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FINITE DIMENSIONAL GROUPS OF LOCAL DIFFEOMORPHISMS

BY

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ABSTRACT

We are interested in classifying groups of local biholomorphisms (or even formal diffeomorphisms) that can be endowed with a canonical structure of algebraic groups and their subgroups. Such groups are called finite-dimensional. We obtain that cyclic groups, virtually polycyclic groups, finitely generated virtually nilpotent groups and connected Lie groups of local biholomorphisms are finite-dimensional. We provide several methods to identify finite-dimensional groups and build examples.

As a consequence we generalize results of Arnold, Seigal–Yakovenko and Binyamini on uniform estimates of local intersection multiplicities to bigger classes of groups, including for example virtually polycyclic groups and in particular finitely generated virtually nilpotent groups.

1. Introduction

We study the action of groups of self-maps on intersection multiplicities. More precisely, given varieties V and W of complementary dimension of an ambient space M and a subgroup G of self-maps of M, we want to identify conditions guaranteeing that $F \mapsto (F(V), W)$ is bounded uniformly over G where $(F(V), W)$ is the intersection multiplicity of $F(V)$ and W. Let us introduce a classical example of an application of such a property. Consider a continuous map $F : M \to M$ and an isolated fixed point P of F. The fixed point

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index of F at P is equal to the topological intersection multiplicity of Δ and $(F \times Id)(\Delta)$ at (P,P) where Δ is the diagonal of $M \times M$. By considering the iterates $(F \times Id)^n$ with $n \in \mathbb{Z}$ we obtain fixed point indexes for the fixed points of the iterates of F^n, i.e., for periodic points. In the context of C^1 maps Shub and Sullivan proved that the intersection index of Δ and $(F \times Id)^n(\Delta)$ at isolated fixed points is uniformly bounded.

Theorem 1.1 ([25]): *Let U be an open subset of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^m$ be a C^1 map such that 0 is an isolated fixed point of F^n for any $n \geq 1$. Then the fixed point index of F^n at 0 is bounded by a constant independent of n.*

As an immediate corollary they show that a C^1 map $F : M \to M$ defined in a compact differentiable manifold M has infinitely many periodic points if the sequence of Lefschetz numbers $(L(F^n))_{n \geq 1}$ is unbounded.

We denote by $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ the group of germs of biholomorphisms defined in a neighborhood of the origin in $\mathbb{C}^n$. We are interested in uniform intersection results in the local holomorphic setting. More precisely, we want to identify subgroups G of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ satisfying that the set

$$\{(\phi(V), W) : \phi \in G \text{ and } (\phi(V), W) < \infty\}$$

of intersection multiplicities (cf. Definition 5.1) is bounded for any pair of germs of holomorphic varieties V, W defined in a neighborhood of 0 in $\mathbb{C}^n$.

The first result in this direction is due to Arnold.

Theorem 1.2 ([1, Theorem 1]): *Let $\phi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$. Consider germs of submanifolds V, W of $(\mathbb{C}^n, 0)$ of complementary dimension. Suppose that the intersection multiplicity $\mu_n := (\phi^n(V), W)$ is finite for any $n \in \mathbb{Z}$. Then the sequence $(\mu_n)_{n \in \mathbb{Z}}$ is bounded.*

The proof is a consequence of the Skolem–Mahler–Lech theorem on roots of quasipolynomials [26].

The previous result was generalized to the finitely generated abelian case by Seigal and Yakovenko. We denote by $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ the group of formal diffeomorphisms (cf. Definition 2.5).

Theorem 1.3 ([23, Theorem 1]): *Let G be an abelian subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ generated by finitely many cyclic and one-parameter groups. Consider formal subvarieties V, W. Then the set*

$$\{(\phi(V), W) : \phi \in G \text{ and } (\phi(V), W) < \infty\}$$

is bounded.

The group $\mathrm{Diff}\,(\mathbb{C}^n,0)$ is a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ and hence Theorem 1.3 holds for subgroups of $\mathrm{Diff}\,(\mathbb{C}^n,0)$ and germs of subvarieties V and W. In contrast with Theorem 1.2 notice that it is not necessary to require that all intersection multiplicities are finite. The proof relies on a noetherianity argument (cf. section 5).

An analogous result was proved by Binyamini for the case in which the subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is embedded in a group of formal diffeomorphisms that has a natural Lie group structure.

Theorem 1.4 ([3, Theorem 5]): *Let G be a Lie subgroup (cf. Definition 3.5) of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ with finitely many connected components. Consider formal subvarieties V, W. Then the set*

$$\{(\phi(V),W) : \phi \in G \text{ and } (\phi(V),W) < \infty\}$$

is bounded.

Theorem 1.3 has more natural hypotheses (commutativity and finite generation) but Theorem 1.4 is somehow more general. Indeed Binyamini shows that any finitely generated abelian subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is a subgroup of a Lie group with finitely many connected components [3]. Thus it is interesting to study how to find an extension of a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ that is also a Lie group. In this paper we characterize the subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ that can be embedded in a Lie group (with finitely many connected components) in a natural way. Moreover, we show that every such group can be canonically embedded in an algebraic matrix group. We call these groups finite-dimensional.

Let us be more precise. We define a Zariski-closure $\overline{G}$ of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ (cf. Definition 2.12); it is a projective limit of algebraic matrix groups and hence it has a natural definition of dimension. We will say that G is finite-dimensional if $\overline{G}$ is finite dimensional (cf. Definition 3.1). If G is finite-dimensional then $\overline{G}$ is isomorphic to one of its subgroups of k-jets and hence $\overline{G}$ can be interpreted as an algebraic group (Proposition 3.2). Equivalently the group G is finite-dimensional if and only if there exists $k_0 \in \mathbb{N}$ such that the coefficients of degree greater than k_0 in the Taylor expansion at the origin of the elements of G are polynomial functions on the coefficients of degree less than or equal to k_0 (Remark 3.4).

Finite-dimensional subgroups satisfy uniform local intersection properties. We define $\hat{\mathcal{O}}_n$ as the ring of formal power series with complex coefficients in n

variables. We define the intersection multiplicity (I, J) of ideals I, J of $\hat{\mathcal{O}}_n$ as the dimension of the complex vector space $\hat{\mathcal{O}}_n/(I+J)$.

THEOREM 1.5: *Let G be a finite-dimensional subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Consider ideals I and J of $\hat{\mathcal{O}}_n$. Then the set*

$$\{(\phi^*(I), J) : \phi \in G \text{ and } (\phi^*(I), J) < \infty\}$$

is bounded.

Since an algebraic group is a complex Lie group with finitely many connected components, Theorem 1.5 can be seen as a consequence of Binyamini's Theorem 1.4. Anyway, the finite-dimensional hypothesis provides a simplification of the proof. On the other hand, Theorem 1.5 implies Theorem 1.4. More precisely, there is no group that satisfies the hypotheses of Theorem 1.4 but does not satisfy the hypotheses of Theorem 1.5: every subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$, that is the image by a morphism of a real Lie group with finitely many connected components (cf. Definition 3.5), is necessarily finite-dimensional (Theorem 4.1). Thus our canonical approach encloses the results by Seigal–Yakovenko and Binyamini [23, 3]. The definition of finite-dimensional subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ allows us to apply the techniques of the algebraic group theory in the study of local intersection problems.

We will exhibit different methods to find finite-dimensional subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ (cf. Theorems 3.1, 4.1, …). In particular, we identify several algebraic group properties implying that a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is finite dimensional, including some notable ones. Our main result is the following theorem:

THEOREM 1.6: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that it is either*

- *a Lie group with finitely many connected components or*
- *virtually polycyclic or*
- *virtually nilpotent and generated by finitely many cyclic and one-parameter subgroups of G.*

Then G is finite-dimensional. In particular, the set

$$\{(\phi^*(I), J) : \phi \in G \text{ and } (\phi^*(I), J) < \infty\}$$

is bounded for any pair of ideals I and J of $\hat{\mathcal{O}}_n$.

See Definitions 2.19, 4.1 and 2.18 for the definitions of virtual property, polycyclic and nilpotent groups, respectively.

Notice that in particular Theorem 1.6 applies to finitely generated virtually nilpotent subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$, i.e., to subgroups of polynomial growth of formal diffeomorphisms.

We introduce several techniques that allow one to build finite-dimensional subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Every time we identify such a group we obtain an analogue of Theorem 1.6. Instead of writing down the most general possible result, we prefer to highlight some remarkable properties that imply finite-dimensionality.

Our canonical approach makes it simpler to analyze whether or not subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ are embeddable in algebraic groups. We will relate finite-dimensionality with other group properties, namely:

- Finite-determination (cf. Definition 3.2) properties. We will show that finite-determination implies finite-dimension under certain closedness properties (Corollary 3.2).
- Finite-decomposition properties. A subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is finite-dimensional if and only if every element can be written as a word of uniformly bounded length in an alphabet whose letters belong to the union of finitely many cyclic and one-parameter subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ (Theorem 3.1 and Remark 3.10).
- Virtually solvable subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ with suitable finite generation hypotheses are finite-dimensional (Proposition 4.1, Theorems 4.2, 4.3, Corollary 4.3, ...).
- Decomposition of the group in a tower of extensions of the trivial group.

Let us expand on the final item of the previous list. Consider a normal subgroup H of a subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We can define the codimension of H or equivalently the dimension of the extension G/H as the codimension of $\overline{H}$ in $\overline{G}$. Such a definition is interesting because a tower of finite-dimensional extensions is finite-dimensional (Proposition 3.1). In particular, it is possible to decide whether or not a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is finite-dimensional by considering it as a tower of (easier to handle) extensions of the trivial group. We will exhibit some classes of extensions that are finite-dimensional, namely:

- Finite extensions.
- Finitely generated abelian extensions.
- G/H is a connected Lie group (cf. Definition 3.5).

For instance a virtually polycyclic group is a tower of cyclic and finite extensions of the trivial group and hence it is always finite-dimensional. Hence we can apply Theorem 1.5 to show Theorem 1.6. The extension approach provides a method to build examples of finite-dimensional subgroups of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$.

2. Pro-algebraic groups

Let us explain some of the basic properties of the pro-algebraic subgroups of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. Pro-algebraic groups of formal diffeomorphisms have been used in the study of differential Galois theory by Morales-Ruiz–Ramis–Simó [17]. Most of the results in this section can be found in [14] and [19]. We explain them here for the sake of clarity and completeness.

2.1. Formal vector fields and diffeomorphisms. Let us introduce some notations.

Definition 2.1: We denote by $\mathcal{O}_n$ the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of germs of holomorphic functions defined in the neighborhood of 0 in $\mathbb{C}^n$. We denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_n$. Analogously we define $\hat{\mathcal{O}}_n$ as the ring of formal power series with complex coefficients in n variables whose maximal ideal will be denoted by $\hat{\mathfrak{m}}$.

Next, we define formal vector fields as a generalization of local vector fields.

Definition 2.2: We denote by $\mathfrak{X}(\mathbb{C}^n, 0)$ the Lie algebra of germs of holomorphic vector fields defined in the neighborhood of 0 in $\mathbb{C}^n$ that are singular at 0.

Remark 2.1: An element X of $\mathfrak{X}(\mathbb{C}^n, 0)$ is of the form

$$X = f_1(z_1, \ldots, z_n)\frac{\partial}{\partial z_1} + \cdots + f_n(z_1, \ldots, z_n)\frac{\partial}{\partial z_n}$$

where $f_1, \ldots, f_n$ belong to the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_n$. Analogously X can be interpreted as a derivation of the $\mathbb{C}$-algebra $\mathfrak{m}$.

Definition 2.3: We define the Lie algebra $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ as the set of derivations of the $\mathbb{C}$-algebra $\hat{\mathfrak{m}}$. Analogously we can identify an element X of $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ with the expression

$$X = X(z_1)\frac{\partial}{\partial z_1} + \cdots + X(z_n)\frac{\partial}{\partial z_n}$$

where the coefficients of the vector field belong to $\hat{\mathfrak{m}}$.

Let us apply the same program to diffeomorphisms.

Definition 2.4: We denote by $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ the group of germs of biholomorphism defined in the neighborhood of 0 in $\mathbb{C}^n$.

Remark 2.2: An element ϕ of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ is of the form

$$\phi(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$$

where $f_1, \ldots, f_n \in \mathfrak{m}$ and its linear part $D_0\phi$ at the origin is an invertible linear map.

Definition 2.5: We say that ϕ belongs to the group $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ of formal diffeomorphisms if it is of the form

$$\phi(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$$

where $f_1, \ldots, f_n \in \hat{\mathfrak{m}}$ and $D_0\phi$ is an invertible linear map.

We will use the Krull topology (the $\hat{\mathfrak{m}}$-adic topology) in our spaces of formal objects.

Definition 2.6: The sets of the form $f + \hat{\mathfrak{m}}^j$ for any choice of $f \in \hat{\mathcal{O}}_n$ and $j \geq 0$ are a base of open sets of a topology in $\hat{\mathcal{O}}_n$, the so called **$\hat{\mathfrak{m}}$-adic** (or **Krull**) **topology**. Since we can interpret formal vector fields and diffeomorphisms as n-uples of elements in $\hat{\mathfrak{m}}$, we can define the Krull topology in $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ and $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$.

Remark 2.3: A sequence $(f_k)_{k\geq 1}$ of elements of $\hat{\mathcal{O}}_n$ converges to $f \in \hat{\mathcal{O}}_n$ in the Krull topology if for any $j \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that $f - f_k \in \hat{\mathfrak{m}}^j$ for any $k \geq k_0$. Convergence of sequences in $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ and $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is analogous.

Remark 2.4: It is clear that $\hat{\mathcal{O}}_n$, $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ and $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ are the closures in the Krull topology of $\mathcal{O}_n$, $\mathfrak{X}(\mathbb{C}^n, 0)$ and $\mathrm{Diff}\,(\mathbb{C}^n, 0)$, respectively.

It is difficult to work with $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ since it is an infinite-dimensional space. Anyway we can understand $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ as a projective limit $\varprojlim_{k\in\mathbb{N}} D_k$ where every D_k is a finite-dimensional matrix group for $k \in \mathbb{N}$ [19, Lemma 2.2]. We should interpret D_k as the group of k-jets of elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Next, let us explain how to define rigorously the groups D_k for $k \in \mathbb{N}$ and how this allows us to apply the theory of linear algebraic groups to the groups of formal diffeomorphisms.

Given $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ and $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ we can associate $X_k, \phi_k \in \text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ respectively for any $k \in \mathbb{N}$. They are given by

$$\begin{array}{ccc} \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1} & \stackrel{X_k}{\to} & \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1} \\ f + \hat{\mathfrak{m}}^{k+1} & \mapsto & X(f) + \hat{\mathfrak{m}}^{k+1}, \end{array} \qquad \begin{array}{ccc} \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1} & \stackrel{\phi_k}{\to} & \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1} \\ f + \hat{\mathfrak{m}}^{k+1} & \mapsto & f \circ \phi + \hat{\mathfrak{m}}^{k+1}. \end{array}$$

The linear map X_k (resp. ϕ_k) determines and is determined by the k-jet of X (resp. ϕ). Moreover $L_k := \{X_k : X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)\}$ is the Lie algebra of the group $D_k := \{\phi_k : \phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)\}$. It is an algebraic subgroup of $\text{GL}(\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1})$ since it satisfies

$$D_k = \{\alpha \in \text{GL}(\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1}) : \alpha(fg) = \alpha(f)\alpha(g)\ \forall f, g \in \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1}\};$$

cf. [14, section 3], [19, Lemma 2.1].

Definition 2.7: Given $k \geq l \geq 1$ we define the maps $\pi_k : \widehat{\text{Diff}}\,(\mathbb{C}^n, 0) \to D_k$ and $\pi_{k,l} : D_k \to D_l$ given by $\pi_k(\phi) = \phi_k$ and $\pi_{k,l}(\phi_k) = \phi_l$.

Since D_k is the group of truncations of elements of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ up to level k, we can interpret $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ as the projective limit of the projective system $(\varprojlim_{k \in \mathbb{N}} D_k, (\pi_{k,l})_{k \geq l \geq 1})$ of algebraic groups and morphisms of algebraic groups [19, Lemma 2.2]. Analogously $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is the projective limit $\varprojlim L_k$.

2.2. Exponential map. Given $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ we can define its exponential $\exp(X)$. Indeed given $(X_k)_{k \geq 1} \in \varprojlim L_k = \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ the family $(\exp(X_k))_{k \geq 1}$ defines an element $\exp(X)$ of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0) = \varprojlim D_k$. Equivalently we consider a sequence $(X_j)_{j \in \mathbb{N}}$ of convergent vector fields that converges to X in the Krull topology and then we define $\exp(X)$ as the limit in the Krull topology of $(\exp(X_j))_{j \in \mathbb{N}}$ where $\exp(X_j)$ is the time 1 flow of X_j for $j \in \mathbb{N}$.

Definition 2.8: We say that a formal vector field $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is **nilpotent** if its linear part $D_0 X$ is nilpotent. We denote by $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ the subset of $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ of formal nilpotent vector fields.

Definition 2.9: We say that a formal diffeomorphism $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is **unipotent** if its linear part $D_0\phi$ is unipotent. We denote by $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ the subset of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ of formal unipotent diffeomorphisms. Given a subgroup G of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ we denote by G_u the subset of unipotent elements of G. We say that G is unipotent if $G = G_u$.

Proposition 2.1 (cf. [5, 16], [9, Th. 3.17]): *The map*

$$\exp : \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$$

is a bijection.

Definition 2.10: Given $\phi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ we define its **infinitesimal generator** $\log \phi$ as the unique formal nilpotent vector field such that $\phi = \exp(\log \phi)$. We denote $\phi^t = \exp(t \log \phi)$ for $t \in \mathbb{C}$.

2.3. Zariski-closure of a group of formal diffeomorphisms.

Definition 2.11: Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Given $k \in \mathbb{N}$ we define $G_k^* = \{\phi_k : \phi \in G\}$ and G_k as the Zariski-closure of G_k^* in $\mathrm{GL}(\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1})$.

Let us remark that since D_k is algebraic, G_k is a subgroup of D_k for any $k \in \mathbb{N}$.

Remark 2.5: Given $k \geq l \geq 1$ the image of the algebraic closure of G_k^* by $\pi_{k,l}$ is the algebraic closure of the image G_l^* (cf. [4, 2.1 (f), p. 57]). Hence we obtain $\pi_{k,l}(G_k) = G_l$ for any $k \geq l \geq 1$. In particular, $(\varprojlim_{k \in \mathbb{N}} G_k, (\pi_{k,l})_{k \geq l \geq 1})$ is a projective system.

Definition 2.12: Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We define the Zariski-closure $\overline{G}$ of G as $\varprojlim_{k \in \mathbb{N}} G_k$ or, in other words,

$$\overline{G} = \{\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) : \phi_k \in G_k \ \forall k \in \mathbb{N}\}.$$

2.4. Definition of pro-algebraic group.

Definition 2.13: Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We say that G is **pro-algebraic** if $G = \overline{G}$.

Remark 2.6: Since $\pi_{k,l}(G_k) = G_l$ for any $k \geq l \geq 1$, the natural projection $(\pi_k)_{|\overline{G}} : \overline{G} \to G_k$ is surjective for any $k \in \mathbb{N}$ (cf. [19, Lemma 2.5], [21, Corollary 3.25]). Thus the Zariski-closure of $\overline{G}$ coincides with $\overline{G}$ and $\overline{G}$ is pro-algebraic. It is the minimal pro-algebraic group containing G.

We can characterize pro-algebraic subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$.

Proposition 2.2 ([19, Proposition 2.2], cf. [21, Proposition 3.26]): *Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then G is pro-algebraic if and only if G_k^* is an algebraic subgroup of $\mathrm{GL}(\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^{k+1})$ for any $k \in \mathbb{N}$ and G is closed in the Krull topology.*

2.5. Lie algebra of a pro-algebraic group. Pro-algebraic groups have a connected component of Id whose properties are analogous to the connected component of Id of an algebraic matrix group. This is a particular instance of a more general situation: many analogues of concepts involving algebraic groups can be transferred to the pro-algebraic setting.

Definition 2.14: Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We define $G_{k,0}$ as the connected component of Id of G_k for $k \in \mathbb{N}$. We define $\overline{G}_0 = \varprojlim_{k\in\mathbb{N}} G_{k,0}$ or, equivalently,

$$\overline{G}_0 = \{\phi \in \overline{G} : \phi_k \in G_{k,0} \ \forall k \in \mathbb{N}\}.$$

We say that $\overline{G}_0$ is the **connected component** of Id of $\overline{G}$. If G is pro-algebraic, then we denote $G_0 = \overline{G}_0$.

Remark 2.7: The group $\overline{G}_0$ is a finite index normal pro-algebraic subgroup of $\overline{G}$ [19, Proposition 2.3 and Remark 2.9], cf. [21, Proposition 3.35 and Remark 3.37].

Proposition 2.3 ([14, Proposition 2]): *Let G be a subgroup of* $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. *We consider*

$$\mathfrak{g} = \{X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : \exp(tX) \in \overline{G} \ \forall t \in \mathbb{C}\}.$$

Then $\mathfrak{g}$ is a Lie algebra and $\overline{G}_0$ is generated by the set $\exp(\mathfrak{g})$.

Definition 2.15: We say that $\mathfrak{g}$ is the Lie algebra of $\overline{G}$.

It is natural to consider $\overline{G}_0$ as the connected component of Id of $\overline{G}$ since it is a finite index normal subgroup of $\overline{G}$ that is generated by the exponential of the Lie algebra of $\overline{G}$.

The Zariski-closure of a cyclic subgroup of $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ is connected and one-dimensional.

Remark 2.8 ([19, Remark 2.11], cf. [21, Remark 3.30]): Let ϕ be a unipotent element of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then $\overline{\langle\phi\rangle}$ is equal to $\{\phi^t : t \in \mathbb{C}\}$. In particular, the Lie algebra of $\overline{\langle\phi\rangle}$ is the one-dimensional complex vector space generated by $\log\phi$.

The next property is well-known in the finite-dimensional setting.

Lemma 2.1: *Let H be a finite index subgroup of a pro-algebraic subgroup G of* $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. *Then H contains G_0.*

Proof. Given an element X in the Lie algebra $\mathfrak{g}$ of G, its one-parameter group $\{\exp(tX) : t \in \mathbb{C}\}$ is contained in G. Since H is a finite index subgroup of G, $\{\exp(tX) : t \in \mathbb{C}\}$ is contained in H. Any element of G_0 is of the form $\exp(X_1) \circ \cdots \circ \exp(X_m)$ for some $X_1, \ldots, X_m \in \mathfrak{g}$ by Proposition 2.3. Hence G_0 is contained in H. ∎

The next result will be used later on to identify pro-algebraic groups.

THEOREM 2.1 (Chevalley, cf. [4, section I.2.2, p. 57]): *The group generated by a family of connected algebraic matrix groups is algebraic.*

2.6. NORMAL SUBGROUPS OF PRO-ALGEBRAIC GROUPS. The next results relate the properties of normal subgroups with those of their algebraic closures.

LEMMA 2.2: *Let H be a normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then $\overline{H}$ is a normal subgroup of $\overline{G}$. Moreover, H_k is a normal subgroup of G_k for any $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. We have $AH_k^* A^{-1} = H_k^*$ for any $A \in G_k^*$. We deduce $AH_k A^{-1} = H_k$ for any $A \in G_k^*$. The normalizer of the algebraic subgroup H_k in the algebraic group G_k is algebraic and contains G_k^*. Hence it is equal to G_k and then H_k is a normal subgroup of G_k for any $k \in \mathbb{N}$. As a consequence $\overline{H}$ is a normal subgroup of $\overline{G}$. ∎

LEMMA 2.3: *Let H be a finite index subgroup of a subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then H is pro-algebraic if and only if G is pro-algebraic.*

Proof. Since G_k^* and H_k^* are images of G and H respectively by the morphism of groups $\pi_k : \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0) \to D_k$, H_k^* is a finite index subgroup of G_k^* for any $k \in \mathbb{N}$.

Suppose H is pro-algebraic. Then H_k^* is algebraic for any $k \in \mathbb{N}$ by Proposition 2.2. Hence G_k^* is algebraic for any $k \in \mathbb{N}$. In order to prove that G is pro-algebraic, it suffices to show that G is closed in the Krull topology. Since G is the union of finitely many left cosets of H and they are all closed in the Krull topology, we deduce that G is closed in the Krull topology.

Suppose G is pro-algebraic. Then G_0 is pro-algebraic by Remark 2.7. Moreover, G_0 is contained in H by Lemma 2.1. Since G_0 is a finite index subgroup of G and then of H, H is pro-algebraic by the first part of the proof. ∎

LEMMA 2.4: *Let H be a finite index normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then $\overline{H}$ is a finite index normal subgroup of $\overline{G}$. Moreover, H_k is a finite index normal subgroup of G_k for any $k \in \mathbb{N}$.*

Proof. Consider $\phi_1, \ldots, \phi_m \in G$ such that $G/H = \{\phi_1 H, \ldots, \phi_m H\}$. We define the group $J = \langle \overline{H}, \phi_1, \ldots, \phi_m \rangle$; it satisfies $J \subset \overline{G}$. Let us show that $J = \overline{G}$ and that $\overline{H}$ is a finite index normal subgroup of J.

Since $\overline{H}$ is a normal subgroup of J by Lemma 2.2, every element ψ of J is of the form

$$\psi = \phi_{i_1}^{\pm 1} \circ \cdots \circ \phi_{i_l}^{\pm 1} \circ h$$

where $h \in \overline{H}$. The choice of $\phi_1, \ldots, \phi_m$ implies the existence of $1 \leq j \leq m$ and $h' \in H$ such that $\psi = \phi_j \circ (h' \circ h)$. In particular, the natural map $G/H \to J/\overline{H}$ is surjective and hence $\overline{H}$ is a finite index normal subgroup of J. The group J is pro-algebraic by Lemma 2.3. Since $G \subset J \subset \overline{G}$, we deduce $J = \overline{G}$ and $\overline{H}$ is a finite index normal subgroup of $\overline{G}$. Since G_k and H_k are images of $\overline{G}$ and $\overline{H}$ respectively by the morphism $\pi_k : \overline{G} \to G_k$, H_k is a finite index normal subgroup of G_k for any $k \in \mathbb{N}$. ∎

2.7. Algebraic properties of the Zariski-closure. The groups G and $\overline{G}$ share many algebraic properties.

Definition 2.16: Let G be a group. Given $f, g \in G$, we define by $[f,g] = fgf^{-1}g^{-1}$ the commutator of f and g.

Given subgroups H, L of G we define $[H, L] = \langle [h,l] : h \in H, \, l \in L \rangle$ as the subgroup generated by the commutators of elements of H and elements of L.

Definition 2.17: Let G be a group. By induction we define the subgroups

$$G^{(0)} = G, \; G^{(1)} = [G^{(0)}, G^{(0)}], \ldots, \; G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}], \ldots$$

of the derived series of G. We say that $G^{(\ell)}$ is the ℓ-th derived group of G. We use sometimes the notation G' instead of $G^{(1)}$ for the derived group of G.

We say that G is **solvable** if there exists $\ell \in \mathbb{N} \cup \{0\}$ such that $G^{(\ell)} = \{1\}$. We define the **derived length** of G as the minimum $\ell \in \mathbb{N} \cup \{0\}$ with such a property.

Definition 2.18: Let G be a group. By induction we define the subgroups

$$\mathcal{C}^0 G = G, \; \mathcal{C}^1 G = [\mathcal{C}^0 G, G], \ldots, \; \mathcal{C}^{\ell+1} G = [\mathcal{C}^\ell G, G], \ldots$$

of the descending central series of G. We say that G is **nilpotent** if there exists $\ell \in \mathbb{N} \cup \{0\}$ such that $\mathcal{C}^\ell G = \{1\}$. We define the **nilpotence class** of G as the minimum $\ell \in \mathbb{N} \cup \{0\}$ with such a property.

LEMMA 2.5: *Let G be a subgroup of* $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. *We have:*

- *G is abelian if and only if $\overline{G}$ is abelian.*
- *G is solvable if and only if $\overline{G}$ is solvable.*
- *G is nilpotent if and only if $\overline{G}$ is nilpotent.*

The first two properties were proved in [14, Lemma 1]. The proof of the last one is completely analogous. All these properties are a consequence of a simple principle: the derived length (resp. the nilpotence class) does not change when we take the Zariski-closure of a matrix group.

Definition 2.19: Let G be a group and P a group property. We say that G is **virtually** P if there exists a finite index subgroup H of G that satisfies P.

Remark 2.9: If the property P is subgroup-closed (for instance, solubility or nilpotence), then we can suppose that the group H is a finite index normal subgroup of G (cf. [22, 1.6.9, p. 36]).

LEMMA 2.6: *Let G be a subgroup of* $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. *Then the following properties are equivalent:*

(1) *G is virtually nilpotent (resp. solvable).*
(2) *$\overline{G}$ is virtually nilpotent (resp. solvable).*
(3) *$\overline{G}_0$ is nilpotent (resp. solvable).*

Proof. Let us show the result in the virtually nilpotent case. The other case is analogous.

Let us show (1) $\Longrightarrow$ (2). Let H be a finite index normal nilpotent subgroup of G. Then $\overline{H}$ is a finite index normal nilpotent subgroup of $\overline{G}$ by Lemmas 2.4 and 2.5. Hence $\overline{G}$ is virtually nilpotent.

Let us prove (2) $\Longrightarrow$ (3). There exists a finite index normal nilpotent subgroup J of $\overline{G}$. The group J contains $\overline{G}_0$ by Lemma 2.1 and thus $\overline{G}_0$ is nilpotent.

Let us see that (3) implies (1). Since $\overline{G}_0$ is a finite index normal subgroup of $\overline{G}$ by Remark 2.7, $\overline{G}$ is virtually nilpotent. The group G is a subgroup of $\overline{G}$ and hence also virtually nilpotent. ∎

2.8. JORDAN DECOMPOSITION OF FORMAL DIFFEOMORPHISMS. Let us consider the multiplicative Jordan decomposition of formal diffeomorphisms in commuting semisimple (or equivalently diagonalizable) and unipotent parts. It was constructed by Martinet in [15]. The analogous decomposition for algebraic matrix groups is called Jordan–Chevalley decomposition since Chevalley showed the following

Theorem 2.2 (Chevalley, cf. [4, section I.4.4, p. 83]): *Let H be an algebraic matrix group. Then the semisimple and unipotent parts of the elements of H also belong to H.*

We will see that Chevalley's theorem also guarantees the closedness of the Jordan decomposition for pro-algebraic subgroups of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$.

Definition 2.20: We say that $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is **semisimple** if ϕ_k is semisimple for any $k \in \mathbb{N}$.

Remark 2.10: By definition, $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is unipotent if and only if ϕ_1 is unipotent. It is not difficult to show that ϕ is unipotent if and only if ϕ_k is unipotent for any $k \in \mathbb{N}$ (cf. [21, Proposition 3.12]).

Remark 2.11: It is well-known that ϕ is semisimple if and only if ϕ is formally conjugated to a linear diagonal map (cf. [20, Lemma 2.9] [21, Proposition 3.13]).

Given $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ we consider the multiplicative Jordan decomposition of ϕ_k for $k \in \mathbb{N}$. The semisimple and unipotent parts $\phi_{k,s}$ and $\phi_{k,u}$ of ϕ_k belong to the algebraic group D_k by Chevalley's Theorem 2.2. Moreover, since $\pi_{k,l}(\phi_{k,s})$ is semisimple, $\pi_{k,l}(\phi_{k,u})$ is unipotent and

$$\phi_l = \pi_{k,l}(\phi_k) = \pi_{k,l}(\phi_{k,s})\pi_{k,l}(\phi_{k,u}) = \pi_{k,l}(\phi_{k,u})\pi_{k,l}(\phi_{k,s}),$$

we deduce $\pi_{k,l}(\phi_{k,s}) = \phi_{l,s}$ and $\pi_{k,l}(\phi_{k,u}) = \phi_{l,u}$ for any $k \geq l \geq 1$ by uniqueness of the Jordan–Chevalley decomposition. Hence $(\phi_{k,s})_{k\in\mathbb{N}}$ and $(\phi_{k,u})_{k\in\mathbb{N}}$ define elements ϕ_s and ϕ_u in $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0) = \varprojlim D_k$ respectively. This leads to the next well-known result.

Proposition 2.4: *Let $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. There exist unique elements ϕ_s and ϕ_u in $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ such that*

$$\phi = \phi_s \circ \phi_u = \phi_u \circ \phi_s,$$

ϕ_s is semisimple and ϕ_u is unipotent.

Chevalley's Theorem 2.2 implies

Proposition 2.5: *Let G be a subgroup of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. Then $\overline{G}$ contains the semisimple and the unipotent parts of every element of G.*

Analogously there exists an additive Jordan decomposition for formal vector fields.

PROPOSITION 2.6: *Let $X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)$. There exist unique elements X_s and X_N in $\hat{\mathfrak{X}}(\mathbb{C}^n,0)$ such that*

$$X = X_s + X_N \quad \textit{and} \quad [X_s, X_N] = 0,$$

X_s is semisimple (i.e., formally conjugated to a linear diagonal vector field) and X_N is nilpotent.

Remark 2.12: It is clear that if $X = X_s + X_N$ is the additive Jordan decomposition of $X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)$, then $\exp(X) = \exp(X_s) \circ \exp(X_N)$ is the multiplicative Jordan decomposition of $\exp(X)$.

Remark 2.13: Given a semisimple $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ it is easy to calculate $\overline{\langle\phi\rangle}$. Indeed ϕ is of the form $\phi(z_1,\ldots,z_n) = (\lambda_1 z_1,\ldots,\lambda_n z_n)$ in some formal system of coordinates. In such coordinates $\overline{\langle\phi\rangle}$ coincides with the Zariski-closure of the group $\langle \mathrm{diag}(\lambda_1,\ldots,\lambda_n)\rangle$ in $\mathrm{GL}(n,\mathbb{C})$. It can be described in terms of characters. We have

$$\overline{\langle\phi\rangle} = \left\{ \mathrm{diag}(\mu_1,\ldots,\mu_n) : (\mu_1,\ldots,\mu_n) \in \bigcap_{\underline{a}\in\mathbb{Z}^n,\ (\lambda_1,\ldots,\lambda_n)\in\ker(\chi_{\underline{a}})} \ker(\chi_{\underline{a}}) \right\}$$

where given $\underline{a} = (a_1,\ldots,a_n) \in \mathbb{Z}^n$ we consider the character $\chi_{\underline{a}} : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by $\chi_{\underline{a}}(\mu_1,\ldots,\mu_n) = \mu_1^{a_1}\cdots\mu_n^{a_n}$ (cf. [18, Theorem 5, Chapter 3.2.3], [8, 16.1]).

Let us calculate the algebraic closure of a cyclic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$.

Definition 2.21: Given a complex manifold (resp. complex vector space) M we denote its dimension by $\dim M$.

LEMMA 2.7: *Let $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then $\overline{\langle\phi\rangle}$ is an abelian group that is isomorphic to the product $\overline{\langle\phi_s\rangle} \times \overline{\langle\phi_u\rangle}$. Moreover, ${\langle\phi\rangle}_k$ is isomorphic to the product $\langle\phi_s\rangle_k \times \langle\phi_u\rangle_k$ and $\dim {\langle\phi\rangle}_k = \dim\langle\phi_s\rangle_k + \dim\langle\phi_u\rangle_k$ for any $k \in \mathbb{N}$.*

Proof. The group $\overline{\langle\phi\rangle}$ is abelian by Lemma 2.5. By Proposition 2.5, the formal diffeomorphisms ϕ_s and ϕ_u belong to $\overline{\langle\phi\rangle}$ and then $\overline{\langle\phi\rangle}$ contains the group $H := \langle\overline{\langle\phi_s\rangle}, \overline{\langle\phi_u\rangle}\rangle$. We claim $\overline{\langle\phi\rangle} = H$; it suffices to show that H is pro-algebraic.

Remark 2.13 implies that $\overline{\langle\phi_s\rangle}$ consists of semisimple elements and is closed in the Krull topology. Hence $\langle\phi_s\rangle_k$ is composed of semisimple elements for any $k \in \mathbb{N}$. Analogously $\overline{\langle \phi_u \rangle}$ is contained in $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$, is closed in the Krull topology by Remark 2.8 and $\langle \phi_u \rangle_k$ consists of unipotent elements for any $k \in \mathbb{N}$.

The group H_k^* is the image of the morphism

$$\begin{array}{ccc} \langle \phi_s \rangle_k \times \langle \phi_u \rangle_k & \stackrel{\iota}{\to} & D_k \\ (\alpha, \beta) & \mapsto & \alpha\beta \end{array}$$

of algebraic groups. Hence H_k^* is an algebraic subgroup of D_k for any $k \in \mathbb{N}$ (cf. [4, 2.1 (f), p. 57]). Since $\langle \langle \phi_s \rangle_k, \langle \phi_u \rangle_k \rangle$ is abelian, the uniqueness of the Jordan decomposition implies ι is injective. Thus H_k is isomorphic to $\langle \phi_s \rangle_k \times \langle \phi_u \rangle_k$ and satisfies

$$\dim H_k = \dim \langle \phi_s \rangle_k + \dim \langle \phi_u \rangle_k$$

for any $k \in \mathbb{N}$. In order to conclude the proof it suffices to show that H is closed in the Krull topology by Proposition 2.2.

Every element η of H is of the form $\psi \circ \rho$ where $\psi \in \overline{\langle \phi_s \rangle}$ and $\rho \in \overline{\langle \phi_u \rangle}$. Since H is abelian, ψ is semisimple and ρ is unipotent, $\psi \circ \rho$ is the multiplicative Jordan decomposition of η. Moreover, H is isomorphic to $\overline{\langle \phi_s \rangle} \times \overline{\langle \phi_u \rangle}$ by uniqueness of the Jordan–Chevalley decomposition. Thus $\overline{\langle \phi_s \rangle}$ (resp. $\overline{\langle \phi_u \rangle}$) is the set of semisimple (resp. unipotent) elements of H. Given a sequence $(\eta_k)_{k \in \mathbb{N}}$ in H that converges in the Krull topology, the sequences $(\eta_{k,s})_{k \in \mathbb{N}}$ and $(\eta_{k,u})_{k \in \mathbb{N}}$ are contained in $\overline{\langle \phi_s \rangle}$ and $\overline{\langle \phi_u \rangle}$, respectively, and both converge in the Krull topology. Since $\overline{\langle \phi_s \rangle}$ and $\overline{\langle \phi_u \rangle}$ are closed in the Krull topology so is H. ∎

3. Finite dimensional groups of formal diffeomorphisms

Our main goal is characterizing the groups G of local diffeomorphisms that can be embedded in finite-dimensional Lie groups. We approach this problem from a canonical point of view. Indeed we provide an invariant $\dim G$ of G such that $\dim G < \infty$ implies that the Zariski-closure $\overline{G}$ of G is algebraic or, more precisely, that the map $\pi_k : \overline{G} \to G_k$ is an isomorphism of groups for some $k \in \mathbb{N}$. In such a case $\pi_k^{-1} : G_k \to \overline{G}$ can be interpreted as an algebraic morphism and $\overline{G}$ as a matrix algebraic group (in particular, as a complex Lie group with finitely many connected components). On the other hand, we will see that a Lie subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ with finitely many connected components (cf. Definition 3.5) is finite-dimensional (Proposition 3.7 and Lemma 3.3).

There are other advantages of working with the Zariski-closure of a group of local diffeomorphisms. For instance, given a normal subgroup H of a group

$G \subset \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ we can naturally define whether the extension is finite-dimensional. A straightforward consequence of the definition is that G is finite-dimensional if and only if it is a tower of finite-dimensional extensions of the trivial group. Hence it is natural to identify finite-dimensional extensions. The following kind of extensions are finite-dimensional:

(1) H is a finite index subgroup of G.
(2) G/H is a finitely generated abelian group.
(3) G/H is a connected Lie group.

These items are generalizations of the cases treated in [23, 3] in the context of extensions of groups. Thus a natural strategy to show $\dim G < \infty$ for a subgroup G of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is decomposing it as a tower of extensions of the types (1), (2) and (3) of the trivial group. This method allows one to generalize Theorem 1.6 to much bigger classes of groups.

3.1. Dimensional setting. The first step of our program is defining the dimension of an extension of subgroups of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$.

Lemma 3.1: *Let G be a subgroup of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. Consider a subgroup H of G. Then $\dim G_k - \dim H_k \leq \dim G_{k+1} - \dim H_{k+1}$ for any $k \in \mathbb{N}$.*

Proof. Let $\mathfrak{g}_k$ and $\mathfrak{h}_k$ be the Lie algebra of G_k and H_k respectively for $k \in \mathbb{N}$. Since $\pi_{k+1,k} : G_{k+1} \to G_k$ is surjective and we are working in characteristic 0, we obtain $(d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k$ is surjective for any $k \in \mathbb{N}$ (cf. [4, Chapter II.7, p. 105]). Moreover, $(d\pi_{k+1,k})_{Id}(\mathfrak{h}_{k+1})$ is equal to $\mathfrak{h}_k$ for $k \in \mathbb{N}$. Therefore the linear map $(d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1}/\mathfrak{h}_{k+1} \to \mathfrak{g}_k/\mathfrak{h}_k$ is surjective. Since

$$\dim \mathfrak{g}_k - \dim \mathfrak{h}_k \leq \dim \mathfrak{g}_{k+1} - \dim \mathfrak{h}_{k+1},$$

we deduce $\dim G_k - \dim H_k \leq \dim G_{k+1} - \dim H_{k+1}$ for any $k \in \mathbb{N}$. ∎

Since $(\dim G_k - \dim H_k)_{k \geq 1}$ is increasing we can define the codimension of H in G.

Definition 3.1: Consider a subgroup H of a subgroup G of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. We define $\dim G/H \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as

$$\dim G/H = \lim_{k \to \infty} \dim G_k - \dim H_k.$$

We say that G/H is **finite-dimensional** or that H has finite codimension in G if $\dim G/H < \infty$. We define $\dim G$ as $\dim G = \lim_{k\to\infty} \dim G_k$ for any subgroup G of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$.

Remark 3.1: Notice that the definition does not distinguish between a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ and its Zariski-closure. More precisely, if H is a subgroup of a group $G \subset \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$, we have

$$\dim \overline{G} = \dim G \quad \text{and} \quad \dim \overline{G}/\overline{H} = \dim G/H.$$

The following result is an immediate consequence of the definition. It will be useful to know whether or not a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is finite-dimensional in practical applications since it allows one to divide the problem into simpler ones.

Proposition 3.1: *Consider a sequence $G^1 \subset G^2 \subset \cdots \subset G^m$ of subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then we obtain*

$$\dim G^m/G^1 = \dim G^m/G^{m-1} + \cdots + \dim G^3/G^2 + \dim G^2/G^1.$$

In particular, G^m/G^1 is finite-dimensional if and only if G^{j+1}/G^j is finite-dimensional for any $1 \leq j < m$.

The next proposition provides several characterizations of finite-dimensional extensions.

Proposition 3.2: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Let H be a subgroup of G. The following properties are equivalent:*

(1) *There exists $k_0 \in \mathbb{N}$ such that $\phi \in \overline{G}$ and $\phi_{k_0} \in H_{k_0}$ imply $\phi \in \overline{H}$.*
(2) *There exists $k_0 \in \mathbb{N}$ such that the map $\hat{\pi}_{k_0} : \overline{G}/\overline{H} \to G_{k_0}/H_{k_0}$, induced by $\pi_{k_0} : \overline{G} \to G_{k_0}$, is injective.*
(3) *There exists $k_0 \in \mathbb{N}$ such that the map $\hat{\pi}_{k+1,k} : G_{k+1}/H_{k+1} \to G_k/H_k$ (induced by $\pi_{k+1,k}$) is injective for any $k \geq k_0$.*
(4) *G/H is finite-dimensional.*

Remark 3.2: Since we are not supposing that H is normal we consider left cosets. The proposition can be strengthened if H is normal. Then $\overline{H}$ is normal in $\overline{G}$ and H_k is normal in G_k for any $k \in \mathbb{N}$ by Lemma 2.2. Notice that G_k/H_k is an algebraic group for any $k \in \mathbb{N}$ (cf. [4, section II.6.8, p. 98]). Hence $\hat{\pi}_{k+1,k}$ is a morphism of algebraic groups. Since $\hat{\pi}_k$ is always surjective for $k \in \mathbb{N}$, condition (2) is equivalent to $\hat{\pi}_{k_0}$ being an isomorphism of groups from $\overline{G}/\overline{H}$ onto the algebraic matrix group G_{k_0}/H_{k_0}. Condition (3) is equivalent to $\hat{\pi}_{k+1,k}$ being a bijective morphism of algebraic matrix groups and then an isomorphism of algebraic groups (cf. [18, Theorem 6, Chapter 3.1.4]).

Proof. The first two properties are clearly equivalent.

Let us show (2) $\Longrightarrow$ (3). We have $\hat{\pi}_{k_0} = \hat{\pi}_{k,k_0} \circ \hat{\pi}_{k+1,k} \circ \hat{\pi}_{k+1}$ for $k \geq k_0$. Since the maps $\hat{\pi}_{k,k_0}$, $\hat{\pi}_{k+1,k}$, $\hat{\pi}_{k+1}$ are surjective and $\hat{\pi}_{k_0}$ is injective, $\hat{\pi}_{k,k_0}$, $\hat{\pi}_{k+1,k}$, $\hat{\pi}_{k+1}$ are also injective for any $k \geq k_0$.

Let us show (3) $\Longrightarrow$ (2). The map $\hat{\pi}_{k_0}$ is equal to $\hat{\pi}_{k,k_0} \circ \hat{\pi}_k$ for $k \geq k_0$. Since $\hat{\pi}_{k,k_0} = \hat{\pi}_{k_0+1,k_0} \circ \cdots \circ \hat{\pi}_{k,k-1}$ is a composition of injective maps by hypothesis, $\hat{\pi}_{k,k_0}$ is injective for $k \geq k_0$. Given left cosets $\phi\overline{H}$ and $\eta\overline{H}$ such that $\hat{\pi}_{k_0}(\phi\overline{H}) = \hat{\pi}_{k_0}(\eta\overline{H})$, we obtain $\hat{\pi}_k(\phi\overline{H}) = \hat{\pi}_k(\eta\overline{H})$ for any $k \geq k_0$. We deduce $(\eta^{-1}\phi)_k \in H_k$ for any $k \geq k_0$. In particular, we have $\eta^{-1}\phi \in \overline{H}$ and hence $\phi\overline{H} = \eta\overline{H}$. Thus $\hat{\pi}_{k_0}$ is injective.

Let us show (3) $\Longrightarrow$ (4). The map $\hat{\pi}_{k+1,k}$ is an isomorphism of algebraic manifolds for any $k \geq k_0$ by the universal mapping property of quotient morphisms (cf. [4, Chapter II.6]). We deduce

$$\dim G_{k+1} - \dim H_{k+1} = \dim G_k - \dim H_k$$

for any $k \geq k_0$.

Let us show (4) $\Longrightarrow$ (3). Let $\mathfrak{g}_k$ and $\mathfrak{h}_k$ be the Lie algebra of G_k and H_k respectively for $k \in \mathbb{N}$. There exists $k_0 \in \mathbb{N}$ such that

$$\dim G_k - \dim H_k = \dim G_{k_0} - \dim H_{k_0}$$

for any $k \geq k_0$. The linear map $(d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k$ is surjective for any $k \in \mathbb{N}$ by the proof of Lemma 3.1. Since $(d\pi_{k+1,k})_{Id}(\mathfrak{h}_{k+1}) = \mathfrak{h}_k$ for any $k \in \mathbb{N}$ the linear map $(d\hat{\pi}_{k+1,k})_{Id} : \mathfrak{g}_{k+1}/\mathfrak{h}_{k+1} \to \mathfrak{g}_k/\mathfrak{h}_k$ is well-defined and surjective for any $k \in \mathbb{N}$. Since both complex vector spaces $\mathfrak{g}_{k+1}/\mathfrak{h}_{k+1}$ and $\mathfrak{g}_k/\mathfrak{h}_k$ have the same dimension, the map $(d\hat{\pi}_{k+1,k})_{Id}$ is a linear isomorphism for any $k \geq k_0$.

Fix $k \geq k_0$. Let us show that $\hat{\pi}_{k+1,k}$ is injective. Let $A \in G_{k+1}$ such that $\hat{\pi}_{k+1,k}(AH_{k+1}) = H_k$. We have $\pi_{k+1,k}(A) \in H_k$. The restriction $(\pi_{k+1,k})_{|H_{k+1}} : H_{k+1} \to H_k$ is surjective by Remark 2.5, hence there exists $B \in H_{k+1}$ such that $\pi_{k+1,k}(A) = \pi_{k+1,k}(B)$. We obtain $\pi_{k+1,k}(B^{-1}A) = Id$. There exists $\phi \in \overline{G}$ such that $\phi_{k+1} = B^{-1}A$ since $\pi_{k+1} : \overline{G} \to G_{k+1}$ is surjective by Remark 2.6. Since $\phi_k \equiv Id$, the linear part $D_0\phi$ of ϕ at 0 is equal to Id and thus $\log\phi$ belongs to the Lie algebra $\mathfrak{g}$ of $\overline{G}$ (Remark 2.8) and satisfies $(\log\phi)_k \equiv 0$. The property $(d\pi_{k+1,k})_{Id}((\log\phi)_{k+1}) = 0$ and the injective nature of $(d\hat{\pi}_{k+1,k})_{Id}$ imply $(\log\phi)_{k+1} \in \mathfrak{h}_{k+1}$. Since $B^{-1}\ A = \exp((\log\phi)_{k+1})$, we obtain $B^{-1}A \in H_{k+1}$ and then $A \in H_{k+1}$. Hence $\pi_{k+1,k}$ is injective for any $k \geq k_0$. ∎

Remark 3.3: The implication in Proposition 3.2 from (4) to any of the other conditions is one of the key points in the paper. Given a projective system $(\varprojlim_{k\in\mathbb{N}} G_k, (\pi_{k,l})_{k\geq l\geq 1})$ of algebraic groups, we have that $\dim G_k = \dim G_{k+1}$ implies that $(d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k$ is an isomorphism and $\pi_{k+1,k}$ is a finite covering. An example is obtained by considering $G_k = \mathbb{C}^*$ and $\pi_{k+1,k}(z) = z^2$ for any $k \geq 1$. In particular, all elements in $\ker(\pi_{k+1,k})$ have finite order. In our setting we obtain a stronger property: the morphism $\pi_{k+1,k}$ is injective. Indeed any element A in $\ker(\pi_{k+1,k})$ is tangent to the identity and hence it is either trivial or $\overline{\langle A\rangle}$ is isomorphic to $\mathbb{C}$. The latter case is impossible since then A does not have finite order. Alternatively, any non-vanishing element in the Lie algebra of $\overline{\langle A\rangle}$, and in particular the infinitesimal generator of A, would be contained in $\ker(d\pi_{k+1,k})_{Id}$.

The equivalence (3)⇔(4) in Proposition 3.2 validates our point of view. We have that $\hat{\pi}_{k+1,k} : G_{k+1}/H_{k+1} \to G_k/H_k$ is bijective if and only if

$$\dim G_k - \dim H_k = \dim G_{k+1} - \dim H_{k+1}.$$

The dimension determines an extension of the form G_k/H_k for $k \in \mathbb{N}$ modulo isomorphism.

Moreover, if (3) holds then $\dim G/H = \dim G_{k_0} - \dim H_{k_0}$. We have

$$\dim G/H = \dim G_{k_0} - \dim H_{k_0}$$

if (2) holds by the proof of (2) $\Longrightarrow$ (3).

Remark 3.4: Let G be a finite-dimensional subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. There exists $k_0 \in \mathbb{N}$ such that $\pi_{k,k_0} : G_k \to G_{k_0}$ is an isomorphism of algebraic matrix groups for any $k \geq k_0$. Consider the Taylor series expansion

$$\phi(z_1, \ldots, z_n) = \left(\sum_{i_1+\cdots+i_n\geq 1} a^1_{i_1\cdots i_n} z_1^{i_1}\cdots z_n^{i_n}, \ldots, \sum_{i_1+\cdots+i_n\geq 1} a^n_{i_1\cdots i_n} z_1^{i_1}\cdots z_n^{i_n} \right)$$

of $\phi \in G$. Given $(i_1, \cdots, i_n; j)$ a multi-index such that $i_1 + \cdots + i_n > k_0$ and $1 \leq j \leq n$, the function $a^j_{i_1\cdots i_n} : G \to \mathbb{C}$ belongs to the affine coordinate ring $\mathbb{C}[G_{k_0}]$ of G_{k_0}. In other words, every coefficient in the Taylor expansion, of an element of G, of degree greater than k_0 is a regular function $P^j_{i_1\cdots i_n}$ on the coefficients of degree less than or equal to k_0.

Reciprocally, suppose that $a^j_{i_1\cdots i_n} : G \to \mathbb{C}$ is a polynomial function of the coefficients of degree less than or equal to k_0 for any multi-index $(i_1, \ldots, i_n; j)$ such that $i_1+\cdots+i_n > k_0$ and $1 \leq j \leq n$ (meaning that there exists $P^j_{i_1\cdots i_n} \in \mathbb{C}[D_{k_0}]$ such that

$$a^j_{i_1\cdots i_n}(\phi) = P^j_{i_1\cdots i_n}((a^m_{l_1\cdots l_n}(\phi))_{l_1+\cdots+l_n\leq k_0,\ 1\leq m\leq n})$$

for any $\phi \in G$). Since all these equations hold true in the Zariski-closure $G_{i_1+\cdots+i_n}$ of $G^*_{i_1+\cdots+i_n}$, we have that $\pi^*_{k,k_0} : \mathbb{C}[G_{k_0}] \to \mathbb{C}[G_k]$ is an isomorphism of $\mathbb{C}$-algebras and, in particular, $\pi_{k,k_0} : G_k \to G_{k_0}$ is an isomorphism of algebraic groups for any $k \geq k_0$. Thus G is finite-dimensional.

Let us relate the finite-dimension property with a much simpler one, namely the finite-determination property.

Definition 3.2: Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We say that G has the **finite-determination property** if there exists $k \in \mathbb{N}$ such that $\phi \in G$ and $\phi_k \equiv Id$ imply $\phi \equiv Id$.

Remark 3.5: Let us compare the finite-determination and the finite-dimension properties. On the one hand, a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ has the finite-determination property if there exists $k \in \mathbb{N}$ such that the projection $\pi_k : G \to D_k$ is injective. On the other hand, G is finite-dimensional if there exists $k \in \mathbb{N}$ such that $\pi_k : \overline{G} \to D_k$ is injective.

Remark 3.6: Notice that a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is finite-dimensional if and only if $\overline{G}$ has the finite-determination property.

Remark 3.7: Every finite-dimensional group has finite-determination but in general the reciprocal does not hold true. We define

$$\phi(j)(x, y) = (x, y + d_j x^2 + x^{j+2}) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$$

for $j \in \mathbb{N}$. Suppose that the subset $S := \{d_1, d_2, \ldots\}$ of $\mathbb{C}$ is linearly independent over $\mathbb{Q}$. We have

$$\log \phi(j) = (d_j x^2 + x^{j+2})\partial/\partial y$$

for $j \in \mathbb{N}$. We denote by G the group generated by $\{\phi(1), \phi(2), \ldots\}$. It is an abelian group. Moreover, since S is linearly independent over $\mathbb{Q}$, the property $\phi \neq Id$ implies $\phi_2 \neq Id$ for any $\phi \in G$. In particular, G has the finite-determination property.

By choice, the complex Lie algebra generated by $\{\log \phi(1), \log \phi(2), \ldots\}$ is infinite-dimensional as a complex vector space. This implies that $\overline{G}$ contains non-trivial elements whose order of contact with the identity is arbitrarily high or, in other words, that the map $\pi_k : \overline{G} \to D_k$ is not injective for any $k \in \mathbb{N}$. Hence $\overline{G}$ does not have the finite-determination property and G is not finite-dimensional.

An example of a finite-determination group that is not finite-dimensional does not exist in dimension 1 (Proposition 4.1).

Remark 3.8: We define

$$\phi(j)(x, y) = (x, y + d_j x^2 + z^{j+2}, z) \in \mathrm{Diff}\,(\mathbb{C}^3, 0)$$

for $j \in \mathbb{N}$ where the subset $S := \{d_1, d_2, \ldots\}$ of $\mathbb{C}$ is linearly independent over $\mathbb{Q}$. Let G be the group generated by $\{\phi(1), \phi(2), \ldots\}$. Analogously as in the previous example, G is finitely-determined but it is not finite-dimensional. We have

$$(\phi(j)^{-1}\{x = y = 0\}, \{y = 0\}) = \dim \frac{\mathcal{O}_3}{(x, y + d_j x^2 + z^{j+2}, y)} = j + 2$$

for any $j \in \mathbb{N}$. As a consequence finite-determination does not suffice to guarantee the uniform intersection property (cf. Theorem 1.5).

Definition 3.3: We say that a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is **algebraic** if G is pro-algebraic and $\dim G < \infty$.

Remark 3.9: An algebraic subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is the image by an algebraic monomorphism of an algebraic matrix group. Given $k_0 \in \mathbb{N}$ such that $\pi_{k_0} : \overline{G} \to G_{k_0}$ is injective, the map $\pi_{k_0}^{-1} : G_{k_0} \to G$ is an isomorphism of groups (Remark 3.2). Moreover, it is algebraic in every jet space since $\pi_k \circ \pi_{k_0}^{-1} : G_{k_0} \to G_k$ is the inverse of the algebraic isomorphism $\pi_{k,k_0} : G_k \to G_{k_0}$ for any $k \geq k_0$ (Remark 3.2).

The characterization of pro-algebraic groups given by Proposition 2.2 provides a characterization of algebraic subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$.

Lemma 3.2: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then G is algebraic if and only if G_k^* is algebraic for any $k \in \mathbb{N}$ and the sequence $(\dim G_k^*)_{k \geq 1}$ is bounded.*

Proof. The group G is pro-algebraic if and only if G_k^* is algebraic for any $k \in \mathbb{N}$ and G is closed in the Krull topology by Proposition 2.2.

The sufficient condition is obvious. Let us show the necessary condition. It suffices to show that G is closed in the Krull topology. There exists $k_0 \in \mathbb{N}$ such that $\pi_{k+1,k} : G_{k+1} \to G_k$ is injective for any $k \geq k_0$ by Remark 3.3. Therefore the map $\pi_{k_0} : G \to G_{k_0}$ is injective. Consider a sequence $(\eta_m)_{m \geq 1}$ of elements of G that converge in the Krull topology. Then there exists $m_0 \in \mathbb{N}$ such that $(\eta_m)_{k_0} \equiv (\eta_{m_0})_{k_0}$ if $m \geq m_0$. We deduce $\eta_m \equiv \eta_{m_0}$ for any $m \geq m_0$ and the sequence converges to $\eta_{m_0} \in G$. We obtain that G is closed in the Krull topology. ∎

Let us provide the first examples of finite-dimensional groups. Indeed we will see that cyclic groups and one-parameter groups are always finite-dimensional.

Proposition 3.3: *Let $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We have* $\dim\langle\phi\rangle \leq n$.

Proof. We have

$$\dim \langle\phi\rangle_k = \dim \langle\phi_s\rangle_k + \dim \langle\phi_u\rangle_k \tag{1}$$

for any $k \in \mathbb{N}$ by Lemma 2.7. It suffices to show that $\langle\phi_s\rangle$ and $\langle\phi_u\rangle$ are finite-dimensional.

Since ϕ_s is semisimple, we can suppose up to a formal change of coordinates that $\overline{\langle\phi_s\rangle}$ is contained in the group of diagonal matrices (Remark 2.13). We deduce $\dim\langle\phi_s\rangle \leq n$.

Since ϕ_u is unipotent, we obtain $\overline{\langle\phi_u\ \rangle} = \{\phi_u^t : t \in \mathbb{C}\}$ and, in particular, $\langle\phi_u\ \rangle_k = \{\exp(t \log \phi_{u,k}) : t \in \mathbb{C}\}$ for any $k \in \mathbb{N}$ by Remark 2.8. We deduce $\dim\langle\phi_u\rangle = 1$ if $\phi_u \not\equiv Id$ and $\dim\langle Id\rangle = 0$. We get $\dim\langle\phi\rangle \leq n+1$ by Equation (1). In order to show $\dim\langle\phi\rangle \leq n$ let us prove that $\dim\langle\phi_s\rangle = n$ implies $\phi_u \equiv Id$. Indeed in such a case $\overline{\langle\phi_s\rangle}$ is the linear group of diagonal transformations. Every element of such a group commutes with ϕ_u by Lemma 2.7 and hence ϕ_u is linear and diagonal. Since ϕ_u is both semisimple and unipotent, it is equal to the identity map. ∎

The finite dimension of one-parameter groups can be obtained by reduction to the cyclic case. More precisely, we will use that every one-parameter group G of formal diffeomorphisms has cyclic subgroups whose Zariski-closure coincides with $\overline{G}$.

Proposition 3.4: *Let $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$. Then there exists $t_0 \in \mathbb{R}$ such that*

$$\{\exp(tX) : t \in \mathbb{C}\} \subset \overline{\langle\exp(t_0 X)\rangle}.$$

In particular, we obtain

$$\dim\{\exp(tX) : t \in \mathbb{C}\} \leq n.$$

Proof. Consider the Jordan decomposition $X = X_s + X_N$ as a sum of commuting formal vector fields such that X_s is formally diagonalizable and X_N is nilpotent. We can suppose $X_s = \sum_{k=1}^{n} \mu_k z_k \partial/\partial z_k$ where $\mu_1, \ldots, \mu_n \in \mathbb{C}$ up to a formal change of coordinates. We denote

$$D = \left\{ \underline{a} \in \mathbb{Z}^n : \sum_{k=1}^{n} a_k \mu_k \neq 0 \right\}$$

where $\underline{a} = (a_1, \ldots, a_n)$. Given $\underline{a} \in D$ we define

$$C_{\underline{a}} = \left\{ t \in \mathbb{C} : t \sum_{k=1}^{n} a_k \mu_k \in 2\pi i \mathbb{Z} \right\};$$

it is a countable set. Consider an element t_0 in the complement of the countable set $\bigcup_{\underline{a} \in D} C_{\underline{a}}$ in $\mathbb{R}^*$. We define

$$\eta(z_1, \ldots, z_n) = \exp(t_0 X_s) = (e^{t_0 \mu_1} z_1, \ldots, e^{t_0 \mu_n} z_n) \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0).$$

The group $\mathcal{C}$ of characters $\chi_{\underline{a}}(w_1, \ldots, w_n) = w_1^{a_1} \cdots w_n^{a_n}$ with $\underline{a} \in \mathbb{Z}^n$ defined by

$$\mathcal{C} = \{ \chi_{\underline{a}} : (e^{t_0 \mu_1}, \ldots, e^{t_0 \mu_n}) \in \ker(\chi_{\underline{a}}) \}$$

satisfies $\mathcal{C} = \{ \chi_{\underline{a}} : \sum_{k=1}^{n} a_k \mu_k = 0 \}$ by our choice of t_0. The group $\overline{\langle \eta \rangle}$ consists of the linear diagonal maps $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $\chi_{\underline{a}}(\lambda_1, \ldots, \lambda_n) = 1$ for any $\chi_{\underline{a}} \in \mathcal{C}$ by Remark 2.13. Since $(e^{t\mu_1}, \ldots, e^{t\mu_n}) \in \ker(\chi_{\underline{a}})$ for all $\chi_{\underline{a}} \in \mathcal{C}$ and $t \in \mathbb{C}$, the one-parameter group $\{\exp(tX_s) : t \in \mathbb{C}\}$ is contained in $\overline{\langle \eta \rangle}$. We denote $\rho = \exp(t_0 X_N)$; it satisfies

$$\overline{\langle \rho \rangle} = \{ \exp(tX_N) : t \in \mathbb{C} \}$$

by Remark 2.8. We denote $\phi = \exp(t_0 X) = \eta \circ \rho$. Since $\overline{\langle \phi \rangle}$ contains

$$\overline{\langle \phi_s \rangle} \cup \overline{\langle \phi_u \rangle} = \overline{\langle \eta \rangle} \cup \overline{\langle \rho \rangle},$$

it also contains $\{\exp(tX) : t \in \mathbb{C}\}$. Hence $\dim\{\exp(tX) : t \in \mathbb{C}\} \leq n$ is a consequence of Proposition 3.3. ∎

The finite-dimensional nature of a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is related to properties of finite decomposition of the elements of the group in terms of generators. The following result illustrates how a finite writing property allows one to decide whether or not $\dim G < \infty$ by solving simpler problems.

We denote $H_1 \cdots H_m = \{ h_1 \circ \cdots \circ h_m : h_j \in H_j \ \forall 1 \leq j \leq m \}$ for a family $H_1, \ldots, H_m$ of subgroups of a group G.

Proposition 3.5: *Let $H_1, \ldots, H_m$ and G be subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose $G \subset H_1 \cdots H_m$. Then we have*

$$\dim G \leq \sum_{1 \leq j \leq m} \dim H_j.$$

Moreover, given $1 \leq j \leq m$ such that $H_j \subset G$ we obtain

$$\dim G/H_j \leq \sum_{k \neq j} \dim H_k.$$

Proof. Fix $k \in \mathbb{N}$. Consider the map

$$\begin{aligned} \tau_k : H_{m,k} \times H_{m-1,k} \times \cdots \times H_{1,k} &\to D_k \\ (B_m, B_{m-1}, \ldots, B_1) &\mapsto B_m B_{m-1} \cdots B_1. \end{aligned}$$

The map τ_k is algebraic even if it is not in general a morphism of groups. As a consequence $\mathrm{Im}(\tau_k)$ is a constructible set whose dimension is less than or equal to $\sum_{1 \leq j \leq m} \dim H_{j,k}$. The Zariski-closure of $\mathrm{Im}(\tau_k)$ is an algebraic set containing G_k^* and then G_k. We deduce $\dim G_k \leq \sum_{1 \leq j \leq m} \dim H_{j,k}$ for any $k \in \mathbb{N}$. The results are a direct consequence of Definition 3.1. ∎

Theorem 3.1: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose there exist $\psi_1, \ldots, \psi_l \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$, $X_1, \ldots, X_m \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ and $p \in \mathbb{N}$ such that every $\phi \in G$ is of the form $\phi = \phi_1 \circ \cdots \circ \phi_q$ where $q \leq p$ and*

$$\phi_r \in \bigcup_{j=1}^{l} \langle \psi_j \rangle \cup \bigcup_{k=1}^{m} \{\exp(tX_j) : t \in \mathbb{C}\}$$

for any $1 \leq r \leq q$. Then G is finite-dimensional.

Proof. The result is a straightforward consequence of Propositions 3.5, 3.3 and 3.4. ∎

Remark 3.10: Given a real connected Lie group G its Iwasawa–Malcev decomposition (cf. [10, Theorem 6], [13]) provides a finite decomposition. Given a maximal (connected) compact subgroup K of G, there exist Lie subgroups $H_1, \ldots, H_r$ of G isomorphic to $\mathbb{R}$ such that any element g of G can be written uniquely and continuously in the form

$$g = h_1 \cdots h_r k$$

where $h_j \in H_j$ for any $1 \leq j \leq r$ and $k \in K$. Let $\{X_1, \ldots, X_m\}$ be a basis of the Lie algebra of K and denote $J_j = \{\exp(tX_j) : t \in \mathbb{R}\}$. The set $J_1 \cdots J_m$

contains a neighborhood of Id in K. Since K is compact and connected, we have

$$G \subset H_1 \cdots H_r \underbrace{J_1 \cdots J_m \cdots J_1 \cdots J_m}_{p \text{ times}}$$

for some $p \in \mathbb{N}$. In particular, a real connected Lie group (and even a Lie group with finitely many connected components) admits a decomposition in words of uniform length whose letters are taken from the elements of a finite set of cyclic and one-parameter subgroups. The Iwasawa–Malcev decomposition is applied in a similar spirit in Binyamini's proof of Theorem 1.4 [3]. Since every algebraic group of formal diffeomorphisms is isomorphic to an algebraic matrix group, finite-dimension can be interpreted as a finite-decomposition property.

This remark can be used to show that Lie groups of formal diffeomorphisms (with finitely many connected components) are finite-dimensional. They have a finite-decomposition property and then are finite-dimensional by Theorem 3.1.

3.2. Extensions of groups of formal diffeomorphisms. In this section we study extensions that are always finite-dimensional. First, we deal with finite extensions.

Lemma 3.3: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Consider a finite index subgroup H of G. Then we obtain $\dim G/H = 0$.*

Proof. There exists a subgroup J of H such that J is a finite index normal subgroup of G. Since J_k is a finite index normal subgroup of G_k by Lemma 2.2, we obtain $\dim J_k = \dim G_k$ for any $k \in \mathbb{N}$. We deduce $\dim G/J = 0$ and then $\dim G/H = 0$ since $J \subset H$. ∎

Corollary 3.1: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Consider subgroups H, K of G such that $H \subset K$ and K is a finite index subgroup of G. Then $\dim G/H = \dim K/H$.*

Proof. We have $\dim G/H = \dim G/K + \dim K/H = \dim K/H$ by Lemma 3.3. ∎

Next, we will consider finitely generated abelian extensions. First let us discuss the finite generation hypothesis. A positive dimensional connected Lie group is not finitely generated since it is not countable. Anyway, it is finitely generated by a finite number of one-parameter groups whose infinitesimal generators are the elements of a basis of the Lie algebra. This idea inspires an alternative definition of a finitely generated subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ in which generators can be elements of the group or one-parameter flows.

Definition 3.4: Let H be a subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We say that G is **finitely generated in the extended sense** over H if there exist elements $\phi_1, \ldots, \phi_l \in G$ and formal vector fields $X_1, \ldots, X_m \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ such that

$$G = \left\langle H, \phi_1, \ldots, \phi_l, \bigcup_{j=1}^{m} \{\exp(tX_j) : t \in \mathbb{R}\} \right\rangle.$$

We say that G/H is finitely generated in the extended sense if H is normal in G. If G is of the form $\langle H, \phi_1, \ldots, \phi_l \rangle$, we say that G is finitely generated over H.

We are interested in calculating the dimension of subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. In this context the new definition of a finitely generated group can be reduced to the usual one via the next lemma.

Lemma 3.4: *Let H be a subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G is finitely generated over H in the extended sense. Then there exists a subgroup G_+ of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $H \subset G_+ \subset G$, $\overline{G}_+ = \overline{G}$ and G_+ is finitely generated over H.*

Proof. Suppose $G = \langle H, \phi_1, \ldots, \phi_l, \bigcup_{j=1}^{m} \{\exp(tX_j) : t \in \mathbb{R}\} \rangle$. Given $1 \leq j \leq m$, there exists $t_j \in \mathbb{R}$ such that $\psi_j := \exp(t_j X_j)$ satisfies $\{\exp(tX_j) : t \in \mathbb{C}\} \subset \overline{\langle \psi_j \rangle}$ by Proposition 3.4. We define

$$G_+ = \langle H, \phi_1, \ldots, \phi_l, \psi_1, \ldots, \psi_m \rangle.$$

It is clear that $H \subset G_+ \subset G$. The choice of ψ_j for $1 \leq j \leq m$ implies $G \subset \overline{G}_+$. Since $G_+ \subset G$, we obtain $\overline{G} = \overline{G}_+$. ∎

Proposition 3.6: *Let H be a normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose G/H is abelian and G/H is finitely generated in the extended sense. Then G/H is finite-dimensional.*

Proof. We have $G = \langle H, \phi_1, \ldots, \phi_l, \bigcup_{j=1}^{m} \{\exp(tX_j) : t \in \mathbb{R}\} \rangle$. We denote

$$H_j = \langle \phi_j \rangle$$

for $1 \leq j \leq l$ and

$$J_k = \{\exp(tX_k) : t \in \mathbb{R}\}$$

for $1 \leq k \leq m$. Since H is normal in G and G/H is abelian, we obtain $G = H_1 \cdots H_l J_1 \cdots J_m H$. This implies

$$\dim G/H \leq \sum_{1 \leq j \leq l} \dim H_j + \sum_{1 \leq k \leq m} \dim J_k \leq (l+m)n$$

by Propositions 3.5, 3.3 and 3.4. ∎

Remark 3.11: Proposition 3.6 implies in particular that Theorem 1.5 can be applied to a finitely generated (in the extended sense) abelian subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. So Seigal–Yakovenko's Theorem 1.3 can be understood as a consequence of the finite-dimension of finitely generated abelian subgroups of formal diffeomorphisms.

Next, let us consider the third type of extensions, namely extensions that are real connected Lie groups. Of course the first task is finding a proper definition of a Lie group for an extension since it is not clear a priori.

Definition 3.5: Let H be a normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We say that G/H is a (connected) Lie group if there exists a (connected) Lie group L and a surjective morphism of groups $\sigma : L \to G/H$ such that the map $\sigma_k : L \to D_k/H_k$ induced by σ is a morphism of differentiable manifolds for any $k \in \mathbb{N}$.

Notice that D_k/H_k is a smooth algebraic manifold. The map $\pi_k : G \to D_k$ induces a map $\hat{\pi}_k' : G/H \to D_k/H_k$. The map σ_k is equal to $\hat{\pi}_k' \circ \sigma$.

Remark 3.12: The previous definition of a Lie group coincides with the usual one for a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$, i.e., in the case $H = \{Id\}$ (cf. [3]).

Let us see that connected Lie group extensions are finite-dimensional (Proposition 3.8). Proposition 3.7 is a corollary of Proposition 3.8.

PROPOSITION 3.7: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G is a connected Lie group. Then G is finite-dimensional.*

PROPOSITION 3.8: *Let H be a normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G/H is a connected Lie group. Then G/H is finite-dimensional.*

Proof. Let us explain the idea of the proof. Suppose $H = \{Id\}$ and L (cf. Definition 3.5) is a connected complex Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$. If L is abelian, i.e., $[L, L] = \{Id\}$, then $\overline{L}$ is also abelian (cf. Lemma 2.5). In general we have

$[L, L] = [\overline{L}, \overline{L}]$ and, in particular, L' is always the derived group of an algebraic group and thus algebraic (cf. [18, Chapter 3.3.3]). We can think of L as an algebraic-by-finitely generated commutative group since L/L' is abelian and finitely generated in the extended sense. Since all these extensions are finite-dimensional in a natural way, hence the image of L is finite-dimensional.

Consider the real Lie group L and the surjective morphism of groups $\sigma : L \to G/H$ provided by Definition 3.5. Fix $k \in \mathbb{N}$. Let $\mathfrak{g}_k$ and $\mathfrak{h}_k$ be the Lie algebras of G_k and H_k. The set G_k/H_k is an algebraic group for any $k \in \mathbb{N}$ (cf. [4, section II.6.8, p. 98]). Let $\mathfrak{g}$ be the Lie algebra of L. Consider the map $(d\sigma_k)_{Id} : \mathfrak{g} \to \mathfrak{g}_k/\mathfrak{h}_k$ induced by σ_k for $k \in \mathbb{N}$. We define by $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified of the Lie algebra $\mathfrak{g}$. We denote by $(d\overline{\sigma}_k)_{Id} : \tilde{\mathfrak{g}} \to \mathfrak{g}_k/\mathfrak{h}_k$ the morphism of complex Lie algebras induced by $(d\sigma_k)_{Id}$. Let $\tilde{L}$ be a connected simply connected complex Lie group whose Lie algebra is equal to $\tilde{\mathfrak{g}}$. Then there exists a unique morphism $\tilde{\sigma}_k : \tilde{L} \to G_k/H_k$ of complex Lie groups such that $(d\tilde{\sigma}_k)_{Id} = (d\overline{\sigma}_k)_{Id}$ for any $k \in \mathbb{N}$ (cf. [18, Chapter 1.2.8]). Notice that $\tilde{\sigma}_k(\tilde{L})$ contains $\sigma_k(L)$.

We denote by $\rho_k : G_k \to G_k/H_k$ the morphism of algebraic groups given by the projection. Since $\tilde{\sigma}_k(\tilde{L})'$ is the derived group of the connected complex Lie group of matrices $\tilde{\sigma}_k(\tilde{L})$, it is an algebraic subgroup of G_k/H_k (cf. [18, Chapter 3.3.3]). Hence $\rho_k^{-1}(\tilde{\sigma}_k(\tilde{L})')$ is an algebraic subgroup of G_k such that

$$\dim \rho_k^{-1}(\tilde{\sigma}_k(\tilde{L})') - \dim H_k \leq \dim \tilde{L}' \leq \dim \tilde{L} \leq \dim_{\mathbb{R}} L$$

where $\dim_{\mathbb{R}} L$ is the dimension of the real Lie group L. Since $\langle G', H \rangle_k^*$ is contained in $\rho_k^{-1}(\tilde{\sigma}_k(\tilde{L}'))$ and the latter group is algebraic, we obtain

$$\langle G', H \rangle_k \subset \rho_k^{-1}(\tilde{\sigma}_k(\tilde{L}'))$$

and then

$$\dim \langle G', H \rangle_k - \dim H_k \leq \dim_{\mathbb{R}} L$$

for any $k \in \mathbb{N}$. Therefore the extension $\langle G', H \rangle / H$ is finite-dimensional. It suffices to show that $G/\langle G', H \rangle$ is finite dimensional by Proposition 3.1.

Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{g}$ (as a real Lie algebra). Fix $k \in \mathbb{N}$. Analogously as in the proof of Proposition 3.4 there exists a countable subset A_k of $\mathbb{R}$ such that the algebraic closure of $\langle \sigma_k(\exp(tX_1)) \rangle$ in G_k/H_k contains the one-parameter group $\sigma_k\{\exp(sX_1) : s \in \mathbb{C}\}$ for any $t \in \mathbb{R}^* \setminus A_k$. We choose $t \in \mathbb{R}^* \setminus \bigcup_{k \geq 1} A_k$ and a representative $\psi_1 \in G$ of the class in G/H defined by $\sigma(\exp(tX_1))$. Analogously we define $\psi_2, \ldots, \psi_m$. It is clear that the extension $\langle G', H, \psi_1, \ldots, \psi_m\rangle / \langle G', H\rangle$ is abelian and finitely generated. It suffices to show $\overline{\langle G', H, \psi_1, \ldots, \psi_m\rangle} = \overline{G}$ by Proposition 3.6. We will prove

$$\overline{\langle H, \psi_1, \ldots, \psi_m\rangle} = \overline{G}.$$

Fix $k \in \mathbb{N}$. We denote $J = \langle H, \psi_1, \ldots, \psi_m\rangle$. The image of the Zariski-closure J_k of J_k^* by the morphism ρ_k is equal to the closure of $\langle J_k^*, H_k\rangle / H_k$ in G_k/H_k. The Zariski-closure of $\langle \rho_k(\psi_{j,k})\rangle$ contains $\sigma_k(\{\exp(sX_j) : s \in \mathbb{C}\})$ for any $1 \leq j \leq m$ by the choice of ψ_j. Hence the closure of $\langle J_k^*, H_k\rangle / H_k$ in G_k/H_k contains $\langle G_k^*, H_k\rangle / H_k$. We deduce that the closure of $\rho_k(J_k^*)$ in G_k/H_k is equal to G_k/H_k. Thus $\rho_k(J_k)$ is equal to G_k/H_k. Since J_k contains H_k, we get $J_k = G_k$ for any $k \in \mathbb{N}$. Hence we obtain $\overline{\langle H, \psi_1, \ldots, \psi_m\rangle} = \overline{G}$. ∎

Remark 3.13: We can recover Binyamini's Theorem 1.4 in the context of the theory of finite-dimensional groups of formal diffeomorphisms by applying Theorem 1.5, Proposition 3.7 and Lemma 3.3.

Binyamini proves that the matrix coefficients of G belong to a noetherian ring by using the Iwasawa–Malcev decomposition of a connected Lie group (cf. [10, Theorem 6]). The same ideas imply that G is finite-dimensional almost for free. Indeed, by Definition 3.5 the image by σ of a one-parameter group is also a one-parameter group. Hence G admits a finite decomposition of the form $G = L_1 \cdots L_m$ where every L_j is either a cyclic group or a one-parameter group by Remark 3.10. Theorem 3.1 implies that G is finite-dimensional. Our choice of proof is intended to stress the efficacy of the approach through towers of extensions. Indeed given a connected Lie group $G \subset \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$, we showed that its derived group G' is finite-dimensional by using classical results of Lie group theory. This allowed us to reduce the problem to treat the finitely generated (in the extended sense) and abelian extension G/G'.

Remark 3.14: In the proof of Proposition 3.8 it suffices to consider a linear independent subset $\{X_1, \ldots, X_m\}$ of $\mathfrak{g}$ whose image in $\mathfrak{g}/\mathfrak{g}'$ is a basis. As a consequence we obtain

$$\dim \sigma(L) \leq \dim \tilde{L}' + (\dim \tilde{L} - \dim \tilde{L}')n = \dim \tilde{L} + (\dim \tilde{L} - \dim \tilde{L}')(n-1).$$

This formula recalls the formula in Theorem 4 of [3] in which the role of a maximal connected compact subgroup of L is replaced with the derived group.

Remark 3.15: Subgroups of Lie groups (with finitely many connected components) of formal diffeomorphisms are always subgroups of algebraic groups of formal diffeomorphisms by Proposition 3.7 and Lemma 3.3. Given a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ the existence of an embedding of G in a Lie group of formal diffeomorphisms implies that G is contained in the image by an algebraic monomorphism of an algebraic matrix group (Remark 3.9).

The following theorem about finite-determination properties of Lie subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is due to Baouendi et al.

THEOREM 3.2 ([2, Proposition 5.1]): *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G is a Lie group with a finite number of connected components. Then G has the finite-determination property.*

We include a proof since it is an extremely easy application of our techniques.

Proof. The group G is finite-dimensional by Proposition 3.7 and Lemma 3.3. Hence it has the finite-determination property. ∎

In general the implication finite-determination $\implies$ finite-dimension does not hold for subgroups G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ (cf. Remark 3.7). Anyway, it is interesting to explore in which conditions it is true since the former property is much easier to verify. Next, we see that the implication is satisfied, even for extensions, under a property of algebraic closedness for cyclic subgroups.

THEOREM 3.3: *Let H be a normal subgroup of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that the map $\hat{\pi}_{k_0} : \overline{G}/\overline{H} \to G_{k_0}/H_{k_0}$ satisfies that $(\hat{\pi}_{k_0})_{|\langle G, \overline{H}\rangle/\overline{H}}$ is injective for some $k_0 \in \mathbb{N}$. Furthermore suppose $\overline{\langle \phi \rangle} \subset \langle G, \overline{H}\rangle$ for any $\phi \in G$. Then G/H is finite-dimensional.*

Proof. Let T be the subgroup of $\langle G, \overline{H}\rangle$ generated by $\bigcup_{\phi \in G} \overline{\langle \phi \rangle}_0$. The property $\overline{\langle \phi \rangle} \subset \langle G, \overline{H}\rangle$ implies that the group $T_k^* = \langle \bigcup_{\phi \in G} \langle \phi \rangle_{k,0} \rangle$ is contained in $\langle G_k^*, H_k\rangle$. Moreover, it is a (connected) algebraic group since it is generated by a family of connected algebraic matrix groups (Theorem 2.1) for any $k \in \mathbb{N}$. Theorem 2.1 also implies that $\langle T_k^*, H_{k,0}\rangle$ is algebraic. Since H_k is normal in G_k, we deduce that $\langle T_k^*, H_{k,0}\rangle$ is a finite index subgroup of $\langle T_k^*, H_k\rangle$ and, in particular, $\langle T_k^*, H_k\rangle$ is algebraic for any $k \in \mathbb{N}$. The group G_k^* normalizes the algebraic group T_k^*. Thus T_k^* is a normal subgroup of G_k. As a consequence $\langle T_k^*, H_k\rangle$ is a subgroup of $\langle G_k^*, H_k\rangle$ that is normal in G_k for any $k \in \mathbb{N}$.

The group $\overline{\langle\phi\rangle}_0$ is a finite index normal subgroup of $\overline{\langle\phi\rangle}$ and $\langle\phi\rangle_{k,0}$ is a finite index normal subgroup of $\langle\phi\rangle_k$ for all $\phi \in G$ and $k \in \mathbb{N}$. We obtain that all elements of the subgroup $\langle G_k^*, H_k\rangle/\langle T_k^*, H_k\rangle$ of the algebraic matrix group $G_k/\langle T_k^*, H_k\rangle$ have finite order. Suppose that the Zariski-closure of a group of matrices of elements of finite order consists only of semisimple elements; we will prove this later on (Lemma 3.5). Since $G_k/\langle T_k^*, H_k\rangle$ is the Zariski-closure of $\langle G_k^*, H_k\rangle/\langle T_k^*, H_k\rangle$, the group $G_k/\langle T_k^*, H_k\rangle$ consists of semisimple elements.

Let us show that $\hat{\pi}_{k,k_0} : G_k/H_k \to G_{k_0}/H_{k_0}$ is injective for any $k \geq k_0$. Such a property implies that G/H is finite-dimensional by Proposition 3.2. The hypothesis implies that $(\hat{\pi}_{k,k_0})_{|\langle G_k^*, H_k\rangle/H_k}$ is injective. Let $\phi_k H_k$ be an element of the kernel of $\hat{\pi}_{k,k_0}$ where $\phi \in \overline{G}$. Since $\phi_{k_0} \in H_{k_0}$, there exists $\eta \in \overline{H}$ such that $\phi_{k_0} \equiv \eta_{k_0}$ and, in particular,

$$(\phi \circ \eta^{-1})_{k_0} \equiv Id.$$

The formal diffeomorphism $\phi \circ \eta^{-1}$ is unipotent and so is $(\phi \circ \eta^{-1})_k$. Thus the class of $(\phi \circ \eta^{-1})_k$ in $G_k/\langle T_k^*, H_k\rangle$ is unipotent. Since it is also semisimple by the previous discussion, we obtain $(\phi \circ \eta^{-1})_k \in \langle T_k^*, H_k\rangle$ and then $\phi_k \in \langle T_k^*, H_k\rangle \subset \langle G_k^*, H_k\rangle$. Since $(\hat{\pi}_{k,k_0})_{|\langle G_k^*, H_k\rangle/H_k}$ is injective, we obtain $\phi_k \in H_k$. In particular, $\hat{\pi}_{k,k_0} : G_k/H_k \to G_{k_0}/H_{k_0}$ is injective for any $k \geq k_0$. ∎

Corollary 3.2: *Let G be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ that has the finite-determination property. Suppose that $\overline{\langle\phi\rangle}$ is contained in G for any $\phi \in G$. Then G is finite-dimensional.*

Let us remark, regarding Corollary 3.2, that the condition $\overline{\langle\phi\rangle} \subset G$ is easy to verify if we know the Jordan–Chevalley decomposition of the elements of G.

Lemma 3.5: *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$ such that all its elements have finite order. Then all elements of $\overline{G}$ are semisimple.*

Proof. The Tits alternative [27] implies that either G is virtually solvable or it contains a non-abelian free group. Since clearly the second option is impossible, G is virtually solvable. Hence $\overline{G}_0$ is solvable. Since it is also connected, the group $\overline{G}_0$ is upper triangular up to a linear change of coordinates by Lie–Kolchin's theorem (cf. [8, section 17.6, p. 113]).

We denote $H = G \cap \overline{G}_0$; it is a finite index normal subgroup of G. The derived group H' of H consists of unipotent upper triangular matrices. They are also semisimple by hypothesis and, as a consequence, H' is the trivial group and H is abelian. Moreover, H consists of semisimple elements; hence H and then $\overline{H}$ are diagonalizable. Since H is a finite index normal subgroup of G, $\overline{H}$ is a finite index normal subgroup of $\overline{G}$. An element A of $\overline{G}$ satisfies $A^k \in \overline{H}$ for some $k \in \mathbb{N}$. Since A^k is semisimple, A is semisimple for any $A \in \overline{G}$. ∎

4. Families of finite-dimensional groups

The finite-dimensional subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ can be characterized; they are the solvable groups.

Proposition 4.1: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$. Then the following conditions are equivalent:*

(1) *G is solvable.*
(2) *G has the finite-determination property.*
(3) *G is finite-dimensional.*

Proof. The items (1) and (2) are equivalent (cf. [12, Théorème 1.4.1]).

Let us show (1) $\Longrightarrow$ (3). The group $\overline{G}$ is solvable by Lemma 2.5. Since (1) $\Longrightarrow$ (2), $\overline{G}$ has the finite determination property and then G is finite-dimensional by Remark 3.6.

Let us prove (3) $\Longrightarrow$ (1). Since G is finite-dimensional, $\overline{G}$ has the finite-determination property by Remark 3.6. The property (2) $\Longrightarrow$ (1) implies that $\overline{G}$ and then G are solvable. ∎

Remark 4.1: Any solvable subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ satisfies

$$\dim G = \dim \overline{G} \leq 2$$

by the formal classification of such groups (cf. [21, Theorem 5.3] [9, section $6B_3$]).

Remark 4.2: Notice that solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ are not in general finite-dimensional for $n \geq 2$ (cf. Remark 3.6).

It is very easy to use the extension theorems in section 3 to find a big class of examples of finite-dimensional subgroups of formal diffeomorphisms.

Theorem 4.1: *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G has a subnormal series*

$$\{Id\} = G^0 \lhd G^1 \lhd \cdots \lhd G^m = H \tag{2}$$

such that G^{j+1}/G^j is either

- *finite or*
- *abelian and finitely generated in the extended sense or*
- *a connected Lie group (cf. Definition 3.5)*

for any $0 \leq j < m$. Then G is finite-dimensional.

Proof. The result is a straightforward consequence of Proposition 3.1, Lemma 3.3 and Propositions 3.6 and 3.8. ∎

For instance, a finite-by-cyclic-by-cyclic-by-finite-by-cyclic group of formal diffeomorphisms is finite-dimensional. Anyway we think that it is interesting to apply the extension approach laid out in section 3 to show that several distinguished classes of groups are always finite-dimensional. Our first targets are polycyclic groups.

Definition 4.1: Let G be a group. We say that G is **polycyclic** if it has a subnormal series as in Equation (2) such that G^{j+1}/G^j is cyclic for any $0 \leq j < m$.

Remark 4.3: A group G is polycyclic if and only if it is solvable and every subgroup of G is finitely generated (cf. [22, Theorem 5.4.12]).

Theorem 4.2: *Let G be a virtually polycyclic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then G is finite-dimensional.*

Proof. A virtually polycyclic group is a cyclic-by-$\cdots$-by-cyclic-by-finite group. Therefore G is finite-dimensional by Theorem 4.1. ∎

Definition 4.2: We say that a group G is **supersolvable** if it has a normal series in which all the factors are cyclic groups.

Remark 4.4: The definition is very similar to Definition 4.1. Anyway supersolubility is stronger since we require the groups G^j in Equation (2) to be normal in G for $0 \leq j \leq m$.

Corollary 4.1: *Let G be a virtually supersolvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then G is finite-dimensional.*

Let us focus now on nilpotent groups.

Theorem 4.3: *Let G be a virtually nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Suppose that G is finitely generated in the extended sense. Then G is finite-dimensional. In particular subgroups of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ of polynomial growth are finite-dimensional.*

Proof. There exists a subgroup G_+ of G such that G_+ is finitely generated and $\overline{G}_+ = \overline{G}$ by Lemma 3.4. Since G_+ is a subgroup of G, it is virtually nilpotent. Thus up to replacing G with G_+ we can suppose that G is finitely generated.

Let H be a finite index normal nilpotent subgroup of G. Since H is a finite index subgroup of a finitely generated group, it is finitely generated (cf. [22, Theorem 1.6.11]). Any finitely generated nilpotent group is polycyclic (cf. [11, Theorem 17.2.2]). Therefore G is virtually polycyclic and then finite-dimensional by Theorem 4.2.

By a theorem of Gromov [7], the groups of polynomial growth are exactly the finitely generated virtually nilpotent groups. Hence every subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ of polynomial growth is finite-dimensional. ∎

We provide a sort of reciprocal of the previous theorem in the setting of unipotent subgroups of formal diffeomorphisms.

Lemma 4.1: *Let G be a unipotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Suppose that G has the finite-determination property. Then G is nilpotent.*

Proof. Since G has the finite-determination property, there exists $k \in \mathbb{N}$ such that $\pi_k : G \to G_k^*$ is an isomorphism of groups. Moreover, G_k^* is a unipotent algebraic matrix group. Unipotent groups of matrices are always triangularizable and then nilpotent by Kolchin's theorem (cf. [24, chapter V, p. 35]). Hence G is nilpotent. ∎

Corollary 4.2: *Let G be a unipotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Suppose that G is finitely generated in the extended sense and finitely determined. Then G is finite-dimensional.*

Proof. The group G is nilpotent by Lemma 4.1. Thus it is finite-dimensional by Theorem 4.3. ∎

Remark 4.5: Theorems 4.2, 4.3 and Corollary 4.1 admit straightforward generalizations to extensions. For instance, a finitely generated (in the extended sense) virtually nilpotent extension of groups of formal diffeomorphisms is finite-dimensional.

Let us focus on solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ that are not necessarily polycyclic. In order to show $\dim G < \infty$ for a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ it suffices to consider finite generation properties on the derived groups of G.

PROPOSITION 4.2: *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Suppose that $G^{(\ell)}/G^{(\ell+1)}$ is finitely generated in the extended sense for any $\ell \in \mathbb{Z}_{\geq 0}$. Then G is finite-dimensional.*

Proof. Every extension of the form $G^{(\ell)}/G^{(\ell+1)}$ is abelian and then finite-dimensional by Proposition 3.6. Since there exists ℓ such that $G^{(\ell)} = \{1\}$, G is finite-dimensional by Theorem 4.1. ∎

In the following our goal is weakening substantially the finite-generation hypotheses in Proposition 4.2. We will see that under certain hypotheses a virtually solvable subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is finite-dimensional if and only if G_u (cf. Definition 2.9) is finite-dimensional.

THEOREM 4.4: *Let G be a virtually solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ such that G is finitely generated over G_u in the extended sense. Then G/G_u is finite-dimensional. In particular, G is finite-dimensional if and only if G_u is finite-dimensional.*

COROLLARY 4.3: *Let G be a finitely generated in the extended sense virtually solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ such that G_u is finitely generated in the extended sense and nilpotent. Then G is finite-dimensional.*

Remark 4.6: Notice that the nilpotence of G_u is necessary in Corollary 4.3 by Lemma 4.1. The main advantage of Corollary 4.3 is that we are replacing a property of finite generation for every derived subgroup of G by the analogous property for just G and G_u.

Proof of Theorem 4.4. First let us show that G_u is a subgroup of G if G is virtually solvable. Since $\overline{G}_0$ is solvable (by Lemma 2.6) and $G_{1,0}$ is connected, we can suppose that all elements of $\overline{G}_0$ have linear parts that belong to the group of upper triangular matrices by Lie–Kolchin's theorem (cf. [8, 17.6, p. 113]). Notice that G_u is contained in $\overline{G}_0$. Since the set of unipotent upper triangular matrices is a group that contains the linear part of G_u, the set G_u is also a group. Thus G_u is a normal subgroup of G.

There exists a subgroup J of G such that $G_u \subset J$, $\overline{J} = \overline{G}$ and J is finitely generated over G_u by Lemma 3.4. It suffices to show $\dim(J/G_u) < \infty$ since $\dim(G/G_u) = \dim(J/G_u)$.

We denote $K = J \cap \overline{G}_0$. The group K is a finite index normal subgroup of J. Since K/G_u is a finite index normal subgroup of J/G_u, the group K/G_u is finitely generated. The elements of the derived group K' have linear parts that are unipotent upper triangular matrices. Thus K' is contained in G_u and K/G_u is abelian. We deduce $\dim(K/G_u) < \infty$ by Proposition 3.6 and then $\dim(J/G_u) < \infty$ by Corollary 3.1. ∎

Proof of Corollary 4.3. Since G_u is a normal subgroup of G by the proof of Theorem 4.4, it suffices to show $\dim G_u < \infty$ by Theorem 4.4 and Proposition 3.1. The group G_u is finite-dimensional by Theorem 4.3. ∎

4.1. Examples of infinitely dimensional groups. So far we have exhibited distinguished families of virtually solvable groups whose members are finite-dimensional. Now let us consider the problem of finding families of infinite-dimensional solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$.

Remark 4.7: Consider the subgroup $\langle \phi, \eta \rangle$ of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$ generated by

$$\phi(x, y) = (x, y(1 + x)) \quad \text{and} \quad \eta(x, y) = (x, y + x^2).$$

Since $\langle \phi, \eta \rangle'$ is contained in the group

$$H_1 := \{(x, y + b(x)) : b \in \mathbb{C}\{x\} \cap (x^2)\},$$

we get $\langle \phi, \eta \rangle^{(2)} = \{Id\}$. In particular, $\langle \phi, \eta \rangle$ is a finitely generated unipotent solvable subgroup of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$. Since $[(x, y - x^k), \phi] = (x, y + x^{k+1})$ for any $k \in \mathbb{N}$, the group $\langle \phi, \eta \rangle$ is not nilpotent. Thus $\langle \phi, \eta \rangle$ is neither finitely determined nor finite-dimensional by Lemma 4.1.

Next we see that solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ of high derived length are never finite-dimensional.

Proposition 4.3: *Let G be a solvable group contained in $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ whose derived length is greater that n. Then G does not have the finite-determination property. In particular G is not finite-dimensional.*

Remark 4.8: Such groups always exist if $n \geq 2$. Indeed the maximum of the derived lengths of the solvable unipotent subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is $2n-1$ [14].

Remark 4.9: Notice that given a solvable group G, its derived length is the supremum of the derived lengths of all its finitely generated subgroups. Hence there exists a finitely generated subgroup H of G with the same derived length. In particular, we can suppose that the examples provided by Proposition 4.3 for $n \geq 2$ are finitely generated.

Remark 4.10: Let us provide an example of a group that satisfies the hypotheses of Proposition 4.3 [14]. We denote $\phi(x,y) = (x, y(1+x))$, $\eta(x,y) = (x, y+x^2)$ and $\psi(x,y) = (\frac{x}{1-x}, y)$. Consider the subgroup $G := \langle \phi, \eta, \psi \rangle$ of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$. We define the subgroup

$$H_0 = \{x, y(1+a(x)) + xb(x)) : a, b \in \mathbb{C}\{x\} \cap (x)\}$$

of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$. It is clear that G' is contained in H_0 and $G^{(2)}$ is contained in the abelian group H_1 defined in Remark 4.7. Hence G is a finitely generated unipotent solvable subgroup of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$ whose derived length is at most 3. Since $[\eta^{-1}, \phi] = (x, y + x^3)$,

$$[\psi, \phi] = \left(x, y\frac{1+2x}{(1+x)^2}\right) \quad \text{and} \quad [[\psi, \phi], [\eta^{-1}, \phi]] = \left(x, y - \frac{x^5}{(1+x)^2}\right),$$

the diffeomorphism $[[\psi, \phi], [\eta^{-1}, \phi]]$ belongs to $G^{(2)} \setminus \{Id\}$ and hence the derived length of G is equal to 3.

Proof of Proposition 4.3. Suppose that G has the finite-determination property. Hence G is nilpotent by Lemma 4.1. Therefore the derived length of G is less than or equal to n [14, Theorem 5], obtaining a contradiction. ∎

5. Local intersection theory

Let us explain in this section why Theorem 1.5 holds. A priori we could use directly Binyamini's theorem [3] since an algebraic group is a Lie group with finitely many connected components. Anyway we think that it is instructive to apply our canonical approach to the ideas introduced by Seigal–Yakovenko in [23] (to show Theorem 1.5 for finitely generated in the extended sense abelian subgroups of formal diffeomorphisms).

Consider two formal subschemes I and J of the scheme spec $\hat{\mathcal{O}}_n$. We can identify I and J with two ideals of the ring $\hat{\mathcal{O}}_n$ of formal power series.

Definition 5.1: We define the **intersection multiplicity** (I, J) as

$$(I, J) = \dim_{\mathbb{C}} \hat{\mathcal{O}}_n/(I + J).$$

Remark 5.1: This definition of intersection multiplicity coincides with the usual one if I and J are complete intersections of complementary dimension. It is finite if and only if the usual intersection multiplicity is finite. Moreover, it provides an upper bound for the usual intersection multiplicity of I and J viewed with their associated cycle structures (cf. [6, Proposition 8.2]). Therefore, by showing Theorem 1.5 with Definition 5.1 it will be automatically satisfied for the usual intersection multiplicity.

Next let us show Theorem 1.5. Since we follow Seigal–Yakovenko's ideas we refer to their paper [23] for details. We are interested in stressing how their point of view fits in the context of the theory of finite-dimensional groups of formal diffeomorphisms.

Proof of Theorem 1.5. Let V and W be formal subschemes of spec $\hat{\mathcal{O}}_n$. Suppose that V is given by the ideal K of $\hat{\mathcal{O}}_n$. Given $\phi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$, the subscheme $\phi^{-1}(V)$ is given by the ideal

$$\phi^* K = \{f \circ \phi : f \in K\}.$$

There exists $k \in \mathbb{N}$ such that $\hat{\pi}_k : \overline{G} \to G_k$ is an isomorphism of groups by Proposition 3.2. The map $\pi_{m,k} : G_m \to G_k$ is an isomorphism of algebraic groups for any $m \geq k$. In particular, the affine coordinate rings $\mathbb{C}[G_k]$ and $\mathbb{C}[G_m]$ are isomorphic as $\mathbb{C}$-algebras for any $m \geq k$.

Given an ideal J of $\hat{\mathcal{O}}_n$ the property $\dim_{\mathbb{C}} \hat{\mathcal{O}}_n/J > m$ is equivalent to a system of algebraic equations on the coefficients of the m-th jets of the generators of J [23, Lemma 3]. In particular, $(\phi^{-1}(V), W) > m$ holds for $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ if and only if the coefficients of the m-th jet of ϕ satisfy a certain system of algebraic equations. More intrinsically we can say that

$$S_m := \{\phi \in \overline{G} : (\phi^{-1}(V), W) > m\}$$

defines an ideal I_m of the affine coordinate ring $\mathbb{C}[G_m]$. It also defines an ideal, that we denote also by I_m, in $\mathbb{C}[G_k]$ for any $m \geq k$. We can suppose $I_m \subset I_{m'}$ for all $m' \geq m \geq k$ by replacing I_m with $I_k + \cdots + I_m$ for $m \geq k$. Since $\mathbb{C}[G_k]$ is noetherian, there exists $m_0 \geq k$ such that $I_m = I_{m_0}$ for any $m \geq m_0$. In particular, $(\phi^{-1}(V), W) > m_0$ implies $(\phi^{-1}(V), W) = \infty$ for any $\phi \in \overline{G}$. ∎

Remark 5.2: The key point of the proof is showing that the increasing sequence of ideals $I_1 \subset I_2 \subset \cdots$ is contained in a noetherian ring. Seigal and Yakovenko show that it is contained in a ring of quasipolynomials in their setting [23] whereas Binyamini includes them in a noetherian subring of continuous functions of G [3]. We use the fact that the coefficients of degree greater than k of the Taylor expansion of the elements of G are regular functions on the coefficients of degree less than or equal to k if G is finite-dimensional (Remark 3.4). This allows us to write all equations defining the ideals I_m in terms of the coefficients of $\phi \in G$ of degree less than or equal to k. In particular, Theorem 1.5 is an immediate consequence of the noetherianity of polynomial rings in finitely many complex variables. More precisely, in the finite-dimensional setting the noetherian ring $\mathbb{C}[G_k]$ containing all the ideals I_m for $m \in \mathbb{N}$ is an affine coordinate ring of an algebraic matrix group canonically associated to G.

Proof of Theorem 1.6. The hypothesis implies that G is finite-dimensional by Theorems 4.2 and 4.3. Hence the conclusion is a consequence of Theorem 1.5. ∎

References

[1] V. I. Arnol'd, *Bounds for Milnor numbers of intersections in holomorphic dynamical systems*, in *Topological Methods in Modern Mathematics (Stony Brook, NY, 1991)*, Publish or Perish, Houston, TX, 1993, pp. 379–390.

[2] M. S. Baouendi, L. P. Rothschild, J. Winkelmann and D. Zaitsev, *Lie group structures on groups of diffeomorphisms and applications to CR manifolds*, Université de Grenoble. Annales de l'Institut Fourier **54** (2004), 1279–1303.

[3] G. Binyamini, *Finiteness properties of formal Lie group actions*, Transformation Groups **20** (2015), 939–952.

[4] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, New York, 1991.

[5] J. Écalle, *Théorie itérative: introduction à la théorie des invariants holomorphes*, Journal de Mathématiques Pures et Appliquées **54** (1975), 183–258.

[6] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 2, Springer, Berlin, 1998.

[7] M. Gromov, *Groups of polynomial growth and expanding maps. Appendix by Jacques Tits*, Institut des Hautes Études Scientifiques. Publications Mathématiques **53** (1981), 53–78.

[8] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 21, Springer, New York–Heidelberg, 1995.

[9] Y. Ilyashenko and S. Yakovenko, *Lectures on Analytic Differential Equations*, Graduate Studies in Mathematics, Vol. 86, American Mathematical Society, Providence, RI, 2008.

[10] K. Iwasawa, *On some types of topological groups*, Annals of Mathematics **50** (1949), 507–558.

[11] M. I. Kargapolov and Ju. I. Merzljakov, *Fundamentals of the theory of groups*, Graduate Texts in Mathematics, Vol. 62, Springer-Verlag, New York, 1979.

[12] F. Loray, *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*, `https://hal.archives-ouvertes.fr/hal-00016434`, 2006.

[13] A. I. Mal'tsev, *On the theory of the Lie groups in the large*, Matematicheskii Sbornik **16** (1945), 163–190.

[14] M. Martelo and J. Ribón, *Derived length of solvable groups of local diffeomorphisms*, Mathematische Annalen **358** (2014), 701–728.

[15] J. Martinet, *Normalisation des champs de vecteurs holomorphes (d'après A.-D. Brjuno)*, in *Bourbaki Seminar, Vol. 1980/81*, Lecture Notes in Mathematics, Vol. 901, Springer, Berlin–New York, 1981, pp. 55–170.

[16] J. Martinet and J.-P. Ramis, *Classification analytique des équations differentielles non linéaires résonnantes du premier ordre*, Annales Scientifiques de l'École Normale Supérieure **16** (1983), 571–621.

[17] J. J. Morales-Ruiz, J.-P. Ramis and C. Simo, *Integrability of Hamiltonian systems and differential Galois groups of higher variational equations*, Annales Scientifiques de l'École Normale Supérieure **40** (2007), 845–884.

[18] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990.

[19] J. Ribón, *The solvable length of groups of local diffeomorphisms*, Journal für die Reine und Angewandte Mathematik, to appear, doi: 10.1515/crelle-2016-0066.

[20] J. Ribón, *Embedding smooth and formal diffeomorphisms through the Jordan–Chevalley decomposition*, Journal of Differential Equations **253** (2012), 3211–3231.

[21] J. Ribón, *Algebraic properties of groups of complex analytic local diffeomorphisms*, in *VIII Escuela Doctoral Intercontinental de Matemáticas*, Pontificia Universidad Católica del Perú, Lima, 2015, pp. 185–230.

[22] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, Vol. 80, Springer-Verlag, New York, 1996.

[23] A. L. Seigal and S. Yakovenko, *Local dynamics of intersections: V. I. Arnold's theorem revisited*, Israel Journal of Mathematics **201** (2014), 813–833.

[24] J.-P. Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics, Vol. 1500, Springer-Verlag, Berlin, 1992.

[25] M. Shub and D. Sullivan, *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.

[26] Th. Skolem, *Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen*, 8. Skand. Mat.-Kongr., 163–188 (1935).

[27] J. Tits, *Free subgroups in linear groups*, Journal of Algebra **20** (1972), 250–270.