ISRAEL JOURNAL OF MATHEMATICS **229** (2019), 307–339
DOI: 10.1007/s11856-018-1800-3

MEASURABLY ENTIRE FUNCTIONS AND THEIR GROWTH

BY

ADI GLÜCKSAM*

School of Mathematical Sciences, Sackler Faculty of Sciences
Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel
e-mail: adiglucksam@gmail.com

ABSTRACT

In 1997 B. Weiss introduced the notion of measurably entire functions and proved that they exist on every arbitrary free $\mathbb{C}$-action defined on a standard probability space. In the same paper he asked about the minimal possible growth rate of such functions. In this work we show that for every arbitrary free $\mathbb{C}$-action defined on a standard probability space there exists a measurably entire function whose growth rate does not exceed $\exp(\exp[\log^p |z|])$ for any $p > 3$. This complements a recent result by Buhovski, Glücksam, Logunov and Sodin who showed that such functions cannot have a growth rate smaller than $\exp(\exp[\log^p |z|])$ for any $p < 2$.

1. Introduction

A measure space $(X, \mathcal{B}, \mu)$ is called a **standard probability space** if $\mu(X) = 1$ and there exists a topology τ such that (X, τ) is metrizable as a topological space, $\mathcal{B}$ is the completion of the σ-algebra generated by the open sets of τ, and for every $\varepsilon > 0$ there exists a compact set K such that $\mu(K) > 1 - \varepsilon$.

Let $(X, \mathcal{B}, \mu)$ be a standard probability space. A map $f : X \to X$ is called **probability preserving** if for every $B \in \mathcal{B}$, $\mu(B) = \mu(f^{-1}(B))$. We denote by $PPT(X)$ the group of all invertible probability preserving transformations from $(X, \mathcal{B}, \mu)$ to itself. We use the standard topology on this group, defined

* Supported in part by ERC Advanced Grant 692616 and ISF Grant 382/15.
Received January 24, 2018 and in revised form March 21, 2018

by the pull back of the weak operator topology restricted to unitary operators on $L_2(X,\mathcal{B},\mu)$ by the Koopman representation associated with the action, $T \mapsto U_T f$, where $[U_T f](x) = f(Tx)$ (see [2, Page 61]).

A **probability preserving action** of $\mathbb{C}$ (a $\mathbb{C}$-action in short) is a continuous homomorphism $T : \mathbb{C} \to PPT(X)$. A $\mathbb{C}$-action $T : \mathbb{C} \to PPT(X)$ is called **free** if for μ-almost every $x \in X$, $T_z x = x$ implies that $z = 0$. In other words, there are no periodic points almost surely.

Let $\mathcal{E}$ denote the space of entire functions endowed with the local uniform topology, and let $\mathcal{B}$ denote the Borel structure associated with it. The complex plane acts on $(\mathcal{E},\mathcal{B})$ by translations defined by

$$(T_w f)(z) = f(z+w).$$

Whether there exists a probability measure λ defined on $(\mathcal{E},\mathcal{B})$ such that T is a $\mathbb{C}$-action on $(\mathcal{E},\mathcal{B},\lambda)$ is not a trivial fact. In fact, it was not known until Weiss showed such measures exist using notions from dynamical systems, which we shall introduce now:

Definition 1.1: Let $(X,\mathcal{B},\mu)$ be a standard probability space, and suppose $T : \mathbb{C} \to PPT(X)$ is a $\mathbb{C}$-action. A map $F : X \to \mathbb{C}$ is called **measurably entire** if it is a non-constant measurable function and for μ-almost every $x \in X$ the map $F_x : \mathbb{C} \to \mathbb{C}$ defined by $F_x(z) := F(T_z x)$ is entire.

The existence of measurably entire functions is closely related to the question of existence of translation invariant random entire functions. On one hand, the space of entire functions, $\mathcal{E}$, endowed with the topology of local uniform convergence is a Polish space, and so the existence of a translation invariant probability measure on $\mathcal{E}$ is an example of a measurably entire function. On the other hand, the existence of a measurably entire function produces a translation invariant random entire function by defining the measure

$$\mu_F(A) := \mu(\{x \in X;\ F_x \in A\}), \quad A \subset \mathcal{E},\ \text{measurable}.$$

Some years ago Mackey asked the following question:

Question 1.2 (Mackey): Does every probability preserving free action of $\mathbb{C}$ on a standard probability space admit a measurably entire function?

Weiss answered Mackey's question in 1997:

THEOREM 1.3 (Weiss 1997, [5]): *For every free probability preserving action of $\mathbb{C}$ on a standard probability space there exists a measurably entire function.*

Weiss' paper gives rise to an abundance of measurably entire functions and in particular answers Mackey's question positively. In his paper Weiss raised several questions, one of them was about the possible growth rate of such functions, measured by the asymptotic growth of the function $M_f(R) := \max_{z\in\overline{R\mathbb{D}}}|f(z)|$, where $\overline{R\mathbb{D}} := \{|z| \le R\}$. There are two possible interpretations for this question:

(i) What is the minimal growth rate of a measurably entire function of a $\mathbb{C}$-action on a standard probability space $(X, \mathcal{B}, \mu)$?
(ii) Given a $\mathbb{C}$-action on a standard probability space $(X, \mathcal{B}, \mu)$, what is the minimal growth rate of a measurably entire function?

We recently proved in a joint work with L. Buhovsky, A. Logunov, and M. Sodin the following theorem, which gives an almost full answer to the first interpretation. We state this theorem using the terminology of measurably entire functions, where $\log^\alpha x := (\log x)^\alpha$.

THEOREM 1.4 ([1, Theorem 1]):

(a) *There exists a standard probability space $(X, \mathcal{B}, \mu)$ with a free $\mathbb{C}$-action, T, for which there exists a measurably entire function F such that for μ almost every $x \in X$, and for every $\varepsilon > 0$,*

$$\limsup_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}}|F(T_z x)|}{\log^{2+\varepsilon} R} = 0.$$

(b) *For every standard probability space $(X, \mathcal{B}, \mu)$ for every measurably entire function $F : X \to \mathbb{C}$ μ-almost every x, either $z \mapsto F(T_z x)$ is a constant function or for every $\varepsilon > 0$*

$$\lim_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}}|F(T_z x)|}{\log^{2-\varepsilon} R} = \infty.$$

While Weiss' paper tells us such functions always exist, part (b) of Theorem 1.4 gives a lower bound for the minimal possible growth rate of measurably entire functions defined for a general free $\mathbb{C}$-action defined on a standard probability space, but not an upper bound.

We would like to emphasize the difference between the two interpretations. While in the first interpretation one may choose the measure space (and therefore the action) as well as the measurably entire function, in the second one the action is given to us, and one may only choose the measurably entire function.

In this paper we will construct a measurably entire function with bounded growth rate for a general free action:

Theorem 1.5: *Let $(X, \mathcal{B}, \mu)$ be a standard probability space, and suppose $T : \mathbb{C} \to PPT(X)$ is a free action. Then there exists a measurably entire function $F : X \to \mathbb{C}$ such that for μ-almost every $x \in X$ for every $\varepsilon > 0$*

$$\lim_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}} |F(T_z x)|}{\log^{3+\varepsilon} R} = 0. \tag{1}$$

This theorem gives an upper bound for the minimal growth rate of measurably entire functions defined on a general free $\mathbb{C}$-action. Nevertheless, note that there is still a gap between the lower and upper bounds known to us so far:

Question 1.6: Is the gap between the lower bound given by Theorem 1.4 and upper bound given by Theorem 1.5 justified? Namely, does there exists a $\mathbb{C}$-action on a standard probability space $(X, \mathcal{B}, \mu)$ and $p \in (2, 3)$ such that for every measurably entire function $F : X \to \mathbb{C}$ for μ-almost every $x \in X$

$$\lim_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}} |F(T_z x)|}{\log^{p} R} = \infty.$$

1.1. Notation. Given $a > 0$, we denote by S_a the square centered at the origin of edge length $2a$, namely $S_a = [-a, a]^2$.

Let $A \subset \mathbb{C}$ and $\omega \in \mathbb{C}$. We define by $A(\omega) := \omega + A$ the translation of the set A by ω.

For a set $\Omega \subset \mathbb{C}$ we define the sets

$$\Omega^{+\varepsilon} := \{z \in \mathbb{C},\, d(z, \Omega) < \varepsilon\}, \quad \Omega^{-\varepsilon} := \{z \in \mathbb{C},\, d(z, \Omega^c) > \varepsilon\}.$$

Acknowledgments. The author would like to thank her PhD adviser, Mikhail Sodin, for acquainting her with this most intriguing question, conducting many interesting discussions, and reading many revisions of this paper with great patience. The author is grateful to Jon Aaronson for several helpful discussions, and Alon Nishry for insightful editorial remarks. Last but not least, the author would like to thank the referee for a careful review, and for useful comments and corrections.

2. Preliminary lemmas

2.1. Complex analysis lemmas. In this subsection we will state and prove lemmas using tools from complex analysis. Throughout this section we will use the letters λ and μ to denote elements of $\mathbb{C}$ (and not measures). The first lemma, proven in this subsection, is a lemma that creates a non-negative subharmonic function with 'windows', i.e., rectangles where $v = 0$.

Lemma 2.1: *For every $C \geq 1$ and for every set $\Lambda \subset \mathbb{C}$, such that for every $\lambda \neq \mu \in \Lambda$, $\|\mu - \lambda\|_\infty > 2$, there exists a subharmonic function v such that:*

- (P_1) *For every $\lambda \in \Lambda$, define $D_\lambda := \{v = 0\} \cap S_1(\lambda)$; then $D_\lambda^{+\frac{1}{C}} \subset S_1(\lambda)$ while $S_1(\lambda) \setminus D_\lambda$ is a union of at most 20 rectangles of edge length at most 2 and edge width at most $\frac{2}{C}$. In particular,*

$$\frac{m(D_\lambda)}{m(S_1)} \geq 1 - \frac{80}{C}.$$

- (P_2) *For every $z \in \mathbb{C}$, $v(z) \leq \exp(2\pi C)\exp(\frac{\pi C}{2}|z|)$.*
- (P_3) *For every $\lambda \in \Lambda$, $v|_{D_\lambda^{+\frac{5}{3C}} \setminus D_\lambda^{+\frac{1}{3C}}} \geq \frac{1}{2}$.*

Proof. Given $C \geq 1$ we define the subharmonic function

$$b_C(z) = b_C(x+iy) = \begin{cases} \cos(\frac{\pi C}{2} \cdot y)\cosh(\frac{\pi C}{2} \cdot x), & |y| < \frac{1}{C}, \\ 0, & \text{otherwise.} \end{cases}$$

This function is 0 outside an infinite horizontal strip of width $\frac{2}{C}$. Given $\lambda \in \Lambda$ we define the **window function assigned to** λ by

$$v_\lambda(z) := \max\{b_C(iz - \lambda + 1), b_C(z - \lambda + i), b_C(iz - \lambda - 1), b_C(z - \lambda - i)\}.$$

The set $\{z,\ v_\lambda(z) \neq 0\}$ looks like a window, whose cornices have 'infinite tails' (see Figure 1). In addition, note that $v_\lambda|_{S_1^{-\frac{1}{C}}(\lambda)} = 0$, while $v_\lambda|_{S_3(\lambda)} \leq e^{\frac{3\pi C}{2}}$.

We would like to take a maximum over window functions assigned to $\lambda \in \Lambda$. Formally, we would like to define $v(z) = \sup\{v_\lambda(z), \lambda \in \Lambda\}$. The problem is that it is not clear that locally we take a supremum over a finite set, and even if we do, the elements of Λ are not necessarily aligned in the sense that $S_1^{-\frac{1}{C}}(\lambda)$ may intersect 'infinite tails' of many elements $\mu \neq \lambda \in \Lambda$. We get that the 'infinite tails' of windows that were created for different elements of Λ might intrude into the window area of other elements, and the number of 'intruders' is not necessarily bounded and might cover the whole window, making property (P_1)

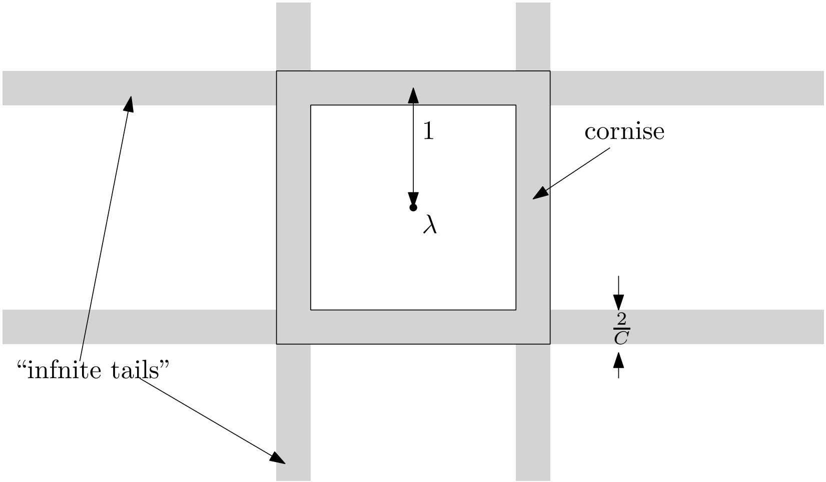

Figure 1. The gray area is where $v_\lambda \neq 0$.

impossible to satisfy. To overcome this problem, we create a grid using the same base function b_C, and then take a maximum over 'window functions' assigned only to elements inside each grid component, bounding the number of possible 'intruders' in each window.

Formally, we define the sets $\mathbb{Z}_{\text{odd}} := \{2n+1,\, n \in \mathbb{Z}\}$, $\mathbb{Z}_{\text{even}} := \{2n,\, n \in \mathbb{Z}\}$ and define the function

$$v_0(z) := e^{2\pi C} \max\{b_C(i\omega + z), b_C(\omega + iz);\ \omega \in \mathbb{Z}_{\text{odd}}\}.$$

For every $\omega \in \mathbb{Z}_{\text{odd}}$ fixed for every $z \in \mathbb{C}$,

$$b_C(i\omega + z) \leq \exp\left(\frac{\pi C}{2} \cdot |Re(z)|\right),$$
$$b_C(\omega + iz) \leq \exp\left(\frac{\pi C}{2} \cdot |Im(z)|\right),$$

independently of ω. We get that v_0 is bounded by

$$\exp\left(\frac{\pi C}{2} \max\{|Re(z)|, |Im(z)|\} + 2\pi C\right).$$

In addition, locally this function is a maximum of at most two subharmonic functions, and therefore it is subharmonic (see Figure 2).

For every $\lambda \in \Lambda$ we define the set

$$A_\lambda := \{\omega \in \mathbb{Z}^2_{\text{even}}, S_1(\lambda) \cap S_1(\omega) \neq \emptyset\};$$

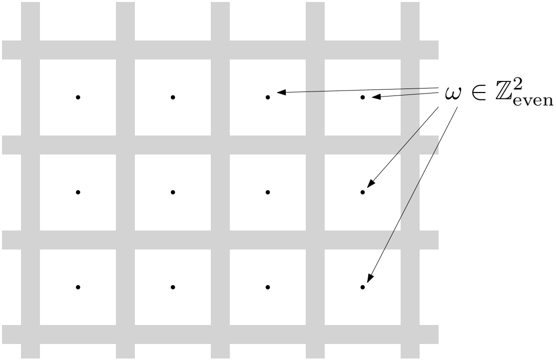

Figure 2. The grid: The gray area is the set where $\{v_0 \neq 0\}$ while the white area is the set where $\{v_0 = 0\}$.

here A_λ is the set of elements $\omega \in \mathbb{Z}^2_{\mathrm{even}}$ such that a square of edge 2 centered at ω intersects a square of edge 2 centered at λ (see Figure 3). For every $\lambda \in \Lambda$, $\#A_\lambda \leq 4$, since the squares are disjoint, aligned, and have the same edge length, and therefore every such intersection creates a rectangle such that at least one of its corners belongs to $S_1(\lambda)$ (see Figure 4).

Symmetrically, for every $\omega \in \mathbb{Z}^2_{\mathrm{even}}$ the set defined by

$$B^\omega := \{\lambda \in \Lambda, S_1(\lambda) \cap S_1(\omega) \neq \emptyset\}$$

also contains at most 4 elements.

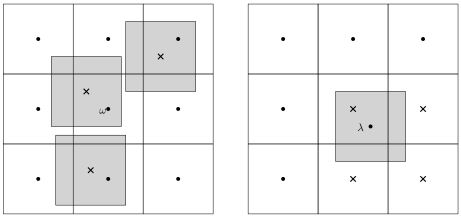

Figure 3. The right-hand picture: the points marked by x represent elements of $\mathbb{Z}^2_{\mathrm{even}}$ which belong to A_λ. The left-hand picture: the points marked by x represent elements of Λ which belong to B^ω.

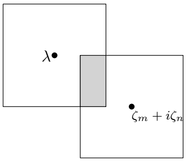

Figure 4. Since the squares are aligned and have the same edge length, every intersection creates a rectangle such that at least one of its corners belongs to $S_1(\lambda)$.

As mentioned before, for every $\lambda\in\Lambda$, v_λ is subharmonic, and $v_\lambda|_{S_1^{-\frac{1}{C}}(\lambda)}=0$, while $v_\lambda|_{S_3(\lambda)}\le e^{\frac{3\pi C}{2}}$. Define

$$v(z):=\max\{v_0(z),\max_{\lambda\in B^\omega}v_\lambda(z)\},\quad \omega\in\mathbb{Z}^2_{\text{even}},\ z\in S_1(\omega).$$

We will first show that this function is well defined and subharmonic. For every $\omega\in\mathbb{Z}^2_{\text{even}}$ and every $z\in S_1^{+\frac{2}{3C}}(\omega)\setminus S_1^{-\frac{2}{3C}}(\omega)$,

$$\begin{aligned}v_0(z)\ge&\cos\left(\frac{\pi C}{2}\cdot\frac{2}{3C}\right)\exp\left(\frac{\pi C}{2}\cdot\min\{|Re(z)|,|Im(z)|\}+2\pi C\right)\\ \ge&\frac{1}{2}\exp(2\pi C).\end{aligned}$$

Since $v_\lambda|_{S_3(\lambda)}\le e^{\frac{3\pi C}{2}}$ and $C\ge1$ we get that, for every $\omega\in\mathbb{Z}^2_{\text{even}}$ and every $\lambda\in B^\omega$,

$$v_0|_{S_1(\omega)\setminus S_1^{-\frac{2}{3C}}(\omega)}\ge\frac{e^{2\pi C}}{2}\ge e^{\frac{3\pi C}{2}}\ge v_\lambda|_{S_3(\lambda)}.$$

In particular, v defined above is well defined and is subharmonic since locally it is a maximum over a finite set of subharmonic functions. Moreover, note that for every $\mu\notin B^\omega$ the function v_μ does not affect the definition of v in $S_1(\omega)$ in any way.

Next, for every $\lambda\in\Lambda$ we look at the set $D_\lambda:=\{z\in\mathbb{C},\ v(z)=0\}\cap S_1(\lambda)$. Note that D_λ is in fact $S_1(\lambda)$ once we remove from it strips of width $\frac{2}{C}$ that originated in the base function b_C. By the way b_C was defined, $b_C(x+iy)\ge\frac{1}{2}$ if $|y|\le\frac{2}{3C}$. We get that if $z\in D_\lambda^{+\frac{5}{3C}}\setminus D_\lambda^{+\frac{1}{3C}}$, then z belongs to a translation and/or rotation of the strip $|y|\le\frac{2}{3C}$, and since v is defined as a maximum of such functions, in particular $v(z)\ge\frac{1}{2}$.

It is left to bound the number of 'intruders' for every $\lambda\in\Lambda$, or formally the number of copies of the set $\{b_C\neq 0\}$ intersecting $S_1(\lambda)$. For this it is enough to bound the number of elements in $\bigcup_{\omega\in A_\lambda} B^\omega\setminus\{\lambda\}$. Why is this enough? As we saw above for every $\mu\in\Lambda$ outside the set $\bigcup_{\omega\in A_\lambda} S_1(\omega)$, the definition of the function $v|_{\bigcup_{\omega\in A_\lambda} S_1(\omega)}$ is unchanged whether or not $\mu\in\Lambda$, and in particular if $A_\mu\cap A_\lambda=\emptyset$, then whether or not $\mu\in\Lambda$ does not change the way v is defined inside $S_1(\lambda)$. We conclude that it is enough to bound the number of elements in the set $\bigcup_{\omega\in A_\lambda} B^\omega\setminus\{\lambda\}$, but the latter is bounded by 16 as the number of elements in A_λ is at most 4 and the number of elements in B^ω is at most 4 as well. Adding the 4 rectangles created by v_λ itself we get 20 'intruding' rectangles as needed. Note that though every intersection of $S_1(\lambda)$ with $S_1(\omega)$ contributes two potentially 'intruding' rectangles, one horizontal and one vertical, only one of them can intersect $S_1(\mu)^{-\frac{1}{C}}$ for $S_1(\mu)$ intersecting $S_1(\omega)$. The reason is that $S_1(\lambda)$ and $S_1(\mu)$ are disjoint, and so one can be positioned either to the left/right with respect to the other (thus intersecting the horizontal rectangle) or above/below (thus intersecting the vertical rectangle), but not both. ∎

An application of this lemma allows us to 'glue' together several subharmonic functions $\{u_\lambda\}$ restricted to disjoint compact subsets of $\mathbb{C}$, $S_1(\lambda)$:

Lemma 2.2: *Let $C>7$, and let $\Lambda\subset\mathbb{C}$ be such that for every $\lambda\neq\mu\in\Lambda$, $\|\mu-\lambda\|_\infty>2$. Assume that for every $\lambda\in\Lambda$ there exists $u_\lambda:\mathbb{C}\to[0,\infty)$ subharmonic such that for a positive constant $\mathcal{M}$*

$$\max_{\lambda\in\Lambda}\max_{z\in S_1} u_\lambda(z)\leq\mathcal{M}.$$

Then there exists a subharmonic function u such that:

- (SH$_1$) *For every $\lambda\in\Lambda$ there exists a set D_λ such that $D_\lambda^{+\frac{1}{C}}\subset S_1(\lambda)$ while $S_1(\lambda)\setminus D_\lambda$ is contained in a union of at most 20 rectangles of edge length at most 2 and edge width at most $\frac{2}{C}$, and for every $z\in D_\lambda$ we have $u(z)=u_\lambda(z-\lambda)$.*
- (SH$_2$) $\max_{z\in S_C} u(z)\leq 2\mathcal{M}e^{\pi C^2}$.
- (SH$_3$) *For every $\lambda\in\Lambda$*

$$\min_{z\in D_\lambda^{+\frac{5}{3C}}\setminus D_\lambda^{+\frac{1}{3C}}} u(z)\geq\mathcal{M}.$$

Proof. Let v denote the subharmonic function obtained by Lemma 2.1 with the set Λ and the constant C. Define for every $\lambda\in\Lambda$ the set $D_\lambda:=\{v=0\}\cap S_1(\lambda)$.

Following property (P_1) of the function v guaranteed by Lemma 2.1, this set satisfies all the properties described in (SH_1). Define the function

$$u(z) = \begin{cases} \max\{2\mathcal{M}\cdot v(z), u_\lambda(z-\lambda)\}, & , z\in D_\lambda^{+\frac{1}{3C}},\ \lambda\in\Lambda,\\ 2\mathcal{M}\cdot v(z), & \text{otherwise,}\end{cases}$$

We will first show that u is subharmonic. Fix $\lambda\in\Lambda$. Following property (P_3) of the function v, for every $z\in D_\lambda^{+\frac{5}{3C}}\setminus D_\lambda^{+\frac{1}{3C}}$ we have $v(z)\geq\frac{1}{2}$, while $\max_{z\in S_1}u_\lambda(z)\leq\mathcal{M}$. And so

$$\min_{z\in D_\lambda^{+\frac{5}{3C}}\setminus D_\lambda^{+\frac{1}{3C}}} 2\mathcal{M}\cdot v(z)\geq\frac{2\mathcal{M}}{2}=\mathcal{M}\geq\max_{z\in S_1}u_\lambda,$$

which implies that u defined above is well defined and subharmonic as locally it is a maximum between two subharmonic functions. This also proves property (SH_3).

To see that property (SH_1) holds, note that since $u_\lambda\geq 0$, for every $z\in D_\lambda$ we have $u_\lambda(z)=u(z-\lambda)$ as needed.

To see that property (SH_2) holds we observe that for $C>7$,

$$\max_{z\in S_C}u=2\mathcal{M}\cdot\max_{z\in S_C}v\leq 2\mathcal{M}\cdot\exp(2\pi C)\exp\left(\frac{\pi C}{2}\cdot\max_{z\in S_C}|z|\right)\leq 2\mathcal{M}e^{\pi C^2},$$

concluding our proof. ∎

The next lemma is an extension of the previous lemma for 'glueing' several entire functions.

Lemma 2.3: *Let C and B be sufficiently large constants and let $\Lambda\subset S_C$ be such that for every $\lambda\neq\mu\in\Lambda$, $\|\mu-\lambda\|_\infty>2$. Assume that for every $\lambda\in\Lambda$ there exists f_λ analytic in S_1 such that for some $\mathcal{M}>40\log C$,*

$$\max_{\lambda\in\Lambda}\max_{z\in S_1}|f_\lambda(z)|\leq\exp(2^{1-B}\mathcal{M}).$$

Then there exists an entire function f with the following properties:

(E_1) *For every $\lambda\in\Lambda$ define the set*

$$A_\lambda=S_1(\lambda)\cap\left\{z,\ |f(z)-f_\lambda(z-\lambda)|<\exp\left(-\frac{\mathcal{M}}{4}\right)\right\}.$$

Then for every $\varepsilon>0$, $m(S_1(\lambda)\setminus A_\lambda^{-\varepsilon})=O(\frac{1}{C}+\varepsilon)$.[1]

(E_2) $\max_{z\in S_C}|f(z)|\leq\exp(2^{1-B}\mathcal{M}\cdot e^{\pi C^2})$.

[1] In fact, $O(\frac{1}{C}+\varepsilon)=\frac{160}{C}+200\varepsilon$.

Proof. Let u be the subharmonic function constructed in Lemma 2.2 with the set Λ, the constant C, and the functions $u_\lambda = \log_+ |f_\lambda|$. Recall the sets $D_\lambda \subset S_1(\lambda)$ defined for every $\lambda \in \Lambda$ in Lemma 2.2. These sets were defined so that for every $z \in D_\lambda$, $u(z) = u_\lambda(z-\lambda)$, and $S_1(\lambda) \setminus D_\lambda$ is a union of at most 20 rectangles of edge length at most 2 and edge width at most $\frac{2}{C}$. Let $\chi : \mathbb{C} \to [0,1]$ be a smooth function with the following properties:

(a) For every $\lambda \in \Lambda$, $\chi|_{D_\lambda^{+\frac{1}{4C}}} = 1$.
(b) $\chi|_{\mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} D_\lambda^{+\frac{3}{4C}}} = 0$.
(c) For every $z \in \mathbb{C}$, $|\nabla \chi(z)| \le 100C$.

For example, we take a convolution of the normalization of a rescaling of the bump function

$$\phi(z) = \begin{cases} \exp(-\frac{1}{1-|z|^2}), & |z| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

by $\frac{1}{4C}$ so that its integral is one, with the function

$$\psi(z) = \sum_{\lambda \in \Lambda} \mathbf{1}_{D_\lambda^{+\frac{1}{2C}}(\lambda)}(z).$$

Define the function $g_0 : \bigcup_{\lambda \in \Lambda} S_1(\lambda) \to \mathbb{C}$ by

$$g_0(z) := \sum_{\lambda \in \Lambda} f_\lambda(z-\lambda) \cdot \mathbf{1}_{S_1(\lambda)}(z).$$

As $D_\lambda^{+\frac{1}{C}} \subset S_1(\lambda)$ and the collection $\{S_1(\lambda)\}_{\lambda \in \Lambda}$ is a collection of disjoint squares, g_0 is holomorphic where it is defined, as locally it is just f_λ for one particular $\lambda \in \Lambda$. Define

$$g(z) = g_0(z) \cdot \chi(z).$$

Note that g is well defined as the area where $\chi = 0$ separates $S_1(\lambda)$ from $S_1(\mu)$, for $\lambda \neq \mu$ (see Figure 5). Next we define the entire function

$$f(z) = g(z) - \alpha(z),$$

where α is Hörmander's solution [3, Theorem 4.2.1] to the $\bar{\partial}$-equation

$$\bar{\partial} g(z) = \bar{\partial}\chi(z) \cdot g_0(z) = \bar{\partial}\alpha(z),$$

satisfying

$$\int_{\mathbb{C}} |\alpha(z)|^2 \frac{e^{-u(z)}}{(|z|^2+1)^2} dz \le \frac{1}{2} \int_{\mathbb{C}} |\bar{\partial} g(z)|^2 e^{-u(z)} dz.$$

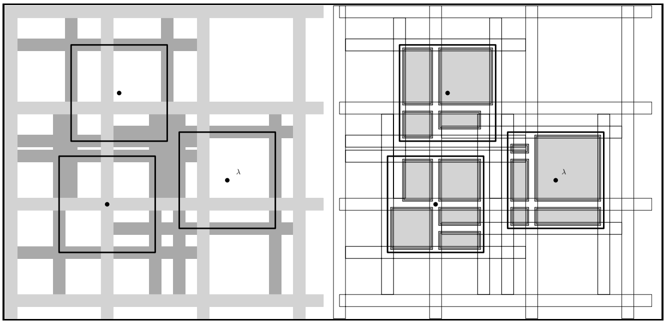

Figure 5. On the picture to the left, the white area is the area where $\{u = 0\}$. The corridors created by the grid are colored in light gray, while the ones created by elements of Λ are colored in dark gray. For every element of Λ in the picture, the sets D_λ are the white areas within the relevant square.
The picture to the right is the same picture, but the light gray area now describes the set where $\chi \equiv 1$, the white area describes the set where $\chi \equiv 0$, and the dark gray area describes the set where the transition occurs.

First of all, let us bound the right hand side of this inequality: by the definition of χ and property (SH$_3$) of u,

$$\begin{aligned}
&\int_{\mathbb{C}} |\bar\partial g(z)|^2 e^{-u(z)} dz \\
&\qquad = \sum_{\lambda\in\Lambda} \int_{D_\lambda^{+\frac{3}{4C}} \setminus D_\lambda^{+\frac{1}{4C}}} |\bar\partial g(z)|^2 e^{-u(z)} dz \\
&\qquad \le \max_{z\in\mathbb{C}} |\nabla\chi(z)|^2 \exp(2^{2-B}\mathcal{M}) \cdot e^{-\mathcal{M}} \cdot m\left(\bigcup_{\lambda\in\Lambda} D_\lambda^{+\frac{3}{4C}} \setminus D_\lambda^{+\frac{1}{4C}}\right) \qquad (2)\\
&\qquad \le 10^4 C^2 \exp(\mathcal{M}(2^{2-B}-1)) \cdot 4C^2 \cdot \frac{40\cdot 2}{2C} \le C^4 \exp\left(-\frac{\mathcal{M}}{2}\right) \\
&\qquad\quad \Rightarrow \int_{\mathbb{C}} |\bar\partial g(z)|^2 e^{-u(z)} dz \le C^4 \exp\left(-\frac{\mathcal{M}}{2}\right),
\end{aligned}$$

provided that B, C, and $\mathcal{M}$ are large enough.

Next, let us find an upper bound for f on S_C: Fix $c_0 = \frac{1}{16C} < \frac{1}{4C}$, then by Cauchy's integral formula

$$\begin{aligned}|f(z)|^2 \leq & \frac{1}{\pi c_0^2}\int_{B(z,c_0)} |f(w)|^2 dm(w)\\ \leq & \frac{2}{\pi c_0^2}\int_{B(z,c_0)} |g(z)|^2 + |\alpha(w)|^2 dm(w) = I_1 + I_2.\end{aligned}$$

To bound I_1 note that by the way g is defined

$$I_1 \leq 2\max_{S_C}|g|^2 \leq 2\max_{\lambda\in\Lambda}\max_{S_1}|f_\lambda|^2 \leq 2\exp(2^{2-B}\mathcal{M}) \leq \frac{1}{2}\exp(2^{2-B}\mathcal{M}\cdot e^{\pi C^2}).$$

On the other hand, using (2) and property (SH$_2$) of u in Lemma 2.2,

$$\begin{aligned}I_2 =& \frac{2}{\pi c_0^2}\int_{B(z,c_0)} |\alpha(w)|^2 dm(w)\\ \leq & \frac{4}{\pi c_0^2}\exp(\max_{z\in S_C} u) C^4 \int_{\mathbb{C}} |\alpha(w)|^2 \frac{e^{-u(w)}}{(|w|^2+1)^2} dm(w)\\ \leq & \frac{64C^2}{\pi}\exp(2^{2-B}\mathcal{M}e^{\pi C^2})C^8 \cdot \exp\left(-\frac{\mathcal{M}}{2}\right)\\ \leq & C^{11}\exp(2^{2-B}\mathcal{M}\cdot e^{\pi C^2})\exp\left(-\frac{1}{2}\mathcal{M}\right)\\ \leq & \frac{1}{2}\exp(2^{2-B}\mathcal{M}\cdot e^{\pi C^2}),\end{aligned}$$

provided that $\mathcal{M}$ is big enough so that $e^{-\frac{\mathcal{M}}{2}}\cdot C^{11} < \frac{1}{2}$. Combining the two estimates we get that

$$|f(z)| \leq \sqrt{I_1 + I_2} < \exp(2^{1-B}\mathcal{M}e^{\pi C^2}).$$

We conclude that property (E$_2$) holds.

Finally, to see property (E$_1$), note that:

(a) For every $z \in D_\lambda^{-\frac{1}{4C}}$, $B(z,c_0) \subset D_\lambda$, which implies that $T_\lambda u(z) = u_\lambda(z)$ by property (SH$_1$) of u, and therefore $\max_{w\in B(z,c_0)} e^u \leq \exp(2^{1-B}\mathcal{M})$.

(b) By the way f was defined, for every $w \in D_\lambda^{+\frac{1}{4C}}$ we have

$$f(w) = f_\lambda(w-\lambda) - \alpha(w).$$

By Cauchy's integral formula applied to $z \in D_\lambda^{-\frac{1}{4C}}$, and the function

$$(f(z) - T_{-\lambda}f_\lambda(z))$$

which is holomorphic in $B(z,c_0)$, and by using the bound given by (2)

$$\begin{aligned}|f(z)-T_{-\lambda}f_\lambda(z)|^2 =& \left|\frac{1}{\pi c_0^2}\int_{B(z,c_0)}(f(w)-T_{-\lambda}f_\lambda(w))dm(w)\right|^2\\ \overset{(b)}{=}& \left|\frac{1}{\pi c_0^2}\int_{B(z,c_0)}\alpha(w)dm(w)\right|^2\\ \leq& \frac{1}{\pi c_0^2}\int_{B(z,c_0)}|\alpha(w)|^2dm(w)\\ \leq& C^7\max_{w\in\mathbb{D}(z,c_0)}e^u\cdot\int_{\mathbb{C}}|\alpha(z)|^2\frac{e^{-u(z)}}{(|z|^2+1)^2}dm(z)\\ \overset{(a)}{\leq}& C^7\cdot\exp(2^{1-B}\mathcal{M})\cdot C^4\exp\left(-\frac{\mathcal{M}}{2}\right)\\ =& C^{11}\cdot\exp\left(\mathcal{M}\left(2^{1-B}-\frac{1}{2}\right)\right)\\ \leq& \exp\left(-\frac{\mathcal{M}}{4}\right)\end{aligned}$$

for B, C, and $\mathcal{M}$ large enough. We obtain that

$$D_\lambda^{-\frac{1}{4C}}\subset A_\lambda\Rightarrow D_\lambda^{-\frac{1}{4C}-\varepsilon}\subset A_\lambda^{-\varepsilon},$$

but since $S_1(\lambda)\setminus D_\lambda$ is a union of at most 20 rectangles, we get by the inclusion of the sets that

$$m(S_1(\lambda)\setminus A_\lambda^{-\varepsilon})\leq m(S_1(\lambda)\setminus D_\lambda^{-\frac{1}{4C}-\varepsilon})\leq 40\cdot\left(5\varepsilon+\frac{4}{C}\right)=O\left(\varepsilon+\frac{1}{C}\right),$$

concluding our proof. ∎

2.2. Measure theoretic lemmas. In this subsection we will present lemmas related to ergodic theory and dynamics.

Definition 2.4: Let $(X,\mathcal{B},\mu)$ be a standard probability space, and suppose $T:\mathbb{C}\to PPT(X)$ is a free $\mathbb{C}$-action. Let $S\subset\mathbb{C}$ be a compact set. A set $B\in\mathcal{B}$ is called an S**-set** if:

- (F_1) For every $z\neq w\in S$, $T_zB\cap T_wB=\emptyset$.
- (F_2) For every $B'\subset B\subset X$ measurable, and every $A\subset S\subset\mathbb{C}$ measurable, the set $AB':=\bigcup_{z\in A}T_zB'$ is a measurable subset of X.

In the definition above, the set S is the set of 'shifts' by the action of $\mathbb{C}$, marked T, while $B\subset X$ is a very small set that we 'shift' by elements of S in the space X. For our purpose, S will be a two dimensional square.

We are interested in S-sets, for $S \subset \mathbb{C}$ a square, because these sets allow us to assign for every $x \in B$ a function $f_x : S \to \mathbb{C}$, which is holomorphic in S, creating a measurably entire function $f : SB \to \mathbb{C}$ defined by $f(T_z x) = f(z,x) = f_x(z)$ without worrying about inconsistencies in the definition of f. Note that as B is an S-set, the map $T_z x \mapsto (z,x)$ is well defined, and so is our function f. We would therefore like to approximate our space X by a sequence of sets $\{S_{a_n}B_n\}_{n=1}^{\infty}$ where B_n is an S_{a_n}-set, and $a_n \nearrow \infty$.

Remark 2.5: For every square $S \subset \mathbb{C}$ and every $B \in \mathcal{B}$, an S-set

$$\mu(AB) := \mu(\{T_z B,\ z \in A\}) = \frac{m(A)}{m(S)} \cdot \mu(SB), \quad \text{for every } A \subset S \text{ measurable.}$$

Explanation: Because the measure μ is a translation invariant measure, we may assume without loss of generality that $S = [-a,a]^2$ for some $a > 0$. Every such cube $S \subset \mathbb{C}$ is also a topological group, with the group action defined by

$$\tau_w z = (w+z) \mod a.$$

This group is a Polish group (i.e., it is a separable completely metrizable topological space), and by Haar's theorem there exists a unique measure (up to multiplication by constants) which is invariant under the group's action. In this case, this is just Lebesgue's measure restricted to S. For every S-set, B, define the measure ν_B on measurable subsets of S by $\nu_B(A) := \mu(AB)$. Since B is an S-set, this is indeed a well defined measure. In addition, since μ is translation invariant, we get that this measure is invariant under the group's action. We conclude that $\nu_B = c_B \cdot m_S$, where m_S denotes the two dimensional Lebesgue measure restricted to S, and c_B is some constant. For a crude bound on c_B note that

$$c_B = \frac{\nu_B(S)}{m_S(S)} = \frac{\mu(SB)}{m(S)} \leq \frac{1}{m(S)}.$$

Lemma 2.6 (The Nested Towers Lemma): *Let $(X, \mathcal{B}, \mu)$ be a standard probability space, and suppose $T : \mathbb{C} \to PPT(X)$ is a free $\mathbb{C}$-action. Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} < \frac{1}{2}$. Then there exists a sequence of sets $\{B_n\} \subset \mathcal{B}$ such that:*

(N_1) *B_n is an S_{a_n}-set.*

(N_2) *$S_{a_n}B_n \subset S_{a_{n+1}}B_{n+1}$.*

(N_3) *$\mu(S_{a_n}B_n) \nearrow 1$ as $n \to \infty$.*

This lemma, originally proven by Weiss in [5], is a natural extension of Rokhlin's lemma about approximating actions of $\mathbb{Z}$ by Rokhlin towers. Rokhlin's lemma for $\mathbb{C}$-action states that for every rectangle $R \subset \mathbb{C}$ and for every $\varepsilon > 0$ there exists a set B which is an R-set such that $\mu(RB) > 1 - \varepsilon$. This lemma is not enough as we would like the approximation of the space X to be monotone.

The version of Rokhlin's lemma for $\mathbb{C}$-actions was proven by Lind in [4]. We note that Weiss' definition for an S-set admits a weaker property than property (F_2). Nevertheless, his proof of The Nested Towers Lemma extends to our definition of an S-set. For the reader's convenience the proof of this lemma can be found in the appendix.

Given a metric space (Y, d) we let $\mathcal{K}(Y)$ denote the set of all compact subsets of Y. For every $A, B \in \mathcal{K}(Y)$ define the metric

$$d_H(A, B) := \inf\{\varepsilon > 0, A \subset B^{+\varepsilon} \text{ and } B \subset A^{+\varepsilon}\};$$

d_H is called the **Hausdorff distance induced** by (Y, d).

We say that $(\mathcal{K}(Y), d_H)$ is the **induced Hausdorff space** of (Y, d).

Let $(X, \mathcal{B}, \mu)$ be a standard probability space, and let $\{B_n\}$ be a sequence of sets given by The Nested Towers Lemma, Lemma 2.6. Assume that $\mathcal{P}_{n-1}$ is a finite partition of B_{n-1} into measurable sets $\{B_{n-1}^j\}_{j=1}^{k_{n-1}}$. For every $x \in B_n$ and $1 \le \ell \le k_{n-1}$ define the set

$$R_n^\ell(x) := \{z \in S_{a_n};\ T_z x \in B_{n-1}^\ell\}.$$

Since B_{n-1} is an $S_{a_{n-1}}$-set, for every $z \neq w \in R_n^\ell(x)$ we have that

$$S_{a_{n-1}} T_z x \cap S_{a_{n-1}} T_w x = \emptyset$$

which implies $\|z - w\|_\infty > 2a_{n-1}$. In particular, $R_n^\ell(x)$ is a finite set and therefore compact.

Given $\delta > 0$, we say a partition $\mathcal{P}_n = \{B_n^j\}_{j=1}^{k_n}$ is a **δ-fine partition consistent** with $\mathcal{P}_{n-1}$ if it is a finite measurable partition of B_n, and for every $1 \le j \le k_n$ for every $x, y \in B_n^j$, $d_H(R_n^\ell(x), R_n^\ell(y)) < \delta$ for every $1 \le \ell \le k_{n-1}$.

Lemma 2.7: *Let $\{B_n\}$ be a sequence of sets given by The Nested Towers Lemma. For every sequence of positive numbers $\{\delta_n\}$ there exists a sequence of partitions $\{\mathcal{P}_n\}$ such that for every n, $\mathcal{P}_n$ is a δ_n-fine partition consistent with $\mathcal{P}_{n-1}$.*

Proof. We will prove it by induction on n, where for the base step one could take any partition $\mathcal{P}_1$. Assume $\mathcal{P}_{n-1}$ was already defined and for every $1 \le \ell \le k_{n-1}$ define the function $f_\ell : B_n \times B_n \to \mathbb{R}_+$ by

$$f_\ell(x,y) = d_H(R_n^\ell(x), R_n^\ell(y)).$$

For every $x \in B_n$ let

$$D_n^\ell(x) = \Big\{ y \in B_n,\ f_\ell(x,y) < \frac{\delta_n}{2} \Big\}.$$

As f_ℓ is a measurable function, these sets are measurable and form a cover for B_n. We will first show there exists a finite sub-cover of $\{D_n^\ell(x)\}_{x\in B_n}$ for B_n. If no such sub-cover exists, then there exists a subsequence $\{x_m\}$ such that for every $k \neq m$ we have

$$f_\ell(x_m, x_k) \ge \frac{\delta_n}{2}$$

creating an infinite separated set in $\mathcal{K}(Y)$, which is a contradiction to the fact that the induced Hausdorff space of a totally bounded set is itself totally bounded (see Claim 2.8).

Let Q_ℓ denote the finite set of elements $x \in B_n$ forming the finite cover of B_n, and let $\mathcal{Q}_\ell$ denote the partition that we obtain by the collection $\{D_n^\ell(x)\}_{x\in Q_\ell}$. Note that for every $A \in \mathcal{Q}_\ell$ there exists $\xi \in Q_\ell$ such that $A \subseteq D_n^\ell(\xi)$, and therefore for every $x, y \in A$,

$$f_\ell(x,y) \le f_\ell(x,\xi) + f_\ell(\xi,y) < \delta_n.$$

We define $\mathcal{P}_n$ to be the refinement of all the partitions $\mathcal{Q}_\ell$,

$$\mathcal{P}_n = \Big\{ \bigcap_{\ell=1}^{k_{n-1}} A_\ell,\ A_\ell \in \mathcal{Q}_\ell \Big\}.$$

Let $A \in \mathcal{P}_n$, then for every $x, y \in A$ for every $1 \le \ell \le k_{n-1}$ there exists $B \in \mathcal{Q}_\ell$ such that $x, y \in B$, and therefore $f_\ell(x,y) < \delta_n$, which implies that $\mathcal{P}_n$ is a δ_n-fine partition consistent with $\mathcal{P}_{n-1}$, concluding our proof. ∎

CLAIM 2.8: *Let (Y,d) be a metric space. If Y is totally bounded, then the induced Hausdorff space, $(\mathcal{K}(Y), d_H)$, is totally bounded.*

For the reader's convenience we add a proof of this fact:

Proof. Let $\varepsilon > 0$. Since Y is totally bounded there exists a finite sub-cover $\{B_j\}_{j=1}^N$, such that for every j, $\operatorname{diam}(B_j) < \varepsilon$. For every $A \in \mathcal{K}(Y)$ we define the sequence

$$x_j(A) = \begin{cases} 1, & A \cap B_j \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Note that since the sequences' elements are only $\{0,1\}$ and their length is finite, then the number of possible sequences is finite. Next, let A and B be sets such that they have the same sequence. Fix $a \in A$; then there exists j such that $a \in B_j$ as $\{B_j\}$ is a cover for Y. Since the sequences of A and B are the same, $1 = x_j(A) = x_j(B)$ and so there exists $b \in B \cap B_j$, and in particular, as $\operatorname{diam}(B_j) < \varepsilon$, we get that $d(a,b) < \varepsilon$. This shows that $A \subset B^{+\varepsilon}$. By a symmetric argument, $B \subset A^{+\varepsilon}$ as well. We conclude that $d_H(A,B) \leq \varepsilon$.

For every $1 \leq j \leq N$ we arbitrarily choose an element $b_j \in B_j$, and define for every sequence $\{x_j\}$ the set

$$D_{\{x_j\}} := \{b_j; x_j = 1\}.$$

The collection $\{D_{\{x_j\}}\}$ forms a finite ε-net for the space $(\mathcal{K}(Y), d_H)$, concluding the proof. ∎

3. Construction of a special sequence

In this section we will use all the lemmas proven in the previous sections to construct a special sequence of functions that will be used in the proof of Theorem 1.5. Beyond this section the only thing one needs to keep in mind is the following lemma:

LEMMA 3.1: *Let $(X, \mathcal{B}, \mu)$ be a standard probability space, and suppose $T : \mathbb{C} \to PPT(X)$ is a free $\mathbb{C}$-action. Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, and $a_n = D \cdot n \log^2 n \cdot a_{n-1}$ for every $n \geq 2$ where $D > 0$ is some parameter sufficiently large. There exists a sequence of measurable sets $\{X_n\}$, $X_n \nearrow X$, and a sequence of measurable functions $F_n : X_n \to \mathbb{C}$ with the following properties:*

(i) $\mu\Big(\Big\{x \in X_1,\ |F_1(x)| \leq \frac{1}{4}\Big\}\Big) \geq \frac{1}{25}$ *and*

$\mu\Big(\Big\{x \in X_1,\ |F_1(x)| \geq \frac{3}{4}\Big\}\Big) \geq \frac{1}{25}$.

(ii) *There exists a sequence of measurable sets $\{G_n\}$ such that $G_n \subset X_n$ and:*

(A) $\mu(X_n \setminus G_n) \leq \mu(X_n \setminus X_{n-1}) + \frac{1}{D} \cdot O\Big(\frac{1}{n \log^2 n}\Big)$.

(B) *For every $x \in G_n$:*

(B_1) $S_{a_{n-2}}x \subseteq X_{n-1}$, *implying that* $S_{a_{n-2}}G_n \subseteq X_{n-1} \subseteq X_n$.

(B_2) *The function $F_n^x : \mathbb{C} \to \mathbb{C}$ defined by $F_n^x(z) = F_n(T_z x)$ is holomorphic in $S_{a_{n-2}}$.*

(B_3) $\max_{z \in S_{a_{n-2}}} |F_n(T_z x) - F_{n-1}(T_z x)| < 10^{-2n}$.

(B_4) *For every $1 \leq m \leq n-2$*

$$\max_{z \in S_{a_m}} |F_n(T_z x)| \leq 2 \exp(2^{1-B} \mathcal{M}_B(m+1)),$$

where B is a numerical constant sufficiently large, and

$$\mathcal{M}_B(m) = \exp\left(B \cdot m + \pi D^2 \sum_{j=2}^{m-1} j^2 \log^4 j\right).$$

Proof. Let $\{B_n\}$ be a sequence of sets obtained for the sequence $\{a_n\}$ by Weiss' Nested Towers Lemma, Lemma 2.6, such that $\mu(X \setminus S_{a_1}B_1) < \frac{1}{200}$.

We will set

$$X_n := S_{a_n}B_n,$$

and define the sequence of functions F_n as a linear combination of step functions,

$$F_n(T_z x) = F_n(z,x) = \sum_{j=1}^{k_n} F_n^j(z) \cdot \mathbf{1}_{B_n^j}(x),$$

where $\{F_n^j\}_{j=1}^{k_n}$ are entire, and $\{B_n^j\}_{j=1}^{k_{n-1}}$ is a measurable partition of B_n, denoted $\mathcal{P}_n$. Note that F_n is well defined, since B_n is an S_{a_n}-set and therefore the mapping $T_z x \mapsto (z,x)$ is well defined, as mentioned in Section 2.2.

Formally, we will construct this sequence inductively. Define $F_1 : X_1 \to \mathbb{C}$ by

$$F_1(T_z x) = F_1(x,z) = z;$$

F_1 is measurable, since it is constant with respect to one variable, and continuous with respect to the other. By the way F_1 is defined for every $x \in B_1$ and $|z| < \frac{1}{4}$

$$|F_1(T_z x)| = |z| < \frac{1}{4},$$

and so $\{F_1 \leq \frac{1}{4}\} \supset \frac{1}{4}\mathbb{D}B_1$. Following Remark 2.5:

$$\mu\Big(\Big\{x \in X_1,\ |F_1(x)| \leq \frac{1}{4}\Big\}\Big) \geq \mu\Big(\frac{1}{4}\mathbb{D}B_1\Big) = m\Big(\frac{1}{4}\mathbb{D}\Big) \cdot \frac{\mu(S_{a_1}B_1)}{m(S_{a_1})}$$
$$\overset{\text{as}}{\underset{a_1=1}{=}} \frac{\pi}{4^3} \cdot \mu(S_{a_1}B_1) > \frac{\pi}{4^3} \cdot \frac{199}{200} > \frac{1}{25}.$$

A similar computation shows that $\mu(\{|F_1| \geq \frac{3}{4}\}) > \frac{1}{25}$ as well, and so property (i) holds.

Assume that $F_{n-1} : X_{n-1} \to \mathbb{C}$ was defined as

$$F_{n-1}(T_z x) = F_{n-1}(x, z) = \sum_{j=1}^{k_{n-1}} F_{n-1}^j(z)\mathbf{1}_{B_{n-1}^j}(x),$$

and that property (ii) holds for F_{n-1}. We assume in addition that instead of property (B_4) we have property (B_4'): for the same parameter B

$$\max_{z \in S_{a_m}} |F_n(T_z x)| \leq \exp(2^{1-B}\mathcal{M}_B(m+1)) + \sum_{j=1}^{n} 10^{-2j}.$$

Naturally, property (B_4') implies property (B_4). Moreover, we assume that for every $1 \leq j \leq k_{n-1}$

$$\max_{z \in S_{a_{n-1}}} |F_{n-1}^j(z)| \leq \exp(2^{1-B}\mathcal{M}_B(n)).$$

We refer to this property as property (B_5), and regard it as part of property (ii′), which is property (ii) where (B_4) is replaced by (B_4') and (B_5) is added.

Since F_{n-1}^j is entire for every j fixed, it is uniformly continuous on $S_{a_{n-1}}^{+1}$, and therefore there exists $\delta_n \in (0,1)$ such that for every $1 \leq j \leq k_{n-1}$:

$$\sup_{\substack{z,w \in S_{a_{n-1}}^{+1} \\ |z-w| < \delta_n}} |F_{n-1}^j(z) - F_{n-1}^j(w)| < \frac{10^{-2n}}{2}.$$

Let $\mathcal{P}_n$ be a partition of B_n which is δ_n-fine and consistent with $\mathcal{P}_{n-1} = \{B_{n-1}^\ell\}_{\ell=1}^{k_{n-1}}$, the partition of B_{n-1} used to define F_{n-1}. Such a partition exists by Lemma 2.7.

For every j we use the axiom of choice to choose a representative $x_n^j \in B_n^j$. We will define the function F_n^j by using Lemma 2.3 with the following parameters:

$$\Lambda_n^j = \Big\{\frac{\lambda}{a_{n-1}},\ \lambda \in S_{a_n} \text{ so that } T_\lambda x_n^j \in B_{n-1}\Big\}, \quad C = \frac{a_n}{a_{n-1}} = D \cdot n\log^2 n,$$
$$\mathcal{M} := \exp(2^{1-B}\mathcal{M}_B(n)), \quad f_\lambda(z) := F_{n-1}(T_{a_{n-1}(\lambda+z)}\, x_n^j) : S_1 \to \mathbb{C}.$$

Let us verify that these parameters satisfy the requirements of the lemma. First of all, because B_{n-1} is an $S_{a_{n-1}}$ set, then for every $\lambda \neq \mu \in \Lambda_n^j$ we have that

$$S_{a_{n-1}}(a_{n-1} \cdot \lambda) \cap S_{a_{n-1}}(a_{n-1} \cdot \mu) = \emptyset \iff S_1(\lambda) \cap S_1(\mu) = \emptyset.$$

In particular, for every $\lambda \neq \mu \in \Lambda_n^j$, we have $\|\lambda - \mu\|_\infty > 2$.

Next, for every $n \geq 1$ for every D large enough

$$\begin{aligned}\mathcal{M} &= \exp(2^{1-B}\mathcal{M}_B(n)) = \exp\left(2^{1-B}\exp\left(B \cdot n + D^2\pi \sum_{j=2}^{n-1} j^2 \log^4 j\right)\right)\\ &\geq \exp(2^{1-B}\exp(B \cdot n + D^2\pi \cdot n \log^2 n)).\end{aligned}$$

In fact a more accurate lower bound is

$$\exp(2^{1-B}\exp(B \cdot n + D^2\pi \cdot n^2 \log^4 n)),$$

but the bound indicated above is enough for our use. In particular, for every constant B there exists D large enough so that

$$\mathcal{M} \geq 40 \log(D \cdot n \log^2 n).$$

We conclude that all the requirements of Lemma 2.3 are satisfied.

Define the function

$$F_n(T_z x) = F_n(z, x) = \sum_{j=1}^{k_n} F_n^j(z) \cdot \mathbf{1}_{B_n^j}(x). \tag{3}$$

Note that every summand in the sum is a measurable function as it is an indicator function of the measurable set B_n^j in one variable and the continuous function F_n^j in the other. This implies that F_n is a measurable function since the number of sets in the partition (and therefore the number of summands in the sum) is finite.

To conclude the proof it remains to show that property (ii') holds for F_n as well.

Fix $1 \leq j \leq n_{k-1}$ and $\lambda \in \Lambda_n^j$, and recall the definition of the sets A_λ in Lemma 2.3:

$$A_\lambda = S_1(\lambda) \cap \left\{z,\ |f(z) - f_\lambda(z - \lambda)| < \exp\left(-\frac{\mathcal{M}}{4}\right)\right\},$$

where for us $f = F_n^j$ for some $1 \le j \le n_{k-1}$. Let $D_\lambda = a_{n-1} \cdot A_\lambda$, and define the set

$$G_n := \bigcup_{j=1}^{k_{n-1}} \Big(\bigcup_{\lambda \in \Lambda_n^j} D_\lambda^{-1-a_{n-2}} \Big) B_n^j \subseteq X_n.$$

We will first show that property (A) holds. By the way the partition $\mathcal{P}_n$ was defined,

$$\bigcup_{j=1}^{k_n} \Big(\bigcup_{\lambda \in \Lambda_n^j} S_{a_{n-1}}^{-\delta_n}(a_{n-1} \cdot \lambda) \Big) B_n^j \subseteq X_{n-1} \subseteq \bigcup_{j=1}^{k_n} \Big(\bigcup_{\lambda \in \Lambda_n^j} S_{a_{n-1}}^{+\delta_n}(a_{n-1} \cdot \lambda) \Big) B_n^j \tag{4}$$

(see Figure 6).

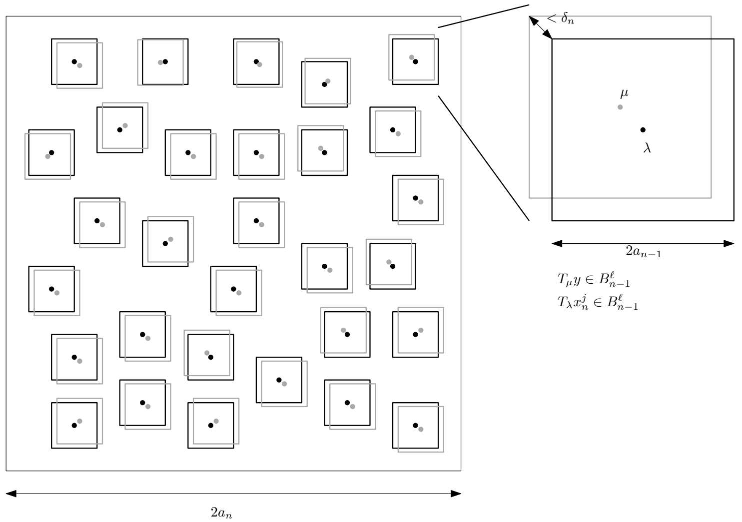

Figure 6. Fix n and $1 \le j \le k_n$. The black points represent the set Λ_n^j. For every $y \in B_n^j$, $y \ne x_n^j$, the gray configuration of squares represents the case where y was chosen to be the representative of B_n^j instead of x_n^j. Thus, the gray configuration of squares is a distortion of the black configuration of squares by at most δ_n, by the way the partition $\mathcal{P}_n$ was defined.

We have

$$\mu(X_n \setminus G_n) \leq \mu\Big(X_n \setminus \bigcup_{j=1}^{k_n}\Big(\bigcup_{\lambda\in\Lambda_n^j} S_{a_{n-1}}(a_{n-1}\cdot\lambda)\Big)B_n^j\Big) + \mu\Big(\bigcup_{j=1}^{k_n}\Big(\bigcup_{\lambda\in\Lambda_n^j}(S_{a_{n-1}}(a_{n-1}\cdot\lambda)\setminus D_\lambda^{-1-a_{n-2}})\Big)B_n^j\Big).$$

We will use Remark 2.5 to bound each of these terms: Remember that following Lemma 2.3,

$$m(S_1(\lambda)\setminus A_\lambda^{-\varepsilon}) = O\Big(\frac{1}{C}+\varepsilon\Big).$$

We obtain that for every $\lambda\in\Lambda_n^j$,

$$m(S_{a_{n-1}}\setminus(a_{n-1}\cdot A_\lambda)^{-1-a_{n-2}}) \leq O(1)\cdot a_{n-1}^2\Big(\frac{a_{n-1}}{a_n}+\frac{1+a_{n-2}}{a_{n-1}}\Big).$$

Define

$$S_j := \Big(\bigcup_{\lambda\in\Lambda_n^j}(S_{a_{n-1}}(a_{n-1}\cdot\lambda)\setminus D_\lambda^{-1-a_{n-2}})\Big).$$

Then since B_n^j is an S_{a_n}-set,

$$\begin{aligned}
\mu\Big(\bigcup_{j=1}^{k_n} S_j B_n^j\Big) &\overset{\text{Remark 2.5}}{=} \sum_{j=1}^{k_n} m(S_j)\cdot\frac{\mu(S_{a_n}B_n^j)}{m(S_{a_n})}\\
&\leq \sum_{j=1}^{k_n}\sum_{\lambda\in\Lambda_n^j} m(S_{a_{n-1}}(a_{n-1}\cdot\lambda)\setminus(a_{n-1}\cdot A_\lambda)^{-1-a_{n-2}})\cdot\frac{\mu(S_{a_n}B_n^j)}{m(S_{a_n})}\\
&\leq \sum_{j=1}^{k_n}\#\Lambda_n^j O(1)\cdot\frac{a_{n-1}^2(\frac{a_{n-1}}{a_n}+\frac{1+a_{n-2}}{a_{n-1}})}{4a_n^2}\cdot\mu(S_{a_n}B_n^j)\\
&\leq \frac{a_n^2}{a_{n-1}^2}\cdot O(1)\cdot\frac{a_{n-1}^2(\frac{a_{n-1}}{a_n}+\frac{1+a_{n-2}}{a_{n-1}})}{4a_n^2}\sum_{j=1}^{k_n}\mu(S_{a_n}B_n^j)\\
&\leq O\Big(\frac{a_{n-2}}{a_{n-1}}\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}\mu\Big(X_n \setminus \bigcup_{j=1}^{k_n}\Big(\bigcup_{\lambda\in\Lambda_n^j} S_{a_{n-1}}(a_{n-1}\cdot\lambda)\Big)B_n^j\Big)\\ &\le \mu\Big(X_n \setminus \bigcup_{j=1}^{k_n}\Big(\bigcup_{\lambda\in\Lambda_n^j} S^{+1}_{a_{n-1}}(a_{n-1}\cdot\lambda)\Big)B_n^j\Big)\\ &\quad + \mu\Big(\bigcup_{j=1}^{k_n}\Big(\bigcup_{\lambda\in\Lambda_n^j} (S^{+1}_{a_{n-1}}\setminus S_{a_{n-1}})(a_{n-1}\cdot\lambda)\Big)B_n^j\Big)\\ &\overset{(4)}{\le} \mu(X_n\setminus X_{n-1}) + \sum_{j=1}^{k_n} \#\Lambda_n^j \cdot \frac{2\cdot a_{n-1}}{a_n^2}\cdot \mu(S_{a_n}B_n^j)\\ &< \mu(X_n\setminus X_{n-1}) + \frac{2}{a_{n-1}}.\end{aligned}$$

Overall, we get that

$$\begin{aligned}\mu(X_n\setminus G_n) &\le \mu(X_n\setminus X_{n-1}) + O\Big(\frac{a_{n-2}}{a_{n-1}}\Big)\\ &= \mu(X_n\setminus X_{n-1}) + \frac{1}{D}\cdot O\Big(\frac{1}{n\log^2 n}\Big),\end{aligned}$$

concluding the proof.

Next, we will show that for every $x\in G_n$ the properties enumerated as (B) hold. Fix $x\in G_n$. There exists $1\le j\le k_n$, and $\lambda\in\Lambda_n^j$, such that $x\in D_\lambda^{-1-a_{n-2}}B_n^j$.

Property (B$_1$) holds: Note that $T_\lambda x_n^j\in B_{n-1}$ by the way Λ_n^j was defined. Similarly, for every $y\in B_n^j$ there exists $\mu\in\mathbb{C}$ such that $|\mu-\lambda|<\frac{\delta_n}{a_{n-1}}<\frac{1}{a_{n-1}}$ and $T_{a_{n-1}\cdot\mu}y\in B_{n-1}$. Let $y\in B_n^j$ be such that $x=T_wy$ for $w\in D_\lambda^{-1-a_{n-2}}$. Then for every $z\in S_{a_{n-2}}$ we have that

$$T_zx = T_{z+w}y\in D_\lambda^{-1}y\subset D_\mu^{-1+\delta_n}y\subset D_\mu y\subset X_{n-1},$$

as needed.

Property (B$_2$) holds: Since F_n^j is entire, we get that $z\mapsto F_n(T_zx)$ is entire for every $x\in G_n$.

Property (B$_3$) holds: We want to show that for every $z\in S_{a_{n-2}}$,

$$|F_n(T_zx)-F_{n-1}(T_zx)|<10^{-2n}.$$

The idea is that δ_n was chosen so that for every j, if the function F_{n-1}^j is perturbed by something smaller than δ_n, then its image is perturbed by something which is bounded by $\frac{10^{-2n}}{2}$. Next, we take a partition which is a δ_n-fine partition consistent with $\mathcal{P}_{n-1}$, which means that for every $y \in B_n^j$, the configuration of squares associated with it is at most a δ_n-distortion of the configuration of squares associated with x_n^j, meaning $\{f_\lambda\}$ used to construct F_n differ by at most $\frac{10^{-2n}}{2}$ from those used if y was the chosen representative. Combining this with the fact that F_n approximates these f_λ to begin with, we get that $|F_n - F_{n-1}|$ is small.

Formally, let $x_0 \in B_{n-1}^\ell \cap S_{a_n} B_n^j$; then there exist $\lambda \in \Lambda_n^j$ and $w \in S_{a_n}$ such that $x_0 = T_w y$, $y \in B_n^j$, and $|\lambda - w| < \delta_n$. By the way F_n is constructed, for every $z \in D_\lambda^{-\delta_n}$

$$\begin{aligned}|F_{n-1}(T_z x_0) - F_n(T_z x_0)| =& |F_{n-1}^\ell(z) - F_n(T_{z+w}T_{-w}x_0)|\\ =& |F_{n-1}^\ell(z) - F_n(T_{z+w}y)|\\ \overset{\text{by (3)}}{=}& |F_{n-1}^\ell(z) - F_n^j(z+w)|\\ \leq& |F_{n-1}^\ell(z) - F_{n-1}^\ell(z+w-\lambda)|\\ &+ |F_{n-1}^\ell(z+w-\lambda) - F_n^j(z+w)|.\end{aligned}$$

Now, by property (E_1) of F_n^j, guaranteed by Lemma 2.3, we know that since $|\lambda - w| < \delta_n$, then $z + w \in D_\lambda$ and so

$$|F_{n-1}^\ell(z+w-\lambda) - F_n^j(z+w)| < \exp\Big(-\frac{\mathcal{M}_B(n)}{4}\Big) < \frac{10^{-2n}}{2}.$$

On the other hand, since $|\lambda - w| < \delta_n$,

$$|F_{n-1}^\ell(z) - F_{n-1}^\ell(z+w-\lambda)| \leq \sup_{\substack{\zeta,\xi \in S_{a_{n-1}}^{+1}\\ |\zeta-\xi|<\delta_n}} |F_{n-1}^\ell(\zeta) - F_{n-1}^\ell(\xi)| < \frac{10^{-2n}}{2}$$

as well. Overall, for every $x_0 \in B_{n-1} \cap S_{a_n} B_n^j$ we have that

$$|F_n(T_z x_0) - F_{n-1}(T_z x_0)| < 10^{-2n}. \tag{5}$$

Next, since $x \in D_\lambda^{-1-a_{n-2}} B_n^j$ there exists $w \in D_\lambda^{-1-a_{n-2}}$ such that $T_{-w}x \in B_n^j$. In addition, since the partition $\mathcal{P}_n$ is δ_n-fine there exists ζ such that $|\zeta - \lambda| < \delta_n$, and $T_{-w-\zeta}x \in B_{n-1}^\ell$. For every $z \in S_{a_{n-2}}$

$$z + w \in D_\lambda^{-1} \Rightarrow z + w - \zeta + \lambda \in D_\lambda^{-1+\delta_n} \subset D_\lambda^{-\delta_n},$$

and by using (5) we get that if $x_0 = T_{-w-\zeta}x \in B_{n-1}^{\ell} \cap S_{a_n} B_n^j$, then

$$\begin{aligned}|F_n(T_z x) - F_{n-1}(T_z x)| &= |F_n(T_{z+w+\zeta}T_{-w-\zeta}x) - F_{n-1}(T_{z+w+\zeta}T_{-w-\zeta}x)| \\ &= |F_n(T_{z+w+\zeta}x_0) - F_{n-1}(T_{z+w+\zeta}x_0)| < 10^{-2n},\end{aligned}$$

and property (B_3) holds.

Property (B_4') holds: Note that for every $m \le n-3$ we have that by property (B_3) and the induction assumption (which holds only for $m \le n-3$),

$$\begin{aligned}\max_{z\in S_{a_m}} |F_n(T_z x)| &\le \max_{z\in S_{a_m}} |F_{n-1}(T_z x)| + 10^{-2n} \\ &\le \exp(2^{1-B}\mathcal{M}_B(m+1)) + \sum_{j=1}^{n} 10^{-2j}.\end{aligned}$$

For $m = n-2$, by property (B_3),

$$\begin{aligned}\max_{z\in S_{a_{n-2}}} |F_n(T_z x)| &\le \max_{z\in S_{a_{n-1}}} |F_{n-1}(T_z x)| + 10^{-2n} \\ &\le \exp(2^{1-B}\mathcal{M}_B(n)) + 10^{-2n}.\end{aligned}$$

Property (B_5) holds: By property (E_2) of Lemma 2.3 for every $1 \le j \le k_n$

$$\begin{aligned}\max_{z\in S_{a_n}} |F_n^j(z)| &\le \exp(2^{1-B} \max_{1\le \ell\le k_{n-1}} \max_{z\in S_{a_{n-1}}} |F_{n-1}^{\ell}(z)| \cdot \exp(\pi C^2)) \\ &\le \exp(2^{1-B}\mathcal{M}_B(n)\exp(\pi\cdot n^2\log^4 n)) \\ &\le \exp(2^{1-B}\mathcal{M}_B(n+1)).\end{aligned}$$

This concludes the proof of the lemma. ∎

4. The Proof of Theorem 1.5

Let $\{F_n\}$ be the sequence constructed in Lemma 3.1.

4.1. The sequence $\{F_n\}$ converges almost surely to a measurably entire function. Let $x \in \bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} G_k$. Then by property ($B_3$), $\{F_n(T_z x)\}$ converges locally uniformly to an entire function, and in particular if $\{F_n\}$ converges almost surely, then it converges to a measurably entire function. It is therefore enough to show that $\mu(\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} G_k) = 1$ to conclude the proof. To see that $\mu(\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} G_k) = 1$ we will show that

$$0 = \mu\Big(\Big(\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} G_k\Big)^c\Big) = \mu\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} G_k^c\Big) = \lim_{m\to\infty}\mu\Big(\bigcup_{k=m}^{\infty} G_k^c\Big).$$

By using property (A) of the sequence $\{F_n\}$ and the fact that $\{X_n\}$ is increasing, we obtain that

$$\begin{aligned}\mu\Big(\bigcup_{k=m}^{\infty} G_k^c\Big) &\overset{G_k\subseteq X_k}{=} \mu\Big(\bigcup_{k=m}^{\infty}(X\setminus X_k)\uplus(X_k\setminus G_k)\Big)\\ &\leq \mu\Big(\bigcup_{k=m}^{\infty}(X\setminus X_k)\Big)+\sum_{k=m}^{\infty}\mu(X_k\setminus G_k)\\ &\overset{\text{Property (A)}}{\leq} \mu(X\setminus X_m)+\sum_{k=m}^{\infty}\Big(\mu(X_k\setminus X_{k-1})+\frac{1}{D}\cdot O\Big(\frac{1}{k\log^2 k}\Big)\Big)\\ &\overset{X_{n-1}\subseteq X_n}{\leq} 2\mu(X\setminus X_{m-1})+\frac{O(1)}{D}\sum_{k=m}^{\infty}\frac{1}{k\log^2 k}.\end{aligned}$$

To conclude the proof, note that the latter tends to zero as m tends to ∞, since the series converges.

4.2. The limiting function F is not constant. Since the sequence $\{F_n\}$ converges in measure to a function which we shall denote by F,

$$\begin{aligned}\mu\Big(\Big\{|F|\leq\frac{1}{3}\Big\}\Big) &=\lim_{n\to\infty}\mu\Big(\Big\{|F_n|\leq\frac{1}{3}\Big\}\Big)\\ \mu\Big(\Big\{|F|\geq\frac{2}{3}\Big\}\Big) &=\lim_{n\to\infty}\mu\Big(\Big\{|F_n|\geq\frac{2}{3}\Big\}\Big).\end{aligned}$$

We will bound each of the quantities above from below by a uniform constant for every n:

$$\begin{aligned}\mu\Big(\Big\{|F_n|\leq\frac{1}{3}\Big\}\Big) &\overset{\substack{\text{property}\\(\text{B}_3)}}{\geq} \mu\Big(\Big\{|F_{n-1}|\leq\frac{1}{3}-10^{-2n}\Big\}\setminus G_n^c\Big)\\ &\ \ \vdots\\ &\overset{\substack{\text{property}\\(\text{B}_3)}}{\geq} \mu\Big(\Big\{|F_1|\leq\frac{1}{3}-\sum_{m=1}^{n}10^{-2m}\Big\}\setminus\bigcup_{m=1}^{n}G_m^c\Big)\\ &\geq \mu\Big(\Big\{|F_1|\leq\frac{1}{4}\Big\}\Big)-\mu\Big(\bigcup_{m=1}^{n}G_m^c\Big)\\ &\overset{(\text{A})}{\geq} \frac{1}{25}-2\mu(X\setminus X_1)-\frac{O(1)}{D}\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}\\ &=\frac{3}{100}-\frac{O(1)}{D}\sum_{n=2}^{\infty}\frac{1}{n\log^2 n},\end{aligned}$$

where D is the constant from the definition of the sequence $\{a_n\}$. If we choose D large enough we get

$$\mu\left(\left\{|F_n| \leq \frac{1}{3}\right\}\right) \geq \frac{1}{100}.$$

A similar computation shows that $\mu(\{|F| \geq \frac{2}{3}\})$ is greater than the same constant, concluding that F is not constant.

4.3. Upper bound for the growth rate of the function. Let $x \in \bigcap_{k=n}^{\infty} G_k$; we will show that (1) holds. For every $k \geq n$, by property (B_4) of the sequence $\{F_n\}$

$$\max_{z \in S_{a_m}} |F(T_z x)| \leq 2\exp(2^{1-B}\mathcal{M}_B(m+1)).$$

In addition, as $\frac{a_{m+1}}{a_m} \sim m \log^2 m$, for every $\varepsilon > 0$ for every $m > m_\varepsilon$ large enough

$$\begin{aligned}
\mathcal{M}_B(m) &= \exp\left(B \cdot m + \pi \sum_{k=2}^{m} \left(\frac{a_k}{a_{k-1}}\right)^2\right) \\
&\leq \exp\left(B \cdot m + O(1) \cdot D \sum_{k=2}^{m} k^2 \log^4 k\right) \\
&\leq \exp(B \cdot m + O(1) \cdot D m^3 \log^4 m) \\
&\leq \exp(O(1) \cdot \log^{3+\frac{\varepsilon}{2}} a_m).
\end{aligned}$$

We conclude that for every $\varepsilon > 0$

$$\frac{\log\log \max_{z \in S_{a_m}} |F(T_z x)|}{\log^{3+\varepsilon} a_m} \leq \frac{O(1)}{\log^{\frac{\varepsilon}{2}} a_m} \xrightarrow[m \to \infty]{} 0.$$

For every R large enough, let m be such that $a_m \leq R < a_{m+1}$. Using the estimate above with $m+1$ and $\frac{\varepsilon}{2}$ instead of ε we get

$$\begin{aligned}
\frac{\log\log \max_{z \in S_R} |F(T_z x)|}{\log^{3+\varepsilon} R} &\leq \frac{\log\log \max_{z \in S_{a_{m+1}}} |F(T_z x)|}{\log^{3+\varepsilon} a_m} \\
&\leq \frac{O(1)}{\log^{\frac{\varepsilon}{2}} a_m} \cdot \left(\frac{\log(a_{m+1})}{\log(a_m)}\right)^{3+\frac{\varepsilon}{2}} \xrightarrow[m \to \infty]{} 0.
\end{aligned}$$

concluding the proof of the theorem.

5. Appendix

For completeness we introduce here a proof of The Nested Towers Lemma:

Lemma 2.6 (The Nested Towers Lemma): *Let $(X, \mathcal{B}, \mu)$ be a standard probability space, and suppose $T : \mathbb{C} \to PPT(X)$ is a free $\mathbb{C}$-action. Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} < \frac{1}{2}$. Then there exists a sequence of sets $\{B_n\} \subset \mathcal{B}$ such that*

(N_1) *B_n is an S_{a_n}-set.*
(N_2) *$S_{a_n} B_n \subset S_{a_{n+1}} B_{n+1}$.*
(N_3) *$\mu(S_{a_n} B_n) \nearrow 1$ as $n \to \infty$.*

We start this appendix with a discussion of a preliminary lemma. This lemma is a version of Rokhlin's lemma for flows. It was proven by Lind in [4]:

Lemma 5.1: *Let T be a free n-dimensional flow on a standard probability space $(X, \mathcal{B}, \mu)$. Then for any rectangle $Q \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exists a Q-set, $F \subset X$ such that*

$$\mu(T_Q F) > 1 - \varepsilon.$$

Remark 5.2: Note that unlike our definition of an S-set, Lind's definition does not require measurability. Namely, his definition lacks condition (F_2) completely. Nevertheless, he proved that the set F found in Lemma 5.1, not only satisfies condition (F_2), but in fact fulfills a stronger condition than the one we impose. For more information see [4, p. 177].

We first describe the idea of the proof. We start with a sequence of sets $\{B_n(0)\}$ obtained by Rokhlin's lemma for a sequence $\{\varepsilon_n\}$, and define for every n a sequence of sets $\{B_n(k)\}_{k=0}^{\infty}$: $B_n(k+1)$ will only include elements of $B_n(k)$ so that their restricted orbit, $S_{a_n} x$, is included in $S_{a_{n+1}} B_{n+1}(k)$. Then we will bound the measure of the sets that we remove to conclude that for $B_n := \bigcap_{k=0}^{\infty} B_n(k)$ we have $S_{a_n} B_n \nearrow X$ as $n \to \infty$.

Proof. Let $\{\varepsilon_n\}$ be a positive monotone decreasing sequence such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. By Rokhlin's lemma for flows, Lemma 5.1, for every n there exists a set $B_n(0)$ such that:

(L_1) $B_n(0)$ is an S_{a_n}-set.
(L_2) $\mu(S_{a_n} B_n(0)) > 1 - \varepsilon_n$.

We inductively define the sets

$$B_j(k+1) = \{x \in B_j(k),\ S_{a_j}x \subset S_{a_{j+1}}B_{j+1}(k)\}.$$

First of all, $B_j(k+1) \subseteq B_j(k)$ and so the set $B_j := \bigcap_{k=1}^{\infty} B_j(k)$ is well defined. Next, every measurable subset of $B_j(0)$ is in itself an S_{a_j}-set, because of property (F_2) of an S_{a_j}-set. If for every k, $B_j(k)$ is measurable, then it is an S_{a_j}-set, and so is B_j. To conclude that property (N_1) holds it is left to show that for every k the set $B_j(k)$ is measurable.

It is clear that the inclusion condition, condition (N_2), holds by the way the sequence $\{B_j\}$ is defined. To prove that (N_1), (N_3) hold we will need the following claim:

Claim: *Let $x \in B_j(k)$. Then $x \in B_j(k+1)$ if and only if there exists $y \in B_{j+1}(k)$ such that $S_{a_j}x \subset S_{a_{j+1}}y$.*

This claim tells us that for every x that we threw away on step k of the construction of the sequence $\{B_j(k)\}$, $x \in B_j(k) \setminus B_j(k+1)$, its restricted orbit, $S_{a_j}x$, is included in the restricted orbit of some element $y \in B_{j+1}(k-1)$ that we threw away on step $(k-1)$ of the construction of the sequence $\{B_{j+1}(k)\}$, $y \in B_{j+1}(k-1) \setminus B_{j+1}(k)$.

Proof of the claim. The 'if' part of the claim is obvious. To prove the other side, assume by contradiction that the set defined by

$$A_x := \{y \in B_{j+1}(k),\ S_{a_j}x \cap S_{a_{j+1}}y \neq \emptyset\}$$

contains at least two elements. Note that A_x may contain at most four elements, for if $\zeta \in S_{a_j}x \cap S_{a_{j+1}}y \neq \emptyset$ then there exists $z \in S_{a_j}$ and $w \in S_{a_{j+1}}$ such that $\zeta = T_z x = T_w y$, meaning that

$$\begin{aligned} S_{a_j}x \cap S_{a_{j+1}}y =& S_{a_j}T_{w-z}y \cap S_{a_{j+1}}y = (S_{a_{j+1}} \cap S_{a_j}(w-z))y \\ =& (S_{a_{j+1}}(z-w) \cap S_{a_j})x. \end{aligned}$$

In particular, there exists a rectangle $R = S_{a_{j+1}}(z-w) \cap S_{a_j}$ such that

$$Rx = S_{a_j}x \cap S_{a_{j+1}}y.$$

As S_{a_j} and $S_{a_j}(z-w)$ are aligned squares, their intersection will give us a rectangle. Since $a_j < a_{j+1}$ and the squares $T_z S_{a_j}$ and $S_{a_{j+1}}$ are aligned, the rectangle R must contain at least one of the corners of S_{a_j} (see Figure 7).

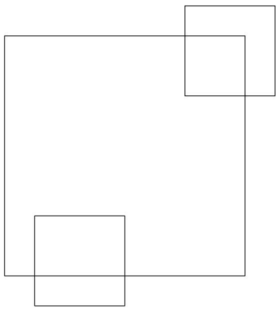

Figure 7. An intersection of aligned squares is a rectangle, so that at least one of its corners belongs to the smaller square in the intersection.

In addition, because $B_{j+1}(k) \subset B_{j+1}(0)$ it fulfills property (F_1) of an $S_{a_{j+1}}$-set, and so each element of the set of 'corners' of $S_{a_j}x$,

$$\{T_{a_j(1+i)}x,\ T_{a_j(1-i)}x,\ T_{a_j(-1+i)}x,\ T_{a_j(-1-i)}x\},$$

belongs to the set $S_{a_{j+1}}y$ for a unique $y \in A_x$. We conclude that A_x cannot contain more than four elements.

Next, note that R is a closed rectangle as an intersection of two closed squares. We get that

$$S_{a_j}x = \biguplus R_\alpha x,$$

where the collection $\{R_\alpha\}$ contains at most four disjoint closed rectangles, and the union is disjoint since $x \in B_j(k)$ for which property (F_1) of an S_{a_j}-set holds. This yields that $S_{a_j} = \biguplus R_\alpha$, which is a contradiction to the fact that a square is a connected set, and thus concludes the proof of the claim.

We will prove the measurability of $B_j(k)$ by induction on k. Recall that an S-set $B \subset X$ satisfies condition (F_2) if for every $B' \subset B$ measurable and every $A \subset S$ measurable the set

$$AB' := \bigcup_{z\in A} T_z B'$$

is a measurable subset of X. Now, for every j the set $B_j(0)$ is measurable by property (F_2). Assume that for every j we know that $B_j(k)$ is measurable.

Following the claim above and the definition of $B_j(k+1)$, for $x \in B_j(k)$

$$\begin{aligned} x \in B_j(k+1) &\overset{\text{def}}{\iff} S_{a_j}x \subset S_{a_{j+1}}B_{j+1}(k) \overset{\text{claim}}{\iff} \exists y \in B_{j+1}(k), S_{a_j}x \subset S_{a_{j+1}}y \\ &\iff \exists y \in B_{j+1}(k), x \in S_{a_{j+1}-a_j}y \iff x \in S_{a_{j+1}-a_j}B_{j+1}(k). \end{aligned}$$

We conclude that

$$B_j(k+1) = B_j(k) \cap S_{a_{j+1}-a_j}B_{j+1}(k),$$

which is measurable as the intersection of two measurable sets, since property (F_2) holds for $B_\ell(k)$ for every ℓ, by the induction assumption.

It is left to show that this sequence saturates the whole space, namely that $\mu(S_{a_n}B_n) \nearrow 1$. Following the claim above, if $x \in B_j(k) \setminus B_j(k+1)$ and $y \in B_{j+1}(k-1)$ is such that $T_z x = y$, then necessarily $y \notin B_{j+1}(k)$. We obtain that

$$S_{a_j}B_j(k) \setminus S_{a_j}B_j(k+1) \subseteq S_{a_{j+1}}B_{j+1}(k-1) \setminus S_{a_{j+1}}B_{j+1}(k). \tag{6}$$

In addition, if $S_{a_j}x \cap S_{a_{j+1}-2a_j}B_{j+1}(0) \neq \emptyset$, then there exists $z \in S_{a_j}$ and $w \in S_{a_{j+1}-2a_j}$ such that $T_{z-w}x \in B_{j+1}(0)$, but then for every $\xi \in S_{a_j}$ we have that $\xi + w - z \in S_{a_{j+1}}$ and so

$$T_\xi x = T_{\xi+w-z}T_{z-w}x \in S_{a_{j+1}}B_{j+1}(0) \Rightarrow S_{a_j}x \subset S_{a_{j+1}}B_{j+1}(0),$$

contradicting the fact that $x \notin B_j(1)$. We conclude that

$$S_{a_j}B_j(0) \setminus S_{a_j}B_j(1) \subseteq X \setminus S_{a_{j+1}-2a_j}B_{j+1}(0). \tag{7}$$

Combining (6) and (7) one can see that

$$\begin{aligned} S_{a_j}B_j(k) \setminus S_{a_j}B_j(k+1) &\overset{(6)}{\subseteq} S_{a_{j+1}}B_{j+1}(k-1) \setminus S_{a_{j+1}}B_{j+1}(k) \\ &\;\vdots \\ &\overset{(6)}{\subseteq} S_{a_{j+k}}B_{j+k}(0) \setminus S_{a_{j+k}}B_{j+k}(1) \\ &\overset{(7)}{\subseteq} X \setminus S_{a_{k+j+1}-2a_{k+j}}B_{k+j+1}(0). \end{aligned}$$

Now, using Remark 2.5, for every k and j we get that for $m = j + k$

$$
\begin{aligned}
\mu(S_{a_j}B_j(k) \setminus S_{a_j}B_j(k+1)) \leq & \mu(X \setminus S_{a_{m+1}-2a_m}B_{m+1}(0)) \\
= & 1 - \mu(S_{a_{m+1}-2a_m}B_{m+1}(0)) \\
= & 1 - m(S_{a_{m+1}-2a_m}) \cdot \frac{\mu(S_{a_{m+1}}B_{m+1}(0))}{m(S_{a_{m+1}})} \\
\leq & 1 - \frac{(a_{m+1} - 2a_m)^2}{a_{m+1}^2}(1 - \varepsilon_m) \\
< & 2\varepsilon_m + \frac{4a_m}{a_{m+1}}.
\end{aligned} \tag{8}
$$

We note that by the triangle inequality

$$
\begin{aligned}
\mu(X \setminus S_{a_j}B_j(n)) \leq & \mu(X \setminus S_{a_j}B_j(0)) + \sum_{k=0}^{n-1} \mu(S_{a_j}B_j(k) \setminus S_{a_j}B_j(k+1)) \\
& \overset{(8)}{\leq} \mu(X \setminus S_{a_j}B_j(0)) + \sum_{k=1}^{n-1} \left(2\varepsilon_{j+k} + \frac{4a_{j+k}}{a_{j+k+1}}\right) \\
& < 2\sum_{k=j}^{\infty} \left(\varepsilon_k + \frac{2a_k}{a_{k+1}}\right).
\end{aligned}
$$

Since the latter is the tail of a converging series it tends to 0, concluding the proof of (N_3), as $\mu(X \setminus B_n) = \lim_{k\to\infty}\mu(X \setminus B_n(k)) \underset{n\to\infty}{\longrightarrow} 0$. ∎

References

[1] L. Buhovsky, A. Glücksam, A. Logunov and M. Sodin, *Translation-invariant probability measures on entire functions*, Journal d'Analyse Mathématique, to appear, arXiv:1703.08101.

[2] P. R. Halmos, *Lectures on Ergodic Theory*, Chelsea Publishing, New York, 1960.

[3] L. Hörmander, *Notions of Convexity*, Modern Birkhäuser Classics, Birkhäuser, Boston, MA, 2007.

[4] D. Lind, *Locally compact measure preserving flows*, Advances in Mathematics **15** (1975), 175–193.

[5] B. Weiss, *Measurable entire functions*, Annals of Numerical Mathematics **4** (1997), 599–605.