ISRAEL JOURNAL OF MATHEMATICS **230** (2019), 563–581
DOI: 10.1007/s11856-018-1823-9

HARMONIC FUNCTIONS VANISHING ON A CONE

BY

Dan Mangoubi and Adi Weller Weiser

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 9190401, Israel
e-mail: dan.mangoubi@mail.huji.ac.il, adi.weller@mail.huji.ac.il

ABSTRACT

Let Z be a quadratic harmonic cone in $\mathbb{R}^3$. We consider the family $\mathcal{H}(Z)$ of all harmonic functions vanishing on Z. Is $\mathcal{H}(Z)$ finite or infinite dimensional? Some aspects of this question go back to as early as the 19th century. To the best of our knowledge, no nondegenerate quadratic harmonic cone exists for which the answer to this question is known. In this paper we study the right circular harmonic cone and give evidence that the family of harmonic functions vanishing on it is, maybe surprisingly, finite dimensional. We introduce an arithmetic method to handle this question which extends ideas of Holt and Ille and is reminiscent of Hensel's Lemma.

1. Introduction

1.1. Background. Consider the family $\mathcal{H}(Z)$ of harmonic functions in the unit ball $B \subset \mathbb{R}^n$ vanishing on a given set $Z \subseteq B$. It was conjectured in [9] and was completely proved by Logunov and Malinnikova in [7] and [8] that $\mathcal{H}(Z)$ possesses compactness properties. More precisely, one can prove a Harnack type inequality for the quotient of two functions in $\mathcal{H}(Z)$. In $\mathbb{R}^2$ the family $\mathcal{H}(Z)$ is locally infinite dimensional (see [9] for examples). In higher dimensions few examples of infinite dimensional $\mathcal{H}(Z)$ are known. In fact, all known examples stem from two dimensional ones (see [7, §4.2]). In particular, in dimension $n = 3$ it is not even known whether there exists an infinite dimensional family $\mathcal{H}(Z)$ where Z is a nondegenerate quadratic harmonic cone (it may be worth

Received July 27, 2017 and in revised form April 18, 2018

mentioning that in dimension $n = 4$ there exists such an example, see [7, §4.2]). It turns out that this question and similar ones attracted the attention of several mathematicians.

Maybe the oldest closely related problem is a classical conjecture by Stieltjes (in a letter to Hermite [11]), which concerns arithmetic properties of harmonic functions in $\mathbb{R}^3$ which are invariant under rotations around some axis. The present work considers the rotationally equivariant cases (see details in §1.2).

Second, an analogous question was raised and solved in the context of Bessel functions. Siegel [10] proved Bourget's hypothesis that no two distinct Bessel functions have common zeros (see also [12, pp. 484–485]). To make the resemblance clear we note that the problem we treat here can be formulated as whether an associated Legendre function P_l^m has a common root with the Legendre polynomial P_2 (see §8).

Third, as a possible application to the wave equation, Agranovsky and Krasnov raised in [1] the conjecture that there exists a quadratic harmonic cone $Z \subset \mathbb{R}^3$ such that $\mathcal{H}(Z)$ is finite dimensional.

Last, a spectral theory point of view of the same problem was given recently by Bourgain and Rudnick in [3]. That work shows that given a curve of positive curvature on the standard two-dimensional flat torus there exist only a finite number of Laplace eigenfunctions vanishing on that curve. In the case of the sphere, an analogous question would be: Let $\gamma \subset S^2$ be a curve of constant latitude which is not the equator. Do there exist only a finite number of eigenfunctions vanishing on γ? This question is still open, and the current work can be considered as treating a special case of it.

The aim of the present paper is to study the family $\mathcal{H}(Z)$ where Z is the right circular harmonic cone in $\mathbb{R}^3$. In some sense, this is the simplest nondegenerate harmonic zero set in $\mathbb{R}^3$. We give evidence that this family is finite dimensional, while introducing a new method for handling this question.

1.2. Results and Methods.

1.2.1. *Main result.* Let us formulate the following Conjecture.

Conjecture 1: *Let $Z = \{x^2 + y^2 - 2z^2 = 0\} \cap B_1$ where $B_1 \subset \mathbb{R}^3$ is the unit ball. Consider the family*

$$\mathcal{H}(Z) = \{u : B_1 \to \mathbb{R} | \Delta u = 0 \text{ and } u|_Z = 0\}.$$

Then $\mathcal{H}(Z)$ is finite dimensional.

Using standard tools of harmonic analysis (see [2] and §8) it is not difficult to show that Conjecture 1 is equivalent to the following one, which concerns the associated Legendre functions P_l^m. The Legendre polynomials ($m = 0$) will simply be denoted by P_l.

Conjecture 1′: *The number of pairs (l, m) such that $P_2 | P_l^m$ is finite.*

Here, for odd m, $P_2|P_l^m$ means that P_2 divides the polynomial

$$\frac{P_l^m}{\sqrt{1-x^2}}.$$

The case $m = 0$ of the preceding conjecture would follow from a conjecture of Stieltjes [11] concerning the irreducibility of the Legendre polynomials over $\mathbb{Q}$. As such, it aroused the interest of several authors and, in fact, was proved by Holt [5] and Ille [6]. The main new contribution of the current work comes in the cases where $m \neq 0$. We prove the following theorem (where we include the case $m = 0$ for completeness).

Theorem 2:

- *For $m = 0$, $P_2|P_l$ if and only if $l = 2$.*
- *For $m = 2$, $P_2|P_l^2$ if and only if $l = 5$.*
- *If m is even, then there exists at most one $l \in \mathbb{N}$ such that $P_2|P_l^m$.*
- *If m is odd, then $P_2 \nmid P_l^m$ for all $l \in \mathbb{N}$.*

In fact, we prove a stronger statement in the case where m is even. We show that in some sense there exists a unique dyadic integer l such that $P_2|P_l^m$. For the precise meaning of this please see §1.2.2 and Theorems 13 and 21.

1.2.2. *Method.* The method we use in this paper consists of two steps. In the first step, following an idea of Holt for the case $m = 0$, we transform P_l^m to a polynomial H_l^m whose coefficients are dyadic integers. As such, this polynomial can be studied using modular arithmetic. The question is whether this polynomial vanishes at the point $z = -2$. In the second step we consider $H_l^m(-2) = H^m(l)$ as a (non-polynomial) function of l. We ask whether $H^m(l) \equiv 0 \pmod{2^N}$. Our analysis shows that, if m is even, H^m is well defined on $\mathbb{Z}/2^N\mathbb{Z}$ and that a solution modulus 2^N can be lifted uniquely to a solution modulus 2^{N+1}. In this way we get a unique dyadic integer l such that $H^m(l) = 0$ (Propositions 12 and 20; Theorems 13 and 21). This idea is reminiscent of Hensel's Lemma. However, we cannot apply Hensel's Lemma in our case since the nature of the coefficients in Taylor's expansion of H^m is unclear.

1.2.3. *Secondary results.* We describe a second approach to Conjecture 1, under significant additional assumptions. We prove

THEOREM 3: *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a harmonic function. Then the product $(x^2 + y^2 - 2z^2) \cdot f(x, y, z)$ is harmonic if and only if*

$$f(x, y, z) = \alpha + \beta xyz + \gamma(x^2 - y^2)z$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Under the same assumption we can also break the rotational symmetry of the quadratic cone and get

THEOREM 4: *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a harmonic function and $b > 1$. Then the product $(x^2 + by^2 - (b + 1)z^2) \cdot f(x, y, z)$ is harmonic if and only if*

$$f(x, y, z) = \alpha + \beta xyz$$

for some $\alpha, \beta \in \mathbb{R}$.

The additional assumption on the harmonicity of f lets us give a proof of Theorems 3 and 4 without arithmetic considerations. Perhaps our assumptions on f can be weakened.

ACKNOWLEDGMENTS. We are grateful to Charles Fefferman from whose discussions with the first author the work on this topic originated. We thank Zeev Rudnick for his interest in the present work and for helpful comments. We thank Eran Asaf, Nir Avni, Alexander Logunov, Eugenia Malinnikova and Amit Ophir for interesting discussions. This paper is part of the second author's research towards a Ph.D. dissertation, conducted at the Hebrew University of Jerusalem. The cases of $m = 2$ and odd m in Theorem 2 together with Theorems 3 and 4 were proved in [13]. We gratefully acknowledge the support of ISF grant no. 753/14.

2. Step 1: Holt–Ille Transformation

In this section we transform the associated Legendre function P_l^m to a polynomial H_l^m of degree $\lfloor \frac{l-m}{2} \rfloor$. The main property of H_l^m is that its coefficients are dyadic integers, making it useful in analyzing whether P_2 and P_l^m share a common root. This idea was developed by Holt [5] and Ille [6] for the case $m = 0$ and we extend it here to the cases where $m \neq 0$.

We recall the following integral representation of the associated Legendre functions:

$$P_l^m(x) = i^m \frac{(l+m)!}{l!} \frac{1}{\pi} \int_0^\pi (x + y\cos\varphi)^l \cos(m\varphi) d\varphi \tag{1}$$

for $x \in [-1, 1]$, where

$$y = i\sqrt{1-x^2}$$

[4, Ch. VII, p. 505]. By Lemma 25 the integral on the right-hand side is of the form $x^\delta y^m Q(x^2, y^2)$ where $\delta \in \{0, 1\}$ with $\delta \equiv (l-m) \pmod 2$ and Q is a real homogeneous polynomial of degree $\lfloor \frac{l-m}{2} \rfloor$. If we define

$$z = \frac{4x^2}{x^2 - 1},$$

then we can express x^2 and y^2 by

$$x^2 = \frac{z}{z-4}, \quad y^2 = \frac{4}{z-4}. \tag{2}$$

Substituting the preceding expressions in Q gives

$$P_l^m(x) = x^\delta y^m \frac{C_l^m}{(z-4)^{\lfloor \frac{l-m}{2} \rfloor}} H_l^m(z), \tag{3}$$

where $H_l^m(z)$ is a polynomial of degree $\lfloor \frac{l-m}{2} \rfloor$ normalized so that $H_l^m(0) = 1$ and the constant C_l^m depends on l and m (for details see proof of Lemma 7 in §9). We note

Proposition 5: *$P_2 | P_l^m$ if and only if $H_l^m(-2) = 0$.*

Proof. $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Hence $P_2 | P_l^m$ if and only if

$$P_l^m\Big(\frac{1}{\sqrt{3}}\Big) = 0.$$

The proposition now follows from (2) and (3). ∎

Notation 6: We denote by $\sigma_l^m(k)$ coefficients such that the following holds

$$H_l^m(z) = \sum_{k=0}^{\lfloor \frac{l-m}{2} \rfloor} (-\frac{z}{2})^k \sigma_l^m(k).$$

The formulas for $\sigma_l^m(k)$ are recorded in the following lemma.

LEMMA 7: *If $l+m$ is even, then*

$$\sigma_l^m(k) = \frac{(-2)^k \binom{\lfloor \frac{l-m}{2} \rfloor}{k} \binom{\lfloor \frac{l+m}{2} \rfloor}{k}}{\binom{2k}{k}}, \tag{4}$$

and if $l+m$ is odd, then

$$\sigma_l^m(k) = \frac{(-2)^k \binom{\lfloor \frac{l-m}{2} \rfloor}{k} \binom{\lfloor \frac{l+m}{2} \rfloor}{k}}{\binom{2k}{k}(2k+1)}. \tag{5}$$

In both cases

$$\sigma_l^m(0) = 1. \tag{6}$$

A proof for these formulas is included in §9.

3. Proof of Theorem 2: The case of odd m

It will be convenient to use the following

Notation: We denote

$$s = \left\lfloor \frac{l}{2} \right\rfloor, \quad t = \left\lfloor \frac{m}{2} \right\rfloor. \tag{7}$$

We first assume that l is odd. We rewrite formula (4) as follows:

$$\sigma_{2s+1}^{2t+1}(k) = (-1)^k \binom{s-t}{k} \frac{(s+t+1)(s+t)\cdots(s+t-k+2)}{1\cdot 3\cdots(2k-1)}.$$

The term $\sigma_l^m(1)$ is even, since either $(s-t)$ or $(s+t+1)$ is even.

For $k \geq 2$ we have that $\sigma_l^m(k)$ is even, since there are at least two consecutive numbers at the nominator of the second factor, and no even numbers in the denominator.

Combined with (6), it follows that

$$H_l^m(-2) = \sum_k \sigma_l^m(k) \equiv 1 \pmod 2.$$

From Proposition 5 we get that $P_2 \nmid P_l^m$.

If l is even, essentially the same argument holds replacing formula (4) by formula (5). We leave the details to the reader.

4. Proof of Theorem 2: The case of $m \equiv 0 \pmod 4$

4.1. Recovering the lowest three bits of l.

Proposition 8: *If* $H_l^m(-2) \equiv 0 \pmod 8$ *and* $m \equiv 0 \pmod 4$, *then*

$$l \equiv 2 \pmod 8.$$

The proof is by ruling out the other possibilities one by one.

Lemma 9: *If* $m \equiv 0 \pmod 4$ *and either* $l \equiv 0 \pmod 4$ *or* $l \equiv 1 \pmod 4$, *then*

$$H_l^m(-2) \equiv 1 \pmod 2.$$

Proof. In these cases both s and t (see (7)) are even. Rewriting formulas (4) and (5) modulus 2, we get

$$\sigma_l^m(k) \equiv \binom{s-t}{k}(s+t)(s+t-1)\cdots(s+t-k+1) \pmod 2.$$

For every $k \geq 1$ we have that $\sigma_l^m(k)$ is even, since it has the even factor $(s+t)$. Summing over $k \geq 0$ we get $H_l^m(-2) \equiv 1 \pmod 2$. ∎

Lemma 10: *If* $m \equiv 0 \pmod 4$ *and* $l \equiv 3 \pmod 4$, *then*

$$H_l^m(-2) \equiv 2 \pmod 4.$$

Proof. Here s is odd and t is even. We calculate $H_{2s+1}^{2t}(-2) \pmod 4$ using formula (5).

$$\sigma_{2s+1}^{2t}(1) = -\frac{(s-t)(s+t)}{3} = -\frac{s^2-t^2}{3} \equiv -\frac{1-0}{-1} \equiv 1 \pmod 4$$

and

$$\sigma_{2s+1}^{2t}(3) = -\sigma_{2s+1}^{2t}(2)\frac{(s-t-2)(s+t-2)}{3\cdot 7} \equiv -\sigma_{2s+1}^{2t}(2) \pmod 4.$$

From basic divisibility properties (Lemma 27) we also have

$$\forall k \geq 4, \quad \sigma_l^m(k) \equiv 0 \pmod 4.$$

Summing over $k \geq 0$ we get $H_{2s+1}^{2t}(-2) \equiv 2 \pmod 4$. ∎

Lemma 11: *If* $m \equiv 0 \pmod 4$ *and* $l \equiv 6 \pmod 8$, *then*

$$H_l^m(-2) \equiv 4 \pmod 8.$$

Proof. We let $l = 8q + 6$. By formula (4)

$$\sigma_l^m(1) = -(4q+3-t)(4q+3+t) \equiv t^2 - 1 \pmod 8,$$
$$\sigma_l^m(2) = -\sigma_l^m(1)\frac{(4q+2-t)(4q+2+t)}{2\cdot 3} \equiv -\sigma_l^m(1)\frac{4-t^2}{2\cdot 3} \pmod 8$$

and

$$\sigma_l^m(3) = -\sigma_l^m(2)\frac{(4q+1)^2-t^2}{3\cdot 5} \equiv \sigma_l^m(2) \pmod 8,$$

where in the last calculation we used the fact that $\sigma_l^m(2) \equiv 0 \pmod 2$ (Lemma 27). From basic divisibility properties (Lemma 27) we have $\sigma_l^m(k) \equiv 0 \pmod 8$ for all $k \geq 4$. Summing over $k \geq 0$ we get $H_l^m(-2) \equiv 4 \pmod 8$. ∎

To be complete we verify the remaining case $l \equiv 2 \pmod 8$.

Proposition 12: *If* $m \equiv 0 \pmod 4$ *and* $l \equiv 2 \pmod 8$, *then*

$$H_l^m(-2) \equiv 0 \pmod 8.$$

Proof. We let $l = 8q + 2$. By formula (4)

$$\sigma_l^m(1) = -(4q+1-t)(4q+1+t) \equiv t^2 - 1 \pmod 8,$$
$$\sigma_l^m(2) = -\sigma_l^m(1)\frac{(4q-t)(4q+t)}{2\cdot 3} \equiv \sigma_l^m(1)\frac{t^2}{2\cdot 3} \pmod 8$$

and

$$\sigma_l^m(3) = -\sigma_l^m(2)\frac{(4q-1)^2-t^2}{3\cdot 5} \equiv \sigma_l^m(2) \pmod 8.$$

From basic divisibility properties (Lemma 27) we have $\sigma_l^m(k) \equiv 0 \pmod 8$ for all $k \geq 4$. Summing over $k \geq 0$ we get $H_l^m(-2) \equiv 0 \pmod 8$. ∎

4.2. Recovering the high bits of l. In this section we introduce an idea in the spirit of Hensel's Lemma to recover the high bits of l.

Theorem 13: *Let* $m \equiv 0 \pmod 4$ *and suppose that* $H_l^m(-2) \equiv 0 \pmod{2^N}$. *Then there exists a unique* $l \leq \tilde{l} < l + 2^{N+1}$ *such that* $H_{\tilde{l}}^m(-2) \equiv 0 \pmod{2^{N+1}}$. *Moreover,* $\tilde{l} \equiv l \pmod{2^N}$.

Using Proposition 8, Theorem 13 is an immediate corollary of

Proposition 14: *Let* $m \equiv 0 \pmod 4$ *and* $l \equiv 2 \pmod 8$. *Fix* $N \geq 3$ *and write* $l = r + 2^N q$ *with* $0 \leq r < 2^N$. *Then*

$$H_l^m(-2) \equiv H_r^m(-2) + 2^N q \pmod{2^{N+1}}.$$

Remark: In particular, it follows that if $l \equiv \tilde{l} \pmod{2^N}$ then

$$H_l^m(-2) \equiv H_{\tilde{l}}^m(-2) \pmod{2^N}.$$

To prove Proposition 14 we first observe

LEMMA 15: *Let $m \equiv 0 \pmod 4$, $l \equiv 0 \pmod 2$ and $k \geq 6$. If $\tilde{l} \equiv l \pmod{2^N}$, then*

$$\sigma_{\tilde{l}}^m(k) \equiv \sigma_l^m(k) \pmod{2^{N+1}}.$$

We postpone the proof of this Lemma to the end of the section.

Proof of Proposition 14. By Lemma 15, for all $k \geq 6$ we have that

$$\sigma_l^m(k) \equiv \sigma_r^m(k) \pmod{2^{N+1}}.$$

For $0 \leq k \leq 5$ we calculate $\sigma_l^m(k)$ explicitly.

From (6) we have

$$\sigma_l^m(0) = \sigma_r^m(0) = 1.$$

Let $\tilde{s} = \frac{r}{2}$. From the assumptions we have $\tilde{s} \equiv 1 \pmod 4$. Using this and the assumptions that $N \geq 3$ and t is even, we get the following expressions for the next terms:

$$\sigma_l^m(1) = -(2^{N-1}q + \tilde{s} - t)(2^{N-1}q + \tilde{s} + t) \equiv 2^N q + \sigma_r^m(1) \pmod{2^{N+1}},$$
$$\sigma_l^m(2) = -\sigma_l^m(1)\frac{(2^{N-1}q + \tilde{s} - t - 1)(2^{N-1}q + \tilde{s} + t - 1)}{2 \cdot 3}.$$

Noticing that $\tilde{s} \equiv 1 \pmod 4$, $N \geq 3$, $\tilde{s} + t - 1$ is even and that $\sigma_r^m(1)$ is odd, elementary manipulations of this expression give

$$\sigma_l^m(2) \equiv 2^{2N-3}q + \sigma_r^m(2) \pmod{2^{N+1}}.$$

Moving on to the next term, using again that $N \geq 3$ and noticing that $\sigma_l^m(2)$ is even, we get

$$\begin{aligned}\sigma_l^m(3) = &-\sigma_l^m(2)\frac{(2^{N-1}q + \tilde{s} - t - 2)(2^{N-1}q + \tilde{s} + t - 2)}{3 \cdot 5}\\ \equiv &2^{2N-3}q + \sigma_r^m(3) \pmod{2^{N+1}}.\end{aligned}$$

At this point we observe that $\sigma_l^m(2)+\sigma_l^m(3)\equiv\sigma_r^m(2)+\sigma_r^m(3) \pmod{2^{N+1}}$ (since $N\geq 3$). A similar circumstance occurs in the next two terms:

$$\begin{aligned}\sigma_l^m(4) =& -\sigma_l^m(3)\frac{(2^{N-1}q+\tilde{s}-t-3)(2^{N-1}q+\tilde{s}+t-3)}{4\cdot 7}\\ \equiv& 2^{2N-5}q(4-t^2)+\sigma_r^m(3)[2^{2N-4}q+2^{N-1}q]+\sigma_r^m(4)\\ \equiv& 2^{2N-3}q+2^{N-1}tq+\sigma_r^m(4) \pmod{2^{N+1}}\end{aligned}$$

and

$$\begin{aligned}\sigma_l^m(5) \equiv& -\sigma_l^m(4)\frac{(2^{N-1}q+\tilde{s}-t-4)(2^{N-1}q+\tilde{s}+t-4)}{5\cdot 9}\\ \equiv& 2^{2N-3}q+2^{N-1}tq+\sigma_r^m(5) \pmod{2^{N+1}}\end{aligned}$$

Summing over $k\geq 0$ we get that $H_l^m(-2)\equiv 2^N q+H_r^m(-2) \pmod{2^{N+1}}$. ∎

It remains to prove Lemma 15.

Proof of Lemma 15. For any $x\in\mathbb{Q}$ let $v_2(x)$ be the dyadic valuation of x (see Notation 26). For k such that $v_2(k!)\geq N+1$ we have that

$$\sigma_l^m(k)\equiv 0 \pmod{2^{N+1}}$$

for all l (see Lemma 27). Hence we may assume that

$$v_2(k!)\leq N. \tag{8}$$

In particular, since $k\geq 6$ it follows that

$$N\geq 4. \tag{9}$$

Let $s=\lfloor\frac{l}{2}\rfloor=2^{N-1}q+\tilde{s}$ with $0\leq\tilde{s}<2^{N-1}$. Collecting terms in (4) according to the powers of q we see that

$$\sigma_l^m(k)=\sum_{i=0}^{k}\sum_{j=0}^{k}\underbrace{\frac{1}{k!(2k-1)!!}2^{(N-1)(i+j)}P_{k-i,k}(\tilde{s}-t)P_{k-j,k}(\tilde{s}+t)}_{a_{i,j,k,l,m}}q^{i+j}$$

where $P_{i,k}(x)$ is the elementary symmetric polynomial of degree i in the k variables $\{x,x-1,\dots,x-k+1\}$.

We now show that all the coefficients $a_{i,j,k,l,m}$ with $i+j>0$ vanish modulus 2^{N+1}. This is enough since $\tilde{s}$ is determined by l modulus 2^N.

Case (i): $i+j\geq 4$. We use (8) and (9) to get

$$v_2(a_{i,j,k,l,m})\geq 4(N-1)-v_2(k!)\geq 3N-4>N+1.$$

Case (ii): $i+j=1$. We may assume $i=0, j=1$. Here

$$v_2(P_{k-i,k}(\tilde{s}-t)) = v_2(P_{k,k}(\tilde{s}-t)) \geq v_2(k!)$$

(see proof of Lemma 27). Since $k \geq 6$ we also have

$$v_2(P_{k-j,k}(\tilde{s}+t)) = v_2(P_{k-1,k}(\tilde{s}+t)) \geq 2.$$

So $v_2(a_{i,j,k,l,m}) \geq N-1+v_2(k!)+2-v_2(k!) \geq N+1$.

Case (iii): $3 \geq i+j \geq 2$. Here a direct examination of the few possibilities, taking into account that $k \geq 6$, shows that

$$v_2(P_{k-i,k}(\tilde{s}-t)) + v_2(P_{k-j,k}(\tilde{s}+t)) \geq 3.$$

We use (8) and get

$$v_2(a_{i,j,k,l,m}) \geq 2(N-1)+3-v_2(k!) \geq 2N+1-N = N+1. \qquad \blacksquare$$

5. Proof of Theorem 2: The case of $m \equiv 2 \pmod 4$

The arguments in this section are similar to those in §4.

5.1. Recovering the lowest three bits of l.

Proposition 16: *If $H_l^m(-2) \equiv 0 \pmod 8$ and $m \equiv 2 \pmod 4$, then*

$$l \equiv 5 \pmod 8.$$

The proof is by ruling out the other possibilities one by one.

Lemma 17: *If $m \equiv 2 \pmod 4$ and either $l \equiv 2 \pmod 4$ or $l \equiv 3 \pmod 4$, then*

$$H_l^m(-2) \equiv 1 \pmod 2.$$

Proof. In these cases both s and t are odd. Repeating the proof of Lemma 9, rewriting formulas (4) and (5) modulus 2, we get

$$\sigma_l^m(k) \equiv \binom{s-t}{k}(s+t)(s+t-1)\cdots(s+t-k+1) \pmod 2.$$

For every $k \geq 1$ we have that $\sigma_l^m(k)$ is even, since it has the even factor $(s+t)$. So $H_l^m(-2) \equiv 1 \pmod 2$. $\blacksquare$

Lemma 18: *If $m \equiv 2 \pmod 4$ and $l \equiv 0 \pmod 4$, then*

$$H_l^m(-2) \equiv 2 \pmod 4.$$

Proof. Here s is even and t is odd. We calculate $H_{2s}^{2t}(-2) \pmod 4$ using formula (4):

$$\sigma_{2s}^{2t}(1) = -(s-t)(s+t) = t^2 - s^2 \equiv 1 \pmod 4$$

and

$$\sigma_{2s}^{2t}(3) = -\sigma_{2s}^{2t}(2)\frac{(s-t-2)(s+t-2)}{3 \cdot 5} \equiv -\sigma_{2s}^{2t}(2) \pmod 4.$$

From basic divisibility properties (Lemma 27) we also have

$$\forall k \geq 4, \quad \sigma_l^m(k) \equiv 0 \pmod 4.$$

Summing over $k \geq 0$ we get $H_{2s}^{2t}(-2) \equiv 2 \pmod 4$. ∎

Lemma 19: *If $m \equiv 2 \pmod 4$ and $l \equiv 1 \pmod 8$, then*

$$H_l^m(-2) \equiv 4 \pmod 8.$$

Proof. We denote $l = 8q+1$ and look at the terms of $H_l^m(-2) \pmod 8$ individually using formula (5). Using the assumption that t is odd we get

$$\sigma_l^m(1) = -\frac{(4q-t)(4q+t)}{3} \equiv 3 \pmod 8,$$

and by Lemma 28

$$\sigma_l^m(3) = -\sigma_l^m(2)\frac{(4q-2)^2 - t^2}{3 \cdot 7} \equiv -\sigma_l^m(2) \pmod 8.$$

From basic divisibility properties (Lemma 27) we have $\sigma_l^m(k) \equiv 0 \pmod 8$ for all $k \geq 4$. Summing over $k \geq 0$ we get $H_l^m(-2) \equiv 4 \pmod 8$. ∎

To be complete we verify the remaining case for $l \equiv 5 \pmod 8$.

Proposition 20: *If $m \equiv 2 \pmod 4$ and $l \equiv 5 \pmod 8$, then*

$$H_l^m(-2) \equiv 0 \pmod 8.$$

Proof. We let $l = 8q+5$. By formula (5)

$$\sigma_l^m(1) = -\frac{(4q+2-t)(4q+2+t)}{3} \equiv -1 \pmod 8,$$

and by Lemma 28

$$\sigma_l^m(3) = -\sigma_l^m(2)\frac{(4q)^2 - t^2}{3 \cdot 7} \equiv -\sigma_l^m(2) \pmod 8.$$

From basic divisibility properties (Lemma 27) we have $\sigma_l^m(k) \equiv 0 \pmod 8$ for all $k \geq 4$. Summing over $k \geq 0$ we get $H_l^m(-2) \equiv 0 \pmod 8$. ∎

5.2. Recovering the high bits of l.

Theorem 21: *Let $m \equiv 2 \pmod 4$ and suppose that $H_l^m(-2) \equiv 0 \pmod{2^N}$. Then there exists a unique $l \le \tilde{l} < l+2^{N+1}$ such that $H_{\tilde{l}}^m(-2) \equiv 0 \pmod{2^{N+1}}$. Moreover, $\tilde{l} \equiv l \pmod{2^N}$.*

Using Proposition 16, Theorem 21 is an immediate corollary of

Proposition 22: *Let $m \equiv 2 \pmod 4$ and $l \equiv 5 \pmod 8$. Fix $N \ge 3$ and write $l = r + 2^N q$ with $0 \le r < 2^N$. Then*

$$H_l^m(-2) \equiv H_r^m(-2) + 2^N q \pmod{2^{N+1}}.$$

To prove Proposition 22 we first observe

Lemma 23: *Let $m \equiv 2 \pmod 4$, $l \equiv 1 \pmod 2$ and $k \ge 6$. If $\tilde{l} \equiv l \pmod{2^N}$, then*

$$\sigma_{\tilde{l}}^m(k) \equiv \sigma_l^m(k) \pmod{2^{N+1}}.$$

Proof. The only difference from the proof of Lemma 15 is a division by an odd number which does not influence the calculations. ∎

Now we move on to the proof of the main proposition of this section.

Proof of Proposition 22. By Lemma 23, for all $k \ge 6$ we have that

$$\sigma_l^m(k) \equiv \sigma_r^m(k) \pmod{2^{N+1}}.$$

For $0 \le k \le 5$ we calculate $\sigma_l^m(k)$ explicitly.

From (6) we have $\sigma_l^m(0) = \sigma_r^m(0) = 1$.

Denote $\tilde{s} = \lfloor \frac{r}{2} \rfloor$; from the assumptions we get that $\tilde{s} \equiv 2 \pmod 4$. Using this and the assumption $N \ge 3$ we get

$$\sigma_l^m(1) = -\frac{1}{3}(2^{N-1}q + \tilde{s} - t)(2^{N-1}q + \tilde{s} + t) \equiv \sigma_r^m(1) \pmod{2^{N+1}}.$$

The next term is

$$\sigma_l^m(2) = -\sigma_l^m(1)\frac{(2^{N-1}q + \tilde{s} - t - 1)(2^{N-1}q + \tilde{s} + t - 1)}{2 \cdot 5}.$$

Noticing that $N \ge 3$, $\tilde{s} - 1 \equiv 1 \pmod 4$ and that $\sigma_r^m(1) \equiv -1 \pmod 4$ (since t is odd), elementary manipulations of this expression give

$$\sigma_l^m(2) \equiv 2^{2N-3}q + 2^{N-1}q + \sigma_r^m(2) \pmod{2^{N+1}}.$$

Moving on to the next term, using again $N \geq 3$ and $\tilde{s} \equiv 2 \pmod 4$, we get

$$\begin{aligned}\sigma_l^m(3) =& -\sigma_l^m(2)\frac{(2^{N-1}q+\tilde{s}-t-2)(2^{N-1}q+\tilde{s}+t-2)}{3\cdot 7}\\ \equiv& 2^{2N-3}q+2^{N-1}q+\sigma_r^m(3) \pmod{2^{N+1}}.\end{aligned}$$

At this point we observe that

$$\sigma_l^m(2)+\sigma_l^m(3) \equiv 2^N q+\sigma_r^m(2)+\sigma_r^m(3) \pmod{2^{N+1}}$$

(since $N \geq 3$). A similar circumstance occurs in the next two terms.

By Lemma 28, $\sigma_r^m(2) \equiv 0 \pmod 4$ so also $\sigma_r^m(3) \equiv 0 \pmod 4$. Hence,

$$\begin{aligned}\sigma_l^m(4) =& -\sigma_l^m(3)\frac{(2^{N-1}q+\tilde{s}-t-3)(2^{N-1}q+\tilde{s}+t-3)}{4\cdot 9}\\ \equiv& 2^{2N-3}q+2^{N-3}q(1-t^2)+2^{N-2}q\sigma_r^m(3)+\sigma_r^m(4)\\ \equiv& 2^{2N-3}q+\sigma_r^m(4) \pmod{2^{N+1}}\end{aligned}$$

and

$$\begin{aligned}\sigma_l^m(5) \equiv& -\sigma_l^m(4)\frac{(2^{N-1}q+\tilde{s}-t-4)(2^{N-1}q+\tilde{s}+t-4)}{5\cdot 11}\\ \equiv& 2^{2N-3}q+\sigma_r^m(5) \pmod{2^{N+1}}.\end{aligned}$$

Summing over $k \geq 0$ we get that $H_l^m(-2) \equiv 2^N q + H_r^m(-2) \pmod{2^{N+1}}$. ∎

6. Proof of Theorem 2: The special cases $m = 0$ and $m = 2$

In these cases we trivially see that $P_2|P_2$ and we can easily check that $P_2|P_5^2$. The general statement for even m shows that these are the only solutions. Note that P_5^2 corresponds to the harmonic polynomials

$$(x^2+y^2-2z^2)xyz \quad \text{and} \quad (x^2+y^2-2z^2)(x^2-y^2)z.$$

7. Harmonic products of two harmonic polynomials

We prove Theorem 3, giving evidence to the validity of Conjecture 1.

Proof of Theorem 3. We let

$$p(x,y,z) = x^2+y^2-2z^2$$

and let

$$h = p \cdot f.$$

We assume that both h and f are harmonic. Harmonic functions are analytic, so they can be represented as infinite sums of homogeneous polynomials. If f is harmonic and $\Delta(pf) = 0$, then every homogeneous component of f, denoted f_d, is also harmonic and has to give $\Delta(pf_d) = 0$. So we can assume f is a homogeneous harmonic polynomial of degree d.

Let

$$f(x, y, z) = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k.$$

The Laplacian of h is

$$\Delta h = 2 \sum_{\alpha \in \{x,y,z\}} \partial_\alpha p \partial_\alpha f = 4 \sum_{i+j+k=d} (i + j - 2k) a_{ijk} x^i y^j z^k.$$

Since the product h is harmonic every coefficient of every monomial of the above expression has to vanish. If $a_{ijk} \neq 0$ then $i + j = 2k$. Combining this with $i + j + k = d$ we get $3k = d$. Hence f can only be of the form

$$f(x, y, z) = \sum_{i=0}^{\frac{2}{3}d} a_i x^i y^{\frac{2}{3}d-i} z^{\frac{d}{3}}.$$

Since f is harmonic,

$$0 = \Delta f = g(x, y) z^{\frac{d}{3}} + \sum_{i=0}^{\frac{2}{3}d} a_i \frac{d}{3}\Big(\frac{d}{3} - 1\Big) x^i y^{\frac{2}{3}d-i} z^{\frac{d}{3}-2}$$

where $g(x, y)$ is a polynomial in x, y.

We assume $f \neq 0$ so $\exists i$ such that $a_i \neq 0$ and we have

$$\frac{d}{3}\Big(\frac{d}{3} - 1\Big) = 0 \Longrightarrow d \in \{0, 3\}.$$

A straightforward calculation gives that the only harmonic homogeneous polynomials f of degree 3 such that the product pf is harmonic are linear combinations of xyz and $x^2 z - y^2 z$, so the harmonic functions f such that pf is harmonic are of the form

$$\alpha + \beta xyz + \gamma(x^2 z - y^2 z)$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. ∎

Proof of Theorem 4. The proof is in the same spirit of the proof of Theorem 3; for details see [13]. ∎

8. The equivalence of Conjectures 1 and 1′

In this section we explain why Conjectures 1 and 1′ are equivalent. We note that this equivalence was already observed by Armitage [2].

Let

$$p(x, y, z) = x^2 + y^2 - 2z^2 \quad \text{and} \quad Z = \{p = 0\}.$$

It is known from [7] that $h \in \mathcal{H}(Z)$ if and only if there exists an analytic function f such that $h = pf$. In view of this, the equivalence of Conjectures 1 and 1′ follows from the following theorem.

THEOREM 24: *Let $l \in \mathbb{N}$. The following statements are equivalent:*

(1) *Let $p(x, y, z) = x^2 + y^2 - 2z^2$. There exists a harmonic polynomial h_l of degree l such that $p|h_l$.*
(2) *$\exists m \in \mathbb{N}$ such that $P_2(x)|P_l^m(x)$.*

Proof. Assume statement 2. Notice that $p = r^2 P_2(\cos\theta)$. If $P_2|P_l^m$ let

$$h_l = r^l P_l^m(\cos\theta)e^{im\varphi}.$$

Conversely, assume h_l is a harmonic polynomial of degree l such that $p|h_l$. Since h_l is harmonic it can be written in spherical coordinates in the form

$$\sum_{k=0}^{l} \sum_{m=-k}^{k} a_{k,m} r^k P_k^m(\cos\theta)e^{im\varphi}$$

with some $a_{k,m} \in \mathbb{R}$. The polynomial h_l vanishes on $\theta_\pm$, where $\cos(\theta_\pm) = \pm\frac{1}{\sqrt{3}}$. Since $\{e^{im\varphi}\}$ and $\{r^k\}$ are each a linearly independent set of functions, we get that

$$P_l^m(\cos\theta_\pm) = 0$$

for some m. ∎

9. Auxiliary lemmas and special notation

LEMMA 25: *For non-negative integers m, n:*

$$\frac{1}{\pi}\int_0^\pi \cos^n\varphi\cos(m\varphi)d\varphi = \begin{cases} \frac{1}{2^n}\binom{n}{\frac{n+m}{2}}, & m \le n \text{ and } m \equiv n \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This becomes easy to check by writing $\cos\psi = \frac{e^{i\psi}+e^{-i\psi}}{2}$ and noting that

$$\Re\left(\frac{1}{\pi}\int_0^{\pi} e^{il\varphi}d\varphi\right) = \delta_{0,l}. \qquad \blacksquare$$

Proof of Lemma 7. We use formula (1) and set $\delta \in \{0,1\}$ such that $(l-m) \equiv \delta$ (mod 2):

$$\begin{aligned}
\frac{m!}{(l+m)!}P_l^m(x) =& i^m\frac{1}{\pi}\int_0^{\pi}(x+y\cos\varphi)^l\cos(m\varphi)d\varphi \\
\underset{\text{Lemma 25}}{=}& i^m\sum_{j=0}^{\lfloor\frac{l-m}{2}\rfloor}\binom{l}{m+2j}\frac{1}{2^{m+2j}}\binom{m+2j}{j}x^{l-m-2j}y^{m+2j} \\
\underset{(2)}{=}& x^\delta(iy)^m\sum_{j=0}^{\lfloor\frac{l-m}{2}\rfloor}\binom{l}{m+2j}\frac{1}{2^{m+2j}}\binom{m+2j}{j} \\
&\times\left(\frac{z}{z-4}\right)^{\lfloor\frac{l-m}{2}\rfloor-j}\left(\frac{4}{z-4}\right)^j \\
=& \frac{x^\delta(iy)^m}{2^m(z-4)^{\lfloor\frac{l-m}{2}\rfloor}}\sum_{j=0}^{\lfloor\frac{l-m}{2}\rfloor}\binom{l}{m+2j}\binom{m+2j}{j}z^{\lfloor\frac{l-m}{2}\rfloor-j} \\
=& \frac{x^\delta(iy)^m}{2^m(z-4)^{\lfloor\frac{l-m}{2}\rfloor}}\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\binom{l}{l-2k-\delta}\binom{l-2k-\delta}{\lfloor\frac{l-m}{2}\rfloor-k}z^k \\
=& \frac{x^\delta(iy)^m}{2^m(z-4)^{\lfloor\frac{l-m}{2}\rfloor}}\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\frac{l!}{(2k+\delta)!(\lfloor\frac{l-m}{2}\rfloor-k)!(\lfloor\frac{l+m}{2}\rfloor-k)!}z^k \\
=& \frac{x^\delta(iy)^m}{2^m(z-4)^{\lfloor\frac{l-m}{2}\rfloor}}\frac{l!}{(\lfloor\frac{l-m}{2}\rfloor)!(\lfloor\frac{l+m}{2}\rfloor)!} \\
&\times\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\frac{(\lfloor\frac{l-m}{2}\rfloor)!(\lfloor\frac{l+m}{2}\rfloor)!}{(2k+\delta)!(\lfloor\frac{l-m}{2}\rfloor-k)!(\lfloor\frac{l+m}{2}\rfloor-k)!}z^k \\
=& \frac{A_l^m x^\delta y^m}{(z-4)^{\lfloor\frac{l-m}{2}\rfloor}}\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\frac{(-2)^k k!k!}{(2k+\delta)!}\binom{\lfloor\frac{l-m}{2}\rfloor}{k}\binom{\lfloor\frac{l+m}{2}\rfloor}{k}\left(-\frac{z}{2}\right)^k.
\end{aligned}$$

Recalling relation (3), the normalization $H_l^m(0) = 1$ and Notation 6, we obtain the desired formulas. $\blacksquare$

Notation 26: The dyadic valuation. For an integer n, denote

$$v_2(n) = \begin{cases} \max\{\lambda \in \mathbb{N} \cup \{0\} \,:\, 2^\lambda | n\}, & n \neq 0, \\ \infty, & n = 0. \end{cases}$$

For a rational number $\frac{m}{n}$ denote

$$v_2\left(\frac{m}{n}\right) = v_2(m) - v_2(n).$$

LEMMA 27: $v_2(\sigma_l^m(k)) \geq v_2(k!)$.

Proof. $\sigma_l^m(k)$ can also be written in the form

$$\sigma_l^m(k) = \binom{x}{y} \frac{z(z-1)\cdots(z-k+1)}{1 \cdot 3 \cdot 5 \cdots (2k \pm 1)}$$

so there are k consecutive numbers in the nominator and no even numbers in the denominator. ∎

LEMMA 28: *For* $m \equiv 2 \pmod 4$ *and* $l \equiv 1 \pmod 4$, $\sigma_l^m(2) \equiv 0 \pmod 4$.

Proof. Here s is even and t is odd. The only factors with powers of two in this term are the following

$$\frac{(s-t-1)(s+t-1)}{2} \equiv 0 \pmod 4. \qquad ∎$$

References

[1] M. L. Agranovsky and Y. Krasnov, *Quadratic divisors of harmonic polynomials in* $\mathbf{R}^n$, Journal d'Analyse Mathématique **82** (2000), 379–395.

[2] D. H. Armitage, *Cones on which entire harmonic functions can vanish*, Proceedings of the Royal Irish Academy. Section A. Mathematical and Physical Sciences **92** (1992), 107–110.

[3] J. Bourgain and Z. Rudnick, *On the nodal sets of toral eigenfunctions*, Inventiones Mathematicae **185** (2011), 199–237.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, Interscience Publishers, New York, 1953.

[5] J. B. Holt, *On the irreducibility of Legendre's polynomials*, Proceedings of the London Mathematical Society **2** (1913), 126–132.

[6] H. Ille, *Zur Irreduzibilität der Kugelfunktionen*, PhD thesis, Universität Berlin, 1924.

[7] A. Logunov and E. Malinnikova, *On ratios of harmonic functions*, Advances in Mathematics **274** (2015), 241–262.

[8] A. Logunov and E. Malinnikova, *Ratios of harmonic functions with the same zero set*, Geometric and Functional Analysis **26** (2016), 909–925.

[9] D. Mangoubi, *A gradient estimate for harmonic functions sharing the same zeros*, Electronic Research Announcements in Mathematical Sciences **21** (2014), 62–71.

[10] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse (1929), Nr. 1; reprint in *On Some Applications of Diophantine Approximations*, Publications of the Scuola Normale Superiore, Vol. 2, Edizioni della Normale, Pisa, 2014, pp. 81–138.

[11] T. J. Stieltjes, *Letter no: 275 of Oct. 2, 1890*, in *Correspondance d'Hermite et de Stieltjes. Volume 2*, Gauthier-Villars, Paris, 1905, pp. 104–106.

[12] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge; The Macmillan Company, New York, 1944.

[13] A. Weller Weiser, *Topics in the study of harmonic functions in three dimensions*, Master's thesis, The Hebrew University of Jerusalem, 2014.