcal{B}$, and preserves the canonical orientation from §2.3, the positively oriented edges of X correspond to double cosets $\Gamma\backslash G/E$, giving an identification of $\Omega^1(X)$ with $L^2(\Gamma\backslash G)^E$, by

$$\begin{aligned} f(ge_0) = f([gv_0, g\sigma v_0]) &= \sqrt{\mu(E)} f(\Gamma g), \\ f([g\sigma v_0, gv_0]) &= -\sqrt{\mu(E)} f(\Gamma g), \end{aligned} \tag{3.2}$$

where $\mu(E) = \frac{\mu(K)}{[K:E]} = \frac{1}{q^2+q+1}$. The scaling by $\sqrt{\mu(E)}$ is needed to make the isomorphism $\Omega^1(X) \cong L^2(\Gamma\backslash G)^E$ an isometry: if $\{g_i e_0\}_{i=1}^{nk_0/2}$ represent the edges of X and $f \in \Omega^1(X)$, then

$$\|f\|^2_{\Omega^1(X)} = \sum_i |f(g_i e_0)|^2 = \sum_i \mu(E) |f(\Gamma g_i)|^2 = \int_{\Gamma\backslash G} |f(\Gamma g)|^2 dg = \|f\|^2_{L^2(\Gamma\backslash G)}.$$

We fix the triangle $t_0 = [v_0, \sigma v_0, \sigma^2 v_0]$, whose pointwise stabilizer is the Iwahori subgroup

$$I = K \cap \sigma K \sigma^{-1} \cap \sigma^2 K \sigma^{-2} = \left\{ \left(\begin{smallmatrix} * & * & * \\ x & * & * \\ y & z & * \end{smallmatrix}\right) \in K \,\middle|\, x, y, z \in \pi\mathcal{O} \right\}.$$

As for edges, G acts transitively on non-oriented triangles, and preserves triangle orientation. Thus, the stabilizer of t_0 as a cell is

$$T := \mathrm{stab}_G\, t_0 = \langle\sigma\rangle I = I \sqcup \sigma I \sqcup \sigma^2 I,$$

and in particular $\langle\sigma\rangle$ and I commute. Again, $f(gt_0) = \sqrt{\mu(T)} f(\Gamma g)$ gives a linear isometry $\Omega^2(X) \cong L^2(\Gamma\backslash G)^T$, where

$$\mu(T) = 3\mu(I) = \frac{3\mu(K)}{[K:I]} = \frac{3}{(q^2+q+1)(q+1)}.$$

Denoting $K_0 = K$, $K_1 = E$, and $K_2 = T$, we have $\Omega^i(X) \cong L^2(\Gamma\backslash G)^{K_i}$.[2] As $I \leq E \leq K$ and $I \leq T$, the three spaces $L^2(\Gamma\backslash G)^{K_i}$ are contained in $L^2(\Gamma\backslash G)^I$. The Iwahori–Hecke algebra $\mathcal{H} = C_c(I\backslash G/I)$ consists of the compactly supported, bi-I-invariant complex functions on G, with multiplication defined by convolution. If (ρ, V) is a representation of G, then $(\overline{\rho}, V^I)$ is a representation of $\mathcal{H}$, where

$$\overline{\rho}(\eta)v := \int_G \eta(g)\rho(g)v\,dg.$$

We proceed to show that the (co-)boundary maps between $L^2(\Gamma\backslash G)^{K_i}$ are given by certain intertwining elements in $\mathcal{H}$.

Proposition 3.1: *The following elements of $\mathcal{H}$:*

$$\begin{aligned} \partial_1 &= \tfrac{1}{\sqrt{\mu(E)}}(\mathbb{1}_{K\sigma^2} - \mathbb{1}_K), & \delta_1 &= \tfrac{1}{\sqrt{\mu(E)}}(\mathbb{1}_{\sigma K} - \mathbb{1}_K), \\ \partial_2 &= \tfrac{1}{\sqrt{\mu(E)\mu(T)}} \cdot \mathbb{1}_{ET}, & \delta_2 &= \tfrac{1}{\sqrt{\mu(E)\mu(T)}} \cdot \mathbb{1}_{TE}, \end{aligned} \tag{3.3}$$

act as the corresponding simplicial operators. Namely, each $\partial_i \in \mathcal{H}$ takes $L^2(\Gamma\backslash G)^{K_i}$ to $L^2(\Gamma\backslash G)^{K_{i-1}}$ and acts as the boundary operator

$$\partial_i : \Omega^i(X) \to \Omega^{i-1}(X)$$

with respect to the identifications of $\Omega^i(X)$ with $L^2(\Gamma\backslash G)^{K_i}$, and likewise for $\delta_i \in \mathcal{H}$ and $\delta_i : \Omega^{i-1} \to \Omega^i$.

Proof. Both ∂_{i+1} and δ_i map any representation V into V^{K_i}, since they are constant on right K_i cosets (note that $\sigma K = E\sigma K$). Let $f \in L^2(\Gamma\backslash G)^E \cong \Omega^1(E)$, and let $K = \coprod_{kE\in K/E} kE$. For any $gv_0 \in X^0$ we have

$$\begin{aligned} (\mathbb{1}_K f)(gv_0) &= (\mathbb{1}_K f)(\Gamma g) = \int_G \mathbb{1}_K(x)(xf)(\Gamma g)dx = \int_K (xf)(\Gamma g)dx \\ &= \int_K f(\Gamma gx)dx = \sum_{kE\in K/E} \int_E f(\Gamma gke)de = \sum_{kE\in K/E} \int_E f(\Gamma gk)de \\ &= \mu(E) \sum_{kE\in K/E} f(\Gamma gk) = \sqrt{\mu(E)} \sum_{kE\in K/E} f(gke_0). \end{aligned}$$

[2] This nice picture only holds for PGL_3. In PGL_d with $d \geq 4$ the group does not act transitively on each dimension, and there are also elements which flip orientations in the middle dimension.

The group K acts transitively on the q^2+q+1 positive edges leaving v_0, so that the positive edge leaving gv_0 (for any $g \in G$) are $\{gke_0\}_{kE\in K/E}$. Therefore,

$$\frac{1}{\sqrt{\mu(E)}}(\mathbb{1}_K f)(gv_0) = \sum_{kE\in K/E} f(gke_0) = \sum_{\substack{\text{orig } e=gv_0\\ e \text{ positive}}} f(e) = -\sum_{\substack{\text{term } e=gv_0\\ e \text{ negative}}} f(e). \tag{3.4}$$

In a similar manner, the positive edges with terminus gv_0 are $\{gk\sigma^2 e_0\}_{k\in K}$, and if $K\sigma^2 E = \coprod_{k\sigma^2 E\in K\sigma^2 E/E} k\sigma^2 E$ then

$$\begin{aligned}(\mathbb{1}_{K\sigma^2} f)(gv_0) &= \int_{K\sigma^2 E} f(\Gamma g x)dx \\ &= \sum_{k\sigma^2 E\in K\sigma^2 E/E} \int_E f(\Gamma g k\sigma^2 e)de \\ &= \sqrt{\mu(E)} \sum_{k\sigma^2 E\in K\sigma^2 E/E} f(gk\sigma^2 e_0) \\ &= \sqrt{\mu(E)} \sum_{\substack{\text{term } e=gv_0\\ e \text{ positive}}} f(e).\end{aligned} \tag{3.5}$$

Together with (3.4), this implies that ∂_1 from (3.3) indeed act as the simplicial ∂_1, justifying the abuse of notation. The reasoning for ∂_2 is similar, save for the fact that $T \nleq E$ (in fact, $E\cap T = I$). However, E acts transitively on the triangles containing e_0, hence for $f \in L^2(\Gamma\backslash G)^T$

$$\begin{aligned}\tfrac{1}{\sqrt{\mu(E)\mu(T)}}(\mathbb{1}_{ET} f)(ge_0) &= \tfrac{1}{\sqrt{\mu(T)}}(\mathbb{1}_{ET} f)(\Gamma g) \\ &= \sqrt{\mu(T)} \sum_{eT\in ET/T} f(\Gamma g e) = \sum_{eT\in ET/T} f(get_0) \\ &= \sum_{\tau\in X^2 \,:\, e_0\in\partial\tau} f(g\tau) = \sum_{\tau\in X^2 \,:\, ge_0\in\partial\tau} f(\tau),\end{aligned}$$

agreeing with $\partial_2 : \Omega^2 \to \Omega^1$. The coboundary operators can be analyzed in a similar manner, or as follows: $\mathcal{H}$ is a $*$-algebra by $\eta^*(g) = \overline{\eta(g^{-1})}$, and a unitary representation ρ of G induces a unitary $\mathcal{H}$-representation, i.e.,

$$\overline{\rho}(\eta)^* = \overline{\rho}(\eta^*)$$

(this uses unimodularity of G). For $V = L^2(\Gamma\backslash G)$ this gives

$$\partial_1^* = \tfrac{1}{\sqrt{\mu(E)}}(\overline{\mathbb{1}_{(K\sigma^2)^{-1}}} - \overline{\mathbb{1}_{K^{-1}}}) = \tfrac{1}{\sqrt{\mu(E)}}(\mathbb{1}_{\sigma K} - \mathbb{1}_K)$$

and similarly for ∂_2^*. ∎

Since Γ is cocompact, $L^2(\Gamma\backslash G)$ decomposes as a sum of irreducible unitary representations,

$$L^2(\Gamma\backslash G) = \bigoplus_\alpha W_\alpha,$$

and

$$\Omega^i(X) \cong L^2(\Gamma\backslash G)^{K_i} = \bigoplus_\alpha W_\alpha^{K_i} \le \bigoplus_\alpha W_\alpha^I.$$

Each W_α^I is a sub-$\mathcal{H}$-representation, so that the operators ∂_i, δ_i decompose with respect to this sum, and thus the Laplacians as well, giving

$$\mathrm{Spec}\Delta_i^\pm = \bigcup_\alpha \mathrm{Spec}\, \Delta_i^\pm|_{W_\alpha^{K_i}},$$

with the correct multiplicities. To understand the spectra it is enough look at the W_α which are Iwahori-spherical, namely, contain I-fixed vectors. Furthermore, the isomorphism type of W_α already determines the spectrum of $\Delta_i^\pm$ on $W_\alpha^{K_i}$. By [Cas80, Prop. 2.6], if $W_\alpha^I \neq 0$ then W_α is embeddable in a principal series representation $V_\mathfrak{z}$. Namely, there exists $\mathfrak{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$ (the Satake parameters) with $z_1 z_2 z_3 = 1$, and

$$V_\mathfrak{z} = \mathrm{uInd}_B^G \chi_\mathfrak{z} = \{f : G \to \mathbb{C} \mid f(bg) = \delta^{-\frac{1}{2}}(b)\chi_\mathfrak{z}(b) f(g)\ \forall b \in B\}, \tag{3.6}$$

where $\chi_\mathfrak{z}$ is the character

$$\chi_\mathfrak{z}(b) = \prod_{i=1}^3 z_i^{\mathrm{ord}_\pi b_{ii}}$$

of the Borel group

$$B := \left\{ \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix} \right) \in G \right\},$$

and $\delta(b) = |b_{33}|^2/|b_{11}|^2$ is the modular character of B. For obvious reasons, it is convenient to introduce the notation

$$\widetilde{\chi}_\mathfrak{z}(b) = \delta^{-\frac{1}{2}}(b)\chi_\mathfrak{z}(b) = \frac{|b_{11}|}{|b_{33}|} \prod_{i=1}^3 z_i^{\mathrm{ord}_\pi b_{ii}} = \left(\frac{z_1}{q}\right)^{\mathrm{ord}_\pi b_{11}} z_2^{\mathrm{ord}_\pi b_{22}} (q z_3)^{\mathrm{ord}_\pi b_{33}}.$$

Having decomposed

$$L^2(\Gamma\backslash G) = \bigoplus_\alpha W_\alpha,$$

and found a $\Delta_i^\pm$-eigenform $f \in W_\alpha^{K_i} \le \Omega^i(X)$, we can lift it to a Γ-periodic eigenform $\tilde{f} \in {}^\Gamma\Omega^i(\mathcal{B})$. For some $\mathfrak{z}$ we have $\Psi : W_\alpha \hookrightarrow V_\mathfrak{z}$, and naturally $\Psi f \in V_\mathfrak{z}^{K_i}$; since $V_\mathfrak{z}$ is defined as a set of complex functions on G, we can think of Ψf as an i-form on $\mathcal{B}$. Thus, both $\tilde{f}$ and Ψf are $\Delta_i^\pm$-eigenforms with the same eigenvalue. However, they are not the same, as $\tilde{f}$ attains finitely many values and Ψf infinitely many, in general. Nevertheless, the matrix coefficients $g \mapsto \langle g\widetilde{f}, \widetilde{f}\rangle$ and $g \mapsto \langle g\Psi f, \Psi f\rangle$ are the same, since Ψ is a unitary embedding. When these matrix coefficients are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, and only then, the representation W_α is weakly contained in $L^2(G)$, which implies that the corresponding eigenvalue is in the L^2-spectrum of $\Delta_i^\pm$ on $\mathcal{B}$ (cf. [CHH88]).

3.2. Analysis of the principal series. Even though W_α is only a subrepresentation of $V_{\mathfrak{z}}$, it is simpler to consider ∂_i, δ_i and $\Delta_i^\pm$ acting on $V_{\mathfrak{z}}$, and later restrict to W_α. The Weyl group of G is S_3 (as permutation matrices), and G decomposes as

$$G = BK = \coprod_{w\in A_3} BwE = BT \sqcup B(1\,2)T = \coprod_{w\in S_3} BwI.$$

From $G = BK$ and (3.6) we see that $\dim V_{\mathfrak{z}}^K \le 1$, and in fact this is an equality since $\chi_{\mathfrak{z}}|_{B\cap K} \equiv 1$, hence $f^K(bk) := \widetilde{\chi}_{\mathfrak{z}}(b)$ is well defined. Similarly, $\dim V_{\mathfrak{z}}^I = 6$, with basis $\{f_w^I\}_{w\in S_3}$ defined by $f_w^I(w') = \delta_{w,w'}$, and $\dim V_{\mathfrak{z}}^E = 3$ with basis $\{f_w^E\}_{w\in A_3}$, where $f_w^E := f_w^I + f_{w\cdot(1\,2)}^I$ satisfies $f_w^E(w') = \delta_{w,w'}$ for $w, w' \in A_3$. Finally, $\dim V_{\mathfrak{z}}^T = 2$ with basis $f_w^T(w') := \delta_{w,w'}$ for $w, w' \in \{(\,), (1\,2)\}$, which satisfy

$$\begin{aligned} f_{(\,)}^T &= f_{(\,)}^I + \frac{1}{qz_3} f_{(3\,2\,1)}^I + \frac{z_1}{q} f_{(1\,2\,3)}^I, \\ f_{(1\,2)}^T &= f_{(1\,2)}^I + z_2 f_{(2\,3)}^I + \frac{1}{qz_3} f_{(1\,3)}^I; \end{aligned} \tag{3.7}$$

indeed, if c_w is the coefficient of f_w^I in $f_{(\,)}^T$, then

$$c_{(1\,2\,3)} = f_{(\,)}^T((1\,2\,3)) = f_{(\,)}^T\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1\end{pmatrix}\sigma^2\right) = f_{(\,)}^T\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1\end{pmatrix}\right) = \widetilde{\chi}_{\mathfrak{z}}\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1\end{pmatrix}\right) = \frac{z_1}{q},$$

and the other coefficients in (3.7) are obtained similarly.

Let $\Omega_{\mathfrak{z}}^i(\mathcal{B})$ be the realization of $V_{\mathfrak{z}}^{K_i}$ as a subspace of $\Omega^i(\mathcal{B})$, given by the explicit construction (3.6). Any $f \in \Omega_{\mathfrak{z}}^0(\mathcal{B})$ is determined by its value on v_0, namely $f = f(v_0) f^K$. Similarly, the values on e_0, $e_1 := (3\,2\,1)e_0$ and $e_2 := (1\,2\,3)e_0$ determine a unique element in $\Omega_{\mathfrak{z}}^1(\mathcal{B})$, and likewise for t_0, $t_1 := (1\,2)t_0$ and $\Omega_{\mathfrak{z}}^2(\mathcal{B})$. As $S_3 \le K$, one can compute the action of $\partial_i|_{\Omega_{\mathfrak{z}}^i(\mathcal{B})}$ and $\delta_i|_{\Omega_{\mathfrak{z}}^{i-1}(\mathcal{B})}$ by evaluation on star(v_0) alone (see Figure 3.1).

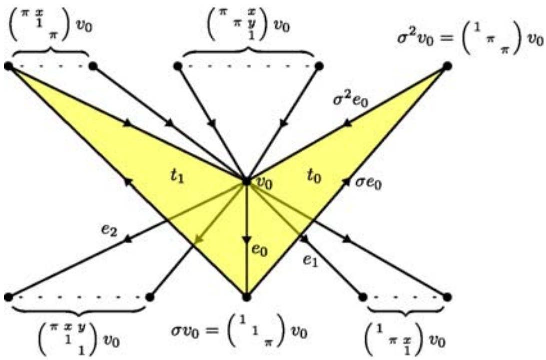

Figure 3.1. The star of v_0 in $\mathcal{B}$.

For the basis $\mathfrak{B}^E = \{f^E_{()}, f^E_{(3\,2\,1)}, f^E_{(1\,2\,3)}\}$ one has

$$[\delta_1|_{\Omega^0_{\mathfrak{z}}(\mathcal{B})}]^{\{f^K\}}_{\mathfrak{B}^E} = \begin{pmatrix} qz_3 - 1 \\ z_2 - 1 \\ \frac{z_1}{q} - 1 \end{pmatrix}; \tag{3.8}$$

for example, $(\delta_1 f^K)(e_1) = f^K((3\,2\,1)\sigma v_0) - f^K(v_0) = f^K(\left(\begin{smallmatrix}1&&\\&\pi&\\&&1\end{smallmatrix}\right)v_0) - 1 = z_2 - 1$, and $(\delta_1 f^K)(e_0), (\delta_1 f^K)(e_2)$ are computed similarly. We turn to ∂_1. The positive edges with origin v_0 are e_0, $\left(\begin{smallmatrix}1&&\\&1&x\\&&1\end{smallmatrix}\right)e_1$ for $x \in \mathbb{F}_q = \mathcal{O}/\pi\mathcal{O}$, and $\left(\begin{smallmatrix}1&x&y\\&1&\\&&1\end{smallmatrix}\right)e_2$ with $x, y \in \mathbb{F}_q$. As $\widetilde{\chi}_{\mathfrak{z}}$ is trivial on upper-triangular unipotent matrices, (3.4) implies that for $f \in \Omega^1_{\mathfrak{z}}(\mathcal{B})$ we have $(\mu(E)^{-\frac{1}{2}}\mathbb{1}_K f)(v_0) = f(e_0) + qf(e_1) + q^2 f(e_2)$. The positive edges entering v_0 are

$$[\sigma^2 v_0, v_0] = \sigma^2 e_0 = \begin{pmatrix}&&1\\ \pi&&\\ &\pi&\end{pmatrix}e_0 = \begin{pmatrix}1&&\\&\pi&\\&&\pi\end{pmatrix}(1\,2\,3)e_0 = \begin{pmatrix}1&&\\&\pi&\\&&\pi\end{pmatrix}e_2,$$

and similarly $\left(\begin{smallmatrix}\pi&x&\\&1&\\&&\pi\end{smallmatrix}\right)e_1$ and $\left(\begin{smallmatrix}\pi&&x\\&\pi&y\\&&1\end{smallmatrix}\right)e_0$ $(x, y \in \mathbb{F}_q)$. By (3.5),

$$\begin{aligned}\Big(\tfrac{1}{\sqrt{\mu(E)}}\mathbb{1}_{K\sigma^2 E} f\Big)(v_0) =& f\left(\begin{pmatrix}1&&\\&\pi&\\&&\pi\end{pmatrix}e_2\right) + \sum_{x\in\mathbb{F}_q} f\left(\begin{pmatrix}\pi&x&\\&1&\\&&\pi\end{pmatrix}e_1\right) \\ &+ \sum_{x,y\in\mathbb{F}_q} f\left(\begin{pmatrix}\pi&&x\\&\pi&y\\&&1\end{pmatrix}e_0\right) \\ =& z_2 q z_3 f(e_2) + q\cdot z_1 z_3 f(e_1) + q^2 \cdot \frac{z_1}{q} z_2 f(e_0),\end{aligned}$$

and in total (see (3.3))

$$[\partial_1|_{\Omega^1_{\mathfrak{z}}(\mathcal{B})}]^{\mathfrak{B}^E}_{\{f^K\}} = \left(\tfrac{q}{z_3} - 1 \quad \tfrac{q}{z_2} - q \quad \tfrac{q}{z_1} - q^2\right). \tag{3.9}$$

As $\Delta_0^+ = \partial_1\delta_1$ and $\Delta_1^- = \delta_1\partial_1$, we can now compute explicitly their action on the $\mathfrak{z}$-principal series. Denoting $\widetilde{\mathfrak{z}} = \sum_{i=1}^{3}(z_i + z_i^{-1})$, we have by (3.8) and (3.9)

$$\Delta_0^+|_{\Omega_{\mathfrak{z}}^0(\mathcal{B})} = (\lambda^K) := (k_0 - q\widetilde{\mathfrak{z}}),$$

$$[\Delta_1^-|_{\Omega_{\mathfrak{z}}^1(\mathcal{B})}]_{\mathfrak{B}^E} = \begin{pmatrix} q^2 - qz_3 - \frac{q}{z_3} + 1 & -q^2 z_3 + \frac{q^2 z_3}{z_2} + q - \frac{q}{z_2} & -q^3 z_3 + \frac{q^2 z_3}{z_1} + q^2 - \frac{q}{z_1} \\ \frac{qz_2}{z_3} - z_2 - \frac{q}{z_3} + 1 & -qz_2 + 2q - \frac{q}{z_2} & -q^2 z_2 + q^2 + \frac{qz_2}{z_1} - \frac{q}{z_1} \\ -\frac{q}{z_3} - \frac{z_1}{q} + \frac{z_1}{z_3} + 1 & q - z_1 - \frac{q}{z_2} + \frac{z_1}{z_2} & q^2 - qz_1 - \frac{q}{z_1} + 1 \end{pmatrix}. \tag{3.10}$$

Observe that Δ_0^+ agrees with the computation of the spectrum of the Hecke operators in [Mac79, CS02, Li04, LSV05a], as

$$\Delta_0^+ = k_0 \cdot I - \sum_{i=1}^{d-1} A_i,$$

where A_i is the i-th Hecke operator on $\mathcal{B}_d$ (loc. cit.). In effect, Δ_1^- can also be understood without the machinery above, as it has the eigenvalue λ^K corresponding to $\delta_1\Omega_{\mathfrak{z}}^0(\mathcal{B})$ (since $\Delta_1^-\delta_1 f^K = \delta_1\Delta_0^+ f^K = \lambda^K\delta_1 f^K$), and two zeros (which come from $\partial_2\Omega_{\mathfrak{z}}^2(\mathcal{B})$). However, this machinery allows us to compute as easily the edge/triangle spectrum: for $f \in \Omega_{\mathfrak{z}}^1(\mathcal{B})$, one has

$$\begin{aligned}(\delta_2 f)(t_0) &= \sum_{i=0}^{2} f(\sigma^i e_0) = f(e_0) + f\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix}(3\,2\,1)e_0\right) + f\left(\begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi \end{pmatrix}(1\,2\,3)e_0\right) \\ &= f(e_0) + qz_3 f(e_1) + z_2 q z_3 f(e_2) \\ (\delta_2 f)(t_1) &= \sum_{i=0}^{2} f((1\,2)\sigma^i e_0) \\ &= f(e_0) + f\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix}(1\,2\,3)e_0\right) + f\left(\begin{pmatrix} \pi & & \\ & 1 & \\ & & \pi \end{pmatrix}(3\,2\,1)e_0\right) \\ &= f(e_0) + qz_3 f(e_2) + z_1 z_3 f(e_1),\end{aligned}$$

which gives

$$[\delta_2|_{\Omega_{\mathfrak{z}}^1(\mathcal{B})}]_{\mathfrak{B}^T}^{\mathfrak{B}^E} = \begin{pmatrix} 1 & qz_3 & qz_2 z_3 \\ 1 & z_1 z_3 & qz_3 \end{pmatrix},$$

where $\mathfrak{B}^T$ is the ordered basis $f_{()}^T, f_{(1\,2)}^T$. The triangles containing e_0 are obtained by adjoining $\sigma^2 v_0$ (which gives t_0) and

$$\begin{pmatrix} \pi & x & \\ & 1 & \\ & & \pi \end{pmatrix} v_0 \quad (x \in \mathbb{F}_q),$$

giving

$$\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} t_1.$$

This yields

$$(\partial_2 f)(e_0) = f(t_0) + qf(t_1),$$

but for e_1, e_2 we need to work a little harder, and use (3.7):

$$\begin{aligned}
(\partial_2 f)(e_1) &= (\partial_2 f)((3\,2\,1)e_0) = f((3\,2\,1)t_0) + \sum_{x\in\mathbb{F}_q} f\Big((3\,2\,1)\left(\begin{smallmatrix}1 & x & \\ & 1 & \\ & & 1\end{smallmatrix}\right)t_1\Big) \\
&= \frac{1}{qz_3} f(t_0) + f((3\,2\,1)t_1) + \sum_{x\in\mathbb{F}_q^\times} f\Big((3\,2\,1)\left(\begin{smallmatrix}1 & x & \\ & 1 & \\ & & 1\end{smallmatrix}\right)(1\,2)t_0\Big) \\
&= \frac{1}{qz_3} f(t_0) + f((2\,3)t_0) + \sum_{x\in\mathbb{F}_q^\times} f\Big(\left(\begin{smallmatrix}-\frac{1}{x} & & 1\\ & 1 & \\ & & x\end{smallmatrix}\right)(3\,2\,1)\left(\begin{smallmatrix}1 & \frac{1}{x} & \\ & 1 & \\ & & 1\end{smallmatrix}\right)t_0 v\Big) \\
&= \frac{1}{qz_3} f(t_0) + z_2 f(t_1) + \sum_{x\in\mathbb{F}_q^\times} f((3\,2\,1)t_0) = \frac{1}{z_3} f(t_0) + z_2 f(t_1), \\
(\partial_2 f)(e_2) &= \frac{z_1}{q} f(t_0) + \sum_{x\in\mathbb{F}_q} f\Big(\left(\begin{smallmatrix}1 & & \\ & 1 & x\\ & & 1\end{smallmatrix}\right)(1\,3)t_0\Big) = \frac{z_1}{q} f(t_0) + \frac{1}{z_3} f(t_1),
\end{aligned}$$

so that

$$[\partial_2|_{\Omega^2_{\mathfrak{z}}(\mathcal{B})}]^{\mathfrak{B}^T}_{\mathfrak{B}^E} = \begin{pmatrix} 1 & q \\ 1/z_3 & z_2 \\ z_1/q & 1/z_3 \end{pmatrix},$$

giving

$$\begin{aligned}
[\Delta_1^+|_{\Omega^1_{\mathfrak{z}}(\mathcal{B})}]_{\mathfrak{B}^E} &= \begin{pmatrix} q+1 & \frac{q}{z_2} + qz_3 & q^2 z_3 + \frac{q}{z_1} \\ z_2 + \frac{1}{z_3} & q+1 & \frac{q}{z_1} + qz_2 \\ \frac{z_1}{q} + \frac{1}{z_3} & \frac{1}{z_2} + z_1 & q+1 \end{pmatrix}, \\
[\Delta_2^-|_{\Omega^2_{\mathfrak{z}}(\mathcal{B})}]_{\mathfrak{B}^T} &= \begin{pmatrix} q+2 & \frac{q}{z_1} + qz_2 + q \\ \frac{1}{z_2} + z_1 + 1 & 2q+1 \end{pmatrix}.
\end{aligned}$$

Recalling that $\lambda^K = k_0 - q\widetilde{\mathfrak{z}} = 2(q^2+q+1) - q(\sum z_i + z_i^{-1})$,

$$\begin{aligned}
\text{Spec}\,\Delta_1^+|_{\Omega^1_{\mathfrak{z}}(\mathcal{B})} &= \{\lambda_0^E, \lambda_\pm^E\} := \Big\{0, \tfrac{3}{2}(q+1) \pm \tfrac{1}{2}\sqrt{(q+1)^2 + 4q(2+\widetilde{\mathfrak{z}})}\Big\} \\
&= \{0, \tfrac{3k_1}{2} \pm \sqrt{(\tfrac{3k_1}{2})^2 - \lambda^K}\},
\end{aligned} \tag{3.11}$$

and again $\text{Spec}\,\Delta_2^-|_{\Omega^2_{\mathfrak{z}}(\mathcal{B})} = \{\lambda_\pm^E\}$ as we have argued for Δ_1^-. For Δ_1^+, $\lambda_0^E = 0$ is obtained on $\delta_1 f^K$ (whose f_w^E coefficients were computed in (3.8)), and $\lambda_\pm^E$ are obtained on

$$f^E_\pm = \begin{pmatrix} 2(z_2^{-2}+\frac{z_1}{z_2})q^2 - 2(z_3+1)q \\ 1-q^2z_1+q(z_1+\frac{2}{z_2}-\frac{2}{z_3}-1) \pm (qz_1-1)\sqrt{9k_1^2-4\lambda^K} \\ qz_1(z_2^{-1}+2z_1+1)-\frac{z_1}{z_2}-z_1-\frac{2}{z_2} \pm (-z_1+\frac{z_1}{z_2})\sqrt{9k_1^2-4\lambda^K} \end{pmatrix}^T \cdot \begin{pmatrix} f^E_{()} \\ f^E_{(3\,2\,1)} \\ f^E_{(1\,2\,3)} \end{pmatrix}$$

$$= 2q\Big(1+z_2+\frac{1}{z_1}\Big)\partial_2 f^T_{()} + \Big(q-1\pm\sqrt{9k_1^2-4\lambda^K}\Big)\partial_2 f^T_{(1\,2)}.$$

3.3. Unitary Iwahori-spherical representations. In general, an irreducible Iwahori-spherical representation is only a subrepresentation of $V_\mathfrak{z}$. Denote by $W_\mathfrak{z}$ this subrepresentation (there is only one such). Tadic [Tad86] classified the Satake parameters for which the representation $W_\mathfrak{z}$ admits a unitary structure, and in [KLW10] the possible $\mathfrak{z}$ for $PGL_3(F)$ are listed, and a basis for $W_\mathfrak{z} \leq V_\mathfrak{z}$ is computed explicitly, using results from [Bor76, Zel80]. It turns out that a unitary E-spherical $W_\mathfrak{z}$ is of one of the following types:

(a) $|z_i| = 1$ for $i = 1,2,3$. In this case $W_\mathfrak{z} = V_\mathfrak{z}$, and $\widetilde{\mathfrak{z}} \in [-3,6]$ gives $\lambda^K \in [k_0-6q, k_0+3q]$ and $\lambda^E_\pm \in \mathcal{I}_\pm$ (see (3.11) and (2.3)).

(b) $\mathfrak{z} = (c^{-2}, cq^a, cq^{-a})$ for some $|c| = 1$ and $0 < a < \frac{1}{2}$. Here too $W_\mathfrak{z} = V_\mathfrak{z}$.

(c) $\mathfrak{z} = (\frac{c}{\sqrt{q}}, c\sqrt{q}, c^{-2})$ for some $|c| = 1$. In this case $W^E_\mathfrak{z}$ is one-dimensional, and spanned by f^E_-, which is proportional to $qf^E_{(3\,2\,1)} - f^E_{(1\,2\,3)}$. It corresponds to

$$\begin{aligned}\lambda^E_- &= \frac{1}{2}\Big(3k_1 - \sqrt{k_1^2+8q+4q\Big(\frac{c}{\sqrt{q}}+\overline{c}\sqrt{q}+c\sqrt{q}+\frac{\overline{c}}{q}+c^{-2}+c^2\Big)}\Big) \\ &= \frac{1}{2}\Big(3k_1 - \sqrt{q^2+8q\sqrt{q}\Re(c)+2q+16q\Re(c)^2+1+8\sqrt{q}\Re(c)}\Big) \\ &= \frac{1}{2}(3k_1-(q+4\sqrt{q}\Re(c)+1)) = k_1 - 2\sqrt{q}\Re(c)\end{aligned}$$

which lies in $\mathcal{I}$ (see (2.4)). As f^E_- is not K-fixed, $W^K_\mathfrak{z} = 0$.

(d) $\mathfrak{z} = (c\sqrt{q}, \frac{c}{\sqrt{q}}, c^{-2})$ for some $|c| = 1$. Here $W^E_\mathfrak{z} = \langle f^E_0, f^E_+\rangle$, where f^E_+ is proportional to $(q+1)f^E_{()} + (c^2+\frac{c}{\sqrt{q}})(f^E_{(3\,2\,1)} + f^E_{(1\,2\,3)})$, and

$$\lambda^E_+ = 2k_1 + 2\sqrt{q}\Re(c)$$

similarly to the computation in type (c). This time f^K is in $W^E_\mathfrak{z}$, and corresponds to

$$\lambda^K = k_0 - 2q\Re\Big(\frac{(q+1)}{\sqrt{q}}c + c^2\Big).$$

(e) $\mathfrak{z} = (q, 1, \frac{1}{q})$; $W_\mathfrak{z}$ is the trivial representation $\rho : G \to \mathbb{C}^\times$, and $W_\mathfrak{z}^E = W_\mathfrak{z}^K$ are spanned by $f^K = f_+^E$. Since f^K is constant and f_+^E is a disorientation we have $\lambda^K = 0$ and $\lambda_+^E = 3k_1$ (alternatively, use (3.10) and (3.11)).

(f) $\mathfrak{z} = (\omega q, \omega, \frac{\omega}{q})$ where $\omega = e^{\pm\frac{2\pi i}{3}}$; $W_\mathfrak{z}$ is the one-dimensional representation $\rho(g) = \omega^{\tau(g)}$, and $W_\mathfrak{z}^K = W_\mathfrak{z}^E = \langle f^K \rangle = \langle f_0^E \rangle$, giving $\lambda^K = \frac{3k_0}{2}$.

Apart from these there is the Steinberg (Stn) representation $\mathfrak{z} = (\frac{1}{q}, 1, q)$. It is not E-spherical, and W^T is spanned by $f_0^T = qf_{()}^T - f_{(1\,2)}^T$, which is always in $\ker \partial_2 = \ker \Delta_2^-$. (In [KLW10] the twisted Steinberg representations

$$\mathfrak{z} = \left(\frac{\omega}{q}, \omega, \omega q\right)$$

are also considered, but they do not contribute to Ω^* as they have no K, E or T-fixed vectors.)

Let $X = \Gamma \backslash \mathcal{B}$ be a non-tripartite Ramanujan complex with

$$L^2(\Gamma\backslash G) \cong \bigoplus_i W_{\mathfrak{z}_i},$$

and denote by $N_{(t)}$ the number of $W_{\mathfrak{z}_i}$ of type (t). These are computed in [KLW10] for the tripartite case, and our arguments are similar. By the Ramanujan assumption every Iwahori-spherical $W_{\mathfrak{z}_i}$ is either tempered, which are the types (a), (c), and (Stn), or finite-dimensional (types (e), (f)), so that $N_{(\mathrm{b})} = N_{(\mathrm{d})} = 0$. The trivial representation (e) always appears once in $L^2(\Gamma\backslash G)$ as the constant functions, so that $N_{(\mathrm{e})} = 1$.[3] Type (f) corresponds to $f \in L^2(\Gamma\backslash G)$ satisfying

$$f(\Gamma g) = (gf)(\Gamma) = \omega^{\tau(g)} f(\Gamma),$$

which is unique up to scaling, and well defined iff $\Gamma \le \ker \tau$, i.e., X is tripartite. Therefore, $N_{(\mathrm{f})} = 0$ and

$$n = \dim \Omega^0(X) = \sum_i \dim W_{\mathfrak{z}_i}^K = N_{(\mathrm{a})} + N_{(\mathrm{e})} + N_{(\mathrm{f})},$$

$$\frac{nk_0}{2} = \dim \Omega^1(X) = \sum_i \dim W_{\mathfrak{z}_i}^E = 3N_{(\mathrm{a})} + N_{(\mathrm{c})} + N_{(\mathrm{e})} + N_{(\mathrm{f})}$$

[3] This explains why $3k_1$ always appear in $\operatorname{Spec} \Delta_1^+$, unlike the graph case, where $2k_0 \in \operatorname{Spec} \Delta_0^+$ only for bipartite quotients of $\mathcal{B}_2$.

together imply $N_{(\mathrm{a})} = n-1$ and $N_{(\mathrm{c})} = n(q^2+q-2)+2$. This is summarized in Table 3.1, together with the tripartite case, and this also completes the proof of Theorem 2.3. For completeness, Table 3.1 also shows W^T for each type. From

$$\frac{nk_0k_1}{6} = \dim \Omega^2(X) = 2N_{(\mathrm{a})} + N_{(\mathrm{c})} + N_{(\mathrm{e})} + N_{(Stn)}$$

one has

$$N_{((\mathrm{Stn}))} = \sum_{i=-1}^{2} (-1)^i |X^i| = \widetilde{\chi}(X),$$

the reduced Euler characteristic of X.

Type		W^K	Δ_0^+ e.v.	W^E	Δ_1^+ e.v.	W^T	mult. $\Gamma \leq \ker \tau$	mult. $\Gamma \not\leq \ker \tau$
(a)	tempered	f^K	$k_0 + q\widetilde{\mathfrak{z}}$	$f_0^E, f_\pm^E$	$0, \frac{3k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda^K}$	$\delta_1 f_\pm^E$	$n-3$	$n-1$
(b)		f^K	$k_0 + q\widetilde{\mathfrak{z}}$	$f_0^E, f_\pm^E$	$0, \frac{3k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda^K}$	$\delta_1 f_\pm^E$	0	0
(c)	tempered	0	-	f_-^E	$k_1 - 2\sqrt{q}\Re(c)$	$\delta_1 f_-^E$	nq^2+nq $-2n+6$	nq^2+nq $-2n+2$
(d)		f^K	$k_0 + q\widetilde{\mathfrak{z}}$	f_0^E, f_+^E	$0, 2k_1 + 2\sqrt{q}\Re(c)$	$\delta_1 f_+^E$	0	0
(e)	trivial	f^K	0	f_+^E	$3k_1$	$\delta_1 f_+^E$	1	1
(f)	fin. dim.	f^K	$\frac{3k_0}{2}$	f_0^E	0	0	2	0
(Stn)	tempered	0	-	0	-	f_0^T	$\widetilde{\chi}(X)$	$\widetilde{\chi}(X)$

Table 3.1. The representations appearing in $L^2(\Gamma\backslash G)$, with the corresponding Laplacian eigenvalues, and the multiplicity of appearance in the tripartite and non-tripartite Ramanujan cases.

4. Combinatorial expansion

4.1. Isoperimetric expansion. The nontrivial spectrum of Δ_0^+ on a non-tripartite Ramanujan complex is highly concentrated, lying in a $k_0 \pm O(\sqrt{k_0})$ strip. The nontrivial Δ_1^+-spectrum on 1-cycles is "almost concentrated": there are $\approx nq^2$ eigenvalues in a $k_1 \pm O(\sqrt{k_1})$ strip, but also $n-1$ eigenvalues at $2k_1 \pm O(1)$ (and the trivial eigenvalue $3k_1$). Nevertheless, having a concentrated vertex spectrum, and edge spectrum bounded away from zero is enough to prove the Cheeger-type inequality in Theorem 1.3(1): For a partition of the vertices into sets A_0, A_1, A_2 of sizes at least ϑn,

$$\frac{|X(A_0, A_1, A_2)|n^2}{|A_0||A_1||A_2|} \geq 2q^3 - 4q^{2.5} - C \cdot \frac{q^2}{\vartheta^3}.$$

Remarks: (1) This should be compared to the pseudo-random expectation: X has $\frac{1}{3!}nk_0k_1$ triangles, so its triangle density is indeed

$$\frac{\frac{1}{3}n(q^2+q+1)(q+1)}{\binom{n}{3}} \approx \frac{2q^3}{n^2}.$$

(2) The restriction $|A_i| \geq \vartheta n$ is essential: If $f(n)$ is any sub-linear function, one can take $A_0 \subseteq X^0$ to be any set of size $f(n)$, A_1 some set containing

$$\partial A_0 = \{v \,|\, \mathrm{dist}(v, A_0) = 1\},$$

and A_2 the rest of the vertices. Assuming n is large enough one has $|A_0|, |A_1|, |A_2| \geq f(n)$, and

$$T(A_0, A_1, A_2) = \varnothing.$$

(3) Another Cheeger constant for complexes was suggested in [PRT16], and studied in [GS14]. However, it is trivial for clique complexes, so we do not address it here.

We prove Theorem 1.3(1) as part of Theorem 1.2, which applies to general dimension.

Proof of Theorem 1.2. For $f \in \Omega^{d-1}(X)$ defined by

$$f([\sigma_0\sigma_1\cdots\sigma_{d-1}]) = \begin{cases} \mathrm{sgn}\,\pi|A_{\pi(d)}| & \exists\pi \in \mathrm{Sym}_{\{0\ldots d\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq d-1, \\ 0 & \text{else, i.e., } \exists k, i \neq j \text{ with } \sigma_i, \sigma_j \in A_k, \end{cases}$$

it is shown in [PRT16, §4.1] that $\|\delta f\|^2 = |X(A_0, \ldots, A_d)|n^2$.[4] For

$$f_B = \mathbb{P}_{B^{d-1}} f \quad \text{and} \quad f_Z = \mathbb{P}_{Z_{d-1}} f,$$

this gives

$$\begin{aligned} |X(A_0,\ldots,A_d)|n^2 &= \|\delta f\|^2 = \|\delta f_Z\|^2 \\ &= \langle \Delta^+_{d-1} f_Z, f_Z\rangle \\ &\geq \lambda_{d-1}\|f_Z\|^2 \\ &= \lambda_{d-1}(\|f\|^2 - \|f_B\|^2). \end{aligned}$$

[4] While [PRT16] assumes that X has a complete skeleton, this claim does not use this assumption.

Denoting $\mathcal{K} = k_0 \cdots k_{d-2}$ and

$$\mathcal{E} = \frac{\mu_0}{k_0} + \cdots + \frac{\mu_{d-2}}{k_{d-2}},$$

we have by [Par17, Thm. 1.3]

$$\begin{aligned}\|f\|^2 &= \sum_{i=0}^{d} |X(A_0,\ldots,\widehat{A_i},\ldots,A_d)||A_i|^2 \\ &\geq \sum_{i=0}^{d}\left[\frac{\mathcal{K}}{n^{d-1}}\prod_{j\neq i}|A_j| - c_{d-1}\mathcal{K}\mathcal{E}\max_{j\neq i}|A_j|\right]|A_i|^2 \\ &\geq \frac{\mathcal{K}}{n^{d-2}}\left(\prod_{i=0}^{d}|A_i|\right) - (d+1)c_{d-1}\mathcal{K}\mathcal{E}n^3 \\ &\geq \mathcal{K}\left(n^{2-d}\prod_{i=0}^{d}|A_i| - (d+1)c_{d-1}\mathcal{E}n^3\right).\end{aligned}$$

Turning to f_B, let us denote

$$\mathfrak{D} = k_0\mathbb{P}_{B^{d-1}} - \Delta^-_{d-1}.$$

Any linear maps $T: V \to W$ and $S: W \to V$ satisfy

$$(\operatorname{Spec} TS)\backslash\{0\} = (\operatorname{Spec} ST)\backslash\{0\},$$

and thus

$$\begin{aligned}\operatorname{Spec}\Delta^-_{d-1}|_{B^{d-1}} &= \operatorname{Spec}\Delta^-_{d-1}\backslash\{0\} \\ &= \operatorname{Spec}\Delta^+_{d-2}\backslash\{0\} = \operatorname{Spec}\Delta^+_{d-2}|_{B_{d-2}} \\ &\subseteq \operatorname{Spec}\Delta^+_{d-2}|_{Z_{d-2}} \\ &\subseteq [k_{d-2}-\mu_{d-2}, k_{d-2}+\mu_{d-2}].\end{aligned}$$

Together with $\Delta^-_{d-1}|_{Z_{d-1}} = 0$ this implies $\|\mathfrak{D}\| \leq \mu_{d-2}$, so that

$$\|f_B\|^2 = \langle \mathbb{P}_{B^{d-1}}f, f\rangle \leq \frac{|\langle \mathfrak{D}f, f\rangle| + |\langle \Delta^-_{d-1}f, f\rangle|}{k_{d-2}} \leq \frac{\mu_{d-2}}{k_{d-2}}\|f\|^2 + \frac{1}{k_{d-2}}\|\partial f\|^2.$$

We note that ∂f is supported on $(d-2)$-cells with vertices in distinct blocks of the partition $\{A_i\}$. For a sequence of sets $B_0,\ldots,B_\ell$, denote by $X^j(B_0,\ldots,B_\ell)$ the set of j-**galleries** in $B_0,\ldots,B_\ell$, namely, sequences of j-cells

$$\tau_i \in X(B_i,\ldots,B_{i+j})$$

such that τ_i and τ_{i+1} intersect in a $(j-1)$-cell. To shorten the formulae, we write $A_{[d]\setminus\{i,j\}}$ for $A_0, \ldots, \widehat{A_i}, \ldots, \widehat{A_j}, \ldots, A_d$.We have

$$
\begin{aligned}
\|\partial f\|^2 &= \sum_{i<j} \sum_{\tau \in X(A_{[d]\setminus\{i,j\}})} (\partial f)(\tau)^2 \\
&= \sum_{i<j} \sum_{\tau} \left| |A_j| \sum_{\rho \in A_i} \delta_{\tau\cup\rho\in X} - |A_i| \sum_{\eta \in A_i} \delta_{\tau\cup\eta\in X} \right|^2 \\
&= \sum_{i<j} [\, |A_j|^2 |X^{d-1}(A_i, A_{[d]\setminus\{i,j\}}, A_i)| \\
&\qquad - 2|A_i||A_j||X^{d-1}(A_i, A_{[d]\setminus\{i,j\}}, A_j)| \\
&\qquad + |A_i|^2 |X^{d-1}(A_j, A_{[d]\setminus\{i,j\}}, A_j)| \,]
\end{aligned} \tag{4.1}
$$

Proposition 1.6 in [Par17] estimates the number of j-galleries in $B_0, \ldots, B_\ell$ when each $j+1$ tuple $B_i, B_{i+1}, \ldots, B_{i+j+1}$ consists of disjoint sets, giving

$$
\begin{aligned}
\left| |A_j|^2 |X^{d-1}(A_i, A_{[d]\setminus\{i,j\}}, A_i)| - \frac{\mathcal{K} k_{d-2}}{n^d} |A_i||A_j| \prod_{k=0}^{d} |A_k| \right| \\
\leq c_{d-2,d} \mathcal{K} k_{d-2} \mathcal{E} n^3,
\end{aligned} \tag{4.2}
$$

and similarly for the other summands in (4.1). Observe that the middle term in (4.2) is the same in all three cases, and thus cancels out in (4.1), giving

$$\|\partial f\|^2 \leq 4 \binom{d+1}{2} c_{d-2,d} \mathcal{K} k_{d-2} \mathcal{E} n^3.$$

In total,

$$
\begin{aligned}
&\frac{|X(A_0, \ldots, A_d)| n^d}{|A_0| \cdots |A_d|} \\
&\geq \frac{\lambda_{d-1} n^{d-2}}{|A_0| \cdots |A_d|} \left(\left(1 - \frac{\mu_{d-2}}{k_{d-2}}\right) \|f\|^2 - \frac{1}{k_{d-2}} \|\partial f\|^2 \right) \\
&\geq \mathcal{K} \lambda_{d-1} \left(\left(1 - \frac{\mu_{d-2}}{k_{d-2}}\right) \left(1 - (d+1) c_{d-1} \mathcal{E} \frac{n^{d+1}}{\prod |A_i|}\right) - 4 \binom{d+1}{2} c_{d-2,d} \mathcal{E} \frac{n^{d+1}}{\prod |A_i|} \right) \\
&\geq \mathcal{K} \lambda_{d-1} \left(1 - \frac{\mu_{d-2}}{k_{d-2}} - \left((d+1) c_{d-1} + 4 \binom{d+1}{2} c_{d-2,d} \right) \frac{\mathcal{E} n^{d+1}}{\prod_{i=0}^{d} |A_i|} \right)
\end{aligned}
$$

and the theorem follows.

In the case of $d = 2$, one can work out the constants c_{d-1} and $c_{d-2,d}$ explicitly, and get the following inequality:

$$\frac{|T(A_0,A_1,A_2)|n^2}{|A_0||A_1||A_2|} \geq \lambda_1\Big(k_0 - \mu_0\Big(1 + \frac{10n^3}{9|A_0||A_1||A_2|}\Big)\Big).$$

This implies Theorem 1.3(1), since by Theorem 1.1 one has

$$k_0 = 2(q^2+q+1),\ \mu_0 = 6q \quad \text{and} \quad \lambda_1 = (q+1) - 2\sqrt{q}. \qquad \blacksquare$$

4.2. Pseudo-randomness. In this section we use not only the lower bound on the edge spectrum, but the fact that it is concentrated in two narrow stripes, to show a pseudo-random behavior of 2-galleries.

Theorem 4.1: *Let X be an n-vertex tripartite triangle complex with vertex and edge degrees k_0 and k_1, such that*

$$\operatorname{Spec}\Delta_0^+|_{Z_0} \subseteq [k_0-\mu_0, k_0+\mu_0] \cup \{\tfrac{3k_0}{2}\} \quad \textit{and}$$
$$\operatorname{Spec}\Delta_1^+|_{Z_1} \subseteq [k_1-\mu_1, k_1+\mu_1] \cup [2k_1-\mu_1, 2k_1+\mu_1] \cup \{3k_1\}.$$

If A, B, C, D are disjoint sets of vertices of sizes a, b, c, d, respectively, and each of $A \cup D$, B and C is contained in a different block of the three-partition of X (see Figure 4.1), then

$$\Big||X^2(A,B,C,D)| - \frac{27k_0k_1^2abcd}{2n^3}\Big|$$
$$\leq \frac{6\mu_0k_1^2\sqrt{abcd}}{k_0 n}\Big(\frac{3k_0(\sqrt{ab}+\sqrt{cd})}{2n} + \mu_0\Big) \tag{4.3}$$
$$+ \Big[\frac{2k_1^2\mu_0}{k_0} + (k_1+\mu_1)\mu_1\Big]\sqrt[4]{abcd}\sqrt{\Big(\frac{3k_0\sqrt{ab}}{2n}+\mu_0\Big)\Big(\frac{3k_0\sqrt{cd}}{2n}+\mu_0\Big)}.$$

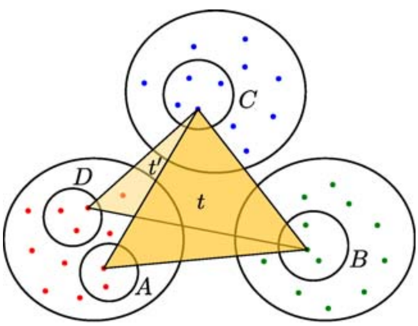

Figure 4.1. A 2-gallery through A, B, C, D in a tripartite triangle complex.

It follows that if $a, b, c, d \leq \vartheta n$ (where $\vartheta \leq \frac{1}{3}$), then the l.h.s. in (4.3) is bounded by

$$\left| |X^2(A,B,C,D)| - \frac{27k_0k_1^2abcd}{2n^3} \right| \leq \vartheta\Big(9\vartheta + \frac{4\mu_0}{k_0}\Big)(k_1\mu_0 + k_0\mu_1)k_1 n,$$

and for Ramanujan complexes $k_0 = 2(q^2+q+1)$, $k_1 = q+1$, $\mu_0 = 6q$ and $\mu_1 = 2\sqrt{q}$, which gives Theorem 1.3(2). The main term in (4.3) agrees with the pseudo-random expectation: given vertices $\alpha, \beta, \gamma, \delta$ in A, B, C, D, respectively, the probability that $\beta\gamma$ is an edge in X is $\frac{3k_0}{2n}$, and the k_1 triangles which contain it have their third vertex in the block containing $A \cup D$. The probability that α and δ are two of these is $\frac{k_1(k_1-1)}{n/3(n/3-1)}$, so that

$$\mathbb{E}(|X^2(A,B,C,D)|) = \frac{3k_0}{2n} \cdot \frac{k_1(k_1-1)}{n/3(n/3-1)} \cdot abcd \approx \frac{27k_0k_1^2abcd}{2n^3}.$$

We shall need a c-partite version of the expander mixing lemma, where we say that a k-regular graph (V,E) on n vertices is c-partite if $V = V_0 \sqcup \cdots \sqcup V_{c-1}$ with $|V_i| = \frac{n}{c}$ so that $E(V_i,V_i) = \varnothing$ and $|E(v,V_j)| = \frac{k}{c-1}$ for $v \in V_i$ and $j \neq i$. The functions $f_j(V_\ell) \equiv \exp(\frac{2\pi i j \ell}{c})/\sqrt{n}$ are orthonormal eigenfunctions of Δ_0^+ with corresponding eigenvalues

$$\lambda_j = \begin{cases} 0, & j = 0, \\ (\frac{c}{c-1})k, & 0 < j < c, \end{cases} \tag{4.4}$$

and we call $\{\lambda_0, \ldots, \lambda_{c-1}\}$ the **partite spectrum**.

Lemma 4.2: *If the non-partite spectrum of a c-partite k-regular graph on n vertices is contained in $[k-\mu, k+\mu]$, and $A \subseteq V_i$, $B \subseteq V_j$ for $i \neq j$, then*

$$\left| |E(A,B)| - \frac{ck|A||B|}{(c-1)n} \right| \leq \mu\sqrt{|A||B|}. \tag{4.5}$$

Proof. Assuming that $A \subseteq V_0$ and $B \subseteq V_1$, and denoting by $\mathbb{P}_W$ the orthogonal projection on $W = \langle f_0, \ldots, f_{c-1}\rangle^\perp$, we have

$$\begin{aligned} |E(A,B)| &= \langle (kI - \Delta_0^+)\mathbb{1}_A, \mathbb{1}_B \rangle \\ &= \sum_{j=0}^{c-1} (k-\lambda_j)\langle \mathbb{1}_A, f_j\rangle\langle \mathbb{1}_B, f_j\rangle + \langle (kI - \Delta_0^+)\mathbb{P}_W \mathbb{1}_A, \mathbb{1}_B\rangle \\ &= \frac{|A||B|}{n} \sum_{j=0}^{c-1} (k-\lambda_j)\exp\Big(\frac{2\pi i j}{c}\Big) + \langle (kI - \Delta_0^+)\mathbb{P}_W \mathbb{1}_A, \mathbb{1}_B\rangle, \end{aligned}$$

and (4.5) follows by (4.4) and $\|(kI - \Delta_0^+)|_W\| \leq \mu$. ∎

Proof of Theorem 4.1. Denote by U^+ the span of the Δ_1^+-eigenforms with eigenvalues in $[k_1 - \mu_1, k_1 + \mu_1] \cup [2k_1 - \mu_1, 2k_1 + \mu_1]$, and by η a normalized $3k_1$-eigenform for Δ_1^+, so that $\Omega^1(X) = B^1 \oplus U^+ \oplus \langle \eta \rangle$. Denoting

$$p(x) = (x - k_1)(x - 2k_1),$$

$p(\Delta_1^+)$ acts on $B^1 \oplus \langle \eta \rangle$ as the scalar $2k_1^2$, and

$$\begin{aligned} \|p(\Delta_1^+)|_{U^+}\| \leq& \max\{|p(\lambda)| \mid \lambda \in [k_1 - \mu_1, k_1 + \mu_1] \cup [2k_1 - \mu_1, 2k_1 + \mu_1]\} \\ =& (k_1 + \mu_1)\mu_1. \end{aligned}$$

Say that two directed edges are **neighbors** if they have a common origin or a common terminus, and their union (as a cell) is in X^2. We denote this by $e \sim e'$, and define $\mathcal{A} : \Omega^1(X) \to \Omega^1(X)$ by

$$(\mathcal{A}f)(e) = \sum_{e' \sim e} f(e').$$

The upper Laplacian satisfies $\Delta_1^+ = k_1 \cdot I - \mathcal{A}$ (see [Par17]), and it follows that $p(\Delta_1^+) = \mathcal{A}^2 + k_1\mathcal{A}$. Define $\mathbb{1}_{AB} \in \Omega^1(X)$ by

$$\mathbb{1}_{AB}(vw) = \begin{cases} 1, & v \in A, w \in B, \\ -1, & v \in B, w \in A, \\ 0, & \text{otherwise}, \end{cases}$$

and similarly $\mathbb{1}_{CD}$. We claim that

$$|X^2(A, B, C, D)| = \langle p(\Delta_1^+)\mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle. \tag{4.6}$$

Indeed, edges in $E(A, B)$ have no neighbors in $E(C, D)$, so that $\langle \mathcal{A}\mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = 0$, and $\mathcal{A}$ is self-adjoint (since Δ_1^+ is), giving

$$\langle p(\Delta_1^+)\mathbb{1}_{AB}, \mathbb{1}_{CD} \rangle = \langle \mathcal{A}\mathbb{1}_{AB}, \mathcal{A}\mathbb{1}_{CD} \rangle = \sum_{\substack{e, e', e'' \in X^1 \\ e' \sim e \sim e''}} \mathbb{1}_{AB}(e')\mathbb{1}_{CD}(e''). \tag{4.7}$$

The nonzero terms in this sum come from edges e which have neighbors $e' \in E(A, B)$ and $e'' \in E(C, D)$, and it follows that $e \in E(B, C)$. Thus,

$$(e' \cup e, e \cup e'')$$

is a 2-gallery in $X^2(A, B, C, D)$, and observing that $\mathbb{1}_{AB}(e') = \mathbb{1}_{CD}(e'')(= \pm 1)$ it contributes one to (4.7). On the other hand, for every gallery

$$(t, t') \in X^2(A, B, C, D),$$

the edges $e' = t\backslash C$, $e = t \cap t'$, $e'' = t\backslash B$ form such a triplet, and we obtain (4.6). On the spectral side,

$$\langle p(\Delta_1^+)\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle = 2k_1^2\langle \mathbb{P}_{B^1\oplus\langle\eta\rangle}\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle + \langle p(\Delta_+^1)\mathbb{P}_{U^+}\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle,$$

and the last term is bounded by

$$\|p(\Delta_+^1)|_{U^+}\|\|\mathbb{1}_{AB}\|\|\mathbb{1}_{CD}\| \le (k_1+\mu_1)\mu_1\sqrt{E_{AB}E_{CD}}, \tag{4.8}$$

where $E_{ST} := |E(S,T)|$. As η has constant sign on $V_0 \to V_1 \to V_2 \to V_0$,

$$2k_1^2\langle \mathbb{P}_{\langle\eta\rangle}\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle = 2k_1^2\langle \mathbb{1}_{AB}, \eta\rangle\langle \eta, \mathbb{1}_{CD}\rangle = \frac{4k_1^2 E_{AB}E_{CD}}{k_0 n}, \tag{4.9}$$

and we are left to analyze $\mathbb{P}_{B^1}\mathbb{1}_{AB}$. As in the non-tripartite case, one has $\operatorname{Spec}\Delta_-^1|_{B^1} = \operatorname{Spec}\Delta_+^0|_{B_0}$, but now the latter comprises not only eigenvalues in $[k_0-\mu_0, k_0+\mu_0]$, but also $\frac{3k_0}{2}$ (twice, see Theorem 2.3). If $\omega = \exp(\frac{2\pi i}{3})$ and

$$\xi(vw) = \begin{cases} \sqrt{2/k_0 n}, & v\in V_0, w\in V_1;\\ -\sqrt{2/k_0 n}, & w\in V_0, v\in V_1;\\ \omega\sqrt{2/k_0 n}, & v\in V_1, w\in V_2;\\ -\omega\sqrt{2/k_0 n}, & w\in V_1, v\in V_2;\\ \overline{\omega}\sqrt{2/k_0 n}, & v\in V_2, w\in V_0;\\ -\overline{\omega}\sqrt{2/k_0 n}, & w\in V_2, v\in V_0, \end{cases}$$

then $\{\xi, \overline{\xi}\}$ is an orthonormal basis for the $\frac{3k_0}{2}$-eigenspace of Δ_1^-. Denote by U^- the space spanned by the Δ_1^--eigenforms with eigenvalue in $[k_0-\mu_0, k_0+\mu_0]$. By the action of each summand in

$$\mathfrak{D}' := k_0\mathbb{P}_{B^1} + \frac{k_0}{2}\mathbb{P}_{\langle\xi,\overline{\xi}\rangle} - \Delta_1^-$$

on each of the terms in the orthogonal decomposition

$$\Omega^1(X) = Z_1 \oplus U^- \oplus \langle\xi,\overline{\xi}\rangle$$

we see that $\|\mathfrak{D}'\| \le \mu_0$. Due to the fact that $\partial_1\mathbb{1}_{AB}$ and $\partial_1\mathbb{1}_{CD}$ are supported on different vertices, $\langle\Delta_1^-\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle$ vanishes, and together with

$$\langle\mathbb{P}_{\langle\xi,\overline{\xi}\rangle}\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle = 2\Re(\langle\mathbb{1}_{AB},\xi\rangle\langle\xi,\mathbb{1}_{CD}\rangle) = -\frac{2E_{AB}E_{CD}}{k_0 n}$$

(and $\mathbb{P}_{B^1} = \frac{1}{k_0}\Delta_1^- - \frac{1}{2}\mathbb{P}_{\langle\xi,\overline{\xi}\rangle} + \frac{1}{k_0}\mathfrak{D}'$) this gives

$$\left|2k_1^2\langle\mathbb{P}_{B^1}\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle - \frac{2k_1^2E_{AB}E_{CD}}{k_0 n}\right| \le \frac{2k_1^2}{k_0}\langle\mathfrak{D}'\mathbb{1}_{AB}, \mathbb{1}_{CD}\rangle \le \frac{2\mu_0 k_1^2\sqrt{E_{AB}E_{CD}}}{k_0}.$$

Combining this with (4.8) and (4.9) we conclude that

$$\left||X^2(A,B,C,D)| - \frac{6k_1^2 E_{AB}E_{CD}}{k_0 n}\right| \leq \left(\frac{2\mu_0 k_1^2}{k_0} + (k_1+\mu_1)\mu_1\right)\sqrt{E_{AB}E_{CD}}.$$

This estimates $|X^2(A,B,C,D)|$ in terms of E_{AB} and E_{CD}. To have an estimate in terms of a, b, c and d we use Lemma 4.2, which gives $|E_{AB} - \frac{3k_0 ab}{2n}| \leq \mu_0\sqrt{ab}$ and similarly for E_{CD}, and the theorem follows. ∎

Various applications of a triangle mixing lemma can be adjusted to use our gallery mixing lemma. We demonstrate this below for chromatic numbers, and other examples are Gromov's overlap property, along the lines of [FGL+12, Par17], and the crossing numbers of complexes, as discussed in [GW13, §8.1]. Nevertheless, the question of triangle pseudorandomness remains interesting, and should give better results if it does hold. The fact that most of the spectrum is concentrated in the strip $\mathcal{I}$ (see Theorem 2.3) gives hope that this can be done by analyzing the combinatorics of eigenforms which occur in the principal series (type (a) in §3.3), and showing that their contribution is negligible.

4.3. Chromatic number. As an application of Theorem 4.1 we prove Theorem 1.3(3), which bounds the chromatic number of non-tripartite Ramanujan complexes.

Proof of Theorem 1.3(3). Write $X = \Gamma\backslash\mathcal{B}_3$ and let $\widehat{\Gamma} = \Gamma \cap \ker\tau$. This is a normal subgroup of Γ of index three, and $\widehat{X} := \widehat{\Gamma}\backslash\mathcal{B}_3 \overset{\pi}{\twoheadrightarrow} X$ is a tripartite three-cover. If the chromatic number of X is χ, we can find a set $N \subseteq X^0$ of size $\frac{n}{\chi}$ with $T(N,N,N) = \varnothing$. Let

$$N_i = \{v \in \widehat{X}^0 \,|\, \pi(v) \in N, \tau(v) = i\},$$

and take $A = N_0$, $B = N_1$, $C = N_2$ and $D \subseteq \{v \in \widehat{X}^0 \,|\, \tau(v) = 0\}\backslash N_0$ such that $|D| = \frac{n}{2}$. Since the set $T(N,N,N)$ is empty, $X^2(A,B,C,D)$ is empty as well. Therefore the l.h.s. of (4.3) reads

$$\frac{q^4 n}{2\chi^3} = \frac{q^4 abcd}{n^3} \leq \frac{27k_0k_1^2 abcd}{2(3n)^3}.$$

Assume to the contrary that $\chi < \frac{\sqrt[3]{q}}{5}$. Then the r.h.s. of (4.3) is bounded by

$$\frac{nq^{3.5}}{\chi^{1.5}\sqrt{2}}\left(2 + \frac{1}{125}(264 + 255\sqrt{2})\right) < \frac{nq^{3.5}}{\chi^{1.5}} \cdot \frac{7}{\sqrt{2}},$$

so that Theorem 4.1 implies $\sqrt{q} < 7\sqrt{2}\chi^{1.5}$, which contradicts the assumption. ∎

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