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GROWTH OF CENTRAL POLYNOMIALS OF VERBALLY PRIME ALGEBRAS

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ABSTRACT

We compute the exact asymptotics of the codimension sequence for the central polynomials of $k \times k$ matrices and show that it is asymptotic to $\frac{1}{k^2}$ times the ordinary cocharacter. For the other verbally prime algebras we show that these sequences are bounded above and below by constants times the ordinary codimensions.

1. Introduction

Let A be a p. i. algebra over the characteristic 0 field F, let $Id(A)$ denote the identities of A, and let $Id^Z(A)$ denote the space of central polynomials. Our convention is that polynomial identities are considered to be central polynomials, and polynomials in $Id^Z(A)$ but not in $Id(A)$ will be called proper central polynomials. The second author studied the codimension growth of central

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polynomials in [7]. The notations are the usual ones for p. i. algebras: V_n is the space of degree n multilinear polynomials in $x_1, \ldots, x_n$ and

$$c_n(A) = \dim \frac{V_n}{V_n \cap Id(A)} \quad \text{and} \quad c_n^Z(A) = \dim \frac{V_n}{V_n \cap Id^Z(A)}.$$

We call these numbers the codimensions and Z-codimensions of A, respectively. It is also of interest to study the gap between these two numbers

$$\delta_n(A) = c_n(A) - c_n^Z(A). \tag{1}$$

If $D_n(A)$ is the quotient space $(V_n \cap Id^Z(A))/(V_n \cap Id(A))$, then $\delta_n(A)$ is the dimension of $D_n(A)$. In [4] Giambruno and Zaicev consider the central polynomials of finite-dimensional algebras. They proved that the limits

$$\lim_{n\to\infty} (c_n^z(A))^{1/n} \quad \text{and} \quad \lim_{n\to\infty} (\delta_n(A))^{1/n}$$

both exist and are integers which they denote $\exp^Z(A)$ and $\exp^\delta(A)$, and moreover that $\exp^Z(A) = \exp(A)$. In the special case of $A = M_k(F))$ this gives Regev's theorem that $\exp^Z(M_k(F))$ equals k^2; see [7].

In the current paper we first use the theory of trace polynomials to determine the exact asymptotics of the Z-codimensions of $M_k(F)$. If $\{a_n\}$ and $\{b_n\}$ are sequences, we say they are asymptotic and write $a_n \simeq b_n$ if $\lim a_n/b_n = 1$. We let V_n^{PTR} and V_n^{MTR} represent, respectively, the spaces of degree n, multilinear pure trace and mixed trace polynomials in $x_1, \ldots, x_n$. For example, $x_1 tr(x_2)$ would be in V_2^{MTR} but not V_2^{PTR} and $tr(x_1)tr(x_2)$ would be in both. Then pure trace and mixed trace codimensions, $c_n^{PTR}(A)$ and $c_n^{MTR}(A)$, are defined to be the quotients of these two spaces by the (pure or mixed) trace identities of A which we denote $Id^{tr}(A)$, namely,

$$c_n^{PTR}(A) = \dim\left(\frac{V_n^{PTR}}{V_n^{PTR} \cap Id^{tr}(A)}\right)$$

and

$$c_n^{MTR}(A) = \dim\left(\frac{V_n^{MTR}}{V_n^{MTR} \cap Id^{tr}(A))}\right).$$

It was shown in [6] that $c_n(M_k(F))$ and $c_n^{MTR}(M_k(F))$ are asymptotically equal. Using the techniques of that paper we will show

Theorem: $c_n^Z(M_k(F)) \simeq c_n^{PTR}(M_k(F)) \simeq \frac{1}{k^2} c_n(M_k(F))$.

This immediately gives that the gap $\delta_n(M_k(F)) \simeq \frac{k^2-1}{k^2} c_n(M_k(F))$. It also completely determines the asymptotics of $c_n^Z(M_k(F))$ and $\delta_n(M_k(F))$ since it was proven in [6] that

$$c_n(M_k(F)) \simeq a(k) n^{-\frac{k^2-1}{2}} k^{2n}, \tag{2}$$

where $a(k)$ was shown in [6] to equal

$$\Big(\frac{1}{\sqrt{2\pi}}\Big)^{k-1}\Big(\frac{1}{2}\Big)^{k^2-1/2} \cdot 1! \cdots (k-1)! \cdot k^{k^2/2}.$$

Turning from $M_k(F)$ to the other verbally prime algebras, $M_{k,\ell}$ and $M_k(E)$, we are able to estimate the Z-codimensions less precisely. It is useful to use the Θ notation, so that $f(n) = \Theta(g(n))$ will mean $C_1 g(n) \le f(n) \le C_2 g(n)$ for unspecified positive constants C_1 and C_2. Our main result is that if A is verbally prime, then

Theorem: $c_n^Z(A),\ \delta_n(A) = \Theta(c_n(A))$.

It was shown in [2] that

$$c_n(M_{k,\ell}) = \Theta(n^{-(k^2+\ell^2-1)/2}(k+\ell)^{2n})$$

and so $\delta(M_{k,\ell})$ and $c_n^Z(M_{k,\ell})$ are each $\Theta(n^{-(k^2+\ell^2-1)/2}(k+\ell)^{2n})$. For $c_n(M_k(E))$ we have only the less precise result $\exp(M_k(E)) = 2k^2$ (see [2]) implying that

$$\exp^Z(M_k(E)) = 2k^2.$$

A bit more precisely, $c_n(M_k(E))$ is bounded below by a constant times $n^{-(2k^2-1)/2}(2k^2)^n$ and above by a constant times $n^{-(k^2-1)/2}(2k^2)^n$ and so the same will be true of $c_n^Z(M_k(E))$ and $\delta_n(M_k(E))$,

$$C_1 n^{-(2k^2-1)/2}(2k^2)^n \le c_n^Z(M_k(E)),\ \delta_n(M_k(E)) \le C_2 n^{-(k^2-1)/2}(2k^2)^n$$

for some positive constants C_1, C_2.

We conclude this introduction with a series of related conjectures.

Conjecture 1: $c_n^Z(M_{k,\ell}) \simeq \frac{1}{(k+\ell)^2} c_n(M_{k,\ell})$.

This conjecture is equivalent to $\delta_n(M_{k,\ell}) \simeq (1 - \frac{1}{(k+\ell)^2}) c_n(M_{k,\ell})$. The next conjecture is more speculative, but is known for $k = 1$; see [7].

Conjecture 2: $c_n^Z(M_k(E)) \simeq \frac{1}{2k^2} c_n(M_k(E))$.

And this is equivalent to $\delta_n(M_k(E)) \simeq (1 - \frac{1}{2k^2}) c_n(M_k(E))$.

And, even more speculatively:

CONJECTURE 3: *For any p. i. algebra A there exists a constant $0 \leq \alpha \leq 1$ such that $c_n^Z(A) \simeq \alpha c_n(A)$.*

If $\alpha \neq 1$, then this conjecture would imply that $\delta_n(A)$ is asymptotic to $(1-\alpha)c_n(A)$. However, in the $\alpha = 1$ case it is possible for $\delta_n(A)$ to have smaller exponential rate of growth than $c_n(A)$; see Corollary 4 of [4].

Our last conjecture is not on the subject on asymptotics, but we think it is interesting and take the opportunity to include it.

CONJECTURE 4: *If A and B each have proper central polynomials, so does $A \otimes B$.*

2. Exact asymptotics for $M_k(F)$

The computation of the asymptotics of $c_n^Z(M_k(F))$ in this section will parallel the computation of $c_n(M_k(F))$ in [6], some key ideas of which come from [3]. Our main result of this section will be to show that $c_n^Z(M_k(F))$ is asymptotic to the pure trace codimensions of $M_k(F)$. Throughout this section we let $A = M_k(F)$.

As in the introduction, let V_n^{PTR} and V_n^{MTR} be, respectively, the spaces of degree n, multilinear pure trace polynomials and mixed trace polynomials in $x_1, \ldots, x_n$. Letting $Id^{tr}(A)$ be the trace identities of $A = M_k(F)$ we define $c_n^{PTR}(A)$ and $c_n^{MTR}(A)$ in the usual way, namely,

$$c_n^{-TR}(A) = \dim \frac{V_n^{-TR}}{V_n^{-TR} \cap Id^{tr}(A)}$$

where we write $-TR$ as a shorthand for either PTR or MTR.

The map

$$f(x_1, \ldots, x_n) \mapsto tr(f(x_1, \ldots, x_n)x_{n+1})$$

is a linear isomorphism $V_n^{MTR} \to V_{n+1}^{PTR}$, and since the trace is non-degenerate it also affords a linear isomorphism

$$\frac{V_n^{MTR}}{V_n^{MTR} \cap Id^{tr}(A)} \to \frac{V_{n+1}^{PTR}}{V_{n+1}^{PTR} \cap Id^{tr}(A)}.$$

Hence,

$$c_n^{MTR}(A) = c_{n+1}^{PTR}(A) \quad \text{for all } n \geq 0. \tag{3}$$

We now use this equation to prove

THEOREM 2.1: $c_n^{PTR}(M_k(F)) \simeq \frac{1}{k^2} c_n(M_k(F))$.

Proof. By (3), $c_n^{PTR}(A) \simeq c_{n-1}^{MTR}(A)$, and it was proved in [6] that

$$c_n(A) \simeq c_n^{MTR}(A).$$

By (2),

$$\begin{aligned} c_{n-1}(A) &\simeq \alpha(k)(n-1)^{-(k^2-1)/2}(k^2)^{n-1} \\ &\simeq \frac{1}{k^2}\alpha(k)n^{-(k^2-1)/2}(k^2)^n \\ &\simeq \frac{1}{k^2}c_n(A) \end{aligned}$$

and the theorem follows. ∎

Definition 2.2: The symmetric group acts on each of V_n, V_n^{PTR} and V_n^{MTR} by permuting the x_i and the intersections

$$V_n \cap Id^Z(A),\ V_n^{-TR} \cap Id^{tr}(A)$$

are submodules. Taking quotients we define the three cocharacters

$$\begin{aligned} \chi_n^Z(A) &= \chi_{S_n}\Big(\frac{V_n}{V_n \cap Id^Z(A)}\Big), \\ \chi_n^{PTR}(A) &= \chi_{S_n}\Big(\frac{V_n^{PTR}}{V_n^{PTR} \cap Id^{tr}(A)}\Big), \\ \chi_n^{MTR}(A) &= \chi_{S_n}\Big(\frac{V_n^{MTR}}{V_n^{MTR} \cap Id^{tr}(A)}\Big). \end{aligned}$$

Each of these characters decomposes into a sum of irreducible S_n-characters. Following [3] we denote the multiplicities of the irreducible components as c_λ, $\bar{c}_\lambda$ and $\bar{r}_\lambda$:

$$\chi_n^Z(A) = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda, \quad \chi_n^{PTR}(A) = \sum_{\lambda \vdash n} \bar{c}_\lambda \chi^\lambda, \quad \chi_n^{MTR}(A) = \sum_{\lambda \vdash n} \bar{r}_\lambda \chi^\lambda.$$

Taking dimensions we get the cocharacters

$$c_n^Z(A) = \sum_{\lambda \vdash n} c_\lambda d_\lambda, \quad c_n^{PTR}(A) = \sum_{\lambda \vdash n} \bar{c}_\lambda d_\lambda, \quad c_n^{MTR}(A) = \sum_{\lambda \vdash n} \bar{r}_\lambda d_\lambda, \tag{4}$$

where d_λ is the degree of χ^λ.

We now collect the theorems which together will imply our main result. Part (2) is Theorem 12 of [3], part (3) is Theorem 16 of [3], part (4) follows from (1) and (2), but was first proven in [5], parts (5) and (6) follow from [6], part (7) occurs in the proof of Theorem 1 in [4], and part (8) is proven in [6].

THEOREM 2.3: *For c_λ, $\bar{c}_\lambda$ and $\bar{r}_\lambda$ as above:*

1. $0 \le c_\lambda \le \bar{c}_\lambda \le \bar{r}_\lambda$ *for all* λ.
2. *The multiplicities* $\bar{c}_\lambda$ *and* $\bar{r}_\lambda$ *are zero unless* λ *has at most* k^2 *parts.*
3. *There exists an integer* d *so that if* $\lambda_{k^2} \ge d$ *then* $c_\lambda = \bar{c}_\lambda$.
4. c_λ *is zero unless* λ *has at most* k^2 *parts.*
5. $c_n(A)$ *is asymptotic to a rational function times* k^{2n}.
6. $c_n(A)$ *and* $c_n^{MTR}(A)$ *are asymptotically equal.*
7. $c^Z(A)$ *is bounded above and below by rational functions times* k^{2n}.
8. *The sum* $\sum \bar{r}_\lambda d_\lambda$ *over* λ *of height at most* k^2 *and* $\lambda_{k^2} \le d$ *is bounded by a rational function times* $(k^2-1)^n$.

Here is our main theorem of this section:

THEOREM 2.4: $c_n^Z(A) \simeq c_n^{PTR}(A) = c_{n-1}^{MTR}(A) \simeq c_{n-1}(A)$.

Proof. Only $c_n^Z \simeq c_n^{PTR}$ requires proof: The equality is (3) and the second asymptotic equation is Theorem 2.3(6).

By Theorem 2.3(7) $c_n^Z(A)$ has exponential rate of growth k^2, and by Theorems 2.1 and 2.3(5) so does $c_n^{PTR}(A)$. Since $c_n^Z(A) \le c_n^{PTR}(A)$ we need only show that $c_n^{PTR}(A) - c_n^Z(A)$ has smaller exponential rate of growth. Using the previous theorem

$$
\begin{aligned}
c_n^{PTR}(A) - c_n^Z(A) &= \sum_{\lambda\vdash n} \bar{c}_\lambda d_\lambda - \sum_{\lambda\vdash n} c_\lambda d_\lambda && \text{(by equation (4))}\\
&= \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2}} \bar{c}_\lambda d_\lambda - \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2}} c_\lambda d_\lambda && \text{(by Theorem 2.3(2), (4))}\\
&= \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2}} (\bar{c}_\lambda - c_\lambda) d_\lambda \\
&= \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2,\ \lambda_{k^2}<d}} (\bar{c}_\lambda - c_\lambda) d_\lambda && \text{(by Theorem 2.3(3))}\\
&\le \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2,\ \lambda_{k^2}<d}} \bar{c}_\lambda d_\lambda \\
&\le \sum_{\substack{\lambda\vdash n\\ ht(\lambda)\le k^2,\ \lambda_{k^2}<d}} \bar{r}_\lambda d_\lambda && \text{(by Theorem 2.3(1))}
\end{aligned}
$$

which has exponential rate of growth at most k^2-1 by part 8 of Theorem 2.3. ∎

3. Asymptotics for $M_{k,\ell}$ and $M_k(E)$

In this section we let A be one of $M_{k,\ell}$ or $M_k(E)$. Our main lemma will be that $c_n^Z(A)$ and $\delta_n(A)$ are each bounded below by a positive constant times $c_n(A)$. Since $c_n^Z(A)$ and $\delta_n(A)$ are each bounded above by $c_n(A)$ by (1), this will show

$$\delta_n(A),\, c_n^Z(A) = \Theta(c_n(A)).$$

We begin with this elementary observation.

Lemma 3.1: *Given $a \in M_{k,\ell}$ non-zero, or given $a \in M_k(E)$ non-zero and homogeneous, and given an index j, there exist $b, c \in A$ such that bac is equal to an element of E_0 (a central element of E) times the matrix unit e_{jj}.*

Proof. Let

$$a = \sum a_{ij} e_{ij}$$

and assume that $a_{s,t} \neq 0$ for a certain s, t. The proof is now slightly different depending on whether A is $M_{k,\ell}$ or $M_k(E)$. If $A = M_{k,\ell}$, then for each $b = \beta e_{js}$, $c = \gamma e_{tj} \in M_{k,\ell}$ we have $bac = \beta a_{st} \gamma e_{jj}$. Since this is a diagonal element of $M_{k,\ell}$, $\beta a_{st} \gamma$ must be in E_0 and we can easily find β and γ so it won't be zero.

Next, if $A = M_k(E)$ we again let $b = \beta e_{js}$ and $c = \gamma e_{tj}$ in $M_k(E)$, so that $bac = \beta a_{st} \gamma e_{jj}$ and since a_{st} is a homogeneous element of E and non-zero, we can choose β and γ so that $\beta a_{st} \gamma$ is central and non-zero. ∎

The next lemma translates the foregoing to the language of polynomial identities. For the sake of concreteness, let the Grassmann algebra E be $F[e_1, e_2, \ldots]$ where the e_i anticommute and each has square zero. Given $a \in A$ we define the support of a to be

$$\mathrm{Supp}(a) = \{i | e_i \text{ occurs in some entry of } a\}.$$

Lemma 3.2: *Given $f(y_1, \ldots, y_n)$ a non-identity for A, and given an index j, and I a finite subset of $\mathbb{N}$, the polynomial $y_0 f(y_1, \ldots, y_n) y_{n+1}$ has an evaluation on A equal to ze_{jj} where z is non-zero and central, and* $\mathrm{Supp}(z) \cap I = \emptyset$.

Proof. Let $E' \subseteq E$ be the Grassmann algebra generated by the e_i for $i \notin I$ and let $A' \subseteq A$ consist of the elements of A with entries in E'. Then A' is p. i. equivalent to A and so $f(y_1, \ldots, y_n)$ has a non-zero evaluation a on A'. By Lemma 3.1 there exist $b, c \in A' \subseteq A$ such that bac is of the form ze_{jj}, and the lemma follows since we can evaluate y_0 to b and y_{n+1} to c. ∎

We denote $V_n \cap Id(A)$ by I_n and $V_n \cap Id^Z(A)$ by Z_n. Let $g(x_1, \ldots, x_d)$ be a multilinear, proper central polynomial of A of degree d and construct a linear map $\Gamma : V_n \to Z_{n+d+2}$ by substituting $x_1 y_0 f(y) y_{d+1}$ for x_1,

$$\Gamma(f(x_1, \ldots, x_n)) = g(x_1 y_0 f(y_1, \ldots, y_d) y_{d+1}, x_2, \ldots, x_n)$$

where we identify $y_0, \ldots, y_{d+1}$ with $x_{n+1}, \ldots, x_{n+d+2}$. Since g is a central polynomial $\Gamma(f)$ will also be central, and Γ is a linear transformation.

LEMMA 3.3: *If $f_1, \ldots, f_t \in V_n$ are linearly independent modulo I_n, then*

$$\Gamma(f_1), \ldots, \Gamma(f_t) \in Z_{n+d+2}$$

are linearly independent modulo I_{n+d+2}.

Proof. Since Γ is a linear transformation, we need only show that if f is not an identity for A then $\Gamma(f)$ is also not an identity for A. Since g is multilinear and non-zero, there exists a homogeneous evaluation

$$x_\alpha \mapsto \bar{x}_\alpha = a_\alpha e_{j_\alpha, j_\alpha}$$

under which g is non-zero. By the previous lemma, $y_0 g(y_1, \ldots, y_d) y_{d+1}$ has a non-zero evaluation $y_i \mapsto \bar{y}_i$ such that

$$\bar{y}_0 g(\bar{y}_1, \ldots, \bar{y}_d) \bar{y}_{d+1} = z e_{i_1, j_1},$$

where z is non-zero and central in E with support disjoint from the union $\bigcup_i \mathrm{Supp}(\bar{x}_i)$. Hence

$$\begin{aligned}\Gamma(f)(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_0, \ldots, \bar{y}_{d+1}) &= f(\bar{x}_1 \cdot z e_{j_1, j_1}, \bar{x}_2, \ldots, \bar{x}_n) \\ &= z f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \neq 0. \qquad \blacksquare\end{aligned}$$

Taking dimensions we get this corollary.

COROLLARY 3.4: $c_{n-d-2}(A) \leq \delta_n(A)$ *for all* $n \geq d+2$.

We can use this lower bound on $\delta_n(A)$ to compute a lower bound on $c_n^Z(A)$ using the following lemma.

LEMMA 3.5: *If $f_1, \ldots, f_t \in Z_n$ are linearly independent modulo I_n, then*

$$x_{n+1} f_1, \ldots, x_{n+1} f_t \in V_{n+1}$$

are linearly independent modulo Z_{n+1}.

Proof. Assume that some linear combination

$$f = \sum \alpha_i x_{n+1} f_i = x_{n+1} \sum \alpha_i f_i$$

was in Z_{n+1}. If the α_i were not all zero, then $\sum \alpha_i f_i$ would be a proper central polynomial and so would have a non-trivial evaluation z in the center of A. But then the assumption that $x_{n+1} \sum \alpha_i f_i$ is central would imply that the ideal Az of A is central, which is impossible. ∎

Corollary 3.6: *$c_{n-d-3}(A) \leq c_n^Z(A)$ for all $n \geq d+3$.*

Here is our main result for the Z-codimensions of verbally prime algebras:

Theorem 3.7: *If A is a verbally prime algebra, then there is a constant k so that*

$$k \cdot c_n(A) \leq c_n^Z(A), \quad \delta_n(A) \leq c_n(A).$$

Hence $c_n^Z(A)$ and $\delta_n(A)$ are each $\Theta(c_n(A))$.

Proof. It follows from Theorem 4.22 of [1] that $c_n(A)$ is asymptotic to a function of the form $Cn^g e^n$ ($e = \exp(A) \neq 2.71\cdots$) and so $c_{n-d-2}(A)$ is asymptotic to $e^{-d-2}c_n(A)$ and $c_{n-d-3}(A)$ is asymptotic to $e^{-d-3}c_n(A)$. Hence, each is bounded below by a constant times $c_n(A)$ and the theorem follows from Corollaries 3.4 and 3.6. ∎

As stated in the introduction, the bounds on $c_n(M_{k,\ell})$ and $c_n(M_k(E))$ from [2] imply that

$$c_n^Z(M_{k,\ell}),\, \delta_n(M_{k,\ell}) = \Theta(n^{-(k^2+\ell^2-1)/2}(k+\ell)^{2n})$$

and

$$C_1 n^{-(2k^2-1)/2}(2k^2)^n \leq c_n^Z(M_k(E)),\, \delta_n(M_k(E)) \leq C_2 n^{-(k^2-1)/1}(2k^2)^n$$

for some $C_1, C_2 > 0$.

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