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DETERMINACY AND MONOTONE INDUCTIVE DEFINITIONS

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ABSTRACT

We prove the equivalence of the determinacy of Σ^0_3 (effectively $G_{\delta\sigma}$) games with the existence of a β-model satisfying the axiom of $\mathbf{\Pi}^1_2$ monotone induction, answering a question of Montalbán [8]. The proof is tripartite, consisting of (i) a direct and natural proof of Σ^0_3 determinacy using monotone inductive operators, including an isolation of the minimal complexity of winning strategies; (ii) an analysis of the convergence of such operators in levels of Gödel's L, culminating in the result that the nonstandard models isolated by Welch [18] satisfy $\mathbf{\Pi}^1_2$ monotone induction; and (iii) a recasting of Welch's [17] Friedman-style game to show that this determinacy yields the existence of one of Welch's nonstandard models. Our analysis in (iii) furnishes a description of the degree of $\mathbf{\Pi}^1_2$-correctness of the minimal β-model of $\mathbf{\Pi}^1_2$ monotone induction.

1. Introduction

In an **infinite perfect information game**, two players, Player I and Player II, take turns choosing the digits $x(i)$ of a real $x \in \omega^\omega$ in sequence, as follows:

I	$x(0)$		$x(2)$	$\dots$	$x(2n)$		$\dots$
II		$x(1)$		$\dots$		$x(2n+1)$	$\dots$

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A **winning condition** is simply a subset $A \subseteq \omega^\omega$; then Player I wins the play x if $x \in A$, and Player II wins if $x \notin A$. We call the game so defined $G(A)$; a game $G(A)$ is determined if one of the players has a winning strategy in the natural sense.

Axioms of determinacy assert that for all sets A of some fixed level of descriptive complexity, the game $G(A)$ is determined. These are well-studied in reverse mathematics, in which the intrinsic strength of theorems are isolated by being shown in a weak base theory to be equivalent to some "natural" mathematical axiom (for a survey, see [19]). The results in this paper are a contribution to this study: We show the statement that all games with Σ^0_3 winning condition are determined as equivalent to the existence of a standard model of the scheme of $\mathbf{\Pi}^1_2$ monotone induction.

There is a great deal of precedent for calibrating determinacy strength in low levels of the Borel hierarchy using axioms of inductive definition. In one of the first studies in reverse mathematics, Steel [14] proved over RCA_0 that ATR_0 is equivalent to both Δ^0_1-DET and Σ^0_1-DET. The relevance of monotone inductive definitions was discovered by Tanaka [15], who showed over ACA_0 that Δ^0_2-DET is equivalent to Π^1_1-TR, and [16] that over ATR_0, Σ^0_2-DET is equivalent to Σ^1_1-MI (see Definition 2.1 below). MedSalem and Tanaka [7] established equivalences over ATR_0 between $k{-}\Pi^0_2$-DET and $[\Sigma^1_1]^k$-ID, an axiom allowing inductive definitions using combinations of k-many Σ^1_1 operators; furthermore, they showed over Π^1_3-TI that Δ^0_3-DET is equivalent to $[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}$, an axiom allowing inductive definition by combinations of transfinitely many Σ^1_1 operators. Further results were given by Tanaka and Yoshii [20] characterizing the strength of determinacy for pointclasses refining the difference hierarchy on Π^0_2, again in terms of axioms of inductive definition.

Beyond these pointclasses we have Σ^0_3, where an exact characterization of strength has been elusive (the precise proof-theoretic strength of this determinacy is Question 28 of Montalbán's [8]). The sharpest published bounds on this strength were given by Welch [17], who showed that although Σ^0_3-DET (and more) is provable in Π^1_3-CA_0, Δ^1_3-CA_0 (even augmented by AQI, an axiom allowing definition by arithmetical quasi-induction) cannot prove Σ^0_3-DET. On the other hand, Montalbán and Shore [9] showed that Σ^0_3-DET (and indeed, any true Σ^1_4 sentence) cannot prove Δ^1_2-CA_0. This situation is further clarified by the same authors in [10], where they show (among other things) that Σ^0_3-DET implies the existence of a β-model of Δ^1_3-CA_0.

Welch [18] went on to give a characterization of the ordinal stage at which winning strategies in Σ^0_3 games are constructed in L. There, the least ordinal γ so that every Σ^0_3 game is determined with a winning strategy definable over L_γ is shown to be the least γ for which there exists an illfounded admissible model $\mathcal{M}$ with an infinite descending sequence of nonstandard levels of L that fully Σ_2-reflect to standard levels below γ, and so that $\mathrm{wfo}(\mathcal{M}) = \gamma$ (see Definition 4.1).

The work of Welch and Montalbán–Shore suggests that Σ^0_3-DET should be equivalent to the existence of a β-model of some natural theory in second order arithmetic, with the minimal such β-model being L_γ, where γ is as above. This is what we show: L_γ is the minimal wellfounded model satisfying the axiom of $\mathbf{\Pi}^1_2$ monotone induction, and indeed, Σ^0_3-DET is equivalent over $\mathbf{\Pi}^1_1$-CA_0 to the existence of such a model.

In Section 3, we show that winning strategies in Σ^0_3 games are definable over any β-model of $\mathbf{\Pi}^1_2$-MI. In Section 4, we prove that Welch's infinite depth Σ_2-nestings furnish us with such β-models. We complete this circle of implications in Section 5 by reproducing Welch's lower bound argument in the base theory $\mathbf{\Pi}^1_1$-CA_0 to show that Σ^0_3 determinacy implies the existence of an infinite depth Σ_2-nesting. We conclude with an analysis of the $\mathbf{\Pi}^1_2$ relations which are correctly computed in L_γ: these are precisely the relations $\neg\Game\mathbf{\Sigma}^0_3$ in parameters from L_γ.

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2. Preliminaries: Games and operators

In what follows, we denote subsets of ω by capital Roman letters X, Y, Z, elements of ω by lowercase Roman letters from i up to n, ordinals by lowercase Greek $\alpha, \beta, \gamma, \ldots$ and reals (elements of Baire space ω^ω) by w, x, y, z. Background on the pointclasses $\mathbf{\Pi}^1_n$, Σ^0_n, the difference between bold- and lightface, relativizations $\Sigma^0_n(z)$, etc., may be found in [11].

We restrict our attention to games with moves in ω. A **tree** on ω is a non-empty set $T \subseteq \omega^{<\omega}$ that is closed under initial segment; $[T]$ denotes the set of infinite branches of T, and for $p \in T$, T_p denotes the subtree of T with stem p, that is, $T_p = \{q \in T \mid q \subseteq p \vee p \subseteq q\}$. For a set $A \subseteq [T]$, the **game on T with**

payoff A, denoted $G(A;T)$, is defined as the infinite perfect information game in which two players, I and II, alternate choosing successive nodes of a branch x of T; we call such an infinite branch a **play**. Player I wins the play if $x \in A$; otherwise, Player II wins. We write $G(A)$ for $G(A;\omega^{<\omega})$.

A **strategy** for I in a game on T is a partial function $\sigma : T \rightharpoonup \omega$ that assigns to an even-length position $s \in T$ a legal move $\sigma(s)$ for I at s, that is, $\sigma(s) \in \omega$ so that $s^\frown\langle\sigma(s)\rangle \in T$. We require the domain of σ to be closed under legal moves by II as well as moves by σ; note then that if T contains terminal nodes, it may be the case that no strategy for I exists (though nothing will be lost if we restrict our attention here to trees without terminal nodes). Strategies for II are defined analogously. If an infinite play x can be obtained by playing against a strategy σ, we say x is **according to** or **compatible with** σ. We say a strategy σ is **winning** for Player I (Player II) in $G(A;T)$ if every play according to σ belongs to A ($[T] \setminus A$, respectively). A game $G(A;T)$ is **determined** if one of the players has a winning strategy. For a pointclass Γ, Γ-DET denotes the statement that $G(A;\omega^{<\omega})$ is determined for all $A \subseteq \omega^\omega$ in Γ.

We furthermore define a **quasistrategy** for Player II in T to be a subtree $W \subseteq T$, again with no terminal nodes, that does not restrict Player I's moves, in the sense that whenever $p \in W$ has even length, then every 1-step extension $p^\frown\langle s\rangle \in T$ belongs to W. A quasistrategy may then be thought of as a multi-valued strategy. (Similar definitions of course can be made for Player I, but at no point will we need to refer to quasistrategies for Player I.)

Quasistrategies are typically obtained in the following fashion: if Player I does not have a winning strategy in $G(A;T)$, then setting W to be the collection of $p \in T$ so that I doesn't have a winning strategy in $G(A;T_p)$, we have that W is a quasistrategy for II in T. This W is called II's **nonlosing quasistrategy** in $G(A;T)$.

We recall the definition of the **game quantifier** $\Game$ (see [11] 6D): for a set $B \subseteq R \times \omega^\omega$, R a Polish space, define

$$\Game B = \{x \in R \mid \text{Player I has a winning strategy in } G(R_x;\omega^\omega)\},$$

where R_x is the x-slice of R,

$$R_x = \{y \in \omega^\omega \mid \langle x, y\rangle \in R\}.$$

Then for pointclasses Γ, $\Game\Gamma$ is the pointclass of sets $\Game B$ with $B \in \Gamma$. In our context, R will be either ω or $\omega \times P(\omega^{<\omega})$, and Γ will typically be Σ^0_3 or one of its relativizations.

The language L_2 of second order arithmetic and the various comprehension schemes Γ-CA_0 and choice schemes Γ-AC_0 are defined as usual; a comprehensive resource is [13]. We recall that an L_2-model is an ω-**model** if its set of type 0 objects is isomorphic to $(\omega, <)$. When $\mathcal{M}$ is a countable ω-model of a sufficiently strong theory, we typically conflate $\mathcal{M}$ with the transitive countable set whose elements are coded by reals in $\mathcal{M}$.

An ω-model $\mathcal{M}$ is a β-**model** if it is Σ^1_1-correct; equivalently, if whenever $T \subseteq \omega^{<\omega}$ is a tree (coded by a real) in $\mathcal{M}$, and $[T]$ is non-empty (in V), then there is a branch $x \in [T]$ (coded) in $\mathcal{M}$. Notice that, given a strategy σ, tree T, and Borel set A, the statement that "σ is winning for I in the game $G(A;T)$" is Π^1_1 in A, T, σ (it asserts the inexistence of a branch in $[\sigma] \cap A$), and therefore is absolute for β-models containing these parameters.

Definition 2.1: Let Γ be a pointclass. We say an operator $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a Γ **operator** if

$$\{\langle n, X\rangle \mid n \in \Phi(X)\} \in \Gamma;$$

Φ is **monotone** if it is $\subseteq$-increasing, that is,

$$(\forall X, Y) X \subseteq Y \to \Phi(X) \subseteq \Phi(Y).$$

The **axiom scheme of Γ monotone induction**, denoted Γ-MI, asserts for each Γ operator $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ that if Γ is monotone, then there exists an ordinal $o(\Phi)$ and sequence $\langle \Phi^\xi \rangle_{\xi \leq o(\Phi)}$ such that, setting

$$\Phi^{<\xi} = \bigcup_{\zeta < \xi} \Phi^\zeta,$$

we have

- for all $\xi \leq o(\Phi)$, $\Phi^\xi = \Phi(\Phi^{<\xi}) \cup \Phi^{<\xi}$,
- $\Phi^{o(\Phi)} = \Phi^{<o(\Phi)}$, and
- $o(\Phi)$ is the least ordinal with this property.

$\Phi^{o(\Phi)}$ is the **least fixed point** of Φ, denoted Φ^∞.

This definition is clearly equivalent to one with an arbitrary countable discrete space in place of ω; in our application, this space will be $\omega^{<\omega}$.

There is a prewellorder $\prec_\Phi$ with field $\Phi^\infty \subseteq \omega$ naturally associated with the sequence $\langle \Phi^\xi \rangle_{\xi \leq o(\Phi)}$. Namely, set $m \prec_\Phi n$ if and only if the least ξ with $m \in \Phi^\xi$ is less than the least ζ with $n \in \Phi^\zeta$.

We are interested in the case where Γ is either $\boldsymbol{\Pi}^1_2$ or $\Pi^1_2(z)$ for some fixed real z. The formalization of $\Pi^1_2(z)$-MI in L_2 is the schema asserting the existence of (a real coding) the prewellorder $\prec_\Phi$, for each $\Pi^1_2(z)$ monotone operator Φ. Note for such Φ, the relation "$X = \prec_\Phi$", as a relation holding of $X \in \mathcal{P}(\omega \times \omega)$, is arithmetical in $\Sigma^1_2(z)$, i.e. is obtained from $\Sigma^1_2(z)$ conditions via Boolean combinations and natural number quantification.

3. Proving determinacy

In this section we prove the following theorem.

Theorem 3.1: *Let $\mathcal{M}$ be a β-model of $\boldsymbol{\Pi}^1_2$-MI. Then for any real $z \in \mathcal{M}$ and $\Sigma^0_3(z)$ set A, either*

1. *Player I wins $G(A)$ with a strategy $\sigma \in \mathcal{M}$; or*
2. *Player II wins $G(A)$ with a strategy $\Delta^1_3(z)$-definable over $\mathcal{M}$.*

It will be shown in Section 5 that the degree of definability in the second item is sharp: If for some real $z \in \mathcal{M}$, $\mathcal{M}$ is the minimal β-model of $\boldsymbol{\Pi}^1_2$-MI containing z, then there will be $\Sigma^0_3(z)$ games which Player II wins but no winning strategy for II belongs to $\mathcal{M}$. Since any model of $\boldsymbol{\Pi}^1_2$-MI is trivially a model of $\boldsymbol{\Pi}^1_2$-CA_0, no strategy as in (2) can be definable by Boolean combinations of $\Sigma^1_2(z)$ conditions over $\mathcal{M}$.

We sketch the idea of the proof, which traces back to Morton Davis's original proof of Σ^0_3 determinacy [2]. Let $A \subseteq \omega^\omega$ be a Σ^0_3 set, so that

$$A = \bigcup_{k\in\omega} B_k$$

for some recursively presented sequence $\langle B_k\rangle_{k\in\omega}$ of Π^0_2 sets. The idea behind the proof that the game $G(A)$ is determined is a simple one: if Player I does not have a winning strategy, then Player II refines to a quasistrategy W_0 so that no infinite plays in W_0 belong to B_0, and so that W_0 doesn't forfeit the game for Player II (in the sense that I has no winning strategy in $G(A; W_0)$). Having done this, Player II plays inside W_0 and at all positions of length 1, refines further to a W_1 which avoids B_1 without forfeiting the game. Then refine to W_2 at positions of length 2, and so on. The ultimate refinement of the sequence $W_0, W_1, W_2, \ldots$ of quasistrategies gives a winning quasistrategy for Player II in $G(A)$, since every infinite play must eventually stay in each W_n, and so avoid each B_n.

The key claim that makes this proof work is Lemma 3.2 below, which asserts that whenever Player I does not have a winning strategy in $G(A;T)$, then for all k, there is such a quasistrategy W_k for II.

Lemma 3.2: *Let z be a real and work in $\Pi^1_2(z)$-MI+$\mathbf{\Pi}^1_2$-CA_0. Suppose $T \subseteq \omega^{<\omega}$ is a tree, recursive in z, with no terminal nodes, and fix $B \subseteq A \subseteq [T]$ with $B \in \Pi^0_2(z)$ and $A \in \Delta^1_1(z)$. If $p \in T$ is such that I does not have a winning strategy in $G(A;T_p)$, then there is a quasistrategy W for II in T_p so that*

- *$[W] \cap B = \emptyset$, and*
- *I does not have a winning strategy in $G(A;W)$.*

In keeping with terminology first established in [2], we say a position p for which such a quasistrategy W exists is **good** and that W is a **goodness-witnessing quasistrategy** for p (relative to T, B, A).

To motivate the proof of Lemma 3.2, we describe a general template for proving the determinacy of a game $G(A;T)$ using iteration of a monotone operator

$$\Phi : P(T) \to P(T).$$

Speaking vaguely, one defines $\Phi(X)$ for $X \subseteq \omega^{<\omega}$ to consist of positions from which it is "easy" for I to either enter the set X, or to win the game $G(A)$ outright. Iterating application of Φ to the empty set of positions, we obtain an increasing sequence

$$\emptyset \subseteq \Phi(\emptyset) \subseteq \Phi(\Phi(\emptyset)) \cdots ,$$

with least fixed point Φ^∞. If the initial position $\emptyset = \langle\rangle$ belongs to Φ^∞, then one argues by induction that I has a winning strategy (I plays to "decrease rank" by aiming to enter positions in Φ^α for smaller α); if on the other hand $\emptyset \notin \Phi^\infty$, one uses the fact that Φ^∞ is a fixed point to argue that it is so "difficult" for I to enter Φ^∞ (and hence to win $G(A)$) that Player II must have a winning strategy. An accessible example in this family of arguments is Wolfe's proof of Σ^0_2 determinacy (see Theorem 6A.3 in [11]).

In the present case, we modify Davis's original proof of Lemma 3.2 so that W_k is obtained by iteration of a certain monotone operator. The complexity of this operator is $\neg\Game\mathbf{\Sigma}^0_3$ in the parameter T, and so in particular $\Pi^1_2(T)$.

We remark that $\mathbf{\Pi}^1_2$-CA_0 implies $\mathbf{\Delta}^1_2$-CA_0, which is equivalent to $\mathbf{\Sigma}^1_2$-AC_0 (see VII.6.9 in [13]); this choice principle will be used several times in the course of the proof.

Proof of Lemma 3.2. Letting B be as in the statement of the lemma, fix a set $U \subseteq \omega \times T$ recursive in z so that, setting

$$U_n = \{p \in T \mid (n,p) \in U\} \quad \text{and} \quad D_n = \{x \in [T] \mid (\exists k) x{\restriction}k \in U_n\},$$

we have $B = \bigcap_{n\in\omega} D_n$. For convenience, we may further assume that each U_n is closed under end-extension in T (i.e., if $p \subseteq q \in T$ and $p \in U_n$, then $q \in U_n$), and that $|p| > n$ whenever $p \in U_n$.

We define an operator $\Phi : \mathcal{P}(T) \to \mathcal{P}(T)$ by setting, for $X \subseteq T$,

$$\begin{aligned} p \in \Phi(X) \iff (\exists n)(\forall \sigma) &\text{ if } \sigma \text{ is a strategy for I in } T, \text{ then} \\ &(\exists x) x \text{ is compatible with } \sigma, x \notin A, \text{ and} (\forall k) x{\restriction}k \notin U_n \setminus X. \end{aligned}$$

The operator Φ is clearly monotone on $\mathcal{P}(T)$, and the relation $p \in \Phi(X)$ is $\Pi^1_2(z)$ because (by $\mathbf{\Sigma}^1_2$-AC_0) this pointclass is closed under existential quantification over ω (see [13], Theorem VII.6.9.1). We can write this more compactly by introducing an auxiliary game where Player I tries either to force the play to belong to A, or to at some finite stage enter the set $U_n \setminus X$: Define for $X \subseteq T$ and $n \in \omega$,

$$E^X_n = A \cup \{x \in [T] \mid (\exists k) x{\restriction}k \in U_n \setminus X\}.$$

Then

$$p \in \Phi(X) \iff (\exists n) \text{ I doesn't have a winning strategy in } G(E^X_n; T_p).$$

Now by $\Pi^1_2(z)$-MI let $\langle \Phi^\alpha \rangle_{\alpha \le o(\Phi)}$ be the iteration of the operator Φ with least fixed point Φ^∞. (Note this is our sole use of the main strength assumption of the lemma, $\Pi^1_2(z)$-MI.) Let $\prec_\Phi$ be the associated prewellorder of $\Phi^\infty \subseteq \omega$; formally, we regard definitions and proofs in terms of $\langle \Phi^\alpha \rangle_{\alpha \le o(\Phi)}$ as being carried out in the theory $\mathbf{\Pi}^1_2$-CA_0 of second order arithmetic, using the real $\prec_\Phi$ as a parameter.

Claim 3.3: *If $p \in T \setminus \Phi^\infty$, then I has a winning strategy in $G(A; T_p)$.*

Proof. For each $q \in T \setminus \Phi^\infty$ and $n \in \omega$, we let $\sigma_{q,n}$ be a winning strategy for I in $G(E^{\Phi^\infty}_n; T_q)$, as is guaranteed to exist by the above remarks and the fact that $q \notin \Phi(\Phi^\infty) = \Phi^\infty$. By $\mathbf{\Sigma}^1_2$-AC_0, we may fix a real $\vec{\sigma}$ coding a sequence of such, so that $(\vec{\sigma})_{\langle q,n \rangle} = \sigma_{q,n}$ for all such pairs q, n.

Supposing now that $p \in T \setminus \Phi^\infty$, we describe a strategy σ for Player I in T_p from the parameter $\vec{\sigma}$ as follows. Set $p_0 = p$. Let n_0 be the least n so that $p_0 \notin U_n$ (such exists by our simplifying assumption that $|q| > n$ whenever $q \in U_n$); note also that no initial segment of p_0 can belong to U_n.

Suppose inductively that we have reached some position $p_i \notin \Phi^\infty$ and have fixed n_i such that $p_i \notin U_{n_i}$. Play according to σ_{p_i,n_i} until, if ever, we reach a position $q \in U_{n_i} \setminus \Phi^\infty$. Then set $p_{i+1} = q$, and let n_{i+1} be least such that $p_{i+1} \notin U_{n_{i+1}}$.

Note the strategy just described is arithmetical in the parameters $z, \vec{\sigma}$, and so exists; call it σ. We claim σ is winning for I in $G(A; T_p)$.

Let $x \in [T_p]$ be a play compatible with σ. Then n_0, p_0 are defined. If n_{i+1} is undefined for some i, then fixing the least such i, we must have that x is compatible with the strategy σ_{p_i,n_i}; since this strategy is winning in $G(E^{\Phi^\infty}_{n_i}; T_{p_i})$, we have $x \in E^{\Phi^\infty}_{n_i}$. Now since n_{i+1} was undefined, we never reached a position in $U_{n_i} \setminus \Phi^\infty$, and so $x \in A$, by definition of the set $E^{\Phi^\infty}_{n_i}$.

On the other hand, if n_i is defined for all i, then by definition of the strategy σ, we have $p_i \subseteq x$ for all i, and for each i, $p_i \in \bigcap_{n<n_i} U_n$ (by definition of n_i, and because the sets U_n are closed under end-extension in T). So

$$x \in \bigcap_{n\in\omega} D_n = B \subseteq A.$$

We have shown σ is winning for Player I in $G(A; T_p)$. ∎

Claim 3.4: *If $p \in \Phi^\infty$, then p is good.*

Proof. The construction of a quasistrategy W^p witnessing goodness of p proceeds inductively on the $\prec_\Phi$-rank of $p \in \Phi^\infty$, that is, on the least ordinal α so that $p \in \Phi^\alpha$. Given such p, α, fix the least n so that I does not have a winning strategy in the game $G(E^{\Phi^{<\alpha}}_n; T_p)$. In W^p, have II play according to II's non-losing quasistrategy in $G(E^{\Phi^{<\alpha}}_n; T_p)$ until, if ever, a position q in U_n is reached. Since this non-losing quasistrategy must avoid $U_n \setminus \Phi^{<\alpha}$ by definition of $E^{\Phi^{<\alpha}}_n$, we must have $q \in \Phi^{<\alpha}$; inductively, we have some goodness-witnessing quasistrategy W^q for q, so have II switch to play according to this quasistrategy.

Here is a more formal definition of the quasistrategy W^p just described. For $p \in \Phi^\infty$, define W^p to be the set of positions $q \in T_p$ for which there exists some sequence $\langle(\alpha_i, n_i)\rangle_{|p|\le i\le |q|}$ so that, whenever $|p| \le i \le |q|$,

- if $i = |p|$, or $i > |p|$ and $q{\restriction}i \in U_{n_{i-1}}$, then
  - α_i is the least α so that $q{\restriction}i \in \Phi^\alpha$;
  - n_i is the least n so that I has no winning strategy in $G(E^{\Phi^{<\alpha_i}}_n; T_{q\restriction i})$;
- if $i > |p|$ and $q{\restriction}i \notin U_{n_{i-1}}$, then $\alpha_i = \alpha_{i-1}, n_i = n_{i-1}$; and
- if $i < |q|$, $q{\restriction}(i+1)$ is in II's non-losing quasistrategy in $G(E^{\Phi^{<\alpha_i}}_{n_i}; T_{q\restriction i})$.

Here quantification over ordinals $\alpha < o(\Phi)$ is tantamount to quantification over natural number codes for such as furnished by the prewellorder $\prec_\Phi$. The most complicated clauses in the above definition are those involving assertions of the form "I has no winning strategy in the game $G(E_n^{\Phi^{<\alpha_i}}; T_{q\upharpoonright i})$" (equivalently, "$q\upharpoonright i$ belongs to II's non-losing quasistrategy" in this game), and these are Π^1_2 in the parameter $\prec_\Phi$. So the criterion for membership in W^p is arithmetical in $\Sigma^1_2(\prec_\Phi)$ conditions, and therefore by $\mathbf{\Pi}^1_2$-CA_0 the set W^p is guaranteed to exist.

We need to verify W^p is a quasistrategy for Player II in T_p. An easy induction shows that for each $q \in W^p$, there is a unique witnessing sequence

$$\langle (\alpha_i, n_i) \rangle_{|p| \le i \le |q|}$$

and this sequence depends continuously on q; that the α_i are non-increasing; and that I has no winning strategy in $G(E_{n_i}^{\Phi^{<\alpha_i}}; T_{q\upharpoonright i})$ whenever $|p| \le i \le |q|$.

For $q \in W^p$, we let α^q, n^q denote the final pair (indexed by $|q|$) in the sequence witnessing this membership. By the above remarks, I has no winning strategy in $G(E_{n^q}^{\Phi^{<\alpha^q}}; T_q)$, and by the final condition for membership in W^p, the one-step extensions $q^\frown\langle l \rangle$ in W^p are exactly the one-step extensions of q in II's non-losing quasistrategy in this game. It follows that W^p is a quasistrategy for II in T_p.

We claim W^p witnesses goodness of p. We first show

$$[W^p] \cap B = \emptyset.$$

Given any play $x \in [W^p]$, we have some least i so that $\alpha_j = \alpha_i$ for all $j \ge i$; then for all $j > i$, we have $x\upharpoonright j$ belongs to II's non-losing quasistrategy in

$$G(E_{n_i}^{\Phi^{<\alpha_i}}; T_{x\upharpoonright i}).$$

In particular, for no k do we have $x\upharpoonright k \in U_{n_i}$. Then $x \notin D_{n_i}$, so $x \notin B$ as needed.

We just need to show I has no winning strategy in $G(A; W^p)$. We show something stronger, namely that I has no winning strategy in $G(A; W^p_q)$ for each $q \supseteq p$ in W^p. This argument is by induction on α^q. So assume that there is no winning strategy for I in $G(A; W^p_r)$ whenever $\alpha^r < \alpha^q$.

Suppose towards a contradiction that σ is a winning strategy for Player I in $G(A; W^p_q)$. Let j be least so that $\alpha^q = \alpha_j$. Then q is in II's non-losing quasistrategy in $G(E_{n_j}^{\Phi^{<\alpha_j}}; T_{q\upharpoonright j})$. We claim no $r \supseteq q$ compatible with σ is in U_{n_j}. For otherwise, we have $r \in \Phi^{<\alpha_j}$, so that $\alpha^r < \alpha_j = \alpha^q$, and σ is a winning strategy for I in $G(A; W^p_r)$. This contradicts our inductive hypothesis.

So σ cannot reach any position in U_{n_j}. By our definition of W^p, we have that the strategy σ stays inside II's non-losing quasistrategy for $G(E_{n_j}^{\Phi^{<\alpha_j}}; T_{q\restriction j})$. But since σ is winning for I in $G(A; T_{q\restriction j})$ and $A \subseteq E_{n_j}^{\Phi^{<\alpha_j}}$, this is a contradiction.

We conclude that I has no winning strategy in $G(A; W_q^p)$; inductively, the claim follows for all $q \in W^p$ extending p, so that in particular, W^p witnesses goodness of p. ∎

The last two claims show that every $p \in T$ is either a winning position for I in $G(A; T)$, or is good. This proves the lemma. ∎

For future reference, let us refer to the W^p defined in the proof as the **canonical** goodness-witnessing quasistrategy for p (relative to T, B, A). We have the following remark, which will be important in computing the complexity of winning strategies for Player II:

Remark 3.5: Since "$\prec_\Phi$ witnesses the instance of $\Pi^1_2(z)$-MI at Φ" is $\Delta^1_3(z)$, the statement "W is the canonical goodness-witnessing quasistrategy for p relative to T, B, A" is likewise $\Delta^1_3(z)$ as a relation on pairs $\langle W, p\rangle$.

Proof of Theorem 3.1. The proof proceeds from Lemma 3.2 as usual (see [2], [5]); we give a detailed account here, in order to isolate the degree of definability of II's winning strategy.

Fix a β-model $\mathcal{M}$ of $\mathbf{\Pi}^1_2$-MI. Suppose A is $\Sigma^0_3(z)$ for some $z \in \mathcal{M}$; say

$$A = \bigcup_{k\in\omega} B_k$$

with the B_k uniformly $\Pi^0_2(z)$. By the previous lemma, whenever $T \in \mathcal{M}$ is a tree in $\mathcal{M}$, and $p \in T$ is a position so that in $\mathcal{M}$, there is no winning strategy for I in $G(A; T_p)$, then p is good relative to T, B_k, A, for all k; that is, for each k there is W_k a quasistrategy for II in T_p so that

- $[W_k] \cap B = \emptyset$;
- I does not have a winning strategy in $G(A; W_k)$.

The idea of the proof is to repeatedly apply the lemma inside $\mathcal{M}$. At positions p of length k, II refines her present working quasistrategy W_{k-1} to one W_k witnessing goodness of p relative to W_{k-1}, B_k, A, so "dodging" each of the $\Pi^0_2(z)$ sets B_k, one at a time.

More precisely: Suppose I does not win $G(A)$, where A is $\Sigma^0_3(z)$ for some $z \in \mathcal{M}$. Then since $\mathcal{M}$ is a β-model, the same must hold in $\mathcal{M}$ (see the remark preceding Definition 2.1). So work in $\mathcal{M}$.

Let $W^\emptyset$ be the canonical goodness-witnessing quasistrategy for $\emptyset$ relative to $\omega^{<\omega}, B_0, A$ as constructed in the proof of Lemma 3.2. Then let $H^\emptyset$ be II's non-losing quasistrategy in $G(A; W^\emptyset)$ (so that for no $p \in H^\emptyset$ do we have that I wins $G(A; W^\emptyset_p)$).

Suppose inductively that for some k, we have subtrees H^p of T, defined for a subset of $p \in T$ with length $\leq k$, so that

(1) each H^p is a quasistrategy for II in T_p and belongs to $\mathcal{M}$;
(2) $[H^p] \cap B_{|p|} = \emptyset$;
(3) for no $q \in H^p$ does I have a winning strategy in $G(A; H^p_q)$;
(4) if $p \subseteq q$, then $H^q \subseteq H^p$ whenever both are defined;
(5) if $|p| < k$ and $p^\frown\langle l\rangle \in H^p$, then $H^{p^\frown\langle l\rangle}$ is defined.

In order to continue the construction, we need to define quasistrategies $H^{p^\frown\langle l\rangle}$, whenever $|p| = k$, H^p is defined, and $p^\frown\langle l\rangle \in H^p$. Given such p and l, we have that I has no winning strategy in $G(A; H^p_{p^\frown\langle l\rangle})$ by (3). So applying Lemma 3.2 inside $\mathcal{M}$, let $W^{p^\frown\langle l\rangle}$ be the canonical goodness-witnessing strategy for $p^\frown\langle l\rangle$ relative to H^p, B_{k+1}, A. Then let $H^{p^\frown\langle l\rangle}$ be II's non-losing quasistrategy in $G(A; W^{p^\frown\langle l\rangle})$. It is easy to see that this quasistrategy satisfies the properties (1)–(4), so we have the desired system of quasistrategies H^q satisfying (5), for $|q| = k+1$.

Now set $p \in H$ if and only if for all $i < |p|$, $H^{p\restriction i}$ is defined and $p \in H^{p\restriction i}$. It follows from (1) and (5) that H is a quasistrategy for II in T, and by (4) we have $H_p \subseteq H^p$ for each $p \in H$. By (2) then, $[H] \cap B_k = \emptyset$ for all $k \in \omega$, so that $[H] \cap A = \emptyset$.

Now, step outside $\mathcal{M}$. Observe that for each $p \in H$, we have that the sequence $\langle H^{p\restriction i}\rangle_{i<|p|}$ exists in $\mathcal{M}$, since it is obtained by a finite number of applications of $\mathbf{\Pi}^1_2$-MI and $\mathbf{\Pi}^1_2$-CA_0. Since $\mathcal{M}$ is a β-model, it really is the case (in V) that $[H] \cap B_k = \emptyset$ for all $k \in \omega$ (by item (2) of our construction, $\mathcal{M}$ satisfies this condition, which is Π^1_1 in parameters in $\mathcal{M}$). Though H need not belong to $\mathcal{M}$, we claim it is nonetheless a $\Delta^1_3(z)$-definable class over $\mathcal{M}$. For $p \in H$ if and only if there exists a sequence $\langle W_i, H_i\rangle_{i<|p|}$, so that for all $i < |p|$,

- W_i is II's canonical goodness-witnessing strategy for $p\restriction i$, relative to H_{i-1}, B_i, A (where we set $H_{-1} = \omega^{<\omega}$);
- H_i is II's non-losing quasistrategy in $G(A; W_i)$ at $p\restriction i$;
- for all $i < |p|$, $p \in H_i$.

This is a $\Sigma^1_3(z)$ condition, by Remark 3.5. And note that $p \notin H$ if and only if there is a sequence $\langle H_i, W_i\rangle_{i\leq l}$, for some $l < |p|$, satisfying the first two conditions for $i \leq l$, but so that $p \notin H_l$. This is likewise $\Sigma^1_3(z)$, so that H is $\Delta^1_3(z)$-definable in $\mathcal{M}$.

Given a $\Delta^1_3(z)$ definition of the quasistrategy H, it is easy to see that the strategy τ for II obtained by taking $\tau(p)$ to be the least l so that $p^\frown\langle l\rangle \in H$ is likewise $\Delta^1_3(z)$ and winning for II in $G(A;T)$. This completes the proof of Theorem 3.1. ∎

4. $\mathbf{\Pi}^1_2$ monotone induction from infinite depth Σ_2-nestings

In this section, the theories of KP and Σ_1-Comprehension are defined in the language of set theory as usual (see, e.g., [1]). We will furthermore make use of the theories KPI_0, which asserts that every set is contained in some admissible set (that is, some transitive model of KP), and KPI, which is the union of KP and KPI_0. KPI_0 is relevant largely because it is a weak theory that is strong enough to prove Shoenfield absoluteness; in particular, $\mathbf{\Pi}^1_2$ expressions are equivalent over KPI_0 to Π_1 statements in the language of set theory.

We remark that all consequences of KPI_0 in second order arithmetic are provable in $\mathbf{\Pi}^1_1$-CA_0 (cf. [13], VII.3.36). Since we primarily work with models in the language of set theory in this section, it is convenient to take KPI_0 as our base theory, but all of the results proved here can be appropriately reformulated as statements about countably coded β-models in second order arithmetic (as in Chapter VII of [13]).

The **wellfounded part** of a model $\mathcal{M} = (M, \varepsilon)$ in the language of set theory is the largest downward ε-closed subset W of M so that $\varepsilon\restriction W$ is wellfounded. For $\mathcal{M}$ an illfounded model of KP, we identify the wellfounded part of $\mathcal{M}$ with its transitive collapse, denote this $\mathrm{wfp}(\mathcal{M})$, and set

$$\mathrm{wfo}(\mathcal{M}) = \mathrm{wfp}(\mathcal{M}) \cap \mathrm{ON}$$

(the **wellfounded ordinal** of $\mathcal{M}$). Recall we say $\mathcal{M}$ is an **ω-model** if

$$\omega < \mathrm{wfo}(\mathcal{M}).$$

The following definition is due to Welch [18].

Definition 4.1: For $\mathcal{M}$ an illfounded ω-model of KP in the language of set theory, an **infinite depth Σ_2-nesting based on $\mathcal{M}$** is a sequence $\langle \zeta_n, s_n \rangle_{n\in\omega}$ of pairs so that for all $n \in \omega$,

(1) $\zeta_n \leq \zeta_{n+1} < \mathrm{wfo}(\mathcal{M})$,
(2) $s_n \in \mathrm{ON}^{\mathcal{M}} \setminus \mathrm{wfo}(\mathcal{M})$,
(3) $\mathcal{M} \models s_{n+1} < s_n$,
(4) $(L_{\zeta_n} \prec_{\Sigma_2} L_{s_n})^{\mathcal{M}}$.

We say that a level L_α of L **Σ_ω-projects** to $\rho \leq \alpha$ if ρ is least so that there is a subset of ρ that is definable over L_α which does not belong to L_α. If this subset is Σ_1 definable in some parameter, we say L_α **projects** to ρ and denote this ordinal ρ_1. If L_α Σ_ω-projects to ρ, then there is a partial function $f : \rho \rightharpoonup L_\alpha$, definable over L_α, that surjects onto L_α (see [4]).

Lemma 4.2: *Suppose $\gamma_1 \leq \gamma_2 < \delta_2 < \delta_1$ are ordinals so that*

(1) $L_{\gamma_1} \prec_{\Sigma_1} L_{\delta_1}$*;*
(2) $L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$*;*
(3) *δ_1 is the least admissible ordinal above δ_2;*
(4) *for all $\alpha \leq \delta_2$, L_α Σ_ω-projects to ω.*

Then L_{γ_2} satisfies $\Pi^1_2(z)$-MI, for all reals $z \in L_{\gamma_1}$.

Note that the hypothesis (2) implies L_{γ_2} is a model of Σ_2-Collection, i.e. for all Σ_2 relations R we have

$$L_{\gamma_2} \models (\forall a)[(\forall x \in a)(\exists y)R(x,y)] \to (\exists b)(\forall x \in a)(\exists y \in b)R(x,y).$$

The elementarity then implies each of $\gamma_1, \gamma_2, \delta_2$ is a limit of Σ_2-admissible ordinals.

Item (4) serves as a simplifying assumption to ensure that every L_α is countable, as witnessed by a surjection $f : \omega \to L_\alpha$ with $f \in L_{\alpha+1}$. Note the least level of L that does not Σ_ω-project to ω is a model of ZF^-; since this is far beyond the strength of the theories considered here, we don't lose anything by assuming (4).

Proof. Let $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be a $\Pi^1_2(z)$ monotone operator in L_{γ_2}, in the sense that there is a $\Pi^0_1(z)$ condition T so that

$$n \in \Phi(X) \iff L_{\gamma_2} \models (\forall x)(\exists y)T(n, X, x, y, z)$$

whenever $n \in \omega$ and $X \in \mathcal{P}(\omega) \cap L_{\gamma_2}$, and whenever $X \subseteq Y \subseteq \omega$ belong to L_{γ_2}, we have $\Phi(X) \subseteq \Phi(Y)$. Notice that for such X

$$n \in \Phi(X) \iff (\forall x \in L_{\gamma_2})(\exists y) T(n, X, x, y, z),$$

by Mostowski absoluteness (see I.8 in [1]) and because γ_2 is a limit of admissible ordinals as remarked above. We therefore regard the operator as defined in this way, so that "$n \in \Phi(X)$" makes sense even for sets $X \notin L_{\gamma_2}$ (though this extended Φ may fail to be monotone).

For each ordinal η, we define the approximation Φ_η as the operator Φ relativized to L_η,

$$n \in \Phi_\eta(X) \iff (\forall x \in L_\eta)(\exists y) T(n, X, x, y, z).$$

The point is that the operator Φ_η is then Σ^1_1 in any real parameter coding the countable set $\mathbb{R} \cap L_\eta$ (for example, $\mathrm{Th}(L_\eta)$, the characteristic function of the theory of L_η under some standard coding), and so each Φ_η will be correctly computed in, e.g., L_α for α a limit of admissibles above η.

Obviously $\Phi = \Phi_{\gamma_2}$, so is monotone in L_{γ_2}. But for $\eta \neq \gamma_2$ we may not even have that the operators Φ_η are monotone on $\mathcal{P}(\omega) \cap L_\eta$. So we instead work with the obvious "monotonizations",

$$\begin{aligned} n \in \Psi_\eta(X) &\iff (\exists X' \subseteq X) n \in \Phi_\eta(X') \\ &\iff (\exists X' \subseteq X)(\forall x \in L_\eta)(\exists y) T(n, X', x, y, z). \end{aligned}$$

This relation is again $\Sigma^1_1(\mathrm{Th}(L_\eta), z)$, and

$$\Psi_{\gamma_2}(X) = \Phi_{\gamma_2}(X) = \Phi(X)$$

for $X \in L_{\gamma_2}$.

Let $\langle \Psi^\xi_\eta \rangle_{\xi \leq o(\Psi_\eta)}$ be the sequence obtained via iterated application of the operator Ψ_η, as in Definition 2.1. The most important properties of these sequences are captured in the following two claims.

Claim 4.3: *If $\eta < \eta'$, then $(\forall X) \Psi_\eta(X) \supseteq \Psi_{\eta'}(X)$.*

Proof. Suppose $n \in \Psi_{\eta'}(X)$; then

$$(\exists X' \subseteq X)(\forall x \in L_{\eta'})(\exists y) T(n, X', x, y, z),$$

and any such X' will likewise be a witness to $n \in \Psi_\eta(X)$, since the latter is defined the same way but with the universal quantifier bounded by the smaller set L_η. ∎

Claim 4.4: *Suppose $\xi < \xi'$ and $\eta < \eta'$. Then*

(1) $\Psi_\eta^\xi \subseteq \Psi_\eta^{\xi'}$;
(2) $\Psi_\eta^\xi \supseteq \Psi_{\eta'}^\xi$.

Proof. (1) is by induction and monotonicity of Ψ_η. (2) follows from induction and the chain of inclusions, for $X \supseteq Y$,

$$\Psi_{\eta'}(Y) \subseteq \Psi_{\eta'}(X) \subseteq \Psi_\eta(X),$$

the first holding by monotonicity of $\Psi_{\eta'}$, the second by the previous claim. ∎

So the array $\langle \Psi_\eta^\xi \rangle$ is increasing in ξ and decreasing in η. Applying this claim with $\xi = \omega_1$, we have $\Psi_\eta^\xi = \Psi_\eta^\infty$, so that $\Psi_\eta^\infty \supseteq \Psi_{\eta'}^\infty$ whenever $\eta < \eta'$.

We now consider definability issues with respect to the operators Ψ_η and the associated sequences, with the aim of showing the levels of L under consideration are sufficiently closed to correctly compute these objects, and ultimately ensuring that the sequences $\langle \Psi_\eta^\xi \rangle_{\xi \leq o(\Psi_\eta)}$ converge to the sequence of interest $\langle \Psi_{\gamma_2}^\xi \rangle_{\xi \leq o(\Psi_{\gamma_2})}$ as $\eta \to \gamma_2$.

Claim 4.5: *Suppose $z \in L_\alpha$ and $L_\alpha \models \mathsf{KPI}$. Then the relation "$n \in \Psi_\eta^\xi$" (as a relation on $\langle n, \xi, \eta \rangle \in \omega \times \alpha \times \alpha$) is $\Delta_1^{L_\alpha}$ in the parameter z. Consequently, for all $\eta < \alpha$ and $\nu < \alpha$, the sequence $\langle \Psi_\eta^\xi \rangle_{\xi < \nu}$ belongs to L_α.*

Proof. The relation $n \in \Psi_\eta(X)$ is, as remarked above, $\Sigma_1^1(z, \mathrm{Th}(L_\eta))$ on n, X, and so is Π_1 over the least admissible set containing z, η. Since every set is contained in some admissible set L_β with $\beta < \alpha$, we have that "$n \in \Psi_\eta(X)$" is $\Delta_1(z)$ over L_α. The last part of the claim then follows from Σ_1-recursion inside L_α, using the $\Delta_1^{L_\alpha}(z)$-definability of the relation $Y = \Psi_\eta(X)$. ∎

Claim 4.6: *Suppose $z \in L_\alpha$, $\eta < \alpha$ and L_α is a model of Σ_1-Comprehension. Then $o(\Psi_\eta) < \alpha$, and $\langle \Psi_\eta^\xi \rangle_{\xi \leq o(\Psi_\eta)} \in L_\alpha$. Moreover, the relation $n \in \Psi_\eta^\infty$ (on $\omega \times \alpha$) is $\Delta_1^{L_\alpha}(z)$.*

Proof. Note such L_α satisfies KPI, so by the previous claim together with Σ_1-Comprehension in L_α, $P_\eta := \{n \in \omega \mid (\exists \xi < \alpha) n \in \Psi_\eta^\xi\} = \bigcup_{\xi < \alpha} \Psi_\eta^\xi \in L_\alpha$. By admissibility, the map on P_η sending n to the least ξ such that $n \in \Psi_\eta^\xi$ is bounded in α, and the claim is immediate. The last assertion holds because in L_α,

$$n \in \Psi_\eta^\infty \iff (\exists \xi) n \in \Psi_\eta^\xi \iff (\forall \xi)(\Psi_\eta^\xi = \Psi_\eta^{\xi+1} \to n \in \Psi_\eta^\xi),$$

which proves the claim. ∎

Claim 4.7: *Suppose $z \in L_\alpha$, and that α is a limit of ordinals β so that L_β is a model of Σ_1-Comprehension. Then the relation $n \in \Psi^\infty_\eta$ is $\Delta_1^{L_\alpha}(z)$.*

Proof. Immediate from the previous claim and the fact that the sequences are correctly computed in models of KPI_0. ∎

Claim 4.8: *If $\xi < \gamma_2$, then for some $\eta_0 < \gamma_2$ we have $\Psi^\xi_{\eta_0} = \Psi^\xi_{\gamma_2}$; furthermore, $\langle \Psi^\zeta_{\gamma_2}\rangle_{\zeta<\xi} \in L_{\gamma_2}$.*

Proof. The set

$$Q_\xi = \{n \in \omega \mid (\exists \eta < \gamma_2) n \notin \Psi^\xi_\eta\}$$

is a member of L_{γ_2} by Σ_1-Comprehension there. Now the map sending $n \in Q_\xi$ to the least η such that $n \notin \Psi^\xi_\eta$ is Δ_1, so by admissibility, is bounded by some $\eta_0 < \alpha$. Recall the sequence $\langle \Psi^\xi_\eta\rangle_{\eta\in\mathrm{ON}}$ is decreasing in η; so

$$\begin{aligned} n \in \Psi^\xi_{\eta_0} \iff L_{\gamma_2} \models (\forall \eta) n \in \Psi^\xi_\eta \iff L_{\delta_2} \models (\forall \eta) n \in \Psi^\xi_\eta \\ \implies n \in \Psi^\xi_{\gamma_2} \implies n \in \Psi^\xi_{\eta_0}. \end{aligned}$$

Note we have used the fact that $L_{\gamma_2} \prec_{\Sigma_1} L_{\delta_2}$. For the last part of the claim, consider the map sending $\zeta < \xi$ to the least η_0 such that $(\forall \eta > \eta_0)\Psi^\zeta_\eta = \Psi^\zeta_{\eta_0}$. This map is Π_1-definable, so by Σ_2-Collection in L_{γ_2}, we have a bound $\bar\eta < \gamma_2$, and for each $\zeta < \xi$, $\Psi^\zeta_{\bar\eta} = \Psi^\zeta_{\gamma_2}$. By Claim 4.5 the sequence $\langle \Psi^\zeta_{\bar\eta}\rangle_{\zeta<\xi} = \langle \Psi^\zeta_{\gamma_2}\rangle_{\zeta<\xi}$ is in L_{γ_2}. ∎

Note that by the proof of Claim 4.8, for this η_0, $\Psi^\xi_{\eta_0} = \Psi^\xi_\eta$ whenever $\xi < \gamma_2$ and $\eta_0 \le \eta < \delta_2$.

Claim 4.9: *For all $\xi < \gamma_2$, $\Psi^\xi_{\gamma_2} = \Psi^\xi_{\delta_2}$; consequently $\Psi^{<\gamma_2}_{\gamma_2} = \Psi^{<\gamma_2}_{\delta_2}$.*

Proof. By using induction on ξ and since $\Psi^{<\xi}_{\gamma_2} \in L_{\gamma_2}$ for all $\xi < \gamma_2$ by the previous claim, it is sufficient to show $\Psi_{\gamma_2}(X) = \Psi_{\delta_2}(X)$ whenever $X \in L_{\gamma_2}$. We already know $\supseteq$ holds by Claim 4.3.

So suppose $n \in \Psi_{\gamma_2}(X)$. Then for some $X' \subseteq X$, $n \in \Phi_{\gamma_2}(X')$ by definition of Ψ_{γ_2}. Then $n \in \Phi_{\gamma_2}(X) = \Phi(X)$, by monotonicity of $\Phi = \Phi_{\gamma_2}$ in L_{γ_2}. So

$$L_{\gamma_2} \models (\forall x)(\exists y) T(n, X, x, y, z)$$

so that by Σ_1-elementarity (this is enough, since Π^1_2 relations are $\Pi_1^{\mathsf{KPI}_0}$), L_{δ_2} models the same. Thus $n \in \Psi_{\delta_2}(X)$ (with witness $X' = X$). ∎

We haven't yet used the full strength of $L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$, nor, for that matter, any of the assumptions on γ_1, δ_1. We appeal to Σ_2-elementarity to show that in fact $o(\Psi_{\delta_2}) \leq \gamma_2$; the assumptions on γ_1, δ_1 will be used to show that $\Psi^\infty_{\delta_2} = \Psi^\infty_{\gamma_2}$, and it will follow that the operator Ψ_{γ_2} (which is equal to Φ) stabilizes inside L_{γ_2}.

Notice that by Claims 4.4 and 4.8, $\Psi^\xi_{\gamma_2} = \bigcap_{\eta<\gamma_2} \Psi^\xi_\eta$ for all $\xi < \gamma_2$. So

$$\Psi^{<\gamma_2}_{\gamma_2} = \bigcup_{\xi<\gamma_2} \Psi^\xi_{\gamma_2} = \{n \in \omega \mid (\exists \xi < \gamma_2)(\forall \eta < \gamma_2) n \in \Psi^\xi_\eta\}.$$

This set is Σ_2-definable over L_{γ_2}. By Claim 4.4, $\Psi^{\gamma_2}_{\delta_2} \subseteq \Psi^{\gamma_2}_\eta$ for all $\eta < \delta_2$, and we have

$$\Psi^{\gamma_2}_{\delta_2} \subseteq \{n \in \omega \mid (\forall \eta < \delta_2) n \in \Psi^{\gamma_2}_\eta\} \subseteq \{n \in \omega \mid (\exists \xi < \delta_2)(\forall \eta < \delta_2) n \in \Psi^\xi_\eta\}.$$

(The second inclusion holds simply because $\gamma_2 < \delta_2$.) By the assumed Σ_2-elementarity $L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$, this last set is precisely $\Psi^{<\gamma_2}_{\gamma_2}$. Putting this together with Claim 4.9,

$$\Psi^{\gamma_2}_{\delta_2} \subseteq \Psi^{<\gamma_2}_{\gamma_2} = \Psi^{<\gamma_2}_{\delta_2} \subseteq \Psi^{\gamma_2}_{\delta_2}$$

so that $\Psi^{\gamma_2}_{\delta_2} = \Psi^{<\gamma_2}_{\delta_2}$ is the least fixed point of Ψ_{δ_2}, and $\Psi^{<\gamma_2}_{\gamma_2} = \Psi^\infty_{\delta_2}$.

Claim 4.10: $\Psi^\infty_{\delta_2} = \Psi^\infty_{\gamma_2}$.

Proof. As usual, Claim 4.4 and $\gamma_2 < \delta_2$ gives us the inclusion $\subseteq$. We have $\Psi^\infty_{\delta_2} = \Psi^{<\gamma_2}_{\gamma_2} \in L_{\delta_2}$ by Claim 4.5. Suppose $n \notin \Psi^\infty_{\delta_2}$. Then

$$L_{\delta_1} \models (\exists \eta)(\exists P)(\forall m \in \omega)(m \in \Psi_\eta(P) \to m \in P) \wedge n \notin P,$$

with witnesses $\eta = \delta_2$ and $P = \Psi^\infty_{\delta_2}$. Recall "$m \in \Psi_\eta(P)$", being a Σ^1_1 statement about $m, \mathrm{Th}(L_\eta), P$, is Π_1 over any admissible set containing η, z, P. Since L_{δ_1} is assumed to be admissible, the relation above is then Σ_1 in L_{δ_1}. It therefore reflects to L_{γ_1} (recall that z, the parameter from which everything is defined, is assumed to belong to L_{γ_1}). But then $n \notin \Psi^\infty_\eta$ for some $\eta < \gamma_1$; hence $n \notin \Psi^\infty_{\gamma_2}$. ∎

So the least fixed points $\Phi^\infty = \Psi^\infty_{\gamma_2}$ and $\Psi^\infty_{\delta_2}$ are equal. The argument just given shows the relation $n \notin \Phi^\infty$ is Σ_1 over L_{δ_1}, hence over L_{γ_1}; in any event, the set Φ^∞ belongs to L_{γ_2} (using Σ_1-Comprehension in L_{γ_2} in the case that $\gamma_1 = \gamma_2$).

Finally, we claim $o(\Phi) < \gamma_2$. The map defined in L_{γ_2} that takes a natural $n \in \Phi^\infty = \Psi^\infty_{\gamma_2}$ to the least ξ such that $(\exists \eta_0)(\forall \eta > \eta_0) n \in \Psi^\xi_\eta$ is Σ_2-definable, and so by Σ_2-Collection in L_{γ_2}, is bounded in γ_2. Since for each $\xi < \gamma_2$ we have $\Phi^\xi = \Psi^\xi_{\gamma_2} = \Psi^\xi_{\eta_0}$ for some $\eta_0 < \gamma_2$, this implies $o(\Phi) < \gamma_2$.

That $\langle\Phi^\xi\rangle_{\xi\leq o(\Phi)}$ belongs to L_{γ_2} follows from the last assertion of Claim 4.8. We have that the desired instance of $\Pi^1_2(z)$-MI holds in L_{γ_2}, which completes the proof. ∎

THEOREM 4.11: *Suppose $\mathcal{M}$ is an illfounded ω-model of* KP *with $\langle\zeta_n, s_n\rangle_{n\in\omega}$ an infinite depth Σ_2-nesting based on $\mathcal{M}$, and that $\mathcal{M}$ is locally countable, in the sense that every $L^{\mathcal{M}}_a$ Σ_ω-projects to ω in $\mathcal{M}$. Then if $\beta = \sup_{n\in\omega}\zeta_n$, we have*

$$L_\beta \models \mathbf{\Pi}^1_2\text{-}\mathsf{MI}.$$

Proof. If $\beta = \zeta_n$ for some $n\in\omega$, then we obtain the result immediately by applying the lemma in M to the tuple $\langle\zeta_n,\zeta_{n+1},s_{n+1},s_n\rangle$. So we can assume $\langle\zeta_n\rangle_{n\in\omega}$ is strictly increasing. Let $\Phi:\mathcal{P}(\omega)\to\mathcal{P}(\omega)$ be $\Pi^1_2(z)$ and monotone in L_β for some $z\in L_\beta$, and let ζ_n be sufficiently large that $z\in L_{\zeta_n}$. Now $L_{\zeta_{n+1}}\prec_{\Sigma_1} L_\beta$ and both models satisfy KPI_0, so that whenever $X\subseteq\omega$ is in $L_{\zeta_{n+1}}$, we have

$$L_\beta\models n\in\Phi(X) \iff L_{\zeta_{n+1}}\models n\in\Phi(X).$$

In particular, $L_{\zeta_{n+1}}$ believes Φ is $\Pi^1_2(z)$ and monotone, so that by the lemma applied to the tuple $\langle\zeta_n,\zeta_{n+1},s_{n+1},s_n\rangle$, we have $o(\Phi)<\zeta_{n+1}$, and the sequence $\langle\Phi^\xi\rangle_{\xi\leq o(\Phi)}$ (which is computed identically in $L_{\zeta_{n+1}}$ and L_β) belongs to $L_{\zeta_{n+1}}$. ∎

Combining Theorems 3.1 and 4.11, we obtain

COROLLARY 4.12: *If $\mathcal{M},\beta$ are as in the previous theorem, then for any $\Sigma^0_3(z)$ set with $z\in L_\beta$, either*

(1) *Player I wins $G(A)$ with a strategy $\sigma\in L_\beta$; or*
(2) *Player II wins $G(A)$ with a strategy $\Delta^1_3(z)$-definable over L_β.*

5. Infinite depth Σ_2-nestings from determinacy

In this section we show in the base theory $\mathbf{\Pi}^1_1$-CA_0 that Σ^0_3-DET implies the existence of models bearing infinite depth Σ_2 nestings. The arguments are mostly cosmetic modifications of those given in Welch's [17]. The most significant adjustment is to the Friedman-style game, Welch's G_ψ, which is here tailored to allow the proof of the implication to be carried out in $\mathbf{\Pi}^1_1$-CA_0.

For α an ordinal, let T_2^α denote the lightface Σ_2-theory of L_α, i.e.,

$$T_2^\alpha = \{\sigma \mid \sigma \text{ is a } \Sigma_2 \text{ sentence without parameters, and } L_\alpha \models \sigma\}.$$

We will also abuse this notation slightly by applying it to nonstandard ordinals b, so that if $b \in \mathrm{ON}^{\mathcal{M}} \setminus \mathrm{wfo}(\mathcal{M})$, T_2^b denotes the Σ_2-theory of $(L_b)^{\mathcal{M}}$. It will always be clear from the context which illfounded model $\mathcal{M}$ this b comes from.

Lemma 5.1: *Suppose $\mathcal{M}$ is an illfounded ω-model of* KP *such that $(L_a)^{\mathcal{M}} \models$ "all sets are countable", for every $a \in \mathrm{ON}^{\mathcal{M}}$. Set $\beta = \mathrm{wfo}(\mathcal{M})$. Suppose for all nonstandard ordinals a of $\mathcal{M}$, there exists some $<^{\mathcal{M}}$-smaller nonstandard $\mathcal{M}$-ordinal b so that $T_2^b \subseteq T_2^\beta$. Then there is an infinite depth Σ_2 nesting based on $\mathcal{M}$.*

Proof. This is essentially shown in Claim (5) in Section 3 of [17]. We outline the shorter approach suggested there.

Suppose b is a nonstandard $\mathcal{M}$-ordinal with $T_2^b \subseteq T_2^\beta$. By the assumption of local countability in levels of $L^{\mathcal{M}}$, we have a uniformly Σ_2-definable Σ_2 Skolem function, which we denote h_2^b (see [3]). The set

$$H = h_2^b[\omega^{<\omega}]$$

is transitive in $\mathcal{M}$, since for any $x \in H$, the $<_L^{\mathcal{M}}$-least surjection of ω onto x is in H, and since $\mathcal{M}$ is an ω-model, the range of this surjection is a subset of H. Since $H \models V = L$, we have by condensation in $\mathcal{M}$ that $H = L_{\gamma_b} \prec_{\Sigma_2} L_b$ for some $\gamma_b \leq^{\mathcal{M}} b$.

We claim that γ_b is a standard ordinal, equivalently, $\gamma_b < \beta$. For suppose not, so there is some nonstandard ordinal c of L_b in L_{γ_b}. Let f be the $<_L^{\mathcal{M}}$-least surjection from ω onto c. Then $f = h_2^b(k)$ for some $k \in \omega$, and for $m, n \in \omega$, the sentences "$h_2(k)$ exists, is a function from ω onto some ordinal, and $h_2(k)(m) \in h_2(k)(n)$" are Σ_2. But since $T_2^b \subseteq T_2^\beta$, this would imply $h_2^\beta(k)(m) \in h_2^\beta(k)(n)$ whenever $f(m) \in f(n)$ in $(L_b)^{\mathcal{M}}$. This contradicts the wellfoundedness of β.

The lemma now follows by choosing some descending sequence $\langle b_n \rangle_{n\in\omega}$ of nonstandard ordinals of $\mathcal{M}$ with $T_2^{b_n} \subseteq T_2^\beta$ for all n, and setting

$$\gamma_n = \gamma_{b_n} = \sup h_2^{b_n}[\omega^{<\omega}] < \beta.$$

Since the γ_n are true ordinals, we can choose some non-decreasing subsequence $\langle \gamma_{n_k} \rangle_{k\in\omega}$, and $\langle \gamma_{n_k}, b_{n_k} \rangle_{k\in\omega}$ is the desired infinite depth Σ_2-nesting. ∎

Our winning condition will make use of one fine structural notion, already referred to in this paper: that of a level of L projecting to ω. We will say an ω-model $\mathcal{M}$ of $V = L$ satisfies $\rho_n = \omega$ if there is some subset of ω that is Σ_n-definable over $\mathcal{M}$ but does not belong to $\mathcal{M}$ (note that for fixed n, this is first order expressible in $\mathcal{M}$). As remarked above, if $\mathcal{M}$ satisfies $\rho_n = \omega$ for some n, then there is $f : \omega \rightharpoonup \mathcal{M}$, an $\mathcal{M}$-definable partial surjection. A straightforward diagonalization gives the following fact.

Fact 5.2: *Suppose $\mathcal{M}$ is an ω-model of $V = L + \rho_n = \omega$ for some n. Then the first order theory of $\mathcal{M}$, regarded as a real $Th(\mathcal{M})$ via some natural coding, cannot be a member of $\mathcal{M}$: $Th(\mathcal{M}) \notin \mathcal{M}$.*

Theorem 5.3: *Work in $\mathbf{\Pi}^1_1$-CA_0. If Σ^0_3-determinacy holds, then there is a model $\mathcal{M}$ for which there exists an infinite depth Σ_2-nesting based on $\mathcal{M}$.*

Corollary 5.4: *Work in $\mathbf{\Pi}^1_1$-CA_0. Σ^0_3-determinacy implies the existence of a β-model of $\mathbf{\Pi}^1_2$-MI; in particular, $L_\gamma \models \mathbf{\Pi}^1_2$-$\mathsf{MI}$ for some countable ordinal γ.*

Proof. Immediate, combining Theorem 5.3 with Theorem 4.11. ∎

Proof of Theorem 5.3. We define a variant of Welch's game G_ψ from [17]. Players I and II play complete consistent theories in the language of set theory, $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively, extending

$$(*) \qquad V = L \ + \ KP \ + \ \rho_1 = \omega.$$

These theories uniquely determine term models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$. Player I loses if $\mathcal{M}_{\mathrm{I}}$ has nonstandard ω; similarly, if $\mathcal{M}_{\mathrm{I}}$ is an ω-model and $\mathcal{M}_{\mathrm{II}}$ is not, then Player II loses. (Note that this is a Boolean combination of Σ^0_2 conditions on $f_{\mathrm{I}}, f_{\mathrm{II}}$.)

The remainder of the winning condition assumes $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are both ω-models. Player I wins if either of the following hold:

(1) $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$, or $f_{\mathrm{I}} = f_{\mathrm{II}}$.
(2) $(\exists \beta \leq \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}})(\exists a \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})(\forall n \in \omega)(\exists \langle a_i, \sigma_i \rangle_{i \leq n})$ so that, for all $i < n$,
   - $a_0 = a$ and $a_i \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$,
   - $(a_{i+1} < a_i)^{\mathcal{M}_{\mathrm{II}}}$,
   - σ_i is the first Σ_2 formula (in some fixed recursive list of all formulas in the language of set theory) so that $L_\beta^{\mathcal{M}_{\mathrm{I}}} \not\models \sigma_i$ and $L_{a_i}^{\mathcal{M}_{\mathrm{II}}} \models \sigma_i$;
   - if a_i is a successor ordinal in $\mathcal{M}_{\mathrm{II}}$, then a_{i+1} is the largest limit ordinal of $\mathcal{M}_{\mathrm{II}}$ below a_i;

- if a_i is a limit ordinal in $\mathcal{M}_{\mathrm{II}}$, and σ_i is the formula $\exists u \forall v \psi(u,v)$ (with ψ Δ_0), then a_{i+1} is $\in^{\mathcal{M}_{\mathrm{II}}}$-least in $\mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$ so that

$$(\exists u \in L_{a_{i+1}})(L_{a_i} \models \forall v \psi(u,v))$$

holds in $\mathcal{M}_{\mathrm{II}}$.

Note that if (2) holds, then $\mathcal{M}_{\mathrm{II}}$ must be an illfounded model: If β, a witness the condition, then there is a uniquely determined infinite sequence $\langle a_i, \sigma_i \rangle_{i \in \omega}$ that is the union of the witnessing sequences $\langle a_i, \sigma_i \rangle_{i \leq n}$. Then $\langle a_i \rangle_{i \in \omega}$ is an infinite descending sequence of $\mathcal{M}_{\mathrm{II}}$-ordinals.

Note also that this condition is Σ^0_3 as a condition on $f_{\mathrm{I}}, f_{\mathrm{II}}$. Strictly speaking, the quantifiers over $\mathcal{M}_{\mathrm{I}}$, $\mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$, etc. should be regarded as natural number quantifiers ranging over the indices of defining formulas (codes) for members of the models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$. Clause (1) is then Σ^0_2, and (2) is Σ^0_3, since each bulleted item there is recursive in codes for the objects $\beta, a, \langle a_i, \sigma_i \rangle_{i \leq n}$ and the pair $\langle f_{\mathrm{I}}, f_{\mathrm{II}} \rangle$.

Denote the set of runs which I wins by F; so F is Σ^0_3.

Claim 5.5: *Player I has no winning strategy in $G(F)$.*

Proof. Suppose instead that I has some winning strategy in this game. By Shoenfield absoluteness (a version of which is provable in $\mathbf{\Pi}^1_1$-CA_0, see [13]) there is such a winning strategy σ in L. Let α be the least admissible ordinal so that $\sigma \in L_\alpha$ (such exists since $\mathbf{\Pi}^1_1$-CA_0 implies the reals are closed under the hyperjump; see [12] Ch. VII). Let f_{II} be the theory of L_α. Note that then L_α projects to ω, since it is the least admissible containing some real; in particular, it satisfies condition $(*)$. Let $f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$ be the theory that σ responds to f_{II} with.

Now σ is winning for I in $G(F)$; so $\mathcal{M}_{\mathrm{I}}$ is an ω-model. Since $\mathcal{M}_{\mathrm{II}}$ is wellfounded, (2) must fail, and again since σ is winning for I, we have (1) holds; that is, either $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$ or $f_{\mathrm{I}} = f_{\mathrm{II}}$. If $f_{\mathrm{I}} = f_{\mathrm{II}}$, then II was simply copying I's play, so that $\sigma \in L_\alpha = \mathcal{M}_{\mathrm{I}}$, implying $f_{\mathrm{I}} \in \mathcal{M}_{\mathrm{I}}$, a contradiction to Fact 5.2.

So $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$. The strategy σ is computable from f_{II}, so must also belong to $\mathcal{M}_{\mathrm{I}}$. But then, since $f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$, we again obtain the contradiction $f_{\mathrm{I}} \in \mathcal{M}_{\mathrm{I}}$. ∎

Claim 5.6: *If Player II has a winning strategy in $G(F)$, then there is a model with an infinite depth Σ_2-nesting.*

Proof. Towards a contradiction, let τ be a winning strategy for II, and suppose there is no model as in Definition 4.1. As in the proof of the previous claim, we may assume $\tau \in L$, and let α be the least admissible with $\tau \in L_\alpha$. Put

$$f_{\mathrm{I}} = \mathrm{Th}(L_\alpha);$$

then f_{I} satisfies the condition $(*)$. Let $f_{\mathrm{II}} = \tau * f_{\mathrm{I}}$ be τ's response.

We claim that if $\mathcal{M}_{\mathrm{II}}$ is the model so obtained, then $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$ (note $\mathbf{\Pi}^1_1$-CA_0 is enough to ensure the existence of (a real coding) the wellfounded ordinal of $\mathcal{M}_{\mathrm{II}}$). Suppose otherwise; then $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) > \alpha$, and then $L_\alpha \in \mathcal{M}_{\mathrm{II}}$. Since $f_{\mathrm{I}} = \mathrm{Th}(L_\alpha)$ and τ belongs to $\mathcal{M}_{\mathrm{II}}$, we have $f_{\mathrm{II}} = \tau * f_{\mathrm{I}}$ does as well. As before, this contradicts the assumption that II wins the play; specifically, by Fact 5.2, f_{II} fails to satisfy condition $(*)$.

So $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$. We claim $\mathcal{M}_{\mathrm{II}}$ is illfounded. Otherwise, either $o(\mathcal{M}_{\mathrm{II}}) = \alpha$, in which case we get $\mathcal{M}_{\mathrm{II}} = L_\alpha = \mathcal{M}_{\mathrm{I}}$, in which case (1) holds and I wins; or else $o(\mathcal{M}_{\mathrm{II}}) < \alpha$, so that $\mathcal{M}_{\mathrm{II}} = L_\gamma$ for some $\gamma < \alpha$, so that $f_{\mathrm{II}} = \mathrm{Th}(L_\gamma) \in L_\alpha = \mathcal{M}_{\mathrm{I}}$, and again (1) holds, contradicting that τ is winning for II.

So $\mathcal{M}_{\mathrm{II}}$ is illfounded with $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$. Set $\beta = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$. If there is no model bearing an infinite depth Σ_2-nesting, then by Lemma 5.1 there exists some nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal a, so that, for every nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal b with $b \leq^{\mathcal{M}_{\mathrm{II}}} a$, we have $T^b_2 \not\subseteq T^\beta_2$. That is, for all such b, there is a Σ_2 sentence σ so that $L_\beta \not\models \sigma$, but $L^{\mathcal{M}_{\mathrm{II}}}_b \models \sigma$.

It is now straightforward to show β, a witness the winning condition (2). Set $a_0 = a$. Suppose inductively that a_i is a nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal with $a_i \leq^{\mathcal{M}_{\mathrm{II}}} a$. Then by choice of a, there is some Σ_2 formula σ so that $L^{\mathcal{M}_{\mathrm{II}}}_b \models \sigma$ and $L_\beta \not\models \sigma$; let σ_i be the least such under our fixed enumeration of formulae.

If a_i is not limit in $\mathcal{M}_{\mathrm{II}}$, take a_{i+1} to be the greatest limit ordinal of $\mathcal{M}_{\mathrm{II}}$ below a_i; note then a_{i+1} is also nonstandard and below a, and σ_{i+1} is defined as above.

Now if a_i is limit in $\mathcal{M}_{\mathrm{II}}$, we have that σ_i is of the form $(\exists u)(\forall v)\psi(u, v)$ for some Δ_0 formula ψ. Let a_{i+1} be least so that for some $x \in L^{\mathcal{M}_{\mathrm{II}}}_{a_i+1}$, we have $L^{\mathcal{M}_{\mathrm{II}}}_{a_i} \models (\forall v)\psi(x, v)$. Then $a_{i+1} <^{\mathcal{M}_{\mathrm{II}}} a_i$, and since $L_\beta \not\models \sigma_i$, we must have that a_{i+1} is nonstandard. Thus the construction proceeds, and we have that I wins the play $\langle f_{\mathrm{I}}, f_{\mathrm{II}} \rangle$ via condition (2). So τ cannot be a winning strategy. ∎

These claims combine to show that if there is no model with an infinite depth Σ_2 nesting, then neither player has a winning strategy in the game $G(F)$. This completes the proof of the theorem. ∎

We have thus shown that Σ^0_3 determinacy implies the existence of a model satisfying $\mathbf{\Pi}^1_2$-MI, and indeed, of some ordinal γ so that $L_\gamma \models \mathbf{\Pi}^1_2$-$\mathsf{MI}$. The meticulous reader will observe, however, that our proof of determinacy in Section 3 really only made use of $\neg\Game\mathbf{\Sigma}^0_3$ monotone inductive definitions. This may at first appear strange, in light of the fact that $\neg\Game\mathbf{\Sigma}^0_3$ is a much smaller class than $\mathbf{\Pi}^1_2$. This situation is clarified somewhat by the following theorem, which shows that if γ is minimal with $L_\gamma \models \mathbf{\Pi}^1_2$-$\mathsf{MI}$, then the $\mathbf{\Pi}^1_2$ relations that are correctly computed in L_γ are precisely the $\neg\Game\mathbf{\Sigma}^0_3$ relations.

Theorem 5.7: *Let γ be the least ordinal so that L_γ satisfies $\mathbf{\Pi}^1_2$-MI. Let z be a real in L_γ, and suppose $\Phi(u)$ is a $\mathbf{\Sigma}^1_2$ formula; say it is $\Sigma^1_2(z)$ for some real z. Then there is a $\Game\Sigma^0_3(z)$ relation Ψ so that, for all reals x of L_γ, we have L_γ satisfies $\Phi(x)$ if and only if $\Psi(x)$ holds (in V, or equivalently, in L_γ). Equivalently,*

$$\{x \in \omega^\omega \cap L_\gamma \mid L_\gamma \models \Phi(x)\} = L_\gamma \cap \Game A$$

for some $\Sigma^0_3(z)$ set $A \subseteq \omega^\omega \times \omega^\omega$.

Proof. Fix such a formula $\Phi(x)$; for simplicity, assume z is recursive. Then there is a recursive tree T on ω^3 so that for all x, $\Phi(x)$ holds if and only if for some y, $T_{\langle x,y\rangle}$ is wellfounded; here $T_{\langle x,y\rangle}$ is defined as usual as the tree

$$T_{\langle x,y\rangle} = \{s \in \omega^{<\omega} \mid \langle s, x\restriction|s|, y\restriction|s|\rangle \in T\}.$$

We define a version of the game from Theorem 5.3. This time, for a fixed real x, each player is required to produce their respective ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ satisfying

$$(**) \qquad V = L(x) \ + \ KP \ + \ \rho_1 = \omega.$$

In addition, $\mathcal{M}_{\mathrm{I}}$ must satisfy the sentence "$(\exists y)T_{\langle x,y\rangle}$ is ranked"; whereas $\mathcal{M}_{\mathrm{II}}$ must satisfy its negation. If a winner has not been decided on the basis of one of these conditions being violated, then Player I wins if either of the conditions (1), (2) from the proof of Theorem 5.3 holds. Let F_x be the set of $f \in \omega^\omega$ so that Player I wins the play of the game on x, where $f_{\mathrm{I}}(n) = f(2n)$, $f_{\mathrm{II}}(n) = f(2n+1)$ for all n. Let $F = \{\langle x, f\rangle \mid f \in F_x\}$. Then F is Σ^0_3; let $\Psi(x)$ be the statement "I has a winning strategy in the game $G(F_x)$".

Suppose $x \in L_\gamma$ is such that $L_\gamma \models \Phi(x)$. We claim $\Psi(x)$ holds; that is, Player I has a winning strategy in $G(F_x)$. Let y be a witness to truth of Φ, and let α be least such that $y \in L_\alpha(x)$ and $L_\alpha(x) \models \mathsf{KP}$. Then by admissibility,

$L_\alpha(x)$ contains a ranking function for $T_{\langle x,y\rangle}$. Let σ be the strategy for I that always produces the theory of $L_\alpha(x)$. We claim σ is winning for Player I.

Suppose towards a contradiction that $\mathcal{M}_{\rm II}$ is the model produced by a winning play by II against σ; we can assume $\mathcal{M}_{\rm II} \in L_\gamma$. Then $\mathcal{M}_{\rm II}$ is an ω-model. It cannot be wellfounded, since then it would be of the form $L_\beta(x)$ for some β; but we can't have $\beta \geq \alpha$ (since $\mathcal{M}_{\rm II}$ cannot contain y, or else it would have a ranking function for $T_{\langle x,y\rangle}$), nor can $\beta < \alpha$ hold (since otherwise (1) is satisfied, and I wins the play). So $\mathcal{M}_{\rm II}$ is illfounded, say with $\text{wfo}(\mathcal{M}_{\rm II}) = \beta$; by the Truncation Lemma (see II.8 in [1]), $\text{wfo}(\mathcal{M}_{\rm II})$ is admissible, so we can't have $y \in \mathcal{M}_{\rm II}$ since then we would have a ranking function for $T_{\langle x,y\rangle}$ in $\mathcal{M}_{\rm II}$, contrary to the requirement on II's model; hence $\beta < \alpha$. Now since I does not win the play, the condition (2) fails, so there must be some infinite depth Σ_2-nesting based on $\mathcal{M}_{\rm II}$, by Lemma 5.1 and the argument of Claim 5.6. But this contradicts the fact that the model $\mathcal{M}_{\rm II}$ belongs to L_γ, by minimality of γ and Theorem 4.11.

Conversely, suppose $\Phi(x)$ fails in L_γ. Suppose towards a contradiction that Player I wins the game $G(F_x)$ (in V); then by Theorem 3.1, there is a winning strategy $\sigma \in L_\gamma$. Let $\mathcal{M}_{\rm II}$ be the least level of $L(x)$ containing σ. Note that $\mathcal{M}_{\rm II} \models (**) +$ "$(\forall y) T_{\langle x,y\rangle}$ is not ranked". By the argument in the proof of Theorem 5.3 (Claim 5.6), we obtain failure of both (1) and (2), so that II wins the play, a contradiction to σ being a winning strategy for I. ∎

References

[1] J. Barwise, *Admissible Sets and Structures*, Perspectives in Mathematical Logic, Springer, Berlin–New York, 1975,

[2] M. Davis, *Infinite games of perfect information*, in *Advances in Game Theory*, Princeton University Press, Princeton, NJ, 1964, pp. 85–101.

[3] S.-D. Friedman, *Parameter-free uniformisation*, Proceedings of the American Mathematical Society **136** (2008), 3327–3330.

[4] R. B. Jensen, *The fine structure of the constructible hierarchy*, Annals of Mathematical Logic **4** (1972), 229–308; erratum, ibid. **4** (1972), 443.

[5] D. A. Martin, *Determinacy*, unpublished manuscript.

[6] D. A. Martin, Π^1_2 *monotone inductive definitions*, in *Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79)*, Lecture Notes in Mathematics, Vol. 839, Springer, Berlin–New York, 1981, pp. 215–233.

[7] M. O. MedSalem and K. Tanaka, *Weak determinacy and iterations of inductive definitions*, in *Computational Prospects of Infinity. Part II. Presented Talks*, Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore, Vol. 15, World Scientific, Hackensack, NJ, 2008, pp. 333–353.

[8] A. Montalbán, *Open questions in reverse mathematics*, Bulletin of Symbolic Logic **17** (2011), 431–454.

[9] A. Montalbán and R. A. Shore, *The limits of determinacy in second-order arithmetic*, Proceedings of the London Mathematical Society **104** (2012), 223–252.

[10] A. Montalbán and R. A. Shore, *The limits of determinacy in second order arithmetic: consistency and complexity strength*, Israel Journal of Mathematics **204** (2014), 477–508.

[11] Y. N. Moschovakis, *Descriptive Set Theory*, Mathematical Surveys and Monographs, Vol. 155, American Mathematical Society, Providence, RI, 2009.

[12] G. E. Sacks, *Higher Recursion Theory*, Perspectives in Mathematical Logic, Springer, Berlin, 1990.

[13] S. G. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009.

[14] J. R. Steel, *Determinateness and subsystems of analysis*, ProQuest LLC, Ann Arbor, MI, 1977, Thesis (Ph.D.)–University of California, Berkeley.

[15] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis. I. Δ^0_2 games*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **36** (1990), 481–491.

[16] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis. II. Σ^0_2 games*, Annals of Pure and Applied Logic **52** (1991), 181–193.

[17] P. D. Welch, *Weak systems of determinacy and arithmetical quasi-inductive definitions*, Journal of Symbolic Logic **76** (2011), 418–436.

[18] P. D. Welch, *$G_{\delta\sigma}$ games*, Isaac Newton Institute Preprint Series (2012), 1–10.

[19] K. Yoshii, *A survey of determinacy of infinite games in second order arithmetic*, Annals of the Japan Association for Philosophy of Science **25** (2017), 35–44.

[20] K. Yoshii and K. Tanaka, *Infinite games and transfinite recursion of multiple inductive definitions*, in *How the World Computes*, Lecture Notes in Computer Science, Vol. 7318, Springer, Heidelberg, 2012, pp. 374–383.