MAXIMUM PRINCIPLES IN SYMPLECTIC HOMOLOGY

BY

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ABSTRACT

In the setting of symplectic manifolds which are convex at infinity, we use a version of the Aleksandrov maximum principle to derive uniform estimates for Floer solutions that are valid for a wider class of Hamiltonians and almost complex structures than is usually considered. This allows us to extend the class of Hamiltonians which one can use in the direct limit when constructing symplectic homology. As an application, we detect elements of infinite order in the symplectic mapping class group of a Liouville domain and prove existence results for translated points.

1. Introduction

Symplectic homology is a natural generalisation of Floer homology to open symplectic manifolds which are convex at infinity. It is constructed via a direct limit of Floer homology groups for Hamiltonians which get steeper and steeper at infinity. This construction—in the case where the symplectic manifold is exact—is due to Viterbo [28]—although there were various previous flavours and incarnations. It has since been generalised to non-exact symplectic manifolds by Ritter [16, 17, 18].

Since the underlying symplectic manifold is non-compact, there are additional technical difficulties in the construction of symplectic homology which are not present in the closed setting. Chief amongst these is the necessity of a maximum principle to prevent Floer solutions from escaping to infinity. The need for such a maximum principle to hold severely limits the type of Hamiltonians which one can take in the aforementioned direct limit. The standard technique is to use Hamiltonians that are radial and linear on the convex end. More precisely, the assumption that our underlying symplectic manifold (M, ω) is convex at infinity means that there is a compact domain $M_1 \subset M$ such that the complement of M_1 is symplectomorphic to the positive part of the symplectisation of a closed contact manifold:

$$(1) \qquad (M \setminus M_1, \omega) \cong (\Sigma \times (1, \infty), d(r\alpha)),$$

where (Σ, α) is a closed contact manifold and r denotes the coordinate on $[1, \infty)$. Then one works with Hamiltonians H_t on M that are of the form

$$(2) \qquad H_t(x, r) = ar + b,$$

for large r. For such Hamiltonians the r-component of a solution of the Floer equation is necessarily subharmonic, and this prevents such solutions from escaping to infinity.

In [19], Ritter extended the class of Hamiltonians to those which are of the form

$$(3) \qquad H_t(x, r) = rh_t(x) + b,$$

for large r, where $h_t : \Sigma \to \mathbb{R}$ is a function which is invariant under the Reeb flow of α. A similar construction was also given independently by the second author in [26], and for time-independent h a different proof was given by Fauck in his thesis [9].

In this paper we further generalise this and remove the hypothesis that h_t must be invariant under the Reeb flow. Namely, we prove:

THEOREM 1.1 (Extended maximum principle): *The maximum principle applies to all Hamiltonians of the form* (3).

Remark 1.2: Theorem 1.1 is stated more precisely in Theorem 3.10 below. Actually, the existence of a maximum principle depends not only on the choice of Hamiltonian but also on the choice of almost complex structure (for simplicity

we suppress this fact during the Introduction). The standard maximum principle requires one to work with an almost complex structure which is of SFT-type at infinity. In addition to extending the class of Hamiltonians for which a maximum principle is applicable, we also extend the class of complex structures to those of twisted SFT-type at infinity (see Definition 2.1 below). As far as applications are concerned, extending the class of almost complex structures is just as important as extending the class of Hamiltonians—see for instance Proposition 4.1 below).

We believe that Theorem 1.1 will be useful in several situations. In the present paper we discuss two of them—in the Outlook at the end of this section we mention further applications that will appear in the sequel to this paper.

Assume now that (M, ω) is a symplectic manifold which is convex at infinity as above, and for which the symplectic homology $\mathrm{SH}_*(M)$ is well defined. This really is an extra assumption: the minimum one needs is to assume that (M, ω) is **weakly$^+$ monotone**, a condition introduced by Hofer and Salamon in [12], which requires that at least one of the following three conditions is fulfilled:

- $\omega|_{\pi_2(M)} = 0$ or $c_1|_{\pi_2(M)} = 0$,
- ω is positively monotone: there exists $\beta > 0$ such that $(\omega - \beta c_1)|_{\pi_2(M)} = 0$,
- the minimal Chern number N of (M^{2n}, ω) is at least $n - 1$.

See Ritter [18] for details of the construction in this setting. At the other end of the spectrum, the "simplest" type of symplectic manifolds for which symplectic homology can be defined are Liouville domains. Here one starts with a compact exact symplectic manifold $(M_1, \omega_1 = d\lambda_1)$ with boundary $\Sigma = \partial M_1$, which in addition has the property that the vector field Y_1 defined implicitly by requiring that $\lambda_1 = \omega(Y_1, \cdot)$ points strictly outwards along Σ. Consequently there is a smooth function r defined on a neighbourhood of Σ in M_1 with values in $(0, 1]$ such that $\Sigma = r^{-1}(1)$ and such that $dr(Y_1) = r$. This allows us to identify a neighbourhood of Σ in M_1 with $\Sigma \times (0, 1]$. One then "completes" (M_1, ω_1) into a non-compact symplectic manifold (M, ω) by setting

$$M := M_1 \cup_\Sigma (\Sigma \times \mathbb{R}^+).$$

We then extend λ_1 to a one-form λ on M by setting $\lambda = r\alpha$ on $\Sigma \times \mathbb{R}^+$. Setting $\omega = d\lambda$, the manifold (M, ω) is then obviously convex at infinity. This is the class of symplectic manifolds Viterbo [28] originally worked with.

Symplectic homology is oftentimes an infinite-dimensional theory. The most famous early result, originally proved by Viterbo [27] (and independently by Salamon–Weber [21], Abbondandolo–Schwarz [1], and Abouzaid [3]) is that the symplectic homology of a cotangent bundle $(T^*N, d\lambda_{\mathrm{can}})$ equipped with its standard symplectic form is equal to the singular homology of the free loop space:

$$\mathrm{SH}_*(T^*N) \cong \mathrm{H}_*(\Lambda(N); \mathbb{Z}_2).$$

Nevertheless, motivated by the Seidel representation [23], in [18] Ritter discovered that the existence of a loop of Hamiltonian diffeomorphisms with positive slope at infinity forces symplectic homology to be finite-dimensional. Here is the precise statement.

THEOREM 1.3 (Ritter [18]): *Assume (M, ω) is a weakly$^+$ monotone symplectic manifold which is convex at infinity. If there exists a loop of Hamiltonian diffeomorphisms generated by a Hamiltonian $H_t : M \to \mathbb{R}$ which for r large is of the form*

$$H_t(x, r) = r h_t(x) + b, \quad h_t > 0,$$

such that

(4)   *h_t is invariant under the Reeb flow,*

then the summand of symplectic homology coming from contractible orbits can be seen as a localisation (i.e., quotient) of the quantum homology of M.

The only reason the h_t was required to be invariant under the Reeb flow was that this was the only class for which the maximum principle was known to hold. Our Theorem 1.1 removes this restriction.

COROLLARY 1.4: *The assumption (4) in Theorem 1.3 is not necessary.*

We conclude this Introduction by discussing applications of our maximum principle. The first extends previous work of the second author [26] and resolves a conjecture of Biran–Giroux [7]. Assume now that $(M, \omega = d\lambda)$ is the completion of a Liouville domain $(M_1, \omega_1 = d\lambda_1)$ with boundary $\Sigma = \partial M_1$. Consider the subgroup

$$\mathcal{G} := \{\phi \in \mathrm{Symp}(M, \omega) \mid \phi^*\lambda - \lambda = df, \ \mathrm{supp}(f) \subset \mathrm{int}(M_1))\},$$

and the subset

$$\mathcal{G}_c := \{\phi \in \mathcal{G} \mid \mathrm{supp}(\phi) \subset \mathrm{int}(M_1)\}.$$

Clearly $\mathcal{G}_c$ is a subgroup of the group $\mathrm{Symp}_c(M_1, \omega_1)$ of compactly supported symplectomorphisms of (M_1, ω_1). In fact, the inclusion $\mathcal{G}_c \hookrightarrow \mathrm{Symp}_c(M, \omega)$ is a homotopy equivalence. This is due to Biran–Giroux [7]; a proof can be found in [26, Lemma 3.3].

Next, if we denote by α the restriction of the one-form λ_1 on M_1 to the boundary Σ, then α is a contact form on Σ. Let us denote the associated contact distribution by ξ, and the group of contactomorphisms by $\mathrm{Cont}(\Sigma, \xi)$. An element $\phi \in \mathcal{G}$ naturally defines an element of $\mathrm{Cont}(\Sigma, \xi)$. Indeed, for such a ϕ there exists $T > 0$ such that on $\Sigma \times (T, \infty)$, one has

$$\phi(x, r) = \Big(\varphi(x), \frac{r}{\kappa(x)}\Big),$$

where $\varphi \in \mathrm{Cont}(\Sigma, \xi)$ and $\kappa : \Sigma \to (0, \infty)$ satisfies $\varphi^*\alpha = \kappa\alpha$. Thus there is a well defined map, called the **ideal restriction map**, given by

$$\Theta : \mathcal{G} \to \mathrm{Cont}(\Sigma, \xi), \quad \Theta(\phi) := \varphi.$$

Moreover, Biran and Giroux [7] discovered that the map Θ is a Serre fibration over $\mathrm{Cont}(\Sigma, \xi)$ with fibre $\mathcal{G}_c$. This implies there is a long exact sequence

$$\pi_k(\mathrm{Symp}_c(M_1, \omega_1)) \longrightarrow \pi_k(\mathcal{G}) \longrightarrow \pi_k(\mathrm{Cont}(\Sigma, \xi)) \overset{\Delta}{\longrightarrow} \pi_{k-1}(\mathrm{Symp}_c(M_1, \omega_1)).$$

Denoting the **symplectic mapping class group** of (M_1, ω_1) by

$$\mathrm{SMCG}(M_1, \omega_1) := \pi_0(\mathrm{Symp}_c(M_1, \omega_1)),$$

the connecting homomorphism

$$\Delta : \pi_1(\mathrm{Cont}(\Sigma, \xi)) \to \mathrm{SMCG}(M_1, \omega_1)$$

is given as follows. Given a loop $\varphi^t : \Sigma \to \Sigma$, $t \in S^1$, of contactomorphisms based at the identity, choose a lift $[0, 1] \ni t \mapsto \phi^t \in \mathcal{G}$ such that $\Theta(\phi^t) = \varphi^t$. Then set

(5) $$\Delta([\varphi^t]) := [\phi^1],$$

where in both cases $[\cdot]$ denotes the appropriate equivalence relation. Explicitly, if φ^t has contact Hamiltonian $h_t : \Sigma \to \mathbb{R}$ (see (13) below for the definition of h_t), define $H_t : M \to \mathbb{R}$ by first setting

$$H_t(x, r) = \beta(r) r h_t(x), \quad \forall (x, r) \in \Sigma \times \mathbb{R}^+,$$

where

$$(6) \qquad \beta : (0, \infty) \to [0, 1], \quad \beta(r) = \begin{cases} 0, & r \in (0, 1/4), \\ 1, & r \in (3/4, \infty), \end{cases} \qquad \beta'(r) \geq 0,$$

and then extending H_t to all of M by setting $H_t = 0$ on the rest of M. Then if ϕ_H^t denotes the Hamiltonian flow of H, one has

$$\Theta(\phi_H^t) = \varphi^t,$$

and hence $\Delta([\varphi^t]) = [\phi_H^t]$.

Biran and Giroux conjectured that if the symplectic homology of M is sufficiently rich and φ^t is a **positive** loop (this means that the contact Hamiltonian h_t is everywhere positive), then the element $\Delta([\varphi^t])$ should be of infinite order inside $\mathrm{SMCG}(M_1, \omega_1)$. In the following theorem, the total dimension of the symplectic homology of M is denoted by $\dim \mathrm{SH}_*(M)$.

THEOREM 1.5: *Let (M_1^{2n}, ω_1) be a Liouville domain with completion (M, ω). Let $\varphi_t : \Sigma \to \Sigma$ be a positive loop of contactomorphisms on the boundary $\Sigma := \partial M$. If*

$$\sum_{k=1}^{2n} \dim \mathrm{H}_k(M_1; \mathbb{Z}_2) < \dim \mathrm{SH}_*(M),$$

then the element $\Delta([\varphi^t])$ from (5) is of infinite order in the symplectic mapping class group of (M_1, ω_1).

As remarked above, a special case of the theorem, in which the loop φ^t is strict, was proved in [26]. In addition, recently progress has been made in several related directions by other authors: Seidel [25], Chiang, Ding and van Koert [14, 15], and Barth, Geiges and Zehmisch [6]. See Remarks 4.1 and 4.6 in [6] for a more detailed historical overview.

Another application of the maximum principle pertains to translated points. Given a contactomorphism $\varphi : \Sigma \to \Sigma$ of a contact manifold (Σ, α), a **translated point** of φ is a point x such that $\varphi(x)$ and x both lie on the same Reeb orbit of α, and such that $\varphi^* \alpha|_x = \alpha_x$. This notion was introduced by Sandon in [22], although it can be seen as a special case of a leaf-wise intersection point as introduced by Moser in [13]. Using Rabinowitz Floer homology, the first author and Albers proved:

PROPOSITION 1.6 ([5]): *Let (M_1^{2n}, ω_1) be a Liouville domain with completion (M, ω). Suppose $\mathrm{SH}_*(M)$ is infinite-dimensional. Then any contactomorphism of the boundary isotopic to the identity through contactomorphisms either has infinitely many translated points or a translated point on a closed Reeb orbit.*

In Section 5 we give a new proof of this result using the new maximum principle.

OUTLOOK. The present paper concludes by briefly explaining how to define the analogue of positive symplectic homology in this setting. This gives rise to an invariant $\mathrm{SH}^+(\varphi^t, M)$ associated to a path of contactomorphisms on the boundary of a Liouville domain, which we call the positive symplectic homology for the contact isotopy φ^t. It is defined whenever $\varphi^1 \neq \mathrm{Id}$, and, up to a grading shift, depends only on the time-1 map φ^1. For φ^t equal to a small piece of the Reeb flow this group agrees with the usual positive symplectic homology $\mathrm{SH}^+(M)$. In general though this is not the case, and the groups $\mathrm{SH}^+(\varphi^t, M)$ can be used to obtain more refined existence results on translated points. We will study these groups thoroughly in a sequel to the present paper.

A further variant on this idea is to construct groups $\mathrm{SH}(U, M)$ associated to an open set U of the boundary Σ, by using contact Hamiltonians associated to contact isotopies that are supported inside U. This gives applications to contact embedding and squeezing problems, and will again be discussed in this paper's sequel.

Finally, we remark that Groman [11] has recently developed a novel approach to obtaining L^∞-bounds for Floer solutions on non-compact manifolds. It would be interesting to see whether his methods can recover the results proved here.

2. Preliminaries on symplectisations

Although we are primarily interested in non-compact manifolds that are convex at infinity, almost all of our arguments will take place in the symplectisation of a closed contact manifold. Thus we begin by fixing some notation in this setting.

Suppose (Σ, ξ) is a closed co-orientable contact manifold of dimension $2n-1$. Let $S\Sigma := \Sigma \times \mathbb{R}^+$ denote the **symplectisation** of Σ. A choice of contact form α on Σ supporting ξ gives rise to a symplectic form $d\lambda$ on $S\Sigma$. The one-form λ is defined by $\lambda = r\alpha$, where r is the coordinate on $\mathbb{R}^+$. Given $r > 0$, we abbreviate by $S\Sigma(r)$ the non-compact open manifold $\Sigma \times (r, \infty)$.

We denote by $Y = r\partial_r$ the **Liouville vector field** on $S\Sigma$, and we denote by R the **Reeb vector field** on Σ, which we also think of as a vector field on $S\Sigma$. Note that the symplectic complement of the 2-plane field spanned by R and Y is exactly the contact distribution ξ.

Suppose j is an almost complex structure on the symplectic vector bundle $(\xi, d\alpha) \to \Sigma$ which is compatible[1] with $d\alpha$. The almost complex structure j uniquely determines an almost complex structure J on $S\Sigma$ itself, which is defined by

$$(7) \qquad J(aR + bY + v) = aY - bR + jv, \quad a, b \in \mathbb{R}, \ v \in \xi.$$

We say that J is of **SFT-type**.

We now generalise this notion. Suppose we are given a smooth positive function $q : \Sigma \to \mathbb{R}^+$. Let $Q : S\Sigma \to \mathbb{R}^+$ be defined by

$$Q(x, r) = rq(x).$$

Observe that the Hamiltonian vector field[2] X_Q of Q takes the form

$$(8) \qquad X_Q(x, r) = X_q(x) - dq(x)R(x)Y(r),$$

where X_q is the vector field on Σ defined by

$$\alpha(X_q) = q, \quad d\alpha(X_q, \cdot) = dq(R)\alpha - dq.$$

Let N_Q denote the symplectic complement of the 2-plane field spanned by X_Q and Y in $T(S\Sigma)$. Explicitly,

$$N_Q(x, r) = \left\{ \left(v, -\frac{rdq(x)v}{q(x)} \right) \mid v \in \xi_x \right\}.$$

Thus given $v \in \xi_x$ there is a unique vector $\zeta_Q(v) \in N_Q(x, r)$ such that the first component of $\zeta_Q(v)$ is v.

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[1] Throughout we use the sign convention that an almost complex structure I is compatible with a symplectic form Ω if $\Omega(I\cdot, \cdot)$ is a positive-definite symmetric form.

[2] Our sign convention is that $d\lambda(X_Q, \cdot) = -dQ$.

Definition 2.1: Let $q \in C^\infty(\Sigma, \mathbb{R}^+)$ denote a smooth positive function, and set $Q = rq$ as above. Let j denote a compatible almost complex structure on ξ. The pair (q, j) determines an almost complex structures[3] J_Q on $S\Sigma$ which, following Albers–Frauenfelder [4], we will call of **twisted** SFT-type, via the formula

$$(9) \qquad J_Q(aX_Q + bY + \zeta_Q(v)) = bX_Q - aY + \zeta_Q(jv), \quad a, b \in \mathbb{R}, \ v \in \xi.$$

Observe that taking $q = 1$ recovers the original notion (7) of an SFT-type almost complex structure (in this case $X_Q = R$). Given $\eta \in T_{(x,r)}(S\Sigma)$, write $\eta = w + cY$, where $w \in T\Sigma$ and $c \in \mathbb{R}$. Setting

$$(10) \qquad a = \frac{\lambda_{(x,r)}(\eta)}{Q(x,r)}, \quad b = \frac{dQ(x,r)\eta}{Q(x,r)},$$

and

$$v := w - aX_q(x),$$

we also have

$$\eta = aX_Q + bY + \zeta_Q(v).$$

If $\eta = a'X_Q + b'Y + \zeta_Q(v')$ is another tangent vector to $S\Sigma$, then

$$(11) \quad \begin{aligned} &d\lambda(J_Q\eta, \eta') \\ &= (dr \wedge \alpha + rd\alpha)(J_Q(aX_Q + bY + \zeta_Q(v)), a'X_Q + b'Y + \zeta_Q(v')) \\ &= (dr \wedge \alpha + rd\alpha)(-bX_Q + aY + \zeta_Q(jv), a'X_Q + b'Y + \zeta_Q(v')) \\ &= dr(-bX_Q + aY)\alpha(a'X_Q) - dr(a'X_Q + b'Y)\alpha(-bX_Q) + rd\alpha(jv, v') \\ &= (brdq(R) + ra)a'q - (-a'rdq(R) + rb')(-bq) + rd\alpha(jv, v') \\ &= Q(aa' + bb') + rd\alpha(jv, v'), \end{aligned}$$

which shows that J_Q is compatible with $d\lambda$. From now on we abbreviate

$$|\eta|_{J_Q}^2 := d\lambda(J_Q\eta, \eta) \geq 0.$$

Let us now fix a "background" Riemannian metric g_Σ on Σ. We denote by $|\cdot|_\Sigma$ the associated norm on $T\Sigma$. Since Σ is compact, it follows from (11) that for any pair (q, j) there exists a constant $\varepsilon_Q > 0$ such that for any $\eta = w + cY \in T_{(x,r)}(S\Sigma)$, one has

$$(12) \qquad |\eta|_{J_Q}^2 \geq \varepsilon_Q r|w|_\Sigma^2.$$

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[3] This notation is slightly inconsistent, since J_Q depends on both Q and j. In general we will suppress the dependence of j in all our notation.

3. The extended maximum principle

Definition 3.1: A connected non-compact $2n$-dimensional symplectic manifold (M, ω) is said to be **modelled on Σ at infinity** if the complement of a compact subset of M is symplectomorphic to part of the symplectisation $S\Sigma$ of Σ. Explicitly, this means there exists a number $0 < r_M < 1$ and a compact domain $M_* \subset M$ with compact closure such that $M \setminus M_*$ is symplectomorphic to $S\Sigma(r_M)$:

$$\iota : M \setminus M_* \to S\Sigma(r_M), \quad \iota^*(d\lambda) = \omega.$$

In particular, $\partial M_* \cong \Sigma$, and outside of M_* the symplectic form ω is exact (although this may not be the case on all of M). More generally, a non-compact symplectic manifold is said to be **convex at infinity** if there exists a closed contact manifold Σ such that M is modelled on Σ at infinity.

Remark 3.2: The assumption that $r_M < 1$ is, of course, not important, and can always be achieved by rescaling. In particular, this notion is equivalent to the one discussed at the start of the Introduction.

Let us now fix a symplectic manifold (M, ω) which is modelled on Σ at infinity. In all of the following we will suppress the ι from our notation. For $r \geq r_M$ we denote $M_r := M \setminus S\Sigma(r)$, and think of M as being the disjoint union $M_r \cup S\Sigma(r)$. Thus $M_* = M_{r_M}$.

Definition 3.3: Let $\mathcal{H} \subset C^\infty(M \times S^1)$ denote the set of time-dependent Hamiltonians $H_t(z)$ on M with the property that there exists $r_M < r_H < 1$ such that the restriction of H to $S\Sigma(r_H)$ is, up to a constant, 1-homogeneous in r:

$$H_t(x, r) = r h_t(x) + c_H, \quad h_t : \Sigma \to \mathbb{R}, \quad \forall (x, r) \in S\Sigma(r_H).$$

Just as with X_Q above, the Hamiltonian vector field X_{H_t} of H_t satisfies

$$X_{H_t}(x, r) = X_{h_t}(x) - dh_t(x) R(x) Y(r), \quad (x, r) \in S\Sigma(r_H),$$

where X_h is the (now time-dependent) vector field on Σ defined by

(13) $$\alpha(X_{h_t}) = h_t, \quad d\alpha(X_{h_t}, \cdot) = dh_t(R)\alpha - dh_t.$$

The vector field X_{h_t} generates a path $\varphi_h^t : \Sigma \to \Sigma$ of contactomorphisms isotopic to the identity. We say that h_t is the **contact Hamiltonian** associated to φ_h^t. The **conformal factor** of φ_h^t is the function $\kappa_h^t : \Sigma \to \mathbb{R}^+$ defined by

$$(\varphi_h^t)^* \alpha = \kappa_h^t \alpha.$$

If $\kappa_h^t = 1$, then we say φ_h^t is a **strict** contactomorphism. The flow ϕ_H^t of the Hamiltonian vector field of H_t is given by

$$(14) \qquad \phi_H^t(x, r) = \left(\varphi_h^t(x), \frac{r}{\kappa_h^t(x)} \right).$$

For later use, let us note that if $Q = rq$ and $H = rh$, then the Poisson bracket is given by

$$(15) \qquad \begin{aligned} \{Q, H\} &:= dQ(X_H) \\ &= (qdr + rdq)(X_h - dh(R)Y) \\ &= -Qdh(R) + rdq(X_h). \end{aligned}$$

From (13) we have

$$\begin{aligned} dq(X_h) &= dq(R)h - d\alpha(X_q, X_h) \\ &= dq(R)h + d\alpha(X_h, X_q) \\ &= dq(R)h + dh(R)q - dh(X_q), \end{aligned}$$

and hence we also obtain

$$(16) \qquad \{Q, H\} = Hdq(R) - rdh(X_q).$$

Definition 3.4: Let $\mathcal{H}_\circ$ denote the set of $H \in \mathcal{H}$ that in addition satisfies the following two conditions:

(1) The contactomorphism φ_h^1 has no fixed points.
(2) All the fixed points of ϕ_H^1 are non-degenerate (i.e., for every fixed point $z \in M$ of ϕ_H^1, 1 is not an eigenvalue of $D\phi_H^1(z) : T_z M \to T_z M$).

Note that by (1), any such fixed point is necessarily contained in the compact manifold M_1.

Definition 3.5: Let $\mathcal{J}$ denote the set of families $J = (J_t)$ of almost complex structures on M which are compatible with ω and which have the property that there exists $r_M < r_J < 1$ and a smooth family q_t of positive functions on Σ and an almost complex structure j on ξ such that on $S\Sigma(r_J)$, J_t agrees with the corresponding family J_{Q_t} of almost complex structures of twisted SFT-type.

Let $\mathcal{L}(M) := C^\infty(S^1, M)$ denote the free loop space. Given $J \in \mathcal{J}$ we define an L^2 scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_J$ on $\mathcal{L}(M)$ by

$$\langle\!\langle \eta_1, \eta_2 \rangle\!\rangle_J := \int_{S^1} \langle \eta_1(t), \eta_2(t) \rangle_{J_t} \, dt, \quad v \in \mathcal{L}(M), \ \eta_1, \eta_2 \in C^\infty(S^1, v^* TM),$$

where $\langle \cdot, \cdot \rangle_{J_t} := \omega(J_t \cdot, \cdot)$. Given $H \in \mathcal{H}$ there is a well-defined one-form $\mathfrak{a}_H$ on the free loop space $\mathcal{L}(M)$, given for $v \in \mathcal{L}(M)$ by

$$\mathfrak{a}_H(v) \cdot \eta = \int_{S^1} \omega_{v(t)}(\eta(t), \dot{v}(t) - X_{H_t}(v(t))) \, dt, \quad \eta \in C^\infty(S^1, v^*TM).$$

This one-form is typically not exact, but becomes exact when we lift it to an appropriate Novikov cover of $\mathcal{L}(M)$, cf. [12]. We will not discuss this here, since it is not important as far as the maximum principle is concerned. The kernel of $\mathfrak{a}_H$ is exactly the one-periodic solutions of the Hamiltonian system determined by H:

$$\ker \mathfrak{a}_H = \{v \in \mathcal{L}(M) \mid \dot{v} = X_H(v)\} \cong \mathrm{Fix}(\phi_H^1).$$

We denote by $\mathfrak{m}_{H,J}$ the vector field on $\mathcal{L}(M)$ dual to $\mathfrak{a}_H$:

$$\mathfrak{a}_H(v) \cdot \eta = \langle\!\langle \mathfrak{m}_{H,J}(v), \eta \rangle\!\rangle_J.$$

Explicitly,

$$\mathfrak{m}_{H,J}(v)(t) = J_t(v)(\dot{v} - X_{H_t}(v(t))).$$

The following lemma will be crucial for our maximum principle.

LEMMA 3.6: *Suppose $H \in \mathcal{H}_\circ$ and $J \in \mathcal{J}$. There exists $B, \varepsilon > 0$ depending on H and J such that if a loop $v : S^1 \to M$ satisfies*

$$\|\mathfrak{m}_{H,J}(v)\|_J < \varepsilon,$$

then $v(S^1) \subset M_B$.

Proof. By assumption on $S\Sigma(1)$ we can write $J_t = J_{Q_t}$ and $H_t = rh_t + c_H$. Set

$$\delta := \min_{(x,t) \in \Sigma \times S^1} q_t(x), \quad \Delta := \max_{(x,t) \in \Sigma \times S^1} q_t(x),$$

so that $0 < \delta \le \Delta$. Define

(17) $\qquad m : [0,1] \times \Sigma \to \mathbb{R}, \quad m(t,x) := \dfrac{h_t(x)dq_t(x)R(x) - dh_t(x)X_{q_t}(x)}{q_t(x)}.$

By compactness, there exists $c \le C$ such that

$$c \le m(t,x) \le C$$

for all (t,x). Choose $A, B > 1$ such that

(18) $\qquad\qquad\qquad\qquad B > \dfrac{1 + \Delta e^C A}{\delta e^c};$

note in particular this implies $B > \frac{\Delta}{\delta}A$. Next, recall the fixed background norm $|\cdot|_\Sigma$ defined just before (12). Since φ_h^1 has no fixed points and Σ is compact, there exists ε_0 such that

$$\int_{S^1} |\dot{x} - X_{h_t}(x)|_\Sigma^2 \, dt \geq \varepsilon_0$$

for any loop x in Σ. If $v(S^1)$ is entirely contained in $S\Sigma(A)$, then writing $v(t) = (x(t), r(t))$ and using (12),

$$\|\mathfrak{m}_{H,J}(v)\|_J^2 \geq \int_{S^1} \varepsilon_{Q_t} r(t) |\dot{x}(t) - X_{h_t}(x(t))|_\Sigma^2 \, dt$$
$$\geq \varepsilon_A := A\varepsilon_0 \max_{t \in S^1} \varepsilon_{Q_t}.$$

If instead $v(S^1)$ is entirely contained in M_B, then since M_B has compact closure and H only has non-degenerate one-periodic orbits there exists ε_B such that if

$$\|\mathfrak{m}_{H,J}(v)\|_J \leq \varepsilon_B$$

then $v(S^1)$ is contained in a small neighbourhood of the periodic orbits of H (cf. [20]), and hence also in M_1. Set

$$\varepsilon_{A,B} := \min\{\sqrt{\varepsilon_A}, \varepsilon_B\}.$$

Then if $\|\mathfrak{m}_{H,J}(v)\|_J < \varepsilon_{A,B}$, then there must exist an interval $[a,b] \subset S^1$ such that $r(a) = A$ and $r(b) = B$, and such that $r(t) \in (A,B)$ for all $t \in (A,B)$. Let

$$\rho(t) := Q_t(v(t)) = q_t(x(t))r(t).$$

Then $\rho(a) \leq A\Delta$ and $\rho(b) \geq \delta B$. Thus by (18) there exists $[a', b'] \subset [a, b]$ such that

$$\rho(a') = \Delta A, \quad \rho(b') = \delta B.$$

Abbreviate $m(t) = m(t, x(t))$, where m was defined in (17). We now look at the Y-component of $\dot{v} - X_{H_t}(v)$. Using (10) and (16) this is

$$\frac{dQ_t(v(t))(\dot{v} - X_{H_t}(v))}{Q_t(v(t)} = \frac{\dot{\rho} - \{Q, H\}}{\rho} = \frac{\dot{\rho} - m\rho}{\rho}.$$

Thus we can estimate

$$\|\mathfrak{m}_{H,J}(v)\|_J \geq \int_{a'}^{b'} \frac{1}{\sqrt{\rho}} |\dot{\rho} + m\rho| \, dt \geq \frac{1}{\sqrt{\delta B}} \int_{a'}^{b'} |\dot{\rho} + m\rho| \, dt.$$

Let

$$n(t) := \exp\left(\int_0^t m(s) \, ds\right),$$

and

$$s(t) := n(t)\rho(t),$$

so that $\dot{s} = \dot{n}\rho + n\dot{\rho} = n(\dot{\rho} + m\rho)$. Thus

$$\|\mathfrak{m}_{H,J}(v)\|_J \geq \frac{1}{e^C\sqrt{\delta B}} \int_{a'}^{b'} |\dot{s}|\, dt$$

$$= \frac{1}{e^C\sqrt{\delta B}}(s(b') - s(a'))$$

$$\geq \frac{1}{e^C\sqrt{\delta B}}(e^c\delta B - e^C\Delta A)$$

$$\geq \frac{1}{e^C\sqrt{\delta B}},$$

where the last line used (18). The lemma follows with B as specified and

$$\varepsilon := \min\left\{\varepsilon_{A,B}, \frac{1}{e^C\sqrt{\delta B}}\right\}.$$

This completes the proof.   ∎

Definition 3.7: Let us say that a homotopy $\mathbf{H} = (H^s) \subset \mathcal{H}$, $s \in \mathbb{R}$ is a **continuation homotopy** of Hamiltonians if:

(1) $\mathbf{H}$ is asymptotically constant, i.e.,

$$H^s = H^0, \quad \forall\, s \leq 0,$$
$$H^s = H^1, \quad \forall\, s \geq 1.$$

(2) The asymptotes H^0, H^1 both belong to $\mathcal{H}_\circ$.
(3) $\mathbf{H}$ is monotonically increasing in s:

$$\frac{\partial H^s}{\partial s} \geq 0.$$

Definition 3.8: Let us say that a homotopy $\mathbf{J} = (J^s) \subset \mathcal{J}$, $s \in \mathbb{R}$ is a **continuation homotopy** of almost complex structures if J^s is asymptotically constant, i.e.,

$$J^s = J^0, \quad \forall\, s \leq 0,$$
$$J^s = J^1, \quad \forall\, s \geq 1.$$

Definition 3.9: Given continuation homotopies $\mathbf{H}$ and $\mathbf{J}$ and a real number $E \geq 0$, we denote by

$$\mathcal{M}_{\mathbf{H},\mathbf{J}}(E)$$

the space of all smooth maps $u : \mathbb{R} \times S^1 \to M$ that are negative flow lines of the s-dependent vector field $\mathfrak{m}_{H^s, J^s}$:

$$\partial_s u + \mathfrak{m}_{H^s, J^s}(u(s, \cdot)) = 0, \tag{19}$$

and which have **energy**

$$\mathbb{E}_{\mathbf{J}}(u) := \int_{-\infty}^{\infty} \langle\!\langle \partial_s u, \partial_s u \rangle\!\rangle_{J^s} \, ds \leq E.$$

Explicitly, (19) means that u solves the partial differential equation

$$\partial_s u + J_t(u)\partial_t u = J_t(u)X_{H_t^s}(u).$$

The main result of this section is the following a priori L^∞ bound on elements of $\mathcal{M}_{\mathbf{H}, \mathbf{J}}(E)$. This result was stated in the Introduction as Theorem 1.1.

THEOREM 3.10 (The Extendend Maximum Principle): *Suppose we are given continuation homotopies* $\mathbf{H}$ *and* $\mathbf{J}$ *and* $E \geq 0$. *Then there exists a constant* $r_0 = r_0(E, \mathbf{H}, \mathbf{J}) > 0$ *such that*

$$u(\mathbb{R} \times S^1) \cap S\Sigma(r_0) = \emptyset, \quad \forall u \in \mathcal{M}_{\mathbf{H}, \mathbf{J}}(E).$$

The proof will take some time, and we will proceed in several stages. Let us write $J_t = J_{Q_t^s}$ and $H_t = rh_s^t + c_{H^s}$ on $S\Sigma(1)$, where $h_t^s : \Sigma \to \mathbb{R}$ is a family of smooth functions such that $\frac{\partial h_s^t}{\partial s} \geq 0$ and $q_t^s : \Sigma \to \mathbb{R}^+$ is a family of smooth positive functions. Both h_t^s and q_t^s only depend on s in $[0, 1]$.

First, we prove that a flow line can only spend a finite time outside of a compact set.

LEMMA 3.11: *There exists* $T > 1$ *and* $L > 0$ *such that if* $u \in \mathcal{M}_{\mathbf{H}, \mathbf{J}}(E)$ *and* Ω *is a connected component of* $u^{-1}(S\Sigma(T))$, *then* Ω *is contained in* $I \times S^1$ *for* $I \subset \mathbb{R}$ *an interval of length at most* L.

Proof. By Lemma 3.6 there exists $\varepsilon_i, B_i > 0$ for $i = 0, 1$ such that if a loop v satisfies

$$\|\mathfrak{m}_{H^i, J^i}(v)\|_{J^i} < \varepsilon_i$$

for $i = 0, 1$, then $v \subset M_{B_i}$. Let $T := \max B_i$ and let $\varepsilon = \min \varepsilon_i$. Thus if $u(s, \cdot) \subset S\Sigma(T)$ for all $s \in [a, b]$, where either $[a, b] \subset (-\infty, 0]$ or $[a, b] \subset [1, \infty)$, then we have

$$E = \int_{\mathbb{R}} \|\partial_s u(s, \cdot)\|_{J^s}^2 \, ds \geq |b - a|\varepsilon^2,$$

and hence $|b - a| \leq E\varepsilon^{-2}$. Thus if Ω is a connected component of $u^{-1}(S\Sigma(T))$, decomposing Ω as

$$\Omega = \Omega^- \cup \Omega_0 \cup \Omega^+,$$

where $\Omega^- = \Omega \cap (-\infty, 0]$, $\Omega_0 = \Omega \cup [0, 1]$, and $\Omega^+ = \Omega \cap [1, \infty)$ (some of which may be empty), we see that Ω has length at most

$$E\varepsilon^{-2} + 1 + E\varepsilon^{-2}.$$

Thus the lemma follows with $L = 2E\varepsilon^{-2} + 1$.   ∎

Let now Ω denote a connected component of $u^{-1}(S\Sigma(T))$. Write

$$u|_\Omega = (w, r),$$

so that $w : \Omega \to \Sigma$ and $r : \Omega \to (T, \infty)$. Our goal is to find a constant r_0 depending on $\mathbf{H}, \mathbf{J}$ and E (but not on u!) such that

$$\sup_{(s,t)\in\Omega} r(s,t) \leq r_0.$$

Consider the function

$$\rho(s,t) := Q_t^s(u(s,t)) = q_t^s(w(s,t))r(s,t).$$

It is also convenient to define the following two families of functions:

$$f_t^s : \Sigma \to \mathbb{R}, \quad g_t^s : \Sigma \to \mathbb{R},$$

where

$$f_t^s(x) := \frac{\partial_s q_t^s(x)}{q_t^s(x)},$$

and

$$g_t^s(x) := \frac{\partial_t q_t^s(x)}{q_t^s(x)} + \left(\frac{dq_t^s(x)(X_{h_t^s}(x))}{q_t^s(x)} - dh_t^s(x)(R(x)) \right).$$

Finally, let k_t^s denote the vector valued function

$$k_t^s : \Sigma \to \mathbb{R}^2, \quad k_t^s = (f_t^s, g_t^s).$$

In the following, to keep the notation free from clutter we will drop the s superscript and the t subscript wherever possible. The key to our argument is the following elliptic differential inequality for $\log \rho$.

PROPOSITION 3.12: *Define*

$$\mu : \Omega \to \mathbb{R}, \quad \mu(s,t) := \log \rho(s,t).$$

Then μ satisfies the second order elliptic differential inequality

$$\Delta\mu + k(w) \cdot \nabla\mu \geq \nabla k(w) \cdot w + k(w) \cdot \nabla w.$$

The proof of Proposition 3.12 is a variation on a standard computation (see, for instance, [24] or [2]). We note however that Proposition 3.12 is not enough to apply the standard maximum principle, since there is no reason why the term on the right-hand side should be uniformly pointwise bounded.

Proof. On Ω we have

$$
\begin{aligned}
\partial_s\rho &= \partial_s q(w) + dQ(u)\partial_s u \\
&= \partial_s q(w)r + dQ(u)(-J(u)(\partial_t u - X_H(u)) \\
&= \frac{\partial_s q(w)}{q(w)}\rho - \lambda(\partial_t u) - \lambda(X_H(u)) \\
&= \frac{\partial_s q(w)}{q(w)}\rho - \lambda(\partial_t u) + H(u).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\partial_t\rho &= \partial_t q(w)r + dQ(u)\partial_t u \\
&= \frac{\partial_t q(w)}{q(w)}\rho + dQ(u)(J(u)\partial_s u + X_H(u)) \\
&= \frac{\partial_t q(w)}{q(w)}\rho + \lambda(\partial_s u) + \{Q, H\}(u) \\
&= \frac{\partial_t q(w)}{q(w)}\rho + \lambda(\partial_s u) + \Big(\frac{dq(X_h(w))}{q(w)} - dh(R(w))\Big)\rho,
\end{aligned}
$$

where the last line used (15). Thus

$$dd^c\rho = u^*\omega - (g(w)\partial_t\rho + f(w)\partial_s\rho + (\partial_t(g(w)) + \partial_s(f(w)))\rho + \partial_s(H(u)))ds \wedge dt.$$

However

$$
\begin{aligned}
u^*\omega =& \omega(\partial_s u, \partial_t u)ds \wedge dt \\
&\times \omega\big(\partial_s u, J(u)\partial_s u + X_H(u)\big)ds \wedge dt \\
&\times (-|\partial_s u|^2 + dH(u)(\partial_s u))ds \wedge dt
\end{aligned}
$$

which means we can alternatively write

$$dd^c\rho = -(|\partial_s u|^2 + g(w)\partial_t\rho + f(w)\partial_s\rho + (\partial_t(g(w)) + \partial_s(f(w)))\rho + \partial_s H(u))ds \wedge dt.$$

Since $dd^c\rho = -\Delta\rho ds \wedge dt$ we obtain

$$(20)\quad \Delta\rho = g(w)\partial_t\rho + f(w)\partial_s\rho + (\partial_t(g(w)) + \partial_s(f(w)))\rho + \partial_s H(u) + |\partial_s u|^2.$$

Next, denoting by $\langle\cdot,\cdot\rangle$ the metric $d\lambda(J_Q\cdot,\cdot)$, we have

$$\begin{aligned}
\langle\partial_s u, X_Q\rangle &= \langle -J(u)(\partial_t u - X_H(u)), X_Q\rangle\\
&= \langle\partial_t u, Y\rangle - \{Q, H\}(u)\\
&= \Big\langle\frac{\partial_t r}{r}Y, Y\Big\rangle - \{Q, H\}(u)\\
&= \partial_t\rho - g(w)\rho
\end{aligned}$$

where we used (11) and (15). Similarly

$$\langle\partial_s u, Y\rangle = \partial_s\rho - f(w)\rho.$$

The norm of $\partial_s u$ can be estimated from below by its projection onto the (X_Q, Y)-plane, which gives us

$$\begin{aligned}
|\partial_s u|^2 &\geq \frac{1}{\rho}\langle\partial_s u, X_Q\rangle^2 + \frac{1}{\rho}\langle\partial_s u, Y\rangle^2\\
&= \frac{1}{\rho}|\nabla\rho|^2 + \rho(g(w)^2 + f(w)^2) - 2(\partial_t\rho \cdot g(w) + \partial_s\rho \cdot f(w))\\
&\geq \frac{1}{\rho}|\nabla\rho|^2 - 2(\partial_t\rho \cdot g(w) + \partial_s\rho \cdot f(w)).
\end{aligned}$$

From the identity

$$\Delta\mu = \frac{1}{\rho}\Delta\rho - \frac{1}{\rho^2}|\nabla\rho|^2,$$

and using $\partial_s H \geq 0$, we obtain from (20) that

$$\Delta\mu + k(w) \cdot \nabla\mu \geq \partial_s(f(w)) + \partial_t(g(w)),$$

which is what we wanted to prove. ∎

We can now complete the proof of Theorem 3.10.

Proof of Theorem 3.10. The Aleksandrov integral version of the weak maximum principle ([10, Theorem 9.1], see also [2, Appendix A]) tells us that

$$\sup_{\Omega} \mu \leq \sup_{\partial \Omega} \mu + C \|\nabla k(w) \cdot w + k(w) \cdot \nabla w\|_{L^2(\Omega)}$$
$$= \log T + C \|\nabla k(w) \cdot w + k(w) \cdot \nabla w\|_{L^2(\Omega)},$$

where C depends on the L^2-norm of $k(w)$ and the diameter of Ω. The diameter of Ω is bounded thanks to Lemma 3.11. Since Σ is compact and H^s and J^s are asymptotically constant, it is clear that $k(w)$ and $\nabla k(w)$ are bounded in $L^2(\Omega)$.

It remains to see that ∇w is bounded in $L^2(\Omega)$. It suffices to show that $\partial_s w$ and $\partial_t w$ are bounded in L^2 using our fixed background norm $|\cdot|_\Sigma$, i.e., that

$$\int_\Omega |\partial_s w|_\Sigma^2 \quad \text{and} \quad \int_\Omega |\partial_t w|_\Sigma^2$$

are bounded. That $\partial_s w$ is bounded is easy, since from (12) we have (temporarily writing all the sub/superscripts)

$$|\partial_s w(s,t)|_\Sigma^2 \leq \frac{1}{\varepsilon_{Q_t^s} r(s,t)} |\partial_s u(s,t)|_{J_{Q_t^s}}^2.$$

Thus setting $\varepsilon = \min_{s,t} \varepsilon_{Q_t^s}$ we have

$$\int_\Omega |\partial_s w|_\Sigma^2 \leq \frac{1}{\varepsilon T} \int_\Omega \|\partial_s u\|^2 \leq \frac{E}{\varepsilon T}.$$

The $\partial_t w$ term is a little trickier: for this we use the Floer equation to write $\partial_t u = J \partial_s u + X_H(u)$, which gives

$$\|\partial_t u\|^2 \leq 2\|\partial_s u\|^2 + 2\|X_H(u)\|^2.$$

Then the same estimate gives

$$|\partial_t w|_\Sigma^2 \leq \frac{1}{\varepsilon r} |\partial_t u|^2$$
$$= \frac{1}{\varepsilon r} |J \partial_s u + X_H(u)|^2$$
$$\leq \frac{2}{\varepsilon r} (|\partial_s u|^2 + |X_H(u)|^2).$$

From (8) and (11) we see that $\frac{1}{r}|X_H(u)|^2$ is bounded, and this completes the proof. ∎

4. Symplectic homology revisited

Let us recall the definition of the symplectic homology $\mathrm{SH}_*(M)$ of an exact symplectic manifold M modelled on a contact manifold Σ at infinity. Our approach is slightly non-standard as we will use Hamiltonians which are of the form (3) at infinity (our main result, Theorem 3.10 implies this is well-defined). For brevity we will work with Liouville manifolds only, rather than the technically more complicated setting of a weak$^+$ monotone symplectic manifold as referred to in Theorem 1.3.

Let $h_t : \Sigma \to \mathbb{R}$ be a 1-periodic contact Hamiltonian such that the time-1 map $\varphi_h^1 : \Sigma \to \Sigma$ of the corresponding contact isotopy has no fixed points. We call such contact Hamiltonians **admissible**.

Floer data for h consists of a (time-dependent) Hamiltonian $H \in \mathcal{H}_\circ$ and an almost complex structure $J \in \mathcal{J}$ such that (up to a constant) $H_t(x, r) = r h_t(x)$ at infinity, and such that the pair (H, J) is **regular** in the sense that the linearisation of the Floer operator

$$u \mapsto \partial_s u + J_t(u) \partial_t u - J_t(u) X_{H_t}(u)$$

at any zero is surjective. Regularity is a generic condition, and it ensures that the moduli spaces of the solutions of the Floer equation are actually manifolds. In the situation above, we will say that h is the **slope** of H.

Given Floer data (H, J), one can construct a chain complex $\mathrm{CF}_*(H, J)$ which is generated (as a $\mathbb{Z}_2$ vector space) by 1-periodic Hamiltonian orbits of H and graded by a (negative) Conley–Zehnder index [20, Section 2.4]. The homology of the chain complex $\mathrm{CF}_*(H, J)$ is denoted by $\mathrm{HF}_*(H, J)$, and called Floer homology. For more details on Floer homology, and proofs of the various assertions made in these paragraphs, see for instance [20].

The Floer homologies $\mathrm{HF}_*(H, J)$ and $\mathrm{HF}_*(H', J')$ associated to two different sets (H, J) and (H', J') for h are canonically isomorphic. This leads to the Floer homology $\mathrm{HF}_*(h)$ which is associated to the admissible contact Hamiltonian h. The dependence on the admissible contact Hamiltonian is essential, namely the Floer homologies corresponding to different contact Hamiltonians are in general not isomorphic.

Given admissible contact Hamiltonians $h_t, h_t' : \Sigma \to \mathbb{R}$ such that $h_t \leq h_t'$, using Theorem 3.10, one can construct a **continuation map**

$$\mathrm{HF}_*(h) \to \mathrm{HF}_*(h').$$

The continuation maps turn the set $\{\mathrm{HF}_*(h) \mid h \text{ admissible}\}$ into a directed system of groups indexed by admissible contact Hamiltonians (with the standard order relation—pointwise $\leq$). The **symplectic homology** is the direct limit of this system

$$\mathrm{SH}_*(M) := \varinjlim_{h} \mathrm{HF}_*(h).$$

Note that this definition coincides with the classical finite-slope definition of symplectic homology [27]. This is due to constant admissible contact Hamiltonians forming a cofinal subset of the set of all admissible contact Hamiltonians.

Apart from continuation maps, there is yet another important class of morphisms between $\mathrm{HF}_*(h)$, the so-called **naturality isomorphisms**. They are associated to the loops of Hamiltonian diffeomorphisms on M. We first introduce some notation: Let $\phi_t : M \to M$ be an isotopy, let $H_t : M \to \mathbb{R}$ be a Hamiltonian, let J_t be a family of almost complex structures on M, and let $\gamma : S^1 \to M$ be a loop on M. Denote

$$(\phi^* H)_t(x) := H_t(\phi_t(x)),$$
$$(\phi^* J)_t := \phi_t^* J_t,$$
$$(\phi^* \gamma)(t) := \phi_t^{-1}(\gamma(t)).$$

PROPOSITION 4.1: *Let $h_t : \Sigma \to \mathbb{R}$ be an admissible contact Hamiltonian, let (H, J) be Floer data for h, and let $G \in \mathcal{H}$ be a 1-periodic Hamiltonian that generates a loop of Hamiltonian diffeomorphisms. Denote by $g_t : \Sigma \to \mathbb{R}$ the slope of G. Then, the contact Hamiltonian $\tilde{h}_t : \Sigma \to \mathbb{R}$ defined by*

$$\tilde{h}_t(x) := \frac{(h_t - g_t)(\varphi_g^1(x))}{\kappa_g^t(x)}$$

is admissible, and $((\phi_K)^* H, (\phi_G)^* J)$ is Floer data for $\tilde{h}$. The linear map

$$CF(H, J) \to CF((\phi_K)^* H, (\phi_G)^* J)$$

defined on generators by

$$\gamma \mapsto (\phi_G)^* \gamma$$

is an isomorphism. We denote the induced isomorphism on the homology level by

$$\mathcal{N}(G) \,:\, HF(h) \to HF(\tilde{h}),$$

and call it the naturality isomorphism.

Proof. See the proof of [26, Lemma 2.27]. ∎

Finally, let us discuss the proofs of Theorem 1.5. It is essentially identical to the argument from [26, Theorem 4.13], modulo use of the new maximum principle. Nevertheless for the reader's convenience we will provide a proof.

Proof of Theorem 1.5. Let $h_t : \Sigma \to \mathbb{R}^+$, $t \in S^1$, be a positive, 1-periodic contact Hamiltonian that generates a loop φ_h^t of contactomorphisms. As in (6), let $\beta : \mathbb{R}^+ \to [0,1]$ be a smooth function

$$\beta : \mathbb{R}^+ \to [0,1], \quad \beta(r) = \begin{cases} 0, & r \in (0, 1/4), \\ 1, & r \in (3/4, \infty), \end{cases} \quad \beta'(r) \geq 0.$$

Denote by $H_t : M \to \mathbb{R}$ the Hamiltonian that is equal to 0 on $M \setminus S\Sigma$ and to $(x, r) \mapsto \beta(r) r h_t(x)$ on $S\Sigma$. The time-1 map ϕ_H^1 represents the class $\Theta([\varphi_h^t])$ in the symplectic mapping class group. Assume, by contradiction, that the class $[\phi_H^1]$ is trivial. This means there exists a compactly supported Hamiltonian $L_t : M \to \mathbb{R}$ such that

$$\phi_H^1 = \phi_L^1,$$

and such that $L_{t+1} = L_t$. Consider the Hamiltonian $G_t : M \to \mathbb{R}$ that generates the isotopy $\phi_H^t \circ (\phi_L^t)^{-1}$. Note that G generates a loop of Hamiltonian diffeomorphisms. Denote by $G^m : M \to \mathbb{R}$ the Hamiltonian

$$G_t^m := \begin{cases} mG_{mt} & t \in [0, \frac{1}{m}], \\ mG_{mt} & t \in [\frac{1}{m}, \frac{2}{m}], \\ \vdots \\ mG_{mt} & t \in [1 - \frac{1}{m}, 1], \end{cases}$$

and by g^m the corresponding slopes. The isotopy of the Hamiltonian G^m is equal to the concatenation of m copies of $\{\phi_G^t\}$. Let $\varepsilon > 0$ be a small enough positive number. Denote by $f^m : \Sigma \to \mathbb{R}$ the admissible (cf. Proposition 4.1) slope

$$f_t^m(x) := \varepsilon \cdot \kappa_{g^m}^t \circ (\varphi_{g^m}^t)^{-1} + g_t^m.$$

The naturality with respect to G^m provides the isomorphism

(21) $\mathrm{HF}(f^m) \to \mathrm{HF}(\varepsilon).$

Hence the groups $\mathrm{HF}(f^m)$ are all isomorphic to the singular homology $\mathrm{H}(M_1, \Sigma; \mathbb{Z}_2)$. Since $\min f^m \geq m \cdot \min h$, and since $\min h > 0$, for each admissible $a \in \mathbb{R}$ the map

$$\mathrm{HF}_*(a) \to \mathrm{SH}_*(M; \mathbb{Z}_2)$$

factors through $\mathrm{HF}_*(f^m)$ for large enough m. Hence

$$\dim \mathrm{SH}(M; \mathbb{Z}_2) \leq \dim \mathrm{H}(M_1, \Sigma; \mathbb{Z}_2) = \dim \mathrm{H}(M; \mathbb{Z}_2).$$

Contradiction! Therefore, the class $\Theta([\varphi_h^t])$ is not trivial. Since the iterates of the loop $\{\varphi_h^t\}$ are generated by positive contact Hamiltonians as well, the same argument shows that the class $\Theta([\varphi_h^t])$ is in fact of infinite order in the symplectic mapping class group. ∎

5. Translated points and the positive symplectic homology of a contactomorphism

Definition 5.1: Let (Σ, α) be a co-oriented contact manifold with a choice of contact form. Let $\theta^t : \Sigma \to \Sigma$ denote the Reeb flow of α. Let $\varphi : \Sigma \to \Sigma$ denote a contactomorphism. Let $\kappa : \Sigma \to \mathbb{R}^+$ denote the conformal factor of φ and α, so that

$$\varphi^* \alpha = \kappa \alpha.$$

A **translated point** of φ and α is a point $x \in \Sigma$ such that there exists $T \in \mathbb{R}$ such that

$$\varphi(x) = \theta^{-T}(x), \quad \kappa(x) = 1.$$

We call the minimal (in absolute value) such T the **period** of x (the word "minimal" is included to deal with the case where the translated point lies on a closed Reeb orbit of α). This notion was introduced by Sandon in [22], although it can be seen as a special case of a leaf-wise intersection point as introduced by Moser in [13].

In this section we show how the maximum principle proved here can be used to prove that if a Liovuille domain has infinite-dimensional symplectic homology, then any contactomorphism of the boundary isotopic to the identity through contactomorphisms has infinitely many translated points. This result is not new; in [5] the same result was obtained using Rabinowitz Floer homology (and symplectic homology is infinite-dimensional if and only if Rabinowitz Floer homology is; cf. [8]). We then go on to explain how by working with the analogue

of positive symplectic homology, we can define a more refined invariant which we call the positive symplectic homology of a contactomorphism. Its properties will be elucidated in a sequel to the present paper.

Let $(M, d\lambda)$ denote an exact symplectic manifold modelled on Σ at infinity as before. Let $\varphi^t : \Sigma \to \Sigma$ be a path of contactomorphisms with $\varphi_0 = \mathrm{Id}$, and let $h_t : \Sigma \to \mathbb{R}$ denote the contact Hamiltonian associated to φ^t.

After reparametrising φ^t if necessary, we may assume h_t is 1-periodic in t. Let $\kappa^t : \Sigma \to \mathbb{R}^+$ denote the conformal factor of φ^t. Denote by $\theta^t : \Sigma \to \Sigma$ the Reeb flow of $\alpha := \lambda|_\Sigma$.

Fix a small $\varepsilon > 0$. Let f^c denote the function on Σ given by

$$f_t^c(x) = h_t(x) + c\kappa_t(x),$$

and let F^c denote a Hamiltonian on M such that

(22) $$F_t^c(x) = r f_t^c(x), \quad (x, r) \in \Sigma \times (\varepsilon, \infty).$$

The 1-periodic orbits of this Hamiltonian contained in $\Sigma \times (\varepsilon, \infty)$ correspond to translated points of φ^1 with period c. Indeed, the Hamiltonian flow of F_t^c is given by

$$\phi_{F^c}^t(x, r) = \left(\varphi^t(\theta^{ct}(x)), \frac{r}{\kappa^t(\theta^{ct}(x))} \right),$$

and hence if $\phi_{F^c}^1(x, r) = (x, r)$ then $\theta^c(x)$ is a translated point of φ^1 with period c.

In particular, if c is not the period of a translated point of φ^1 then f^c is admissible, and hence $\mathrm{HF}(f^c)$ is well-defined. Since κ^t is positive, if $c \le c'$ then $f^c \le f^{c'}$, and hence there is a well-defined continuation map

$$\mathrm{HF}(f^c) \to \mathrm{HF}(f^{c'}).$$

This gives almost immediately the following result.

PROPOSITION 5.2: *Suppose* $\mathrm{SH}_*(M)$ *is infinite-dimensional. Then any contactomorphism of the boundary isotopic to the identity through contactomorphisms either has infinitely many translated points or a translated point on a closed Reeb orbit.*

Proof. If not, then there exists a maximum C such that φ^1 has no translated points with period greater than c. The usual continuation arguments then show that

$$\mathrm{HF}(f^c) \cong \mathrm{HF}(f^{c'}), \quad \forall C < c < c'.$$

Since $\mathrm{HF}(f^c)$ is finitely generated for any finite c (by assumption), it follows that

$$\varinjlim_c \mathrm{HF}(f^c)$$

is finite-dimensional. But since f^c forms a cofinal family, this direct limit agrees with $\mathrm{SH}(M)$, a contradiction. ∎

The group

$$\varinjlim_c \mathrm{HF}(f^c)$$

is not so interesting (as an invariant of φ^t) since it just agrees with the symplectic homology of M. But we can rectify this as follows. Suppose φ^1 has no fixed points, so that $\mathrm{HF}(h)$ is well-defined. Let $p > 0$ be such that any translated point x of φ^1 with period $T > 0$ has $T > p$ (such p exists as φ^1 has no fixed points.)

Let H_t be a Hamiltonian on M which agrees with $rh_t(x)$ on $\Sigma \times (\varepsilon, \infty)$ and is C^2 small and Morse on the rest of M. Let $A > 0$ be such that any critical point $z(t)$ of the action functional

$$\mathcal{A}_H(z) := \int_{S^1} z^* \lambda - \int_{S^1} H(z)\, dt$$

has action $\mathcal{A}_H(z) \leq A$ (this is well defined since φ^1 has no fixed points).

Fix now a number $c > p$ which is not the period of a translated point of φ^1 and choose $\varepsilon < r_0 < r_1$. Consider a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$g(r) = 0, \quad \forall\, 0 \leq r \leq r_0, \quad \text{and} \quad g'(r) = c, \quad \forall\, r \geq r_1,$$

and

$$rg'(r) - g(r) > 0, \quad \forall\, r > r_0.$$

We now modify the function F^c from (22) to a new function $\tilde{F}^c : M \to \mathbb{R}$ by requiring that

$$\tilde{F}_t^c(x, r) = rh_t(x) + g(\kappa_t(x)) \quad \forall\, (x, r) \in \Sigma \times (\varepsilon, \infty).$$

For an appropriate choice of $r_0 < r_1$ (depending on p and A), the critical points of $\mathcal{A}_{\tilde{F}^c}$ come in two forms:

- Critical points $z(t)$ corresponding to translated points of φ^1 with period less than c and action $\mathcal{A}_{\tilde{F}^c}(z) > A$.
- Critical points $z(t)$ of $\mathcal{A}_H$ with action $\mathcal{A}_{\tilde{F}^c}(z) \leq A$.

Let $\mathrm{CF}^+(f^c)$ denote the subcomplex generated by those quotienting out those critical points with action less than or equal to A. Let $\mathrm{HF}^+(f^c)$ denote the associated homology. Thus by construction this homology is generated by the translated points of φ^1 with period less than c. Now set

$$\mathrm{SH}^+(\varphi^t, M) := \varinjlim_c \mathrm{HF}^+(f^c).$$

We call $\mathrm{SH}^+(\varphi^t, M)$ the **positive symplectic homology for the contact isotopy** φ^t. It is defined for any isotopy with $\varphi^1 \neq \mathrm{Id}$. By construction we obtain a long exact sequence

$$\cdots \to \mathrm{HF}_*(h) \to \mathrm{SH}_*^+(\varphi^t, M) \to \mathrm{SH}_*(M) \to \mathrm{HF}_{*-1}(h) \to \cdots.$$

If $\varphi^t = \theta^{at}$ is a piece of Reeb flow, where $a > 0$ is small, then

$$\mathrm{HF}_*(h) = \mathrm{H}_{*+n}(M, \Sigma)$$

and thus $\mathrm{SH}^+(\varphi^t, M)$ agrees with the **positive symplectic homology** $\mathrm{SH}^+(M)$ of M. But in general this is not true. Up to a grading shift, these groups only depend on the homotopy class of the isotopy φ^t. The groups $\mathrm{SH}^+(\varphi^t, M)$ can be used to give more interesting results on the existence of translated points than Proposition 5.2, and we will continue its study in this paper's sequel.

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