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CUSPIDAL REPRESENTATIONS IN THE COHOMOLOGY OF DELIGNE–LUSZTIG VARIETIES FOR GL(2) OVER FINITE RINGS

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ABSTRACT

We define closed subvarieties of some Deligne–Lusztig varieties for GL(2) over finite rings and study their étale cohomology. As a result, we show that cuspidal representations appear in it. Such closed varieties are studied in [Lus2] in a special case. We can do the same things for a Deligne–Lusztig variety associated to a quaternion division algebra over a non-archimedean local field. A product of such varieties can be regarded as an affine bundle over a curve. The base curve appears as an open subscheme of a union of irreducible components of the stable reduction of the Lubin–Tate curve in a special case. Finally, we state some conjecture on a part of the stable reduction using the above varieties. This is an attempt to understand bad reduction of Lubin–Tate curves via Deligne–Lusztig varieties.

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1. Introduction

Let K be a non-archimedean local field and $\mathfrak{o}$ the ring of integers in K. Let $\mathfrak{p}$ denote the maximal ideal of $\mathfrak{o}$. Let p be the characteristic of $k = \mathfrak{o}/\mathfrak{p}$. Let k^{ac} be an algebraic closure of k. Assume that the characteristic of K equals p. Let G be a reductive group over k. In [Lus] and [Lus2], for each $n \geq 1$, Lusztig constructs a variety over k^{ac} whose étale cohomology realizes certain irreducible representations of $G(\mathfrak{o}/\mathfrak{p}^n)$. These representations are attached to a "maximal" torus in G and its characters in general position. We call such a variety a Deligne–Lusztig variety for $G(\mathfrak{o}/\mathfrak{p}^n)$. For $n = 1$, this theory is the Deligne–Lusztig theory for $G(k)$ in [DL]. We call the theory in [Lus] and [Lus2] the **Deligne–Lusztig theory over finite rings**.

In [Lus2, §3], the Deligne–Lusztig variety for $\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p}^2)$ is explicitly studied. In [Lus], a construction in the division algebra case is studied. It seems complicated to study the cohomology of a Deligne–Lusztig variety directly in general, because the cohomology of this variety contains many irreducible representations with lower conductor (cf. [Lus2, §3]).

Let D be the quaternion division algebra over K. Let $\mathcal{O}_D$ be the maximal order in D, and $\mathfrak{p}_D$ the two-sided maximal ideal of $\mathcal{O}_D$. In this paper, for $n \geq 1$, we study certain closed subvarieties $\mathbf{X}_n$ and $\mathbf{X}_n^D$ in Deligne–Lusztig varieties for $G_n^F = \mathrm{GL}_2(\mathfrak{o}/\mathfrak{p}^n)$ and $\mathcal{O}_{2n-1}^\times = (\mathcal{O}_D/\mathfrak{p}_D^{2n-1})^\times$ respectively, and study their étale cohomology. An idea to consider such subvarieties is seen in the case $\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p}^2)$ in [Lus2, §§3.3–3.4]. For each n, the cohomology of $\mathbf{X}_n$ realizes cuspidal representations not factoring through the canonical map $G_n^F \twoheadrightarrow G_m^F$ for any integer $m < n$. All irreducible representations of G_n^F are constructed in [Onn] and [Sta]. In [Onn], more generally, all irreducible representations of an automorphism group of a finite $\mathfrak{o}$-module of rank two are classified. For general $r \geq 2$ and $n \geq 1$, strongly cuspidal representations of $\mathrm{GL}_r(\mathfrak{o}/\mathfrak{p}^n)$ are constructed in [AOPS]. In particular, all cuspidal representations of G_n^F are constructed in [AOPS], [Onn] and [Sta]. Let $q = |k|$. Then $\mathbf{X}_1$ is the curve defined by $(x^q y - xy^q)^{q-1} = 1$, and $\mathbf{X}_1^D$ is a disjoint union of finitely many closed points. The curve is called the Deligne–Lusztig curve for $\mathrm{GL}_2(\mathbb{F}_q)$, which we denote by Z_{DL}. For $n \geq 2$, the varieties $\mathbf{X}_n$ and $\mathbf{X}_n^D$ are affine bundles over a disjoint union of some copies of one point or the curve Z_0 defined by the equation $X^{q^2} - X = Y^{q(q+1)} - Y^{q+1}$ over k^{ac} depending on the parity of n. Furthermore, the product $\mathbf{X}_n \times \mathbf{X}_n^D$ is an affine bundle of relative dimension $n-1$

over a disjoint union of copies of the curve Z_0. We can understand their étale cohomology explicitly in Propositions 3.18 and 4.12. Let K_2 be the quadratic unramified extension over K. The cuspidal representations are attached to certain characters of

$$T_n^F = (\mathcal{O}_{K_2}/\mathfrak{p}_{K_2}^n)^\times.$$

The varieties $\mathbf{X}_n$ and $\mathbf{X}_n^D$ admit actions of T_n^F. Let

$$\Delta \colon T_n^F \hookrightarrow T_n^F \times T_n^F; \quad t \mapsto (t, t^{-1}).$$

By taking the quotient of the product $\mathbf{X}_n \times \mathbf{X}_n^D$ by the subgroup $\Delta(T_n^F)$, we obtain a variety $\mathbb{X}_n$, which admits the action of

$$\mathfrak{G}_n = G_n^F \times \mathcal{O}_{2n-1}^\times \times T_n^F.$$

This variety is an affine bundle over a curve Y_n with $\mathfrak{G}_n$-action. This curve Y_n is isomorphic to the curve Z_{DL} if $n = 1$, and a disjoint union of some copies of Z_0 if $n > 1$. The curve Y_n is introduced in §5.1 and its middle cohomology is studied in Theorem 5.1. To describe the group action on $\mathfrak{G}_n$ on $\mathbb{X}_n$, it is natural to use a notion of linking order given in [W2]. Hence, we recall this notion in §2.2.

The above analysis was motivated by the geometry of the Lubin–Tate curve $\mathbf{X}(\mathfrak{p}^n)$ with Drinfeld level $\mathfrak{p}^n$-structures. Let $\mathbf{C}$ be the completion of an algebraic closure of K. Let I_K denote the inertia subgroup of K. Let $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ denote the base change of $\mathbf{X}(\mathfrak{p}^n)$ to $\mathbf{C}$. As irreducible components in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$, it is known that copies of the smooth compactification of Z_0 appear (cf. [T2] and [W3]). We call these components unramified components. See the beginning of §5 for the reason why we call them unramified. The base change $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ admits an action of $\mathrm{GL}_2(\mathfrak{o}) \times \mathcal{O}_D^\times \times I_K$ (cf. [Ca]). By local class field theory over K_2, we have a surjective map $I_{K_2} \twoheadrightarrow \mathcal{O}_{K_2}^\times$. By composing with the canonical isomorphism $I_K \xleftarrow{\sim} I_{K_2}$, we obtain the surjective homomorphism $I_K \twoheadrightarrow \mathcal{O}_{K_2}^\times$. Then, we have the surjective homomorphism

$$\mathfrak{G} = \mathrm{GL}_2(\mathfrak{o}) \times \mathcal{O}_D^\times \times I_K \twoheadrightarrow \mathfrak{G}_n.$$

For an affinoid $\mathbf{X}$, let $\overline{\mathbf{X}}$ denote its canonical reduction. For a positive integer $n \geq 1$, we conjecture that there exists a $\mathfrak{G}$-stable affinoid subdomain $\mathbf{Y}_n \subset \mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ such that

- the $\mathfrak{G}$-action on $\mathbf{Y}_n$ factors through the map $\mathfrak{G} \twoheadrightarrow \mathfrak{G}_n$, and
- there exists a $\mathfrak{G}_n$-equivariant isomorphism $\overline{\mathbf{Y}}_n \simeq Y_n$

(cf. Conjecture 5.2). By definition, the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ is a stable curve. In general, a stable curve consists of several irreducible components which intersect at ordinary double points. By this conjecture, we can understand an open subscheme of a union of irreducible components in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ (cf. Remark 5.3 (2)). In [W1], Weinstein constructs a concrete stable curve which is a candidate of the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$. In the unramified case, the curve Y_n is very similar to the stable curve constructed in [W1] (cf. (5.2)). Originally, our motivation of this work was to give some Deligne–Lusztig interpretation of the curve. Furthermore, the Weinstein conjecture is justified through the works [W3] and [T2] in some sense. In the case where $n = 1$ and $\mathrm{GL}(r)$ ($r \geq 2$), such things are studied in [Y]. We learned that the inertia action can be interpreted as the action of a maximal torus from [Y]. See [BW] for a generalization of [Y].

In the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ in the case where $p \neq 2$, another type of curve appears as an irreducible component. This is the smooth compactification of the Artin–Schreier curve defined by $a^q - a = s^2$ (cf. [T2] and [W3]). The middle cohomology of these components is related to some characters of $\mathcal{O}_L^{\times}$, where L is a tamely ramified quadratic extension of K. We do not know whether a Deligne–Lusztig type interpretation via these components exists as in this paper. See [Sta2] for a generalization of a Deligne–Lusztig variety in this direction. A Lubin–Tate curve can be regarded as a local model of a modular curve. A modular curve is a special case of Shimura varieties. There are many works which relate bad reduction of Shimura varieties to Deligne–Lusztig varieties (cf. [Ra]). The above conjecture is regarded as an attempt to describe bad reduction of Lubin–Tate curves via Deligne–Lusztig theory.

On the division algebra side, a certain Deligne–Lusztig variety is studied in [Ch] in a quite general setting. In the general linear group case, coverings of Deligne–Lusztig varieties are studied in [Iv]. For arbitrary reductive groups, in [CS], they prove that certain representations appear in the cohomology of Deligne–Lusztig varieties.

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2. Preliminaries

In §2.1, we introduce some notation used in this paper. Throughout the rest of the paper, we fix a non-archimedean local field K and always assume that the characteristic of K is p. In §2.2, we introduce a notion of linking order which will be used in §5. We introduce isomorphisms (2.13) and (2.14) which will be used to describe group action on subvarieties of Deligne–Lusztig varieties in §3.2 and §4.2 respectively.

2.1. Notation. For a non-archimedean local field L, let $\mathfrak{p}_L$ denote the maximal ideal of the ring of integers of L. For an integer $i \geq 1$, we set $U_L^i = 1 + \mathfrak{p}_L^i$. As before, we denote by $\mathfrak{o}$ and $\mathfrak{p}$ the ring of integers in K and its maximal ideal respectively. Let $k = \mathfrak{o}/\mathfrak{p}$ and $q = |k|$. Let K^{ur} be the maximal unramified extension of K in an algebraic closure K^{ac} of K and $\widetilde{K}$ the $\mathfrak{p}$-adic completion of K^{ur}. We write $\widetilde{\mathfrak{o}}$ and $\widetilde{\mathfrak{p}}$ for the ring of integers of $\widetilde{K}$ and its maximal ideal, respectively. Let k_2 be the quadratic extension of k in $k^{\mathrm{ac}} = \widetilde{\mathfrak{o}}/\widetilde{\mathfrak{p}}$. Let K_2 be the unramified quadratic extension of K in K^{ac}, and $\mathfrak{O}$ the ring of integers of K_2. For a positive integer $i \geq 1$, we set

$$\mathfrak{o}_i = \mathfrak{o}/\mathfrak{p}^i, \quad \widetilde{\mathfrak{o}}_i = \widetilde{\mathfrak{o}}/\widetilde{\mathfrak{p}}^i, \quad \mathfrak{O}_i = \mathfrak{O}/\mathfrak{p}_{K_2}^i.$$

2.2. Linking order. We recall the linking order defined in [W2, §4.3]. In this paper, we treat only the unramified case.

Let D be the quaternion division algebra over K and let $\mathcal{O}_D$ be the maximal order of D. Let $\mathfrak{p}_D$ be the maximal two-sided ideal of $\mathcal{O}_D$. For a positive integer i, we set $U_D^i = 1 + \mathfrak{p}_D^i \subset \mathcal{O}_D^\times$ and $\mathcal{O}_i = \mathcal{O}_D/\mathfrak{p}_D^i$. By taking a uniformizer $\varpi \in K$, we fix an isomorphism $K \simeq k((\varpi))$. We choose an element $\varphi \in \mathfrak{p}_D$ such that $\varphi^2 = \varpi$. We have isomorphisms $D \simeq K_2 \oplus \varphi K_2$ and $\mathcal{O}_D \simeq \mathfrak{O} \oplus \varphi\mathfrak{O}$. We regard K_2 as a K-subalgebra of D in this way. We set

$$\begin{aligned} &A_1 = \mathrm{M}_2(K), \quad A_2 = D, \\ &\mathfrak{A}_1 = \mathrm{M}_2(\mathfrak{o}), \quad \mathfrak{A}_2 = \mathcal{O}_D, \quad \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2. \end{aligned}$$

For $\zeta \in k_2 \setminus k$, we consider the K-embedding

$$\iota_\zeta \colon K_2 \hookrightarrow A_1; \quad a + b\zeta \mapsto \begin{pmatrix} a + b(\zeta^q + \zeta) & b \\ -b\zeta^{q+1} & a \end{pmatrix} \tag{2.1}$$

with $a, b \in K$. This is the regular embedding with respect to the ordered basis $\{\zeta, 1\}$ of K_2 over K. Note that $\operatorname{tr} \iota_\zeta(\zeta) = \mathrm{Tr}_{K_2/K}(\zeta) = \zeta^q + \zeta$ and

$\det \iota_\zeta(\zeta) = \mathrm{Nr}_{K_2/K}(\zeta) = \zeta^{q+1}$. Some readers may think it unnatural to consider the ordered basis $\{\zeta, 1\}$ not $\{1, \zeta\}$. However, the action of this subgroup $\iota_\zeta(\mathfrak{O}^\times)$ on a Deligne–Lusztig variety will be related to a torus action on it in Lemma 3.8 (1) later. Hence, we consider the basis here.

We fix $\zeta \in k_2 \setminus k$. Let $\Delta_\zeta \colon K_2 \hookrightarrow A_1 \times A_2$ be the diagonal map. For $i = 1, 2$, let C_i be the orthogonal complement of K_2 in A_i with respect to the standard trace pairing. We set $\mathfrak{C}_i = C_i \cap \mathfrak{A}_i$ (cf. [W2, §4.1]). Then, $\mathfrak{C}_i$ is a left and right $\mathfrak{O}$-module of rank one. We have

$$\mathfrak{A}_i \simeq \mathfrak{O} \oplus \mathfrak{C}_i \tag{2.2}$$

for $i = 1, 2$. Let $\mathrm{Gal}(K_2/K)$ be the Galois group of the extension K_2/K. Let $\sigma \in \mathrm{Gal}(K_2/K)$ be the non-trivial element. We have $xv = vx^\sigma$ for $x \in \mathfrak{O}$ and $v \in \mathfrak{C}_i$. We easily check that

$$\mathfrak{C}_1 = \left\{ h(a,b) = \begin{pmatrix} -a & b \\ a(\zeta^q + \zeta) + b\zeta^{q+1} & a \end{pmatrix} \in \mathfrak{A}_1 \;\middle|\; a, b \in \mathfrak{o} \right\}, \tag{2.3}$$

$$\mathfrak{C}_2 = \varphi \mathfrak{O}. \tag{2.4}$$

Let $n \geq 0$ be a non-negative integer. We set $l = [(n+1)/2]$ and $l' = [n/2]$.

We put

$$V_1^n = \mathfrak{p}_{K_2}^l \mathfrak{C}_1 \subset \mathfrak{A}_1, \quad V_2^n = \mathfrak{p}_{K_2}^{l'} \mathfrak{C}_2 \subset \mathfrak{A}_2.$$

We have $V_i^n V_i^n \subset \mathfrak{p}_{K_2}^n$ for $i = 1, 2$. We set $\mathbf{V}^n = V_1^n \times V_2^n \subset \mathfrak{A}$ and

$$\mathcal{L}_{\zeta,n} = \Delta_\zeta(\mathfrak{O}) + \mathfrak{p}_{K_2}^n \times \mathfrak{p}_{K_2}^n + \mathbf{V}^n \subset \mathfrak{A},$$

which is called the **linking order**. This is actually an order of $\mathfrak{A}$ by

$$\mathbf{V}^n \mathbf{V}^n \subset \mathfrak{p}_{K_2}^n \times \mathfrak{p}_{K_2}^n.$$

Any element $g \in \mathcal{L}_{\zeta,n}$ can be written as

$$g = (x + \varpi^n y + \varpi^l z_1, x + \varpi^{l'} z_2) \quad \text{with } x, y \in \mathfrak{O} \text{ and } z_i \in \mathfrak{C}_i \ (i = 1, 2).$$

We consider the two-sided ideal

$$\mathcal{L}_{\zeta,n}^0 = \Delta_\zeta(\mathfrak{p}_{K_2}) + \mathfrak{p}_{K_2}^{n+1} \times \mathfrak{p}_{K_2}^{n+1} + \mathbf{V}^{n+1} \subset \mathcal{L}_{\zeta,n}.$$

In the following, we consider the quotient $\mathcal{L}_{\zeta,n-1}/\mathcal{L}_{\zeta,n-1}^0$ for a positive integer $n \geq 1$. First, we treat the case $n = 1$. The restriction of the natural projection $\mathfrak{A} \to \mathrm{M}_2(k) \times k_2$ to the subring $\mathcal{L}_{\zeta,0}$ induces an isomorphism

$$\mathcal{L}_{\zeta,0}/\mathcal{L}_{\zeta,0}^0 \xrightarrow{\sim} \mathrm{M}_2(k) \times k_2,$$

which does not depend on the choice of $\zeta \in k_2 \setminus k$. This induces

$$\big(\mathcal{L}_{\zeta,0}/\mathcal{L}^0_{\zeta,0}\big)^\times \xrightarrow{\sim} \mathrm{GL}_2(k) \times k_2^\times. \tag{2.5}$$

Let

$$\begin{aligned} Q &= \left\{ g(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^q & \beta^q \\ & & \alpha \end{pmatrix} \in \mathrm{GL}_3(k_2) \;\middle|\; \alpha,\beta,\gamma \in k_2 \right\}, \\ Q_0 &= \{ g(1,\beta,\gamma) \in Q \}. \end{aligned} \tag{2.6}$$

Note that we have

$$|Q| = q^4(q^2-1).$$

The center $Z(Q_0)$ of Q_0 equals $\{g(1,0,\gamma) \mid \gamma \in k_2\}$, and the quotient $Q_0/Z(Q_0)$ is an abelian group of order q^2. Hence, the group Q_0 is a finite Heisenberg group. Assume that $n \geq 2$. For each $\zeta \in k_2 \setminus k$, we have an isomorphism

$$(\mathcal{L}_{\zeta,n-1}/\mathcal{L}^0_{\zeta,n-1})^\times \simeq Q,$$

which is given in [W2, Proposition 4.3.4 (5)]. We will now show how this isomorphism is defined for n odd and give a similar isomorphism for Q_0. Assume that n is odd. Then we have $n = 2l'+1$ and $l = l'+1$. We set

$$v_0 = \begin{pmatrix} -1 & 0 \\ \zeta^q + \zeta & 1 \end{pmatrix} \in A_1^\times \quad \text{and} \quad V_{1,n} = V_1^{n-1}/V_1^n.$$

Note that $v_0^2 = 1$ and $v_0(a+b\zeta) = (a+b\zeta^q)v_0$ for $a,b \in \mathfrak{o}$. We consider the isomorphism

$$\phi_\zeta \colon V_{1,n} \xrightarrow{\sim} k_2; \quad h(a,b)\varpi^{l'} = (a+b\zeta)\varpi^{l'} v_0 \mapsto a+b\zeta$$

with $a,b \in k$. For $v,w \in V_{1,n}$, we have $vw \in \mathfrak{p}_{K_2}^{n-1}/\mathfrak{p}_{K_2}^n$ by $\mathfrak{C}_1\mathfrak{C}_1 \subset \mathfrak{O}$. Then we have

$$\begin{aligned} \phi_\zeta(xv) &= x\phi_\zeta(v), \quad \phi_\zeta(vx) = \phi_\zeta(v)x^q && \text{for } x \in k_2 \text{ and } v \in V_{1,n}, \\ \varpi^{-(n-1)}vw &= \phi_\zeta(v)\phi_\zeta(w)^q && \text{for } v,w \in V_{1,n}. \end{aligned} \tag{2.7}$$

For an element $x \in \mathfrak{O}$, let $\bar{x}$ denote the image of x by the reduction map $\mathfrak{O} \to k_2$. We have the isomorphism

$$(\mathcal{L}_{\zeta,n-1}/\mathcal{L}^0_{\zeta,n-1})^\times \xrightarrow{\sim} Q; \quad (x + \varpi^{n-1}y + v, x) \mapsto g(\bar{x}, \phi_\zeta(v), \bar{y}), \tag{2.8}$$

where $x,y \in \mathfrak{O}$ and $v \in V_1^{n-1}$.

Assume that n is even. Then we have $n=2l'$ and $l=l'$. We set $V_{2,n}=V_2^{n-1}/V_2^n$. We consider the isomorphism

$$\phi\colon V_{2,n} \xrightarrow{\sim} k_2; \quad \varpi^{l-1}\varphi b \mapsto \bar{b}^q$$

with $b \in \mathfrak{O}$. Similarly as (2.7), we have

$$\begin{aligned} \phi(xv) = x\phi(v), \quad \phi(vx) = \phi(v)x^q \qquad & \text{for } x \in k_2 \text{ and } v \in V_{2,n}, \\ \varpi^{-(n-1)}vw = \phi(v)\phi(w)^q \qquad & \text{for } v, w \in V_{2,n}. \end{aligned}$$

Similarly as (2.8), we have the isomorphism

$$(2.9) \qquad (\mathcal{L}_{\zeta,n-1}/\mathcal{L}^0_{\zeta,n-1})^\times \xrightarrow{\sim} Q; \quad (x, x+\varpi^{n-1}y+v) \mapsto g(\bar{x}, \phi(v), \bar{y}),$$

where $x, y \in \mathfrak{O}$ and $v \in V_2^{n-1}$.

Let $n \geq 1$ be an integer. We write $\overline{\mathcal{L}}_{\zeta,n-1}$ and $\overline{\mathcal{L}}^0_{\zeta,n-1}$ for the images of $\mathcal{L}_{\zeta,n-1}$ and $\mathcal{L}^0_{\zeta,n-1}$ by the canonical homomorphism $\mathfrak{A} \twoheadrightarrow \mathrm{M}_2(\mathfrak{o}_n) \times \mathcal{O}_{2n-1}$ respectively. We can easily check that the kernel of $\mathfrak{A} \to \mathrm{M}_2(\mathfrak{o}_n) \times \mathcal{O}_{2n-1}$ is contained in $\mathcal{L}^0_{\zeta,n-1}$. Hence we have an isomorphism

$$(2.10) \qquad (\mathcal{L}_{\zeta,n-1}/\mathcal{L}^0_{\zeta,n-1})^\times \xrightarrow{\sim} (\overline{\mathcal{L}}_{\zeta,n-1}/\overline{\mathcal{L}}^0_{\zeta,n-1})^\times.$$

In the following, we simply write G_n^F for $\mathrm{GL}_2(\mathfrak{o}_n)$. By (2.5), (2.8), (2.9) and (2.10), we obtain a homomorphism

$$(2.11) \qquad \overline{\mathcal{L}}^\times_{\zeta,n-1} \to \left(\overline{\mathcal{L}}_{\zeta,n-1}/\overline{\mathcal{L}}^0_{\zeta,n-1}\right)^\times \simeq \begin{cases} G_1^F \times k_2^\times & \text{if } n=1, \\ Q & \text{if } n \geq 2. \end{cases}$$

Now we assume that $n \geq 2$. We can check that

$$(2.12) \quad |\overline{\mathcal{L}}^\times_{\zeta,n-1}| = q^{4n}(q^2-1), \quad [G_n^F \times \mathcal{O}^\times_{2n-1} : \overline{\mathcal{L}}^\times_{\zeta,n-1}] = q^{4n-7}(q-1)(q^2-1).$$

We set

$$\begin{aligned} H_{1,\zeta,n} =& \overline{\mathcal{L}}_{\zeta,n-1} \cap (G_n^F \times \{1\}) \subset G_n^F, \\ H_{2,n} =& \overline{\mathcal{L}}_{\zeta,n-1} \cap (\{1\} \times \mathcal{O}^\times_{2n-1}) \subset \mathcal{O}^\times_{2n-1}. \end{aligned}$$

We consider the composites

$$\begin{aligned} f_1\colon H_{1,\zeta,n} \subset \overline{\mathcal{L}}^\times_{\zeta,n-1} \xrightarrow{\text{can.}} (\overline{\mathcal{L}}_{\zeta,n-1}/\overline{\mathcal{L}}^0_{\zeta,n-1})^\times, \\ f_2\colon H_{2,n} \subset \overline{\mathcal{L}}^\times_{\zeta,n-1} \xrightarrow{\text{can.}} (\overline{\mathcal{L}}_{\zeta,n-1}/\overline{\mathcal{L}}^0_{\zeta,n-1})^\times. \end{aligned}$$

We set $H^0_{1,\zeta,n} = \ker f_1$ and $H^0_{2,n} = \ker f_2$. Assume that n is odd and $n \geq 3$. By identifying the target of f_1 with Q through (2.9) and (2.10), we can check that the image of f_1 equals the subgroup Q_0. Hence, we obtain the isomorphism

$$\phi_{1,\zeta}\colon H_{1,\zeta,n}/H^0_{1,\zeta,n} \simeq Q_0. \tag{2.13}$$

Assume that n is even. Similarly as above, we obtain the isomorphism

$$\phi_2\colon H_{2,n}/H^0_{2,n} \simeq Q_0. \tag{2.14}$$

3. Deligne–Lusztig variety for G_n^F

In this section, we define a subvariety of the Deligne–Lusztig variety for G_n^F and analyze its cohomology. As a result we obtain Proposition 3.18.

3.1. Subvariety of the Deligne–Lusztig variety for G_n^F. Let n be a positive integer. Let

$$F\colon \widetilde{\mathfrak{o}}_n \to \widetilde{\mathfrak{o}}_n;\quad \sum_{i=0}^{n-1} x_i \varpi^i \mapsto \sum_{i=0}^{n-1} x_i^q \varpi^i \text{ with } x_i \in k^{\mathrm{ac}}. \tag{3.1}$$

We regard $G_n = \mathrm{GL}_2(\widetilde{\mathfrak{o}}_n)$ as a variety over k^{ac}. Let $\{e_1, e_2\}$ be the canonical basis of $V_n = \widetilde{\mathfrak{o}}_n^{\oplus 2}$. The map F induces the maps

$$F\colon V_n \to V_n, \quad F\colon G_n \to G_n.$$

We have $F(vg) = F(v)F(g)$ for $v \in V_n$ and $g \in G_n$. We set

$$T_n^F = \left\{ \begin{pmatrix} F(t) & 0 \\ 0 & t \end{pmatrix} \in G_n \;\middle|\; t \in \mathfrak{O}_n^\times \right\}.$$

We fix the isomorphism

$$\mathfrak{O}_n^\times \simeq T_n^F;\quad t \mapsto \begin{pmatrix} F(t) & 0 \\ 0 & t \end{pmatrix}. \tag{3.2}$$

Let

$$U_n = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in G_n \;\middle|\; c \in \widetilde{\mathfrak{o}}_n \right\}, \quad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G_n^F.$$

We consider the closed subvariety of G_n

$$X_n = \{ g \in G_n \mid F(g)g^{-1} \in U_n v \},$$

which we call the **Deligne–Lusztig variety** for G_n^F (cf. [Lus2]). Let $G_n^F \times T_n^F$ act on X_n by $g \mapsto t^{-1}gg'$ for $x \in X_n$ and $(g', t) \in G_n^F \times T_n^F$.

LEMMA 3.1: (1) *We have*

$$X_n = \left\{ g = \begin{pmatrix} -F(x) & -F(y) \\ x & y \end{pmatrix} \in G_n \;\middle|\; \det(g) \in \mathfrak{o}_n^\times \right\}$$
$$\xrightarrow{\sim} \mathfrak{S}_n = \{v = xe_1 + ye_2 \in V_n \mid v \wedge F(v) \in \mathfrak{o}_n^\times (e_1 \wedge e_2)\}; \quad g \mapsto e_2 g.$$

(2) *For $v \in \mathfrak{S}_n$, we put*

$$v = \sum_{i=0}^{n-1} v_i \varpi^i$$

with $v_i \in (k^{\mathrm{ac}})^{\oplus 2}$. Then $\mathfrak{S}_n$ is defined by

$$v_0 \wedge F(v_0) \in k^\times (e_1 \wedge e_2), \quad \sum_{j=0}^{i} v_{i-j} \wedge F(v_j) \in k(e_1 \wedge e_2)$$

for $1 \leq i \leq n-1$.

Proof. The second assertion follows from the first one. The first one is directly checked. We omit the details. ∎

Remark 3.2: The above lemma is similar to Lusztig's computation for $\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p}^2)$ in [Lus2, §3.3].

Note that we have $\dim X_n = n$. Recall that we set $l' = [n/2]$.

Definition 3.3: (1) We set

$$Y_n = \{v \in \mathfrak{S}_n \mid v \wedge F^2(v) = 0\} \subset \mathfrak{S}_n \simeq X_n$$

and $X_0 = Y_0 = \operatorname{Spec} k^{\mathrm{ac}}$.

(2) Let $p_n \colon X_n \to X_{l'}$ be the canonical projection induced by $G_n \to G_{l'}$. We set

$$\mathbf{X}_n = p_n^{-1}(Y_{l'}).$$

This variety $\mathbf{X}_n$ is our main object in this paper. For an integer $n \geq 1$, the subvariety Y_n is stable under the action of $G_n^F \times T_n^F$. Hence, $\mathbf{X}_n$ is stable under the action of $G_n^F \times T_n^F$, because p_n is compatible with the canonical homomorphism $G_n^F \times T_n^F \twoheadrightarrow G_{l'}^F \times T_{l'}^F$.

Let $(x,y) \in Y_n$. By $(x,y) \in \mathfrak{S}_n$, we have $y \neq 0$. Since we have $F^2(x/y) = x/y$, we obtain $x/y \in \mathfrak{O}_n^\times$. We set $t = y/x$. By $F(x)y - xF(y) \in \mathfrak{o}_n^\times$, we have

$$(3.3)\qquad \begin{gathered}(t^{-1} - F(t^{-1}))xF(x) = (F(t) - t)yF(y) \in \mathfrak{o}_n^\times,\\ F^2(x) = -x, \quad F^2(y) = -y.\end{gathered}$$

Conversely, if $(x,y) \in \mathfrak{S}_n$ satisfies the condition on the second line in (3.3), we have $F^2(x/y) = x/y$. Hence we have $(x,y) \in Y_n$. Therefore we have

$$(3.4)\qquad Y_n = \{(x,y) \in \mathfrak{S}_n \mid F^2(x,y) = -(x,y)\}.$$

By this, Y_n is zero-dimensional. Note that Y_n is regarded as a generalization of $\mathfrak{S}_{00}$ in the notation of [Lus2, §3.3]. In Definition 3.3, this scheme plays a crucial role to define $\mathbf{X}_n$.

For an integer $i \geq 1$, let $U^i_{K_2,n} \subset T_n^F$ denote the image of $U^i_{K_2} \subset \mathfrak{O}^\times$ by the composite $\mathfrak{O}^\times \to \mathfrak{O}_n^\times \simeq T_n^F$. Since we have $F(t) - t \in \mathfrak{O}_n^\times$ by (3.3), we have $t \in \mathfrak{O}_n^\times \setminus \mathfrak{o}_n^\times U^1_{K_2,n}$. We set

$$B_n = \mathfrak{O}_n^\times \setminus \mathfrak{o}_n^\times U^1_{K_2,n}.$$

By (3.3), we obtain the map

$$\nu_n \colon Y_n \to B_n; \quad (x,y) \mapsto x/y.$$

Let G_n^F act on B_n by

$$(3.5)\qquad g\colon B_n \to B_n; \quad t \mapsto \frac{at+c}{bt+d}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G_n^F$. Let T_n^F act on B_n trivially. Then ν_n is $G_n^F \times T_n^F$-equivariant. For $t \in B_n$, we set $Y_n^t = \nu_n^{-1}(t)$. Then Y_n^t is stable under the action of T_n^F. Note that

$$|T_n^F| = q^{2(n-1)}(q^2-1), \quad |B_n| = q^{2n-1}(q-1).$$

For $\zeta \in k_2 \setminus k$, we consider the homomorphism

$$(3.6)\qquad \Delta_\zeta \colon \mathfrak{O}_n^\times \hookrightarrow G_n^F \times T_n^F; \quad x \mapsto (\iota_\zeta(x), x),$$

where ι_ζ is in (2.1).

Lemma 3.4: (1) *The map ν_n is surjective.*

(2) *For each $t \in B_n$, the action of T_n^F on Y_n^t is simply transitive.*
(3) *The variety Y_n consists of $|G_n^F| = q^{4n-3}(q-1)(q^2-1)$ closed points. The action of G_n^F on Y_n is simply transitive.*
(4) *Let $\zeta \in k_2 \setminus k$. Then, $\Delta_\zeta(\mathfrak{O}_n^\times)$ acts on Y_n^ζ trivially.*

Proof. Let $t \in B_n$. We take an element $y \in \widetilde{\mathfrak{o}}_n$ such that $F^2(y) = -y$ and set $x = ty$. By (3.4) we have $(x, y) \in Y_n$, because

$$F^2(x) = -x, \quad F(F(x)y - xF(y)) = F(x)y - xF(y).$$

By $\nu_n(x, y) = t$, the map ν_n is surjective.

Let $t \in B_n$. By the first assertion we can take an element $(x_0, y_0) \in Y_n^t$. Let $(x, y) \in Y_n^t$. By (3.4) we have

$$x/y = x_0/y_0 = t, \quad F^2(x/x_0) = x/x_0, \quad F^2(y/y_0) = y/y_0.$$

Hence there exists a unique element $\xi \in \mathfrak{O}_n^\times$ such that $(x, y) = (\xi x_0, \xi y_0)$. Therefore the action of T_n^F on Y_n^t is simply transitive.

By the first and the second assertions we have

$$|Y_n| = |T_n^F||B_n| = |G_n^F|. \tag{3.7}$$

Assume that $g \in G_n^F$ fixes $x \in Y_n \subset \mathfrak{S}_n$. It fixes also $F(x) \in Y_n$. Since $\{x, F(x)\}$ forms a basis of V_n, we have $g = 1$. Thus the G_n^F-action on Y_n is free. By (3.7), the G_n^F-action on Y_n is simply transitive. Hence the third assertion follows.

Let $\xi \in \mathfrak{O}_n^\times$. We easily check that $\iota_\zeta(\xi)$ fixes $\zeta \in B_n$ by (3.5). Hence $\iota_\zeta(\xi)$ stabilizes Y_n^ζ. Recall that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_n^F$ acts on Y_n by $(x, y) \mapsto (ax + cy, bx + dy)$ for $(x, y) \in Y_n$. Hence $\iota_\zeta(\xi)$ acts on Y_n^ζ by $(x, y) \mapsto (\xi x, \xi y)$, because $x = \zeta y$. By definition, $\xi \in T_n^F$ acts on Y_n by $(x, y) \mapsto (\xi^{-1}x, \xi^{-1}y)$. Hence the fourth assertion follows. ∎

In the sequel, we introduce coordinates and several functions on $\mathbf{X}_n$ to understand this as in Lemma 3.5. For $v = \sum_{i=0}^{n-1} v_i \varpi^i \in V_n$ we set $v_i = (x_i, y_i) \in (k^{\mathrm{ac}})^2$. We define $t_{i,j}$ by

$$v_{i-j} \wedge F(v_j) = t_{i,j} e_1 \wedge e_2$$

for $1 \leq i \leq n-1$ and $0 \leq j \leq i$. Explicitly, we have

$$t_{i,j} = x_{i-j} y_j^q - y_{i-j} x_j^q.$$

We have

$$v \wedge F(v) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} v_{i-j} \wedge F(v_j) \varpi^i = \sum_{i=0}^{n-1} \sum_{j=0}^{i} t_{i,j} \varpi^i.$$

Hence, by Lemma 3.1 (2), the variety $\mathbf{X}_n$ is defined by

$$\left(\sum_{i=0}^{l'-1} x_i\varpi^i, \sum_{i=0}^{l'-1} y_i\varpi^i\right) \in Y_{l'}, \tag{3.8}$$

$$t_{0,0} \in k^\times, \quad \sum_{j=0}^{i} t_{i,j} \in k \quad \text{for } 1 \le i \le n-1. \tag{3.9}$$

By (3.4) and (3.8), we have

$$\begin{gathered} t_{i,j} \in k_2 \quad \text{for } 0 \le i-j, j \le l'-1, \quad t_{2i,i} \in k \quad \text{for } 0 \le i \le l'-1, \\ t_{i,j}^q = t_{i,i-j} \quad \text{for } 1 \le i \le n-1 \text{ and } 0 \le j \le l'-1. \end{gathered} \tag{3.10}$$

We set

$$s_i = \sum_{j=0}^{[(i-1)/2]} t_{i,j} \tag{3.11}$$

for $1 \le i \le 2l'-1$. By the equality on the second line in (3.10) we have

$$s_i^q + s_i = \begin{cases} \sum_{j=0}^{i} t_{i,j} & \text{if } i \text{ is odd,} \\ \sum_{j=0}^{i} t_{i,j} - t_{i,i/2} & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le 2l'-1$. Hence by (3.9) and the first line in (3.10) we have

$$s_i \in k_2 \quad \text{for } 1 \le i \le 2l'-1. \tag{3.12}$$

By the first assertion in (3.10) we have $t_{l',i} \in k_2$ for $1 \le i \le [(l'-1)/2]$. We set $\zeta = x_0/y_0$. By (3.12) and the definition of $t_{l',0}$ we have

$$t_{l',0} = s_{l'} - \sum_{i=1}^{[(l'-1)/2]} t_{l',i} \in k_2, \tag{3.13}$$

$$y_{l'} = \zeta^{-q} x_{l'} - x_0^{-q} t_{l',0}, \tag{3.14}$$

respectively. We set

$$\Delta_{1,n} = Y_{l'} \times k_2^{l'}. \tag{3.15}$$

By (3.12), we obtain the map

$$\mathbf{p}_n \colon \mathbf{X}_n \to \Delta_{1,n}; \quad x \mapsto (p_n(x), (s_{l'}(x), \ldots, s_{2l'-1}(x))),$$

where p_n is in Definition 3.3 (2). It is not difficult to check $\mathbf{p}_n$ is surjective. We set

$$Z_{P,s} = \mathbf{p}_n^{-1}(P,s) \quad \text{for } (P,s) \in \Delta_{1,n}.$$

Let Z_{DL} be the affine curve defined by $(x^q y - xy^q)^{q-1} = 1$. This curve is called the Deligne–Lusztig curve for $\mathrm{GL}_2(\mathbb{F}_q)$. Note that the affine curve defined by $x^q y - xy^q = 1$ is called the Drinfeld curve (cf. [DL, p. 117]). Let Z_0 be the affine curve defined by $X^{q^2} - X = Y^{q(q+1)} - Y^{q+1}$ over k^{ac}. Note that Z_0 has q connected components. For a non-negative integer i, let $\mathbb{A}^i$ denote an i-dimensional affine space over k^{ac}.

We can completely understand $\mathbf{X}_n$ in the following lemma.

Lemma 3.5: *We have*

$$\mathbf{X}_n = \coprod_{(P,s)\in\Delta_{1,n}} Z_{P,s}$$

and an isomorphism

$$Z_{P,s} \simeq \begin{cases} Z_{\mathrm{DL}} & \text{if } n = 1, \\ \mathbb{A}^{l'} \times Z_0 & \text{if } n > 1 \text{ is odd}, \\ \mathbb{A}^{l'} & \text{if } n \text{ is even} \end{cases}$$

over k^{ac}.

Proof. The first equality is clear. Hence we show the latter isomorphism. The required assertion in the case where $n = 1$ is clear. We assume that $n \geq 2$. We show only the case where n is odd, because the other case is proved similarly. Let $(x, y) = \left(\sum_{i=0}^{n-1} x_i \varpi^i, \sum_{i=0}^{n-1} y_i \varpi^i\right) \in Z_{P,s}$. We put

$$s_{2l'} = -\sum_{i=0}^{l'-1} t_{2l',i} - \frac{x_{l'}}{x_0} t_{l',0}^q. \tag{3.16}$$

We set $\zeta = x_0/y_0 \in k_2 \setminus k$. We show

$$s_{2l'}^q + s_{2l'} + (\zeta^{-q} - \zeta^{-1}) x_{l'}^{q+1} \in k. \tag{3.17}$$

By $t_{l',0} \in k_2$ in (3.13) and the second line in (3.10) we have

$$s_{2l'}^q + s_{2l'} = -\sum_{i=0}^{2l'} t_{2l',i} + t_{2l',l'} - \frac{x_{l'}^q}{x_0^q} t_{l',0} - \frac{x_{l'}}{x_0} t_{l',0}^q. \tag{3.18}$$

By (3.14) we have

$$t_{2l',l'} = (\zeta^{-1} - \zeta^{-q})x_{l'}^{q+1} + \frac{x_{l'}^q}{x_0^q} t_{l',0} + \frac{x_{l'}}{x_0} t_{l',0}^q. \tag{3.19}$$

By the second equation in (3.9) for $i = 2l'$, we have $\sum_{i=0}^{2l'} t_{2l',i} \in k$. Hence by (3.18) and (3.19) we obtain (3.17).

Note that $\zeta^q - \zeta \neq 0$. We set

$$X = \frac{s_{2l'}}{(\zeta^q - \zeta)y_0^{q+1}}, \quad Y = \frac{x_{l'}}{\zeta^q y_0}. \tag{3.20}$$

Thus by (3.17) and $(\zeta^q - \zeta)y_0^{q+1} \in k^\times$, we obtain $X^q + X - Y^{q+1} \in k$. This implies that

$$X^{q^2} - X = Y^{q(q+1)} - Y^{q+1}.$$

By (3.11) and (3.20), there exists an upper triangular matrix $A_{P,s} \in \mathrm{GL}_{l'}(k^{\mathrm{ac}})$ and a vector $\mathbf{a}_{P,s} \in (k^{\mathrm{ac}})^{l'}$ such that

$$(y_{l'}, \ldots, y_{2l'-1}) = (Y, x_{l'+1}, \ldots, x_{2l'-1})A_{P,s} + \mathbf{a}_{P,s}. \tag{3.21}$$

Hence by (3.20), there exists a vector $(a_{l'+1}, \ldots, a_{2l'}, b_1, b_2, c) \in (k^{\mathrm{ac}})^{l'+3}$ such that

$$y_{2l'} = \sum_{i=l'+1}^{2l'} a_i x_i + b_1 X + b_2 Y + c. \tag{3.22}$$

By using (3.20), (3.21) and (3.22), we know that the morphism

$$Z_{P,s} \to \mathbb{A}^{l'} \times Z_0; \quad (x, y) = \left(\sum_{i=0}^{2l'} x_i \varpi^i, \sum_{i=0}^{2l'} y_i \varpi^i \right) \mapsto ((x_i)_{l'+1 \leq i \leq 2l'}, (X, Y))$$

is an isomorphism. ∎

Assume that $n \geq 2$. Let $v, v' \in \mathbf{X}_n$ and $(g, t) \in G_n^F \times T_n^F$. We can check that

$$\mathbf{p}_n(v) = \mathbf{p}_n(v') \Rightarrow \mathbf{p}_n(t^{-1}vg) = \mathbf{p}_n(t^{-1}v'g).$$

Hence $\Delta_{1,n}$ has the action of $G_n^F \times T_n^F$ such that $\mathbf{p}_n$ is $G_n^F \times T_n^F$-equivariant. Let $G_n^F \times T_n^F$ act on $Y_{l'}$ through the homomorphism $G_n^F \times T_n^F \twoheadrightarrow G_{l'}^F \times T_{l'}^F$. The first projection $\Delta_{1,n} \to Y_{l'}$ is $G_n^F \times T_n^F$-equivariant. Let

$$X_{P,s} = \begin{cases} \operatorname{Spec} k^{\mathrm{ac}} & \text{if } n \text{ is even,} \\ Z_0 & \text{if } n \text{ is odd,} \end{cases}$$

$$X(\Delta_{1,n}) = \coprod_{(P,s)\in\Delta_{1,n}} X_{P,s}. \tag{3.23}$$

By Lemma 3.5 we have the projections

$$Z_{P,s} \to X_{P,s},$$
$$\pi_n \colon \mathbf{X}_n \to X(\Delta_{1,n}). \tag{3.24}$$

Let $v, v' \in \mathbf{X}_n$ and $(g,t) \in G_n^F \times T_n^F$. We can check that

$$\pi_n(v) = \pi_n(v') \Rightarrow \pi_n(t^{-1}vg) = \pi_n(t^{-1}v'g).$$

Hence $X(\Delta_{1,n})$ admits the action of $G_n^F \times T_n^F$ such that π_n is $G_n^F \times T_n^F$-equivariant.

We choose a prime number $\ell \neq p$ and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. For a variety X over k^{ac} and $i \geq 0$, we write $H_{\mathrm{c}}^i(X)$ for the i-th étale cohomology group with compact support $H_{\mathrm{c}}^i(X, \overline{\mathbb{Q}}_\ell)$. We put $d_1 = \dim X(\Delta_{1,n})$. Since (3.24) is an affine bundle of relative dimension l' we have

$$H_{\mathrm{c}}^n(\mathbf{X}_n) \simeq H_{\mathrm{c}}^{d_1}(X(\Delta_{1,n})) \tag{3.25}$$

as $G_n^F \times T_n^F$-representations. For a positive integer i, let $U_{\mathfrak{A}_1}^i = 1 + \mathfrak{p}^i \mathfrak{A}_1 \subset \mathfrak{A}_1^\times$. We write N_i for the image of $U_{\mathfrak{A}_1}^i$ by the canonical map $\mathfrak{A}_1^\times \to G_n^F$. Note that N_i equals the kernel of the natural homomorphism $G_n^F \to G_i^F$. For $t \in B_{l'}$, we set

$$\Delta_{1,n}^t = Y_{l'}^t \times k_2^{l'} \subset \Delta_{1,n},$$
$$\mathbf{X}_n^t = \mathbf{p}_n^{-1}(\Delta_{1,n}^t) \subset \mathbf{X}_n.$$

3.2. Group action on $\mathbf{X}_n$. To understand the cohomology of $\mathbf{X}_n$ as $G_n^F \times T_n^F$-representations, we need to explicitly understand some group action on it.

In the following, when we consider an element $\zeta \in k_2 \setminus k$, we always regard $\mathfrak{O}_n^\times$ as a subgroup of G_n^F by ι_ζ. Assume that $n \geq 2$. Let $G_n^F \times T_n^F$ act on $B_{l'}$ through the canonical homomorphism $G_n^F \times T_n^F \twoheadrightarrow G_{l'}^F \times T_{l'}^F$.

Lemma 3.6:

(1) *The action of G_n^F on $B_{l'}$ is transitive. For any $\zeta \in k_2 \setminus k \subset B_{l'}$, the stabilizer of ζ in G_n^F equals $\mathfrak{O}_n^\times N_{l'}$.*
(2) *Let $\zeta \in k_2 \setminus k \subset B_{l'}$. The stabilizer of $\Delta_{1,n}^\zeta$ in $G_n^F \times T_n^F$ equals $\mathfrak{O}_n^\times N_{l'} \times T_n^F$.*

Proof. We show the first assertion. By Lemma 3.4 (1)–(3), the map $\nu_{l'}$ is a $G_{l'}^F$-equivariant surjective map, and $G_{l'}^F$ acts on $Y_{l'}$ transitively. Therefore, the action of G_n^F on $B_{l'}$ is transitive. By (3.5), we know that the subgroup $\mathfrak{O}_n^{\times} N_{l'}$ fixes ζ. Since we have

$$|G_n^F/\mathfrak{O}_n^{\times} N_{l'}| = |G_{l'}^F/\mathfrak{O}_{l'}^{\times}| = |B_{l'}|$$

by (3.7), the last assertion follows.

We show the second assertion. Let $\nu' \colon \Delta_{1,n} \to B_{l'}$ be the composite

$$\Delta_{1,n} \xrightarrow{\mathrm{pr}_1} Y_{l'} \xrightarrow{\nu_{l'}} B_{l'}.$$

Since ν' is $G_n^F \times T_n^F$-equivariant, the stabilizer of $\Delta_{1,n}^{\zeta}$ in $G_n^F \times T_n^F$ equals the stabilizer of $\zeta \in B_{l'}$ in it. Recall that T_n^F acts on $B_{l'}$ trivially. Hence the second assertion follows from the first one. ∎

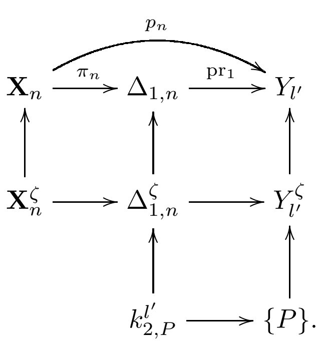
We fix an element $\zeta \in k_2 \setminus k$. In the following, we study actions of subgroups of $\mathfrak{O}_n^{\times} N_{l'} \times T_n^F$ on $\Delta_{1,n}^{\zeta}$.

Lemma 3.7: *The action of T_n^F on $\Delta_{1,n}^{\zeta}$ is transitive. Let $(P, s) \in \Delta_{1,n}^{\zeta}$. The stabilizer of (P, s) in T_n^F equals $U_{K_2,n}^{2l'}$.*

Proof. First, we show that, for each $P \in Y_{l'}^{\zeta}$, the subgroup $U_{K_2,n}^{l'}$ acts on the subset $k_{2,P}^{l'} = \{P\} \times k_2^{l'}$ of $\Delta_{1,n}^{\zeta}$ transitively. Let $P \in Y_{l'}^{\zeta}$ and $t \in U_{K_2,n}^{l'}$. We set

$$t^{-1} = 1 + \sum_{i=l'}^{n-1} a_i \varpi^i \quad \text{with } a_i \in k_2, \quad a = (a_{l'}, \ldots, a_{2l'-1}) \in k_2^{l'}.$$

We consider the cartesian diagram



We take $(x,y)=(\sum_{i=0}^{n-1}x_i\varpi^i,\sum_{i=0}^{n-1}y_i\varpi^i)\in \mathbf{X}_n^{\zeta}$ such that $\pi_n(x,y)=(P,s)$. By definition we have

$$(3.26)\qquad t^*x_{l'+i}=x_{l'+i}+\sum_{j=0}^{i}a_{l'+i-j}x_j,\quad t^*y_{l'+i}=y_{l'+i}+\sum_{j=0}^{i}a_{l'+i-j}y_j$$

for $0\leq i\leq n-l'-1$. By using (3.11) and (3.26), we can directly check that there exists an upper triangular matrix $A_P=(a_{i,j})_{1\leq i,j\leq l'}\in \mathrm{GL}_{l'}(k_2)$ such that the action of t on $k_{2,P}^{l'}$ is given by

$$(3.27)\qquad t\colon k_{2,P}^{l'}\to k_{2,P}^{l'};\quad (P,s)\mapsto (P,s+aA_P).$$

Hence $U_{K_2,n}^{l'}$ acts on $k_{2,P}^{l'}$ transitively. By Lemma 3.4 (2), the group $T_{l'}^F$ acts on $Y_{l'}^{\zeta}$ transitively. Let (P_0,s_0) and (P,s) be elements in $\Delta_{1,n}^{\zeta}$. We take $t\in T_{l'}^F$ such that $P=P_0t$. We take a lifting $\widetilde{t}\in T_n^F$ of t. We set

$$(P,s')=(P_0,s_0)\widetilde{t}.$$

We take $u\in U_{K_2,n}^{l'}$ such that $s'=su$. We have $(P_0,s_0)\widetilde{t}u=(P,s)$. Hence we obtain the first assertion.

Assume that $t\in T_n^F$ stabilizes (P,s). Since P is stabilized by t, we have $t\in U_{K_2,n}^{l'}$ by Lemma 3.4 (2). By (3.27) and the assumption we have $a=0$. Hence we obtain the claim. ∎

We follow the notation in (2.6). Let Q act on Z_0 by

$$g(\alpha,\beta,\gamma)\colon Z_0\to Z_0;\quad (X,Y)\mapsto \Big(X+\frac{\beta^q}{\alpha}Y+\frac{\gamma}{\alpha},\ \alpha^{q-1}\Big(Y+\frac{\beta}{\alpha^q}\Big)\Big)$$

for $g(\alpha,\beta,\gamma)\in Q$. We consider the subgroup

$$k^{\times}\simeq\{g(\alpha,0,0)\in Q\mid \alpha\in k^{\times}\}\subset Q.$$

Then $k^{\times}$ acts on Z_0 trivially. For $\gamma_0\in k_2$, we have the homomorphism

$$(3.28)\qquad f_{\gamma_0}\colon k_2^{\times}\to Q;\quad \alpha\mapsto g(\alpha,(\alpha-\alpha^q)\gamma_0,(\alpha-\alpha^q)\gamma_0^{q+1}).$$

For $\alpha\in k^{\times}$ we have

$$(3.29)\qquad f_{\gamma_0}(\alpha)=g(\alpha,0,0)\in k^{\times}.$$

Let $(P,s)\in\Delta_{1,n}^{\zeta}$. Let Δ_{ζ} be as in (3.6). In the following lemma, we show that $\Delta_{\zeta}(\mathfrak{O}_n^{\times})$ stabilizes $Z_{P,s}$, and describe the action of it on $Z_{P,s}$ with respect to f_{γ_0}. In particular, we know that $\Delta_{\zeta}(\mathfrak{O}_n^{\times})$ acts on $Z_{P,s}$ factoring through $\Delta_{\zeta}(\mathfrak{O}_n^{\times})\to\Delta_{\zeta}(\mathfrak{O}_n^{\times}/\mathfrak{o}_n^{\times}U_{K_2,n}^1)$. This lemma will be used in (3.56).

Lemma 3.8: (1) *The subgroup $\Delta_\zeta(\mathfrak{O}_n^\times)$ acts on $\Delta_{1,n}^\zeta$ trivially.*

(2) *Assume that n is odd. Let $\alpha \in \mathfrak{O}_n^\times$ and $(P,s) \in \Delta_{1,n}^\zeta$. There exists an element $\gamma_0(P,s) \in k_2$ such that*

— *we have the following commutative diagram:*

$$\begin{array}{ccc} Z_{P,s} & \xrightarrow{\Delta_\zeta(\alpha)} & Z_{P,s} \\ \downarrow & & \downarrow \\ X_{P,s} & \xrightarrow{f_{\gamma_0(P,s)}(\bar{\alpha})} & X_{P,s} \end{array}$$

for any $\alpha \in \mathfrak{O}_n^\times$, and

— *$\gamma_0(P,s) = 0$ if $t_{l',0}(P,s) = 0$.*

If $\alpha \in \mathfrak{o}_n^\times U_{K_2,n}^1$, we have $f_{\gamma_0(P,s)}(\bar{\alpha}) \in k^\times$.

Proof. Let $(x,y) = (\sum_{i=0}^{n-1} x_i \varpi^i, \sum_{i=0}^{n-1} y_i \varpi^i) \in \mathbf{X}_n^\zeta$. We have

$$x_i = \zeta y_i \quad \text{for } 1 \leq i \leq l'-1, \tag{3.30}$$
$$y_{l'} = \zeta^{-q} x_{l'} - x_0^{-q} t_{l',0} \quad \text{with } t_{l',0} \in k_2,$$

where the second equality is (3.14). Let $\alpha \in \mathfrak{O}_n^\times$. We set $\alpha = a + b\zeta$ with $a, b \in \mathfrak{o}_n$. On $\mathbf{X}_n^\zeta$ we have

$$\begin{aligned} \Delta_\zeta(\alpha)^* x &= ((a + b(\zeta^q + \zeta))x - b\zeta^{q+1} y)/\alpha, \\ \Delta_\zeta(\alpha)^* y &= (bx + ay)/\alpha. \end{aligned} \tag{3.31}$$

Hence we have

$$\Delta_\zeta(\alpha)^*(x - \zeta^q y) = x - \zeta^q y. \tag{3.32}$$

By Lemma 3.4 (4), y_j is fixed by $\Delta_\zeta(\alpha)$ for $1 \leq j \leq l'-1$. By this and (3.32), for $1 \leq i \leq n-1$ and $0 \leq j \leq [(i-1)/2]$, the function

$$t_{i,j} = x_{i-j} y_j^q - y_{i-j} x_j^q = y_j^q (x_{i-j} - \zeta^q y_{i-j})$$

is fixed by the action of $\Delta_\zeta(\alpha)$. Therefore, for $l' \leq i \leq 2l'-1$, each $s_i \in k_2$ in (3.11) is fixed by $\Delta_\zeta(\alpha)$. The first assertion follows from this and Lemma 3.4 (4).

We prove the second assertion. For $\alpha \in \mathfrak{o}_n^\times U_{K_2,n}^1$ we have $\bar{\alpha} \in k^\times$. Hence the latter assertion follows from (3.29). We show the former assertion. By (3.30) and (3.31) we have

$$\Delta_\zeta(\alpha)^*x = \frac{(a+b(\zeta^q+\zeta))x - b\zeta^{q+1}y}{\alpha} = x + \frac{b\zeta^q}{\alpha}(x-\zeta y)$$
$$\equiv \sum_{i=0}^{l'-1} x_i\varpi^i + \Big(\bar{\alpha}^{q-1}x_{l'} + \frac{\bar{b}\zeta^{q+1}t_{l',0}}{\bar{\alpha}x_0^q}\Big)\varpi^{l'} \mod \varpi^{l'+1}.$$

Hence, by (3.31) and $x_0 = \zeta y_0$, we obtain

$$\Delta_\zeta(\alpha)^*x_{l'} = \bar{\alpha}^{q-1}x_{l'} + \frac{\bar{b}\zeta t_{l',0}}{\bar{\alpha}y_0^q}.$$

By the proof of the first assertion, $t_{2l',i}$ for $0 \le i \le l'-1$ is fixed by $\Delta_\zeta(\alpha)$. By (3.16) and (3.20), we have

$$\Delta_\zeta(\alpha)^*Y = \bar{\alpha}^{q-1}Y + \frac{\bar{b}t_{l',0}}{\bar{\alpha}\zeta^{q-1}y_0^{q+1}},$$
$$\Delta_\zeta(\alpha)^*X = X - \frac{\zeta^{q-1}\bar{b}t_{l',0}^q}{\bar{\alpha}y_0^{q+1}}Y - \frac{\bar{b}t_{l',0}^{q+1}}{\bar{\alpha}y_0^{2(q+1)}(\zeta^q-\zeta)}. \tag{3.33}$$

We set $\gamma_0(P,s) = -t_{l',0}/(\zeta^{q-1}y_0^{q+1}(\zeta^q-\zeta))$. By using $\bar{\alpha}-\bar{\alpha}^q = \bar{b}(\zeta-\zeta^q)$ and $y_0^{q^2} = -y_0$, we can easily check that

$$(\bar{\alpha}-\bar{\alpha}^q)\gamma_0(P,s) = \frac{\bar{b}t_{l',0}}{\zeta^{q-1}y_0^{q+1}}, \quad ((\bar{\alpha}-\bar{\alpha}^q)\gamma_0(P,s))^q = -\frac{\zeta^{q-1}\bar{b}t_{l',0}^q}{y_0^{q+1}},$$
$$(\bar{\alpha}-\bar{\alpha}^q)\gamma_0(P,s)^{q+1} = -\frac{\bar{b}t_{l',0}^{q+1}}{y_0^{2(q+1)}(\zeta^q-\zeta)}.$$

Hence we obtain the claim by (3.33). ∎

For an integer $i \ge 1$, let $\mathfrak{C}_{1,i}$ be the image of $\mathfrak{C}_1$ by $\mathfrak{A}_1 \to \mathrm{M}_2(\mathfrak{o}_i)$. Let $\zeta \in k_2\backslash k$. The decomposition (2.2) induces $\mathrm{M}_2(\mathfrak{o}_i) \simeq \mathfrak{O}_i \oplus \mathfrak{C}_{1,i}$. Let $s_{\zeta,i}\colon \mathrm{M}_2(\mathfrak{o}_i) \to \mathfrak{O}_i$ be the first projection. Explicitly, we have

$$s_{\zeta,i}\colon \mathrm{M}_2(\mathfrak{o}_i) \to \mathfrak{O}_i; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{\zeta^q(b\zeta+d)-(a\zeta+c)}{\zeta^q-\zeta}. \tag{3.34}$$

Let $H^0_{1,\zeta,n} \subset H_{1,\zeta,n}$ be as in §2.2. Explicitly, we have

$$H^0_{1,\zeta,n} = 1 + \mathfrak{p}^l_{K_2}\mathfrak{C}_{1,n-l} \subset H_{1,\zeta,n} = 1 + \mathfrak{p}^{n-1}_{K_2} + \mathfrak{p}^{l'}_{K_2}\mathfrak{C}_{1,n-l'} \subset N_{l'}. \tag{3.35}$$

In the following lemma, we determine the stabilizer of $(P,s) \in \Delta^\zeta_{1,n}$ in G^F_n and describe its action on $Z_{P,s}$. The action of the stabilizer factors through the finite Heisenberg group Q_0 in (2.6). The lemma plays an important role when we will show Lemma 3.12. The property (c) below is important when we relate $\mathbf{X}_n$ to a curve on the right-hand side of (5.2) which admits an action of the multiplicative group of the linking order introduced in §2.2.

LEMMA 3.9: *Let* $(P, s) \in \Delta^{\zeta}_{1,n}$.

(1) *The stabilizer of* (P, s) *in* G_n^F *equals*

$$\begin{cases} H^0_{1,\zeta,n} & \text{if } n \text{ is even,} \\ H_{1,\zeta,n} & \text{if } n \text{ is odd.} \end{cases}$$

(2) *Assume that* n *is odd. Then* $H_{1,\zeta,n}$ *acts on* $Z_{P,s}$ *factoring through* $H_{1,\zeta,n} \to H_{1,\zeta,n}/H^0_{1,\zeta,n}$. *Furthermore, there exists an isomorphism*

$$\phi_{1,\zeta,P,s} \colon H_{1,\zeta,n}/H^0_{1,\zeta,n} \simeq Q_0$$

such that

(a) *for* $g \in H_{1,\zeta,n}/H^0_{1,\zeta,n}$ *we have the commutative diagram*

$$\begin{array}{ccc} Z_{P,s} & \xrightarrow{\quad g \quad} & Z_{P,s} \\ \downarrow & & \downarrow \\ X_{P,s} & \xrightarrow{\phi_{1,\zeta,P,s}(g)} & X_{P,s}, \end{array}$$

(b) $\phi_{1,\zeta,P,s}(g) = g(1, 0, s_{\zeta,1}(g_0))$ *for* $g = 1 + \varpi^{n-1} g_0 \in N_{n-1}$ *with* $g_0 \in \mathrm{M}_2(k)$, *and*

(c) $\phi_{1,\zeta,P,s}$ *corresponds to* $\phi_{1,\zeta}$ *in* (2.13) *for any* $(P, s) \in \Delta^{\zeta}_{1,n}$ *which satisfies*

$$t_{l',0}(P, s) = 0. \tag{3.36}$$

Proof. We prove the first assertion. Let

$$g = 1 + \varpi^{l'} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_{l'}, \quad (x, y) = \left(\sum_{i=0}^{n-1} x_i \varpi^i, \sum_{i=0}^{n-1} y_i \varpi^i \right) \in \mathbf{X}^{\zeta}_n.$$

We have

$$g^* x = x + \varpi^{l'}(ax + cy), \quad g^* y = y + \varpi^{l'}(bx + dy). \tag{3.37}$$

Recall that

$$t_{i,j} = y_j^q (x_{i-j} - \zeta^q y_{i-j}) \quad \text{for } 1 \le i \le n-1 \text{ and } 0 \le j \le [(i-1)/2].$$

We have

$$\varpi^{l'}(x-\zeta y)\equiv 0 \mod \varpi^{2l'}.$$

Let $s_{\zeta,l'}$ be as in (3.34) and $g_0=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. By (3.37) we have

$$\begin{aligned} g^*(x-\zeta^q y) &= x-\zeta^q y+\varpi^{l'}(a\zeta+c-\zeta^q(b\zeta+d))y+\varpi^{l'}(a-b\zeta^q)(x-\zeta y)\\ &= x-\zeta^q y-(\zeta^q-\zeta)s_{\zeta,l'}(g_0)y\varpi^{l'}+\varpi^{l'}(a-b\zeta^q)(x-\zeta y)\\ &= x-\zeta^q y-(\zeta^q-\zeta)s_{\zeta,l'}(g_0)y\varpi^{l'}+\varpi^{2l'}(a-b\zeta^q)(x_{l'}-\zeta y_{l'}). \end{aligned} \tag{3.38}$$

Let $g\in G_n^F$ be an element such that $(P,s)g=(P,s)$. By $P=Pg$ and Lemma 3.4 (3) we have $g\in N_{l'}$. By the assumption g stabilizes each s_i for $l'\leq i\leq 2l'-1$. Let $1\leq i\leq[(l'-1)/2]$ be an integer. Since $t_{l',i}$ is a function of x_j and y_j for $0\leq j\leq l'-1$, the function $t_{l',i}$ is fixed by g. Since $s_{l'}$ is so, $t_{l',0}$ is so by (3.11). Repeating similar arguments, we can check that the function $t_{i,0}=y_0^q(x_i-\zeta^q y_i)$ for any $l'\leq i\leq 2l'-1$ is also stabilized by g. Hence $x_i-\zeta^q y_i$ is so for $l'\leq i\leq 2l'-1$. Therefore we have $g^*(x-\zeta^q y)\equiv x-\zeta^q y \mod \varpi^{2l'}$. Hence, we must have

$$s_{\zeta,l'}(g_0)\equiv 0 \mod \varpi^{l'}$$

by (3.38). Hence the first assertion follows.

We prove the second assertion. Assume that n is odd. Let $h(a,b)$ be as in (2.3). Let $\gamma_0(P,s)$ be as in Lemma 3.8. For

$$g=1+\sum_{i=l'}^{n-1}\varpi^i h(a_i,b_i)+\varpi^{n-1}\xi\in H_{1,\zeta,n} \quad \text{with } a_i,b_i\in k \text{ and } \xi\in k_2,$$

we set

$$\eta(P,s,g)=(a_{l'}+b_{l'}\zeta)\gamma_0(P,s)^q-(a_{l'}+b_{l'}\zeta)^q\gamma_0(P,s)\in k_2,$$
$$\phi_{1,\zeta,P,s}\colon H_{1,\zeta,n}/H^0_{1,\zeta,n}\simeq Q_0;\quad g\mapsto g(1,a_{l'}+b_{l'}\zeta,\eta(P,s,g)+\xi).$$

We check that this satisfies (b) and (c). First, we consider (b). Let $g=1+\varpi^{n-1}g_0$ be as in (b). We have $g=1+\varpi^{n-1}h(a_{n-1},b_{n-1})+\varpi^{n-1}s_{\zeta,1}(g_0)$ with some $a_{n-1},b_{n-1}\in k$. Hence we have the claim. Secondly, we consider (c). Let $(P,s)\in\Delta^{\zeta}_{1,n}$ be an element such that $t_{l',0}(P,s)=0$. We have $\gamma_0(P,s)=0$ by Lemma 3.8 (2). Hence we have the claim.

In the sequel we show the commutativity in (a). Let $(x,y) \in Z_{P,s}$. We have

$$g^* x_{l'} = x_{l'} + (a_{l'} + b_{l'}\zeta)\zeta^q y_0,$$
$$g^*(x_i - \zeta^q y_i) = x_i - \zeta^q y_i \quad \text{for } 0 \le i \le 2l'-1,$$
$$g^*(x_{2l'} - \zeta^q y_{2l'}) = x_{2l'} - \zeta^q y_{2l'} - (a_{l'} + b_{l'}\zeta)^q (x_{l'} - \zeta y_{l'}) - y_0(\zeta^q - \zeta)\xi,$$

where we use (3.37) at the first equality, the second one is proved in the proof of the first assertion, and the third one follows from (3.38). Hence we obtain $g^* t_{2l',i} = t_{2l',i}$ for $1 \le i \le l'-1$. Hence by (3.16), (3.20) and the second equality in (3.30) we have

$$g^* Y = Y + a_{l'} + b_{l'}\zeta,$$
$$g^* X = X + (a_{l'} + b_{l'}\zeta)^q Y + \eta(P,s,g) + \xi.$$

Hence the claim follows. ∎

The following fact will be used in §5.

COROLLARY 3.10: *The action of G_n^F on $\Delta_{1,n}$ is transitive.*

Proof. We take $\zeta \in k_2 \setminus k$ and $\delta_1 = (P,s) \in \Delta_{1,n}^{\zeta}$. Assume that n is odd. By Lemma 3.9 (1), we have the injective map

$$H_{1,\zeta,n}\backslash G_n^F \hookrightarrow \Delta_{1,n}; \quad H_{1,\zeta,n} g \mapsto \delta_1 g. \tag{3.39}$$

By

$$\begin{aligned}|H_{1,\zeta,n}\backslash G_n^F| &= |G_{l'}^F||N_{l'}/H_{1,\zeta,n}| \\ &= |G_{l'}^F||\mathrm{M}_2(\mathfrak{o}_{l'})/\mathfrak{C}_{1,l'}| \\ &= q^{3(n-2)}(q-1)(q^2-1) = |\Delta_{1,n}|\end{aligned}$$

the map (3.39) is surjective. Hence we obtain the claim.

Assume that n is even. By Lemma 3.9 (1) and

$$|H_{1,\zeta,n}^0\backslash G_n^F| = |\Delta_{1,n}| = q^{3(n-1)}(q-1)(q^2-1)$$

the group G_n^F acts on $\Delta_{1,n}$ transitively. ∎

Finally, we write down the group action of $G_1^F \times T_1^F$ on $\mathbf{X}_1 = X_1 \simeq Z_{\mathrm{DL}}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_1^F$ and $t \in T_1^F$. Then (g,t) acts on $\mathbf{X}_1$ by

$$(g,t)\colon \mathbf{X}_1 \to \mathbf{X}_1; \quad (x,y) \mapsto (t^{-1}(ax+cy), t^{-1}(bx+dy)). \tag{3.40}$$

3.3. Preliminaries. We collect some well-known facts on the first cohomology of the curve Z_0. We fix an isomorphism $k_2 \xrightarrow{\sim} Z(Q_0)$; $\gamma \mapsto g(1,0,\gamma)$. For a finite abelian group A, we write $A^\vee$ for $\mathrm{Hom}(A, \overline{\mathbb{Q}}_\ell^\times)$. We regard $k^\vee$ as a subset of $k_2^\vee$ by the dual of the trace map $\mathrm{Tr}_{k_2/k} \colon k_2 \twoheadrightarrow k$. For each character $\psi' \in k_2^\vee \setminus k^\vee$, which is regarded as a character of $Z(Q_0)$, there exists a unique q-dimensional irreducible representation $\tau_{\psi'}$ of Q such that

- $\tau_{\psi'}|_{Z(Q_0)} \simeq \psi'^{\oplus q}$, and
- $\mathrm{Tr}\, \tau_{\psi'}(g(\alpha,0,0)) = -1$ for $\alpha \in k_2 \setminus k$

(cf. [BH, Lemma in §22.2] and [T2, Lemma 4.14]). We regard $k^\times$ as a subgroup of Q by $k^\times \hookrightarrow Q$; $\alpha \mapsto g(\alpha,0,0)$. As $k^\times$-representations we have

$$\tau_{\psi'}|_{k^\times} \simeq \mathbf{1}^{\oplus q}, \tag{3.41}$$

where $\mathbf{1}$ is the trivial character of $k^\times$. We have an isomorphism

$$H^1_{\mathrm{c}}(Z_0) \simeq \bigoplus_{\psi' \in k_2^\vee \setminus k^\vee} \tau_{\psi'} \tag{3.42}$$

as Q-representations (cf. [T2, Lemma 4.16.1]). Let $\gamma_0 \in k_2$. We consider the map (3.28). To understand the restriction $\tau_{\psi'}|_{f_{\gamma_0}(k_2^\times)}$ as in (3.45), we need the following lemma.

Lemma 3.11: *Let $\psi' \in k_2^\vee \setminus k^\vee$. We have*

$$\mathrm{Tr}\, \tau_{\psi'}(f_{\gamma_0}(\alpha)) = -1$$

for all $\alpha \in k_2 \setminus k$.

Proof. For $\xi \in k$, let $Z_{0,\xi}$ be the affine smooth connected curve defined by $X^q + X = Y^{q+1} + \xi$ over k^{ac}. Recall that $Z_0 = \coprod_{\xi \in k} Z_{0,\xi}$. We consider the projective smooth curve

$$\overline{Z}_\xi = \{(S:T:U) \in \mathbb{P}^2_{k^{\mathrm{ac}}} \mid S^q U + S U^q = T^{q+1} + \xi U^{q+1}\}.$$

We have the open immersion $Z_{0,\xi} \hookrightarrow \overline{Z}_\xi$; $(X,Y) \mapsto (X:Y:1)$. We set $\overline{Z} = \coprod_{\xi \in k} \overline{Z}_\xi$, which contains Z_0 as an open subscheme. Let $\eta \in k_2$ and $\alpha \in k_2 \setminus k$, and set $\zeta = \alpha^{q-1} \neq 1$. The action of $g(1,0,\eta) f_{\gamma_0}(\alpha)$ on Z_0 is given by

$$(X,Y) \mapsto (X + (\zeta - 1)\gamma_0^q (Y - \gamma_0) + \eta, \zeta Y + (1-\zeta)\gamma_0).$$

This action naturally extends to the one on $\overline{Z}$. One can check that the multiplicity of any fixed point of $g(1,0,\eta) f_{\gamma_0}(\alpha)$ on $\overline{Z}$ is one. The set of fixed points of $g(1,0,\eta)f_{\gamma_0}(\alpha)$ on Z_0 equals

$$\begin{cases} \coprod_{\xi\in k}\{(X,\gamma_0)\in\mathbb{A}^2_{k^{\mathrm{ac}}}\mid X^q+X=\gamma_0^{q+1}+\xi\} & \text{if } \eta=0,\\ \emptyset & \text{otherwise.}\end{cases}$$

Hence, by [Del, Corollaire 5.4 in Rapport], we have

$$\mathrm{Tr}(g(1,0,\eta)f_{\gamma_0}(\alpha);H^*_{\mathrm{c}}(Z_0)) = \begin{cases} q^2 & \text{if } \eta=0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.43}$$

We set

$$M=\ker\mathrm{Tr}_{k_2/k}.$$

Let $\pi_0(Z_0)$ be the set of connected components of Z_0. As above, we have $\pi_0(Z_0)\simeq k$. Hence we have $H^2_{\mathrm{c}}(Z_0)\simeq\bigoplus_{\chi\in k^\vee}\chi$ as k-representations. We can easily check that $f_{\gamma_0}(\alpha)$ acts on $\pi_0(Z_0)$ trivially, and $g(1,0,\eta)$ acts on it as multiplication by $\mathrm{Tr}_{k_2/k}(\eta)$. Hence we have

$$\mathrm{Tr}(g(1,0,\eta)f_{\gamma_0}(\alpha);H^2_{\mathrm{c}}(Z_0)) = \sum_{\chi\in k^\vee}\chi(\mathrm{Tr}_{k_2/k}(\eta)) = \begin{cases} q & \text{if } \eta\in M,\\ 0 & \text{otherwise.}\end{cases} \tag{3.44}$$

Note that $H^0_{\mathrm{c}}(Z_0)=0$. By (3.43) and (3.44) we obtain

$$\mathrm{Tr}(g(1,0,\eta)f_{\gamma_0}(\alpha);H^1_{\mathrm{c}}(Z_0)) = \begin{cases} -q(q-1) & \text{if } \eta=0,\\ q & \text{if } \eta\in M\setminus\{0\},\\ 0 & \text{otherwise.}\end{cases}$$

We have $\psi'|_M\neq 1$ by the assumption $\psi'\in k_2^\vee\setminus k^\vee$. Let $H^1_{\mathrm{c}}(Z_0)[\psi']$ be the ψ'-isotypic part of $H^1_{\mathrm{c}}(Z_0)$. By (3.42) we have $H^1_{\mathrm{c}}(Z_0)[\psi']\simeq\tau_{\psi'}$. Therefore we have

$$\begin{aligned}\mathrm{Tr}\,\tau_{\psi'}(f_{\gamma_0}(\alpha)) &= \mathrm{Tr}(f_{\gamma_0}(\alpha);\ H^1_{\mathrm{c}}(Z_0)[\psi'])\\ &= \frac{1}{q^2}\sum_{\eta\in k_2}\psi'^{-1}(\eta)\,\mathrm{Tr}(g(1,0,\eta)f_{\gamma_0}(\alpha);H^1_{\mathrm{c}}(Z_0))\\ &= \frac{1}{q^2}\Big(-q(q-1)+q\sum_{\eta\in M\setminus\{0\}}\psi'^{-1}(\eta)\Big) = -1.\end{aligned}$$

Hence the assertion follows. ∎

We write μ_{q+1} for the abelian group $\{x \in k_2^\times \mid x^{q+1} = 1\}$. We regard $\chi \in \mu_{q+1}^\vee$ as a character of $f_{\gamma_0}(k_2^\times)$ via the homomorphism

$$\pi\colon f_{\gamma_0}(k_2^\times) \to \mu_{q+1}; \quad f_{\gamma_0}(x) \mapsto x^{q-1}.$$

The kernel of π equals the subgroup $k^\times \subset Q$. The image of $f_{\gamma_0}(k_2 \setminus k)$ by π equals $\mu_{q+1} \setminus \{1\}$. By (3.41), the action of $f_{\gamma_0}(k_2^\times)$ on $\tau_{\psi'}|_{f_{\gamma_0}(k_2^\times)}$ factors through π. Hence for each $\gamma_0 \in k_2$, we have

$$\tau_{\psi'}|_{f_{\gamma_0}(k_2^\times)} \simeq \bigoplus_{\chi \in \mu_{q+1}^\vee \setminus \{1\}} \chi \tag{3.45}$$

as $f_{\gamma_0}(k_2^\times)$-representations, because both sides have the same trace by Lemma 3.11.

In the sequel, we consider the subgroup $N_l \subset G_n^F$ and describe characters of it. Note that N_l is abelian. We take a K-embedding $K_2 \hookrightarrow \mathrm{M}_2(K)$. We have the isomorphism $N_l \simeq \mathrm{M}_2(\mathfrak{o}_{l'})$; $1 + \varpi^l x \mapsto x \mod \mathfrak{p}^{l'}$. For a character $\chi\colon \mathfrak{o} \to \overline{\mathbb{Q}}_\ell^\times$, the conductor exponent of χ means the least integer $r \geq 0$ such that $\chi|_{\mathfrak{p}^r} = 1$. Let $\psi\colon \mathfrak{o} \to \overline{\mathbb{Q}}_\ell^\times$ be a character of conductor exponent n. For an element $\beta \in \mathrm{M}_2(\mathfrak{o}_{l'})$ we consider the character

$$\psi_\beta\colon N_l \to \overline{\mathbb{Q}}_\ell^\times; \quad g \mapsto \psi(\mathrm{Tr}(\beta(g-1))).$$

We have the isomorphism

$$\kappa\colon \mathrm{M}_2(\mathfrak{o}_{l'}) \xrightarrow{\sim} \mathrm{Hom}(N_l, \overline{\mathbb{Q}}_\ell^\times); \quad \beta \mapsto \psi_\beta.$$

The group G_n^F acts on $\mathrm{M}_2(\mathfrak{o}_{l'})$ by conjugation. By the above isomorphism, G_n^F acts on $\mathrm{Hom}(N_l, \overline{\mathbb{Q}}_\ell^\times)$ by $\psi_\beta \mapsto \psi_\beta^g$ with $\psi_\beta^g(x) = \psi_\beta(g^{-1}xg)$. We have the commutative diagram

$$\begin{array}{ccc}
\mathrm{M}_2(\mathfrak{o}_{l'}) & \xrightarrow[\sim]{\kappa} & \mathrm{Hom}(N_l, \overline{\mathbb{Q}}_\ell^\times) \\
\uparrow & & \downarrow \\
\mathfrak{O}_{l'} & \xrightarrow{\simeq} & \mathrm{Hom}(U_{K_2,n}^l, \overline{\mathbb{Q}}_\ell^\times),
\end{array}$$

where the right vertical arrow is induced by the inclusion $U_{K_2,n}^l \hookrightarrow N_l$.

Let $\omega \in (T_n^F)^\vee$. We take an element $\beta \in \mathfrak{O}_{l'}$ such that $\psi_\beta|_{U_{K_2,n}^l} = \omega|_{U_{K_2,n}^l}$. We define a character σ_ω of $\mathfrak{O}_n^\times N_l$ by

$$\sigma_\omega(xu) = \omega(x)\psi_\beta(u) \tag{3.46}$$

for $x \in \mathfrak{O}_n^\times$ and $u \in N_l$.

3.4. Cohomology of $\mathbf{X}_n$. In the following, we study the étale cohomology of $\mathbf{X}_n$ by using results in §3.2 and §3.3. Our aim is to show Proposition 3.18.

Now we assume that $n \geq 2$. Let $t \in B_{l'}$. Recall that $\mathbf{X}_n^t$ is open and closed in $\mathbf{X}_n$. We set

$$W_t = H_{\mathrm{c}}^n(\mathbf{X}_n^t) \subset H_{\mathrm{c}}^n(\mathbf{X}_n).$$

Let $\zeta \in k_2 \setminus k$. We put

$$\mathbf{G}_{n,\zeta} = \mathfrak{O}_n^{\times} N_{l'} \times T_n^F \subset G_n^F \times T_n^F,$$

where $\mathfrak{O}_n^{\times}$ is regarded as a subgroup of G_n^F by ι_ζ as before. We regard ζ as an element of $B_{l'}$. Recall that $\mathbf{p}_n$ is $G_n^F \times T_n^F$-equivariant. Then $\mathbf{X}_n^\zeta$ admits the action of $\mathbf{G}_{n,\zeta}$ by Lemma 3.6 (2). Hence we can regard W_ζ as a representation of $\mathbf{G}_{n,\zeta}$.

Assume that n is even. By $n = 2l'$, the action of T_n^F on $\Delta_{1,n}^\zeta$ is simply transitive by Lemma 3.7. By Lemma 3.8 (1), we have

$$W_\zeta|_{\mathfrak{O}_n^{\times} \times T_n^F} \simeq \bigoplus_{\omega \in (T_n^F)^\vee} \omega \otimes \omega^{-1} \tag{3.47}$$

as $\mathfrak{O}_n^{\times} \times T_n^F$-representations. We regard

$$\mathrm{Hom}_{T_n^F}(\omega^{-1}, W_\zeta)$$

as a representation of $\mathfrak{O}_n^{\times} N_{l'}$. This is a character of $\mathfrak{O}_n^{\times} N_{l'}$ which is an extension of ω by (3.47). Hence this is isomorphic to σ_ω. Therefore we have

$$W_\zeta \simeq \bigoplus_{\omega \in (T_n^F)^\vee} \sigma_\omega \otimes \omega^{-1} \tag{3.48}$$

as $\mathbf{G}_{n,\zeta}$-representations. By Lemma 3.6 (2), the stabilizer of W_ζ in $G_n^F \times T_n^F$ equals $\mathbf{G}_{n,\zeta}$. The subspaces $\{W_t\}_{t \in B_{l'}}$ are permuted transitively by $G_n^F \times T_n^F$. Hence, by [Se, Proposition 19 in §7.2], we have isomorphisms

$$\begin{aligned} H_{\mathrm{c}}^n(\mathbf{X}_n) &\simeq \bigoplus_{\omega \in (T_n^F)^\vee} \mathrm{Ind}_{\mathbf{G}_{n,\zeta}}^{G_n^F \times T_n^F}(\sigma_\omega \otimes \omega^{-1}) \\ &\simeq \bigoplus_{\omega \in (T_n^F)^\vee} (\mathrm{Ind}_{\mathfrak{O}_n^{\times} N_{l'}}^{G_n^F} \sigma_\omega) \otimes \omega^{-1} \end{aligned} \tag{3.49}$$

as $G_n^F \times T_n^F$-representations.

We assume that n is odd until (3.59). By (3.25), we have

$$W_\zeta \simeq \bigoplus_{(P,s)\in\Delta_{1,n}^{\zeta}} H_{\mathrm{c}}^1(X_{P,s}) \subset H_{\mathrm{c}}^n(\mathbf{X}_n) \simeq \bigoplus_{(P,s)\in\Delta_{1,n}} H_{\mathrm{c}}^1(X_{P,s}). \tag{3.50}$$

For an element $\beta \in \mathrm{M}_2(\mathfrak{o}_{l'})$, we write $\bar{\beta} \in \mathrm{M}_2(k)$ for the image of it by the canonical map $\mathrm{M}_2(\mathfrak{o}_{l'}) \twoheadrightarrow \mathrm{M}_2(k)$. In the following lemma, we understand characters of $N_{l'+1}$ appearing in W_ζ.

Lemma 3.12: *Let $\beta \in \mathrm{M}_2(\mathfrak{o}_{l'})$. Assume that the character ψ_β of $N_{l'+1}$ appears in W_ζ.*

1. *We have $\beta \in \mathfrak{O}_{l'}^\times$ and $\bar{\beta} \in k_2 \setminus k$. The reduction $\bar{\beta}$ is conjugate to the matrix*

$$B = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nr}_{k_2/k}(\bar{\beta}) & \mathrm{Tr}_{k_2/k}(\bar{\beta}) \end{pmatrix} \in \mathrm{M}_2(k).$$

2. *The stabilizer $\{g \in G_n^F \mid \psi_\beta^g = \psi_\beta\}$ equals $\mathfrak{O}_n^\times N_{l'}$.*

Proof. Since n is odd, we have $l = l'+1$ and $n = 2l'+1$. We set $\beta = \beta_0 + \beta_1$ with $\beta_0 \in \mathfrak{O}_{l'}$ and $\beta_1 \in \mathfrak{C}_{1,l'}$. By the former assertion in Lemma 3.9 (2), the subgroup $H_{1,\zeta,n}^0$ acts on W_ζ trivially. By the assumption and (3.35), we have $\psi_\beta(1+\varpi^{l'+1}h) = 1$ for any $h \in \mathfrak{C}_{1,l'}$. By $\mathrm{tr}(\beta_0 h) = 0$, we have

$$\psi(\varpi^{l'+1}\,\mathrm{tr}(\beta_1 h)) = \psi(\mathrm{tr}(\varpi^{l'+1}\beta h)) = \psi_\beta(1+\varpi^{l'+1}h) = 1 \tag{3.51}$$

for any $h \in \mathfrak{C}_{1,l'}$. We put $\beta_1 = h(a,b)$ with $a, b \in \mathfrak{o}_{l'}$ in the notation of (2.3). Assume that $\beta_1 \neq 0$. By $\zeta \in k_2 \setminus k$, we can check that the image of the map

$$\mathfrak{C}_{1,l'} \to \mathfrak{o}_{l'}; \quad h \mapsto \mathrm{tr}(\beta_1 h)$$

equals the ideal (a,b), and this ideal contains $\mathfrak{p}^{l'-1}/\mathfrak{p}^{l'}$ by $\beta_1 \neq 0$. Hence, by (3.51), we have $\psi(\mathfrak{p}^{n-1}) = 1$. Since ψ has conductor exponent n, this is a contradiction. Hence we have $\beta_1 = 0$. Therefore we have $\beta = \beta_0 \in \mathfrak{O}_{l'}$. By (3.42), we have an isomorphism

$$H_{\mathrm{c}}^1(Z_0) \simeq \bigoplus_{\chi \in k_2^\vee \setminus k^\vee} \chi^{\oplus q} \tag{3.52}$$

as $Z(Q) \simeq k_2$-representations. By (3.52), there exists $\chi \in k_2^\vee \setminus k^\vee$ such that

$$\psi_\beta(1+\varpi^{2l'}g_0) = \chi(s_{\zeta,1}(g_0))$$

for $g_0 \in \mathrm{M}_2(k)$ by Lemma 3.9 (2). By $s_{\zeta,1}(x_0) = x_0$ for $x_0 \in k_2$, we have

$$\psi_\beta(1 + \varpi^{2l'} x_0) = \chi(x_0) \tag{3.53}$$

for $x_0 \in k_2$. We identify $\mathfrak{p}^{n-1}/\mathfrak{p}^n$ with k by $\varpi^{n-1}x \mapsto x$ for $x \in k$. We set

$$\psi_0 = \psi|_{\mathfrak{p}^{n-1}/\mathfrak{p}^n \simeq k} \in k^\vee \setminus \{1\}.$$

The left-hand side of (3.53) equals $\psi_0 \circ \mathrm{Tr}_{k_2/k}(\bar{\beta} x_0)$. Hence, by (3.53) and $\chi \in k_2^\vee \setminus k^\vee$, we have $\bar{\beta} \in k_2 \setminus k$. We set $\bar{\beta} = a + b\zeta$ with $a, b \in k$. By $\bar{\beta} \in k_2 \setminus k$ we have $b \in k^\times$. Let $M = \left(\begin{smallmatrix} 1 & 0 \\ a+b(\zeta+\zeta^q) & b \end{smallmatrix}\right) \in G_1^F$. Then, $M\bar{\beta}M^{-1}$ equals B. Therefore the first assertion follows.

The second assertion follows from the first one and [Sta, §2.1]. ∎

The following lemma is a well-known result on representation theory of a finite Heisenberg group.

Lemma 3.13 ([BF, (8.3.3) Proposition]): *Let G be a finite group and N a normal subgroup such that G/N is an elementary abelian p-group. Let χ be a character of N, which is stabilized by G. Define an alternating bilinear form*

$$h_\chi \colon G/N \times G/N \to \overline{\mathbb{Q}}_\ell^\times; \quad (g_1, g_2) \mapsto \chi([g_1, g_2]) = \chi(g_1 g_2 g_1^{-1} g_2^{-1}).$$

Assume that h_χ is non-degenerate. Then there exists a unique up to isomorphism irreducible representation ρ_χ such that $\rho_\chi|_N$ contains χ. The representation ρ_χ has degree $[G : N]^{1/2}$ and the restriction $\rho_\chi|_N$ is a multiple of χ.

Corollary 3.14 ([Sta, §4.2]): *Let ψ_β be a character of N_l appearing in W_ζ. Let $\widetilde{\psi}_\beta$ be a character of $U^1_{K_2,n} N_l$ which is an extension of ψ_β. Then there exists a unique irreducible representation $\rho_{\widetilde{\psi}_\beta}$ of $U^1_{K_2,n} N_{l'}$ of degree q containing $\widetilde{\psi}_\beta$. We have*

$$\rho_{\widetilde{\psi}_\beta}|_{U^1_{K_2,n} N_l} \simeq \widetilde{\psi}_\beta^{\oplus q}.$$

Moreover, every irreducible representation of $U^1_{K_2,n} N_{l'}$ containing ψ_β has this form.

Proof. We set $G = U^1_{K_2,n} N_{l'}$, $N = U^1_{K_2,n} N_l$ and $\chi = \widetilde{\psi}_\beta$. By applying Lemma 3.13 as in [Sta, §4.2] we obtain the assertions. ∎

Definition 3.15: We identify $U^{n-1}_{K_2,n}$ with k_2 by $1 + \varpi^{n-1} x \mapsto x$ for $x \in k_2$. For a character $\omega \in (\mathfrak{O}_n^\times)^\vee$, we say that ω is **strongly primitive** if the restriction $\omega|_{U^{n-1}_{K_2,n}}$ does not factor through the trace map $\mathrm{Tr}_{k_2/k} \colon k_2 \to k$.

In this definition, we follow [AOPS, Definition 5.2]. Note that this definition does not depend on the choice of the uniformizer ϖ. We write $(\mathfrak{O}_n^\times)^\vee_{\mathrm{stp}}$ for the set of all strongly primitive characters of $\mathfrak{O}_n^\times$. Note that

$$|(\mathfrak{O}_n^\times)^\vee_{\mathrm{stp}}| = q^{2n-3}(q-1)(q^2-1).$$

For a strongly primitive character ω, we consider the restriction

$$\sigma_\omega|_{U^1_{K_2,n}N_l} \colon U^1_{K_2,n}N_l \to \overline{\mathbb{Q}}_\ell^\times$$

of σ_ω in (3.46). We obtain the representation $\rho_{\sigma_\omega|_{U^1_{K_2,n}N_l}}$ of $U^1_{K_2,n}N_{l'}$ by Corollary 3.14, for which we simply write ρ_ω. Note that the isomorphism class of ρ_ω depends only on $\omega|_{U^1_{K_2,n}}$.

Let $\Delta_\zeta \colon \mathfrak{O}_n^\times \to \mathbf{G}_{n,\zeta}$ be the diagonal map in (3.6). We consider (3.50). For each $(P,s) \in \Delta^\zeta_{1,n}$, the subspace $H^1_{\mathrm{c}}(X_{P,s})$ of W_ζ is stable under the action of $\Delta_\zeta(\mathfrak{o}_n^\times U^1_{K_2,n})$ by Lemma 3.8 (1). Recall that $k^\times \subset Q$ acts on $X_{P,s}$ trivially. By the latter assertion in Lemma 3.8 (2), the restriction $W_\zeta|_{\Delta_\zeta(\mathfrak{o}_n^\times U^1_{K_2,n})}$ is trivial. We fix the isomorphism

$$\mathfrak{O}_n^\times/\mathfrak{o}_n^\times U^1_{K_2,n} \xrightarrow{\sim} \mu_{q+1}; \quad \alpha \mapsto \bar{\alpha}^{q-1}.$$

By this, the restriction $W_\zeta|_{\Delta_\zeta(\mathfrak{O}_n^\times)}$ can be regarded as a μ_{q+1}-representation. Recall that

$$W_\zeta \simeq \bigoplus_{(P,s)\in\Delta^\zeta_{1,n}} H^1_{\mathrm{c}}(X_{P,s}).$$

By Lemma 3.4 (2) we have $|\Delta^\zeta_{1,n}| = q^{2(n-2)}(q^2-1)$. Hence have

$$|k_2^\vee \setminus k^\vee||\Delta^\zeta_{1,n}| = |(T_n^F)^\vee_{\mathrm{stp}}|.$$

By Lemma 3.8 (2), (3.42) and (3.45), the representation $W_\zeta|_{\Delta_\zeta(\mathfrak{O}_n^\times)}$ is isomorphic to

$$\bigoplus_{\chi\in\mu^\vee_{q+1}\setminus\{1\}} \chi^{|(T_n^F)^\vee_{\mathrm{stp}}|} \tag{3.54}$$

as μ_{q+1}-representations. We identify $U^{n-1}_{K_2,n}$ with k_2 by $1+\varpi^{n-1}x \mapsto x$ for $x \in k_2$. Let $(P,s) \in \Delta^\zeta_{1,n}$. By the latter assertion in Lemma 3.7, we can regard $H^1_{\mathrm{c}}(X_{P,s})$ as a representation of $\{1\} \times U^{n-1}_{K_2,n}$. Note that $W_\zeta|_{\Delta_\zeta(U^{n-1}_{K_2,n})}$ is trivial by (3.54). By the property (b) in Lemma 3.9 (2) and (3.52), we have

$$H^1_{\mathrm{c}}(X_{P,s}) \simeq \bigoplus_{\psi'\in k_2^\vee\setminus k^\vee} \psi'^q$$

as $\{1\} \times U_{K_2,n}^{n-1}$-representations. By the former assertion in Lemma 3.7, we have an isomorphism

$$W_\zeta|_{\{1\}\times T_n^F} \simeq \mathrm{Ind}_{U_{K_2,n}^{n-1}}^{T_n^F} H_{\mathrm{c}}^1(X_{P,s}) \simeq \bigoplus_{\omega\in (T_n^F)^\vee_{\mathrm{stp}}} \omega^{\oplus q} \tag{3.55}$$

as T_n^F-representations. By (3.54) and (3.55), we have an isomorphism

$$W_\zeta|_{\mathfrak{O}_n^\times \times T_n^F} \simeq \bigoplus_{\omega\in (T_n^F)^\vee_{\mathrm{stp}}} \bigoplus_{\chi\in \mu_{q+1}^\vee\setminus\{1\}} \omega\chi\otimes\omega^{-1} \tag{3.56}$$

as $\mathfrak{O}_n^\times \times T_n^F$-representations, where χ is considered as a character of $\mathfrak{O}_n^\times$ through $\mathfrak{O}_n^\times \to \mu_{q+1};\ \alpha\mapsto \bar{\alpha}^{q-1}$. For a strongly primitive character ω we put

$$\widetilde{\sigma}_\omega = \mathrm{Hom}_{T_n^F}(\omega^{-1}, W_\zeta),$$

which is regarded as a representation of $\mathfrak{O}_n^\times N_{l'}$. Since W_ζ contains only strongly primitive characters by (3.56), we have an isomorphism

$$W_\zeta \simeq \bigoplus_{\omega\in (T_n^F)^\vee_{\mathrm{stp}}} \widetilde{\sigma}_\omega\otimes\omega^{-1} \tag{3.57}$$

as $\mathbf{G}_{n,\zeta}$-representations.

Lemma 3.16: *The $\mathfrak{O}_n^\times N_{l'}$-representation $\widetilde{\sigma}_\omega$ is irreducible and satisfies*

- $\widetilde{\sigma}_\omega|_{U_{K_2,n}^1 N_{l'}} \simeq \rho_\omega$ *and*
- $\mathrm{Tr}\,\widetilde{\sigma}_\omega(\zeta') = -\omega(\zeta')$ *for* $\zeta' \in k_2\setminus k$.

Proof. Let $\mathfrak{O}_n^\times \subset G_n^F$. By (3.56), we have an isomorphism

$$\widetilde{\sigma}_\omega|_{\mathfrak{O}_n^\times} \simeq \bigoplus_{\chi\in \mu_{q+1}^\vee\setminus\{1\}} w\chi. \tag{3.58}$$

Let $\zeta' \in k_2\setminus k$. We have $\sum_{\chi\in\mu_{q+1}^\vee\setminus\{1\}} \chi(\zeta'^{q-1}) = -1$ by $\zeta'^{q-1}\neq 1$. By (3.58) we have $\mathrm{Tr}\,\widetilde{\sigma}_\omega(\zeta') = -\omega(\zeta')$. Since $\widetilde{\sigma}_\omega$ is contained in W_ζ, there exists $\beta\in\mathfrak{O}_{l'}^\times\setminus\mathfrak{o}_{l'}^\times$ such that $\widetilde{\sigma}_\omega$ contains the character ψ_β of N_l by Lemma 3.12. By $\dim\widetilde{\sigma}_\omega = q$ and Corollary 3.14, there exists a character $\widetilde{\psi}_\beta$ of $U_{K_2,n}^1 N_l$ which is an extension of ψ_β such that $\widetilde{\sigma}_\omega|_{U_{K_2,n}^1 N_{l'}} \simeq \rho_{\widetilde{\psi}_\beta}$. The irreducibility of $\widetilde{\sigma}_\omega$ follows from the irreducibility of $\widetilde{\sigma}_\omega|_{U_{K_2,n}^1 N_{l'}} \simeq \rho_{\widetilde{\psi}_\beta}$ in Corollary 3.14. We have

$$\widetilde{\sigma}_\omega|_{U_{K_2,n}^1 N_l} \simeq \rho_{\widetilde{\psi}_\beta}|_{U_{K_2,n}^1 N_l} \simeq \widetilde{\psi}_\beta^{\oplus q}.$$

By (3.58), we have $\widetilde{\sigma}_\omega|_{U^1_{K_2,n}} = \omega|^{\oplus q}_{U^1_{K_2,n}}$. Hence we have $\omega|_{U^1_{K_2,n}} = \widetilde{\psi}_\beta|_{U^1_{K_2,n}}$. Therefore, for $x \in U^1_{K_2,n}$ and $y \in N_l$, we have

$$\sigma_\omega(xy) = \omega(x)\psi_\beta(y) = \widetilde{\psi}_\beta(x)\psi_\beta(y) = \widetilde{\psi}_\beta(xy).$$

Hence we obtain $\widetilde{\sigma}_\omega|_{U^1_{K_2,n}N_{l'}} \simeq \rho_\omega$ by the uniqueness in Corollary 3.14. ∎

Remark 3.17: See [AOPS, Lemma 5.6], [BH, Proposition in §19.4] and [Sta, §4.2] for more details on $\widetilde{\sigma}_\omega$.

By the former assertion in Lemma 3.6 (1), we know that the subspaces $\{W_t\}_{t\in B_{l'}}$ are permuted transitively by $G_n^F \times T_n^F$ and the stabilizer of W_ζ equals $\mathbf{G}_{n,\zeta}$. Hence, by (3.57), we have isomorphisms

$$H^n_{\mathrm{c}}(\mathbf{X}_n) \simeq \mathrm{Ind}_{\mathbf{G}_{n,\zeta}}^{G_n^F\times T_n^F} W_\zeta \simeq \bigoplus_{\omega\in (T_n^F)^\vee_{\mathrm{stp}}} (\mathrm{Ind}_{\mathfrak{O}_n^\times N_{l'}}^{G_n^F} \widetilde{\sigma}_\omega) \otimes \omega^{-1} \tag{3.59}$$

as $G_n^F \times T_n^F$-representations.

For each $\omega \in (T_n^F)^\vee_{\mathrm{stp}}$ we set

$$\pi_\omega = \begin{cases} \mathrm{Ind}_{\mathfrak{O}_n^\times N_{l'}}^{G_n^F} \sigma_\omega & \text{if } n \text{ is even,} \\ \mathrm{Ind}_{\mathfrak{O}_n^\times N_{l'}}^{G_n^F} \widetilde{\sigma}_\omega & \text{if } n \text{ is odd.} \end{cases} \tag{3.60}$$

Note that we have $\dim \pi_\omega = q^{n-1}(q-1)$. The isomorphism class of π_ω does not depend on the embedding $\iota_\zeta \colon \mathfrak{O}_n^\times \hookrightarrow G_n^F$. The representation π_ω is called a **strongly cuspidal** representation of G_n^F in [AOPS, §5]. In the case GL(2), strongly cuspidal is equivalent to cuspidal by [AOPS, Theorem A]. Hence, in Introduction, we simply call π_ω cuspidal. This representation is irreducible. This class of representations is described also in [Onn, §6.2] and [Sta, §4.2]. Let $H^n_{\mathrm{c}}(\mathbf{X}_n)_{\mathrm{stp}}$ be the maximal subspace of $H^n_{\mathrm{c}}(\mathbf{X}_n)$ consisting of strongly primitive characters of T_n^F.

Proposition 3.18: *Let $n \geq 2$ be a positive integer. Then we have an isomorphism*

$$H^n_{\mathrm{c}}(\mathbf{X}_n)_{\mathrm{stp}} \simeq \bigoplus_{\omega\in (T_n^F)^\vee_{\mathrm{stp}}} \pi_\omega \otimes \omega^{-1}$$

as $G_n^F \times T_n^F$-representations.

Proof. The required assertion follows from (3.49) and (3.59). ∎

Remark 3.19: (1) If n is odd, as in (3.59), we have $H_c^n(\mathbf{X}_n)_{\mathrm{stp}} = H_c^n(\mathbf{X}_n)$. On the other hand, if n is even, this does not hold as in (3.49).

(2) The above proposition is regarded as a geometric realization of the correspondence in [AOPS, Theorem 5.10] for GL(2) and $\mathfrak{o}$ of characteristic p. The correspondence is a generalization of the Green correspondence $\omega \leftrightarrow \pi_\omega$ in Lemma 3.20 in the case GL(2). See also [AOPS, Introduction].

(3) Let $\sigma \in \mathrm{Gal}(K_2/K)$ be the non-trivial character. Then we have $\pi_\omega \simeq \pi_{\omega^\sigma}$.

Recall the cohomology of $\mathbf{X}_1 = Z_{\mathrm{DL}}$. We regard $(k^\times)^\vee$ as a subgroup of $(k_2^\times)^\vee$ by the dual of the norm map $k_2^\times \to k^\times$. We write $H_c^1(Z_{\mathrm{DL}})_{\mathrm{stp}}$ for the maximal subspace on which $k_2^\times$ acts not factoring through the norm map $k_2^\times \to k^\times$. For any $\omega \in (k_2^\times)^\vee \setminus (k^\times)^\vee$, there exists an irreducible cuspidal representation π_ω (cf. [BH, §6.4]). We identify $k_2^\times \simeq T_1^F$ as before. We set

$$(T_1^F)^\vee_{\mathrm{stp}} = (k_2^\times)^\vee \setminus (k^\times)^\vee.$$

The following is well-known as the Deligne–Lusztig theory for $\mathrm{GL}_2(\mathbb{F}_q)$, which gives a geometric realization of the Green correspondence in this case.

LEMMA 3.20: *We have an isomorphism*

$$H_c^1(\mathbf{X}_1)_{\mathrm{stp}} \simeq \bigoplus_{\omega \in (T_1^F)^\vee_{\mathrm{stp}}} \pi_\omega \otimes \omega^{-1}$$

as $G_1^F \times T_1^F$-representations.

Proof. This is a special case of the Deligne–Lusztig theory in [DL] (cf. (3.40), [T2, §4.3] and [Y]). ∎

Remark 3.21: (1) As in Remark 3.19 (2), we have $\pi_\omega \simeq \pi_{\omega^\sigma}$ for $\omega \in (T_1^F)^\vee_{\mathrm{stp}}$.

(2) See [BH, §6.4] for more details on cuspidal representations of G_1^F.

4. Deligne–Lusztig variety for $\mathcal{O}_{2n-1}^\times$

We use the same notation for the quaternion algebra D at the beginning of §2.2. In this section, we define a closed subvariety of the Deligne–Lusztig variety for $\mathcal{O}_{2n-1}^\times$ and compute its cohomology. Analysis in this section is very analogous to the one in §3. Our main result in this section is Proposition 4.12.

4.1. Deligne–Lusztig variety for $\mathcal{O}_{2n-1}^{\times}$ and its subvariety. Let n be a positive integer. Let G'_n be the group consisting of all 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \in \varpi\widetilde{\mathfrak{o}}_{n-1}$ and $a, d \in \widetilde{\mathfrak{o}}_n^{\times}$ and $b \in \widetilde{\mathfrak{o}}_{n-1}$. We regard this as an affine variety over k^{ac}. By $\varpi\widetilde{\mathfrak{o}}_{n-1} \subset \widetilde{\mathfrak{o}}_n$ we have a determinant map

$$\det\colon G'_n \to \widetilde{\mathfrak{o}}_n^{\times}.$$

Let

$$V_n = \widetilde{\mathfrak{o}}_n \oplus \widetilde{\mathfrak{o}}_{n-1}, \quad V'_n = \varpi\widetilde{\mathfrak{o}}_{n-1} \oplus \widetilde{\mathfrak{o}}_n, \quad V''_n = \widetilde{\mathfrak{o}}_n^{\oplus 2}.$$

These V_n and V'_n admit actions of G'_n by right multiplication. We have the canonical surjective map $V''_n \twoheadrightarrow V_n$ and the injective map $V'_n \hookrightarrow V''_n$. Let $\{e_1, e_2\}$ be the canonical basis of V''_n. Let F be as in (3.1). We define morphisms

$$\begin{aligned} &F'\colon V_n \to V'_n; \quad xe_1 + ye_2 \mapsto \varpi F(y)e_1 + F(x)e_2, \\ &F'\colon G'_n \to G'_n;\ g \mapsto \varphi' F(g) {\varphi'}^{-1}, \end{aligned}$$

where $\varphi' = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. Explicitly, we have

$$F'(g) = \begin{pmatrix} F(d) & F(c)\varpi^{-1} \\ \varpi F(b) & F(a) \end{pmatrix} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'_n.$$

Note that we have

$$\begin{aligned} \det F'(g) &= F(\det g) \qquad \text{for } g \in G'_n, \\ F'(vg) &= F'(v)F'(g) \quad \text{in } V'_n \text{ for } v \in V_n \text{ and } g \in G'_n. \end{aligned}$$

On the other hand, for elements $v \in V_n$ and $w \in V'_n$, we define an element $v \wedge w$ in $\bigwedge^2 V''_n \simeq \widetilde{\mathfrak{o}}_n(e_1 \wedge e_2)$ by $\widetilde{v} \wedge w$ for any lifting $\widetilde{v} \in V''_n$ of v. This is well-defined. In the same manner, for elements $v \in V_n$ and $w \in \varpi V_n$, by considering $\varpi V_n \subset V''_n$, we can define $v \wedge w \in \bigwedge^2 V''_n$.

We set

$$T_n^F = \left\{ \begin{pmatrix} t & 0 \\ 0 & F(t) \end{pmatrix} \in G'_n \;\middle|\; t \in \mathfrak{O}_n^{\times} \right\}$$

and fix an isomorphism

$$\mathfrak{O}_n^{\times} \simeq T_n^F; \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & F(t) \end{pmatrix}. \tag{4.1}$$

This group T_n^F equals the one defined before and is denoted by the same letter. Let U'_n be the group of upper triangular matrices in G'_n with 1's on the diagonal.

Then we set

$$X_n^D = \{g \in G'_n \mid F'(g)g^{-1} \in U'_n\},$$

which we call the **Deligne–Lusztig variety** for $\mathcal{O}_{2n-1}^{\times}$ (cf. [Lus, §2]). Let $G_n'^{F'}$ denote the set of F'-fixed points in G'_n. Then we have

$$G_n'^{F'} = \left\{ [a,b] = \begin{pmatrix} a & F(b) \\ \varpi b & F(a) \end{pmatrix} \in G'_n \;\middle|\; a \in \mathfrak{O}_n^{\times},\ b \in \mathfrak{O}_{n-1} \right\}.$$

Recall that $a\varphi = \varphi F(a)$ for $a \in \mathfrak{O}_n$. We fix an isomorphism

$$G_n'^{F'} \xrightarrow{\sim} \mathcal{O}_{2n-1}^{\times}; \quad [a,b] \mapsto a + \varphi b.$$

Let $\mathcal{O}_{2n-1}^{\times} \times T_n^F$ act on X_n^D by $x \mapsto txd$ for $x \in X_n^D$ and $(d,t) \in \mathcal{O}_{2n-1}^{\times} \times T_n^F$. The reduced norm map $\mathrm{Nrd}_{D/K} \colon D^{\times} \to K^{\times}$ induces

$$\mathrm{Nrd}_{D/K} \colon \mathcal{O}_{2n-1}^{\times} \to \mathfrak{o}_n^{\times}.$$

Lemma 4.1: (1) *We have*

$$\begin{aligned} X_n^D &= \left\{ g = \begin{pmatrix} x & y \\ \varpi F(y) & F(x) \end{pmatrix} \in G'_n \;\middle|\; \det g \in \mathfrak{o}_n^{\times} \right\} \\ &\simeq \mathfrak{S}_n^D = \{v = (x,y) = xe_1 + ye_2 \in V_n \mid v \wedge F'(v) \in \mathfrak{o}_n^{\times}(e_1 \wedge e_2)\}; \\ & g \mapsto e_1 g. \end{aligned}$$

(2) *Let $\mathcal{O}_{2n-1}^{\times} \times T_n^F$ act on $\mathfrak{S}_n^D$ through the isomorphism in* 1. *For $t \in T_n^F$, $v \in \mathfrak{S}_n^D$ and $d \in \mathcal{O}_{2n-1}^{\times}$, we have*

$$\begin{aligned} vd \wedge F'^2(vd) &= \mathrm{Nrd}_{D/K}(d)(v \wedge F'^2(v)), \\ tv \wedge F'^2(tv) &= t^2(v \wedge F'^2(v)). \end{aligned}$$

Proof. The claims follow from direct computations. We omit the details. ∎

Note that we have $\dim X_n^D = n$. As before, we set $l = [(n+1)/2]$ and $l' = [n/2]$.

Definition 4.2: (1) We set

$$Y_n^D = \{v \in \mathfrak{S}_n^D \mid v \wedge F'^2(v) = 0\} \subset \mathfrak{S}_n^D \simeq X_n^D.$$

(2) Let $p_n^D \colon X_n^D \to X_l^D$ be the canonical projection. Then we put

$$\mathbf{X}_n^D = (p_n^D)^{-1}(Y_l^D).$$

Let

$$(x,y) = \left(\sum_{i=0}^{n-1} x_i \varpi^i, \sum_{i=0}^{n-2} y_{i+1} \varpi^i \right) \in \mathfrak{S}_n^D.$$

Explicitly, Y_n^D is defined by

$$x_0 \in k_2^\times, \quad x_i, y_i \in k_2 \quad \text{for } 1 \leq i \leq n-1.$$

Hence, this variety is 0-dimensional and consists of $q^{4(n-1)}(q^2-1)$ closed points. By Lemma 4.1 (2), the variety Y_n^D is stable under the action of $\mathcal{O}_{2n-1}^\times \times T_n^F$. It equals the image of ${G'}_n^{F'} \subset X_n^D$ by the isomorphism $X_n^D \xrightarrow{\sim} \mathfrak{S}_n^D$. Hence, the $\mathcal{O}_{2n-1}^\times$-action on it is simply transitive. We consider the surjective map

$$\nu_n^D : Y_n^D \to B_n^D = \mathfrak{O}_{n-1}; \quad (x,y) \mapsto y/x.$$

Let $\mathcal{O}_{2n-1}^\times$ act on B_n^D by

$$\text{(4.2)} \qquad a + b\varphi \colon B_n^D \to B_n^D; \quad t \mapsto \frac{F(a)t + F(b)}{\varpi b t + a}$$

for $a + \varphi b \in \mathcal{O}_{2n-1}^\times$, where a is regarded as an element of $\mathfrak{O}_{n-1}$ by the canonical map $\mathfrak{O}_n \to \mathfrak{O}_{n-1}$. Let T_n^F act on B_n^D trivially. Then ν_n^D is $\mathcal{O}_{2n-1}^\times \times T_n^F$-equivariant. For $t \in B_n^D$ we set

$$Y_{n,t}^D = (\nu_n^D)^{-1}(t) \subset Y_n^D.$$

The scheme $\mathbf{X}_n^D$ admits an action of $\mathcal{O}_{2n-1}^\times \times T_n^F$, because p_n^D is compatible with the canonical homomorphism $\mathcal{O}_{2n-1}^\times \times T_n^F \twoheadrightarrow \mathcal{O}_{2l-1}^\times \times T_l^F$ and Y_l^D is stable under the action of $\mathcal{O}_{2l-1}^\times \times T_l^F$. Let

$$(x,y) = \left(\sum_{i=0}^{n-1} x_i \varpi^i, \sum_{i=0}^{n-2} y_{i+1} \varpi^i \right) \in V_n.$$

The variety $\mathbf{X}_n^D$ is defined by

$$\text{(4.3)} \qquad \begin{gathered} \sum_{j=0}^{i} x_j^q x_{i-j} - \sum_{j=1}^{i} y_j^q y_{i+1-j} \in k \quad \text{for } 1 \leq i \leq n-1, \\ x_0 \in k_2^\times, \quad x_i, y_i \in k_2 \quad \text{for } 1 \leq i \leq l-1. \end{gathered}$$

We put

$$\text{(4.4)} \qquad s_i = \sum_{j=0}^{[(i-1)/2]} x_j^q x_{i-j} - \sum_{j=1}^{[i/2]} y_j^q y_{i+1-j} \quad \text{for } l' \leq i \leq n-1.$$

Let

$$I = \{i \in \mathbb{Z} \mid l \le i \le 2(l-1)\}.$$

By (4.3), for $l' \le i \le n-1$, we can check that

$$s_i^q + s_i = \begin{cases} \sum_{j=0}^{i} x_j^q x_{i-j} - \sum_{j=1}^{i} y_j^q y_{i+1-j} - x_{i/2}^{q+1} & \text{if } i \text{ is even,} \\ \sum_{j=0}^{i} x_j^q x_{i-j} - \sum_{j=1}^{i} y_j^q y_{i+1-j} + y_{(i+1)/2}^{q+1} & \text{if } i \text{ is odd.} \end{cases} \tag{4.5}$$

Hence we have $s_i \in k_2$ for all $i \in I$ by (4.3). We set

$$\Delta_{2,n} = Y_l^D \times k_2^I. \tag{4.6}$$

We obtain the surjective map

$$\mathbf{p}_n^D \colon \mathbf{X}_n^D \to \Delta_{2,n}; \quad x \mapsto (p_n^D(x), (s_i(x))_{i \in I}).$$

We can check that $\Delta_{2,n}$ admits the action of $\mathcal{O}_{2n-1}^\times \times T_n^F$ such that $\mathbf{p}_n^D$ is $\mathcal{O}_{2n-1}^\times \times T_n^F$-equivariant. We set

$$Z_{P,s}^D = (\mathbf{p}_n^D)^{-1}(P,s) \quad \text{for } (P,s) \in \Delta_{2,n}.$$

Lemma 4.3: *We have*

$$\mathbf{X}_n^D = \coprod_{(P,s) \in \Delta_{2,n}} Z_{P,s}^D$$

and an isomorphism

$$Z_{P,s}^D \simeq \begin{cases} \mathbb{A}^{l-1} \times Z_0 & \text{if } n \text{ is even,} \\ \mathbb{A}^{l-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We prove only the case where n is even. We have $l = l'$ and $n = 2l$. By (4.5) we have

$$s_{2l-1}^q + s_{2l-1} - y_l^{q+1} \in k.$$

By setting

$$X = \frac{s_{2l-1}}{x_0^{q+1}}, \quad Y = \frac{y_l}{x_0}, \tag{4.7}$$

we have $X^q + X - Y^{q+1} \in k$. By (4.4) and (4.7), there exists an upper matrix $A_{P,s} \in \mathrm{M}_{l-1}(k_2)$ and $\mathbf{a}_{P,s} \in k_2^{l-1}$ such that

$$(x_l, \ldots, x_{2l-2}) = (Y, y_{l+1}, \ldots, y_{2l-2}) A_{P,s} + \mathbf{a}_{P,s}. \tag{4.8}$$

By (4.7) and (4.8) there exists a vector $(a_{l+1},\dots,a_{2l-2},b_1,b_2,c)\in k_2^{l+1}$ such that

$$x_{2l-1}=\sum_{i=l+1}^{2l-2}a_iy_i+b_1X+b_2Y+c. \tag{4.9}$$

By (4.7), (4.8) and (4.9), we know that the morphism

$$Z_{P,s}\to\mathbb{A}^{l-1}\times Z_0;\quad \left(\sum_{i=0}^{2l-1}x_i\varpi^i,\sum_{i=0}^{2l-2}y_{i+1}\varpi^i\right)\mapsto((y_i)_{l+1\le i\le 2l-1},(X,Y))$$

is an isomorphism. Hence the required assertion follows. ∎

Remark 4.4: Compare Lemma 4.3 with Lemma 3.5. For varieties X and Y over k^{ac}, we write $X\sim Y$ if $X\simeq Y\times\mathbb{A}^i$ with some non-negative integer i. Let $n>1$ be an integer. By the lemmas we have

$$Z_{P,s}\sim\begin{cases}Z_0 & \text{if } n \text{ is odd,}\\ \operatorname{Spec}k^{\mathrm{ac}} & \text{if } n \text{ is even,}\end{cases}\quad\text{for } (P,s)\in\Delta_{1,n},$$

$$Z^D_{P,s}\sim\begin{cases}Z_0 & \text{if } n \text{ is even,}\\ \operatorname{Spec}k^{\mathrm{ac}} & \text{if } n \text{ is odd,}\end{cases}\quad\text{for } (P,s)\in\Delta_{2,n}.$$

This is asymmetric with respect to the parity of n. This causes the asymmetry mentioned in [BH, §54.8].

For $t\in B^D_n$, we put

$$\Delta^t_{2,n}=Y^D_{l,t}\times k_2^{l-1}\subset\Delta_{2,n},$$
$$\mathbf{X}^{D,t}_n=(\mathbf{p}^D_n)^{-1}(\Delta^t_{2,n})\subset\mathbf{X}^D_n.$$

Let

$$X^D_{P,s}=\begin{cases}\operatorname{Spec}k^{\mathrm{ac}} & \text{if } n \text{ is odd,}\\ Z_0 & \text{if } n \text{ is even,}\end{cases}$$
$$X(\Delta_{2,n})=\coprod_{\delta_2\in\Delta_{2,n}}X^D_{P,s}. \tag{4.10}$$

By Lemma 4.3 we have the projections

$$Z^D_{P,s}\to X^D_{P,s},$$
$$\mathbf{X}^D_n\to X(\Delta_{2,n}). \tag{4.11}$$

We can check that $X(\Delta_{2,n})$ admits the action of $\mathcal{O}_{2n-1}^{\times} \times T_n^F$ such that (4.11) is $\mathcal{O}_{2n-1}^{\times} \times T_n^F$-equivariant. We set $d_2 = \dim X(\Delta_{2,n})$. Since (4.11) is an affine bundle of relative dimension $l-1$, we have an isomorphism

$$H_c^{n-1}(\mathbf{X}_n^D) \simeq H_c^{d_2}(X(\Delta_{2,n}))$$

as $\mathcal{O}_{2n-1}^{\times} \times T_n^F$-representations.

4.2. Group action on $\mathbf{X}_n^D$. We study group action on $\mathbf{X}_n^D$ similarly as in §3.2. Let $\mathcal{O}_{2n-1}^{\times} \times T_n^F$ act on Y_l^D and B_l^D through the canonical homomorphism $\mathcal{O}_{2n-1}^{\times} \times T_n^F \to \mathcal{O}_{2l-1}^{\times} \times T_l^F$.

Lemma 4.5: *The action of $\mathcal{O}_{2n-1}^{\times}$ on B_l^D is transitive. The stabilizer of $0 \in B_l^D$ in $\mathcal{O}_{2n-1}^{\times}$ equals $\mathfrak{O}_n^{\times} U_D^{2l-1}$.*

Proof. The group $\mathcal{O}_{2l-1}^{\times}$ acts on Y_l^D transitively. Since ν_l^D is an $\mathcal{O}_{2l-1}^{\times}$-equivariant surjective map, $\mathcal{O}_{2l-1}^{\times}$ acts on B_l^D transitively. By (4.2), we can know the stabilizer of 0. ∎

Since T_n^F acts on B_l^D trivially, the stabilizer of $\Delta_{2,n}^0$ in $\mathcal{O}_{2n-1}^{\times} \times T_n^F$ equals $\mathfrak{O}_n^{\times} U_D^{2l-1} \times T_n^F$.

Lemma 4.6: *The action of T_n^F on $\Delta_{2,n}^0$ is transitive. For $(P, s) \in \Delta_{2,n}^0$, its stabilizer in T_n^F equals $U_{K_{2,n}}^{2l-1}$.*

Proof. The group T_l^F acts on $Y_{l,0}^D$ transitively. Hence, to prove the first assertion, it suffices to show that, for each $P \in Y_{l,0}^D$, the subgroup $U_{K_{2,n}}^l \subset T_n^F$ acts on the subset $k_{2,P}^I = \{P\} \times k_2^I \subset \Delta_{2,n}^0$ transitively (cf. the proof of Lemma 3.7). Let $P \in Y_{l,0}^D$ and $t \in U_{K_{2,n}}^l$. We put

$$t = 1 + \sum_{i=l}^{n-1} a_i \varpi^i \quad \text{with } a_i \in k_2,$$
$$s = (s_i)_{i \in I}, \quad a = (a_i)_{i \in I} \in k_2^I.$$

We can check that there exists an upper triangular matrix $B_P \in \mathrm{GL}_{l-1}(k_2)$ such that t acts on $k_{2,P}^I$ by

$$k_{2,P}^I \to k_{2,P}^I; \quad (P, s) \mapsto (P, s + aB_P). \tag{4.12}$$

Hence $U_{K_{2,n}}^l$ acts on $k_{2,P}^I$ transitively. Therefore the first assertion follows. If t stabilizes $(P, s) \in \Delta_{2,n}^0$, we have $a = 0$ by (4.12). Hence the latter assertion follows. ∎

By definition, we have

$$y_i = 0 \quad \text{for } 1 \le i \le l-1 \tag{4.13}$$

on $\mathbf{X}_n^{D,0}$.

LEMMA 4.7: (1) *The action of the subgroup* $\mathfrak{O}_n^\times$ *in* $\mathcal{O}_{2n-1}^\times$ *on* $\Delta_{2,n}^0$ *equals the one of* T_n^F.

(2) *Assume that* n *is even. For* $\alpha \in \mathfrak{O}_n^\times$ *and* $(P,s) \in \Delta_{2,n}^0$ *we have the commutative diagram*

$$\begin{array}{ccc} Z_{P,s}^D & \xrightarrow{(\alpha,\alpha^{-1})} & Z_{P,s}^D \\ \downarrow & & \downarrow \\ X_{P,s}^D & \xrightarrow{g(\bar{\alpha},0,0)} & X_{P,s}^D. \end{array}$$

Proof. We simply write α' for $(\alpha,\alpha^{-1}) \in \mathfrak{O}_n^\times \times T_n^F$. We have

$$\alpha'^* x = x, \quad \alpha'^* y = (F(\alpha)/\alpha) y. \tag{4.14}$$

By this, $yF(y)$ is fixed by the action of α'. Hence s_i for $i \in I$ is so. Hence the first assertion follows.

We prove the second assertion. We assume that n is even. By the above argument s_{2l-1} is also fixed by α'. By (4.13) and (4.14) we have $\alpha'^* y_l = \bar{\alpha}^{q-1} y_l$, and hence

$$\alpha'^* X = X, \quad \alpha'^* Y = \bar{\alpha}^{q-1} Y$$

by (4.7). Hence the required assertion follows. ∎

Let $H_{2,n}^0 \subset H_{2,n}$ be as in §2.2. Explicitly, we have

$$H_{2,n}^0 = 1 + \mathfrak{p}_{K_2}^n + \mathfrak{p}_{K_2}^{l'} \mathfrak{C}_2 \subset H_{2,n} = 1 + \mathfrak{p}_{K_2}^{n-1} + \mathfrak{p}_{K_2}^{l-1} \mathfrak{C}_2 \subset U_D^{2l-1}.$$

LEMMA 4.8: *Let* $(P,s) \in \Delta_{2,n}^0$.

(1) *The stabilizer of* (P,s) *in* $\mathcal{O}_{2n-1}^\times$ *equals*

$$\begin{cases} H_{2,n}^0 & \text{if } n \text{ is odd,} \\ H_{2,n} & \text{if } n \text{ is even.} \end{cases}$$

(2) *Assume that* n *is even. The group* $H_{2,n}$ *acts on* $Z_{P,s}^D$ *factoring through* $H_{2,n} \to H_{2,n}/H_{2,n}^0$. *Let* $H_{2,n}/H_{2,n}^0$ *act on* $X_{P,s}^D = Z_0$ *through the*

isomorphism $\phi_2\colon H_{2,n}/H_{2,n}^0 \simeq Q_0$ *in* (2.14). *For each* $d \in H_{2,n}/H_{2,n}^0$, *we have the commutative diagram*

$$\begin{array}{ccc} Z_{P,s}^D & \xrightarrow{\ d\ } & Z_{P,s}^D \\ \downarrow & & \downarrow \\ X_{P,s}^D & \xrightarrow{\ \phi_2(d)\ } & X_{P,s}^D. \end{array}$$

Proof. Let $(x,y) \in X_{P,s}^D$ and

$$d = 1 + \varpi^l a + \varphi^{2l-1} b \in U_D^{2l-1} \quad \text{with } a, b \in \mathfrak{O}.$$

We have

$$d^* x = x + \varpi^l (ax + by), \quad d^* y = y + \varpi^{l-1}(F(b)x + \varpi F(a) y). \tag{4.15}$$

For $i \in I$, by (4.13), we have $s_i = \sum_{j=0}^{[(i-1)/2]} x_j^q x_{i-j}$. We set

$$a = \sum_{i=0}^{\infty} a_i \varpi^i \in \mathfrak{O}, \quad \mathbf{s} = (s_i)_{i \in I}, \quad \mathbf{a} = (a_i)_{i \in I} \in k_2^I.$$

Then, by (4.13) and (4.15), there exists an upper triangular matrix $A_P \in \mathrm{GL}_{l-1}(k_2)$ such that the action of d on $k_{2,P}^I$ is given by

$$k_{2,P}^I \to k_{2,P}^I; \quad (P, \mathbf{s}) \mapsto (P, \mathbf{s} + \mathbf{a} A_P). \tag{4.16}$$

Assume that $d \in \mathcal{O}_{2n-1}^\times$ stabilizes (P, s). Since d stabilizes P we have $d \in U_D^{2l-1}$. By (4.16), we must have $\mathbf{a} = 0$. Hence, we obtain the first assertion.

We prove the second assertion. Assume that n is even. Let

$$d = 1 + \varpi^{2l-1} a + \varphi^{2l-1} b \in H_{2,n} \quad \text{with } a = \sum_{i=0}^{\infty} a_i \varpi^i, \quad b = \sum_{i=0}^{\infty} b_i \varpi^i \in \mathfrak{O}$$

and $(x,y) \in Z_{P,s}^D$. By (4.15), we have

$$d^* x_i = x_i \quad \text{for } l \le i \le 2l-2,$$
$$d^* x_{2l-1} = x_{2l-1} + b_0 y_l + a_0 x_0, \quad d^* y_l = y_l + b_0^q x_0.$$

Note that $s_{n-1} = \sum_{i=0}^{l-1} x_i^q x_{2l-1-i}$. Therefore, by (4.7), we have

$$d^* X = X + b_0 Y + a_0, \quad d^* Y = Y + b_0^q.$$

Hence the required assertion follows. ∎

Corollary 4.9: *The action of* $\mathcal{O}_{2n-1}^\times$ *on* $\Delta_{2,n}$ *is transitive.*

Proof. We take an element $\delta_2 \in \Delta^0_{2,n}$. Assume that n is even. By Lemma 4.8 (1) we have the injective map

$$H_{2,n}\backslash\mathcal{O}^{\times}_{2n-1} \hookrightarrow \Delta_{2,n}; \quad H_{2,n}d \mapsto \delta_2 d. \tag{4.17}$$

By

$$\begin{aligned} |H_{2,n}\backslash\mathcal{O}^{\times}_{2n-1}| =& |\mathcal{O}^{\times}_{2l'-1}||U_D^{2l'-1}/H_{2,n}| \\ =& |\mathcal{O}^{\times}_{2l'-1}||\mathfrak{O}_{l'-1}| \\ =& q^{3(n-2)}(q^2-1) = |\Delta_{2,n}| \end{aligned}$$

the map (4.17) is surjective. Hence we obtain the claim.

Assume that n is odd. By Lemma 4.8 (1) and

$$|H^0_{2,n}\backslash\mathcal{O}^{\times}_{2n-1}| = |\Delta_{2,n}| = q^{3(n-1)}(q^2-1)$$

we obtain the claim in the same way as above. ∎

4.3. Cohomology of $\mathbf{X}^D_n$. In the sequel, we describe characters of the abelian subgroup $U^n_D \subset \mathcal{O}^{\times}_{2n-1}$ similarly as in the end of §3.3. We fix an isomorphism $\mathcal{O}_{n-1} \xrightarrow{\sim} U^n_D$; $x \mapsto 1+\varphi^n x$. Fix a non-trivial additive character $\psi\colon \mathfrak{o} \to \overline{\mathbb{Q}}_\ell^{\times}$ of conductor exponent n. Let $\mathrm{Trd}_{D/K}\colon D \to K$ be the reduced trace map. For any $\beta \in \mathcal{O}_{n-1}$, let

$$\psi^D_\beta\colon U^n_D \to \overline{\mathbb{Q}}_\ell^{\times}; \quad x \mapsto \psi(\mathrm{Trd}_{D/K}(\beta(x-1))).$$

We have the isomorphism $\kappa\colon \mathcal{O}_{n-1} \simeq \mathrm{Hom}(U^n_D, \overline{\mathbb{Q}}_\ell^{\times})$; $\beta \mapsto \psi^D_\beta$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{n-1} & \xrightarrow[\sim]{\kappa} & \mathrm{Hom}(U^n_D, \overline{\mathbb{Q}}_\ell^{\times}) \\ \uparrow & & \downarrow \\ \mathfrak{O}_{l'} & \xrightarrow{\simeq} & \mathrm{Hom}(U^l_{K_2}, \overline{\mathbb{Q}}_\ell^{\times}), \end{array}$$

where the right vertical arrow is induced by the inclusion $U^l_{K_2} \hookrightarrow U^n_D$. Let $\omega \in (T^F_n)^{\vee}$. We write ψ_β with some $\beta \in \mathfrak{O}_{l'}$ for the restriction $\omega|_{U^l_{K_2}}$. Then we obtain a character ψ^D_β of U^n_D. We define a character ω of $\mathfrak{O}^{\times}_n U^n_D$ by

$$\sigma^D_\omega(xu) = \omega(x)\psi^D_\beta(u) \tag{4.18}$$

for $x \in \mathfrak{O}^{\times}_n$ and $u \in U^n_D$. See also [BF, (6.5.2)].

For each $t \in B_n^D$ we set

$$W_t^D = H_c^{n-1}(\mathbf{X}_n^{D,t}) \subset H_c^{n-1}(\mathbf{X}_n^D).$$

First, we consider the case where n is odd. In the same way as (3.48), by using Lemma 4.6 and Lemma 4.7 (1), we have an isomorphism

$$W_0^D|_{\mathfrak{O}_n^\times U_D^n \times T_n^F} \simeq \bigoplus_{\omega \in (T_n^F)^\vee} \sigma_\omega^D \otimes \omega$$

as $\mathfrak{O}_n^\times U_D^n \times T_n^F$-representations, where ω is the character of $\mathfrak{O}_n^\times U_D^n$ in (4.18). In the same way as (3.49), by Lemma 4.5, we obtain an isomorphism

$$H_c^{n-1}(\mathbf{X}_n^D) \simeq \bigoplus_{\omega \in (T_n^F)^\vee} (\mathrm{Ind}_{\mathfrak{O}_n^\times U_D^n}^{\mathcal{O}_{2n-1}^\times} \sigma_\omega^D) \otimes \omega \tag{4.19}$$

as $\mathcal{O}_{2n-1}^\times \times T_n^F$-representations.

Secondly, we consider the case where n is even. In the sequel we analyze the cohomology $H_c^{n-1}(\mathbf{X}_n^D)$. We have an isomorphism

$$H_c^{n-1}(\mathbf{X}_n^D) \simeq \bigoplus_{(P,s) \in \Delta_{2,n}} H_c^1(X_{P,s}^D).$$

We have $\dim H_c^{n-1}(\mathbf{X}_n^D) = q^{3n-4}(q-1)(q^2-1)$.

By Lemmas 4.6 and 4.8 (2) we have an isomorphism

$$W_0^D|_{\{1\} \times T_n^F} \simeq \bigoplus_{\omega \in (T_n^F)^\vee_{\mathrm{stp}}} \omega^{\oplus q}.$$

Hence, by Lemma 4.7, we obtain

$$W_0^D|_{\mathfrak{O}_n^\times \times T_n^F} \simeq \bigoplus_{\omega \in (T_n^F)^\vee_{\mathrm{stp}}} \bigoplus_{\chi \in \mu_{q+1}^\vee \setminus \{1\}} \chi\omega \otimes \omega$$

as $\mathfrak{O}_n^\times \times T_n^F$-representations. Here, $\chi \in \mu_{q+1}^\vee$ is regarded as a character of $\mathfrak{O}_n^\times$ by $\mathfrak{O}_n^\times \to \mu_{q+1}$; $a \mapsto \bar{a}^{q-1}$.

For a strongly primitive character ω we set

$$\widetilde{\sigma}_\omega^D = \mathrm{Hom}_{T_n^F}(\omega, W_0^D).$$

Lemma 4.10: *The representation $\widetilde{\sigma}_\omega^D$ is irreducible and satisfies*

- $\widetilde{\sigma}_\omega^D|_{U_{K_2,n}^1 U_D^n}$ *is a q-multiple of the character* $\sigma_\omega^D|_{U_{K_2,n}^1 U_D^n}$ *and*
- $\mathrm{Tr}\, \widetilde{\sigma}_\omega^D(\zeta) = -\omega(\zeta)$ *for* $\zeta \in k_2 \setminus k$.

Proof. The required assertion is proved in the same way as Lemma 3.16. ∎

Remark 4.11: See [BF, §9] or [BH, Lemma 2 in §54.6 and §54.8] on $\widetilde{\sigma}_\omega^D$.

We have

$$W_0^D \simeq \bigoplus_{\omega\in(T_n^F)^\vee_{\mathrm{stp}}} \widetilde{\sigma}_\omega^D \otimes \omega$$

as $\mathfrak{O}_n^\times U_D^{n-1} \times T_n^F$-representations. By Lemma 4.5, we obtain

$$\begin{aligned} H_{\mathrm{c}}^{n-1}(\mathbf{X}_n^D) &\simeq \mathrm{Ind}_{\mathfrak{O}_n^\times U_D^{n-1}\times T_n^F}^{\mathcal{O}_{2n-1}^\times\times T_n^F} W_0^D \\ &\simeq \bigoplus_{\omega\in(T_n^F)^\vee_{\mathrm{stp}}} (\mathrm{Ind}_{\mathfrak{O}_n^\times U_D^{n-1}}^{\mathcal{O}_{2n-1}^\times} \widetilde{\sigma}_\omega^D) \otimes \omega \end{aligned} \tag{4.20}$$

as $\mathcal{O}_{2n-1}^\times \times T_n^F$-representations.

We set

$$\rho_\omega^D = \begin{cases} \mathrm{Ind}_{\mathfrak{O}_n^\times U_D^{n-1}}^{\mathcal{O}_{2n-1}^\times} \widetilde{\sigma}_\omega^D & \text{if } n \text{ is even,} \\ \mathrm{Ind}_{\mathfrak{O}_n^\times U_D^{n}}^{\mathcal{O}_{2n-1}^\times} \sigma_\omega^D & \text{if } n \text{ is odd.} \end{cases} \tag{4.21}$$

We have $\dim \rho_\omega^D = q^{n-1}$.

PROPOSITION 4.12: *Let $n \geq 1$ be a positive integer. We have an isomorphism*

$$H_{\mathrm{c}}^{n-1}(\mathbf{X}_n^D)_{\mathrm{stp}} \simeq \bigoplus_{w\in(T_n^F)^\vee_{\mathrm{stp}}} \rho_\omega^D \otimes \omega$$

as $\mathcal{O}_{2n-1}^\times \times T_n^F$-representations.

Proof. The required assertion follows from (4.19) and (4.20). ∎

Remark 4.13: Similarly as in Remark 3.19, we note that

$$H_{\mathrm{c}}^{n-1}(\mathbf{X}_n^D)_{\mathrm{stp}} = H_{\mathrm{c}}^{n-1}(\mathbf{X}_n^D)$$

when n is even.

In the lemma below, we check that ρ_ω^D is irreducible by formal arguments on the basis of known results. As a result, we know that the isomorphism in Proposition 4.12 gives an irreducible decomposition of $H_{\mathrm{c}}^{n-1}(\mathbf{X}_n^D)_{\mathrm{stp}}$ as an $\mathcal{O}_{2n-1}^\times \times T_n^F$-representation.

LEMMA 4.14: *The representation ρ_ω^D is irreducible.*

Proof. We have the surjective homomorphism

$$g\colon K_2^\times \mathcal{O}_D^\times = K^\times \mathcal{O}_D^\times \to \mathcal{O}_{2n-1}^\times; \quad \varpi^m x \mapsto \bar{x} \quad \text{with } x \in \mathcal{O}_D^\times,$$

where $\bar{x}$ denotes the image of x by $\mathcal{O}_D^\times \to \mathcal{O}_{2n-1}^\times$. We consider the commutative diagram

$$\begin{array}{ccc} K_2^\times \mathcal{O}_D^\times & \xrightarrow{\ g\ } & \mathcal{O}_{2n-1}^\times \\ \uparrow & & \uparrow \\ K_2^\times U_D^n & \xrightarrow{\ g'\ } & \mathfrak{O}_n^\times U_D^n, \end{array}$$

where g' is the restriction of g to $K_2^\times U_D^n$. Assume that n is even. Let $\widetilde{\sigma}'^D_\omega$ be the inflation of $\widetilde{\sigma}^D_\omega$ by g'. It is known that $\mathrm{Ind}_{K_2^\times U_D^n}^{D^\times} \widetilde{\sigma}'^D_\omega$ is irreducible by [BH, Proposition (1) in §54.4]. We set $\widetilde{\rho}' = \mathrm{Ind}_{K_2^\times U_D^n}^{K_2^\times \mathcal{O}_D^\times} \widetilde{\sigma}'^D_\omega$. Since $\widetilde{\rho}'$ is semisimple, this is irreducible. Let $\widetilde{\rho}^D_\omega$ be the inflation of ρ^D_ω by g. By the Frobenius reciprocity, we have

$$\mathrm{Hom}_{K_2^\times \mathcal{O}_D^\times}(\widetilde{\rho}', \widetilde{\rho}^D_\omega) \simeq \mathrm{Hom}_{\mathcal{O}_{2n-1}^\times}(\rho^D_\omega, \rho^D_\omega) \neq 0.$$

Since $\widetilde{\rho}'$ is irreducible, we have an injective $K_2^\times \mathcal{O}_D^\times$-equivariant homomorphism $\widetilde{\rho}' \to \widetilde{\rho}^D_\omega$. Since both sides have the same dimension, this is an isomorphism. Hence $\widetilde{\rho}^D_\omega$ is irreducible and ρ^D_ω is so. Also in the case where n is odd, we can show that ρ^D_ω is irreducible in the same manner. ∎

5. Conjecture on stable reduction of Lubin–Tate curve

Let $\mathbf{X}(\mathfrak{p}^n)$ be the Lubin–Tate curve with Drinfeld level $\mathfrak{p}^n$-structures. In this section, we state a conjecture on "unramified components" in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$. See Introduction for these components. The cohomology of these components is related to cupspidal representations of $\mathrm{GL}_2(K)$ which are constructed from admissible pairs $(K_2/K, \xi)$, where ξ is some smooth character of $K_2^\times$, in the sense of [BH, Theorem in §20.2]. These cuspidal representations are called unramified in [BH, §20.1]; which we recall the definition in §5.2. In this sense, we call these irreducible components unramified. To state a conjecture, we construct a curve based on $X(\Delta_{1,n})$ and $X(\Delta_{2,n})$ in §5.1. The curve is very similar to a stable curve considered in [W1].

5.1. Construction of curve. Let $n \geq 1$ be a positive integer. We set $l = [(n+1)/2]$ and $l' = [n/2]$ as before. Recall that we set

$$\Delta_{1,n} = Y_{l'} \times k_2^{l'}, \quad \Delta_{2,n} = Y_l^D \times k_2^{l-1}$$

in (3.15) and (4.6) respectively. We set

$$\mathbf{G}_n = G_n^F \times \mathcal{O}_{2n-1}^\times.$$

Let $X(\Delta_{1,n})$ and $X(\Delta_{2,n})$ be as in (3.23) and (4.10) respectively. We write $T_{1,n}^F$ (resp. $T_{2,n}^F$) for T_n^F acting on $X(\Delta_{1,n})$ in §3 (resp. $X(\Delta_{2,n})$ in §4). Note that $T_{1,n}^F$ and $T_{2,n}^F$ are the same group (cf. (3.2) and (4.1)).

We consider the product $X(\Delta_{1,n}) \times X(\Delta_{2,n})$ having the action of

$$\mathbf{G}_n \times T_{1,n}^F \times T_{2,n}^F.$$

Let $\Delta\colon \mathfrak{O}_n^\times \hookrightarrow T_{1,n}^F \times T_{2,n}^F$ be the anti-diagonal map defined by $t \mapsto (t, t^{-1})$ for $t \in \mathfrak{O}_n^\times$. Let

$$Y_n = (X(\Delta_{1,n}) \times X(\Delta_{2,n}))/\Delta(\mathfrak{O}_n^\times).$$

Let $\mathbb{X}_n = (\mathbf{X}_n \times \mathbf{X}_n^D)/\Delta(\mathfrak{O}_n^\times)$ be as in Introduction. Then, as mentioned there, the projection $\mathbb{X}_n \to Y_n$ is an affine bundle. Let $\mathfrak{O}_n^\times$ act on Y_n as $(t,1) \in T_{1,n}^F \times T_{2,n}^F$ for $t \in \mathfrak{O}_n^\times$. Then the curve Y_n admits the action of $\mathbf{G}_n \times \mathfrak{O}_n^\times$. We consider the quotient

$$\boldsymbol{\Delta}_n = (\Delta_{1,n} \times \Delta_{2,n})/\Delta(\mathfrak{O}_n^\times).$$

The action of $\Delta(\mathfrak{O}_n^\times)$ on $\Delta_{1,n} \times \Delta_{2,n}$ is free by Lemmas 3.7 and 4.6, because of $\max\{2l', 2l-1\} \geq n$. Hence we have

$$|\boldsymbol{\Delta}_n| = \begin{cases} 1 & \text{if } n = 1, \\ q^{4n-7}(q-1)(q^2-1) & \text{if } n \geq 2. \end{cases} \tag{5.1}$$

Specifically, Y_n is a disjoint union of $|\boldsymbol{\Delta}_n|$ copies of the curve

$$Z_n = \begin{cases} Z_{\mathrm{DL}} & \text{if } n = 1, \\ Z_0 & \text{if } n \geq 2. \end{cases}$$

The action of $\mathbf{G}_n$ on $\boldsymbol{\Delta}_n$ is transitive by Corollaries 3.10 and 4.9. We take an element $\zeta \in k_2 \setminus k$. Let

- $\delta_1 = (P, s) \in \Delta_{1,n}^\zeta$ such that $t_{l',0}(P,s) = 0$; see (3.36), and
- $\delta_2 \in \Delta_{2,n}^0$.

We write δ for the image of $(\delta_1,\delta_2)\in\Delta_{1,n}\times\Delta_{2,n}$ under the canonical map $\Delta_{1,n}\times\Delta_{2,n}\to\boldsymbol{\Delta}_n$. By Lemmas 3.9 (1) and 4.8 (1), the group $\overline{\mathcal{L}}^{\times}_{\zeta,n-1}$ stabilizes δ. Hence we have the surjective map

$$\overline{\mathcal{L}}^{\times}_{\zeta,n-1}\backslash\mathbf{G}_n\to\boldsymbol{\Delta}_n;\quad \overline{\mathcal{L}}^{\times}_{\zeta,n-1}g\mapsto\delta g.$$

This map is bijective, because of $|\overline{\mathcal{L}}^{\times}_{\zeta,n-1}\backslash\mathbf{G}_n|=|\boldsymbol{\Delta}_n|$ by (2.12) and (5.1). Hence the stabilizer of (δ_1,δ_2) in $\mathbf{G}_n$ equals $\overline{\mathcal{L}}^{\times}_{\zeta,n-1}$. Let $\overline{\mathcal{L}}^{\times}_{\zeta,n-1}$ act on Z_n through the homomorphism (2.11). Let Z_{δ_1,δ_2} be the open and closed subscheme in Y_n labeled by (δ_1,δ_2). By the property (c) in Lemma 3.9 (2) and Lemma 4.8 (2), we have an $\overline{\mathcal{L}}^{\times}_{\zeta,n-1}$-equivariant isomorphism $Z_n\simeq Z_{\delta_1,\delta_2}$. Since the stabilizer of Z_{δ_1,δ_2} in $\mathbf{G}_n$ is $\overline{\mathcal{L}}^{\times}_{\zeta,n-1}$, we have an isomorphism

$$Y_n=\coprod_{(\delta_1',\delta_2')\in\boldsymbol{\Delta}_n}Z_{\delta_1',\delta_2'}\simeq Z_n\times_{\overline{\mathcal{L}}^{\times}_{\zeta,n-1}}\mathbf{G}_n. \tag{5.2}$$

The right hand side of this is similar to Ind $\mathfrak{X}$ when E/F is an unramified quadratic extension in the notation of [W1, §5.1].

For a non-archimedean local field L, let W_L be the Weil group of L. Let $I_L\subset W_L$ be the inertia subgroup of L. Let $\mathbf{a}_L\colon W_L^{\mathrm{ab}}\xrightarrow{\sim}L^{\times}$ be the the Artin reciprocity map normalized such that a geometric Frobenius is sent to a prime element. Composing this with the canonical map $I_L^{\mathrm{ab}}\to W_L^{\mathrm{ab}}$ induces the surjective map $\mathbf{a}_L^0\colon I_L^{\mathrm{ab}}\twoheadrightarrow\mathcal{O}_L^{\times}$. For each $n\geq1$ we consider the composite

$$\mathbf{a}^0_{K_2,n}\colon I_K\simeq I_{K_2}\xrightarrow{\text{can.}}I_{K_2}^{\mathrm{ab}}\xrightarrow{\mathbf{a}^0_{K_2}}\mathfrak{O}^{\times}\xrightarrow{\text{can.}}\mathfrak{O}_n^{\times}.$$

We regard Y_n as a variety with $\mathbf{G}_n\times I_K$-action via the map

$$1\times\mathbf{a}^0_{K_2,n}\colon\mathbf{G}_n\times I_K\to\mathbf{G}_n\times\mathfrak{O}_n^{\times}.$$

Theorem 5.1: *Let $n\geq1$ be a positive integer. Let the notation be as in* (3.60) *and* (4.21).

(1) *We have an isomorphism*

$$H^1_{\mathrm{c}}(Y_n)\simeq\bigoplus_{\omega\in(\mathfrak{O}_n^{\times})^{\vee}_{\mathrm{stp}}}(\pi_\omega\otimes\rho^D_{\omega^{-1}})\otimes\omega^{-1}$$

as $\mathbf{G}_n\times I_K$-representations.

(2) *We have an isomorphism*

$$H^1_{\mathrm{c}}(Y_n)\simeq\mathrm{Ind}_{\overline{\mathcal{L}}^{\times}_{\zeta,n-1}}^{\mathbf{G}_n}H^1_{\mathrm{c}}(Z_0)$$

as $\mathbf{G}_n$-representations.

Proof. We show the first assertion. By Remarks 3.19 (1) and Remark 4.13, we have $H^1_{\mathrm{c}}(Y_n)_{\mathrm{stp}} = H^1_{\mathrm{c}}(Y_n)$. The claim follows from Proposition 3.18, Lemma 3.20 and Proposition 4.12.

The second assertion follows from (5.2). ∎

5.2. Conjecture. Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(K)$. We say that π is unramified if there exists a non-trivial unramified smooth character ϕ of $K^\times$ such that $\pi \otimes (\phi \circ \det) \simeq \pi$ (cf. [BH, §20.1]).

Let $\mathbf{X}(\mathfrak{p}^n)$ be the Lubin–Tate curve with Drinfeld level $\mathfrak{p}^n$-structures (cf. [Ca]). This is a rigid analytic curve over $\widetilde{K}$. Then, $\{\mathbf{X}(\mathfrak{p}^n)\}_{n=1}^\infty$ makes a projective limit. The wide open curve $\mathbf{X}(\mathfrak{p}^n)$ has a stable covering (cf. [CMc, Theorem 2.40]). We state a conjecture on unramified components in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)$, whose cohomology realizes the local Langlands correspondence and the local Jacquet–Langlands correspondence for unramified cuspidal representations of $\mathrm{GL}_2(K)$. For $1 \leq i \leq n$, let $p_{n,i} \colon \mathbf{X}(\mathfrak{p}^n) \to \mathbf{X}(\mathfrak{p}^i)$ be the projection. A morphism of affinoid rigid analytic varieties $f \colon \mathbf{X} \to \mathbf{Y}$ induces the morphism of affine schemes $\bar{f} \colon \overline{\mathbf{X}} \to \overline{\mathbf{Y}}$. Let $\mathbf{C}$ be as in Introduction. For a rigid analytic variety $\mathbf{X}$ over $\widetilde{F}$, let $\mathbf{X}_{\mathbf{C}}$ denote the base change of it to $\mathbf{C}$.

Conjecture 5.2: *For integers $n \geq 1$ and $1 \leq i \leq n$, there exist $\mathbf{G}_n$-stable affinoid subdomains $\mathbf{Y}_{n,i}$ in $\mathbf{X}(\mathfrak{p}^n)$ such that*

- $\mathbf{Y}_{n,i} \cap \mathbf{Y}_{n,j} = \emptyset$ *if $i \neq j$,*
- *there exists a $\mathbf{G}_n \times I_K$-equivariant isomorphism $\overline{\mathbf{Y}}_{n,n,\mathbf{C}} \simeq Y_n$,*
- $p_{n,i}(\mathbf{Y}_{n,i}) = \mathbf{Y}_{i,i}$, *and*
- *the map $\overline{p}_{n,i} \colon \overline{\mathbf{Y}}_{n,i,\mathbf{C}} \to \overline{\mathbf{Y}}_{i,i,\mathbf{C}}$ is a purely inseparable map compatible with $\mathbf{G}_n \times I_K \to \mathbf{G}_i \times I_K$.*

Remark 5.3: (1) If the conjecture is true, an isomorphism

$$H^1_{\mathrm{c}}(\overline{\mathbf{Y}}_{n,i,\mathbf{C}}) \simeq H^1_{\mathrm{c}}(Y_i)$$

as $\mathbf{G}_i \times I_K$-representations holds.

(2) If this conjecture is true, the curve Y_n actually appears as an open subscheme of a disjoint union of irreducible components of the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ by [IT, Proposition 7.11].

(3) In a representation theoretic viewpoint, $\overline{\mathbf{Y}}_{n,i}$ $(i < n)$ is less interesting than $\overline{\mathbf{Y}}_{n,n}$. However, these components $\overline{\mathbf{Y}}_{n,i}$ actually appear in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)_{\mathbf{C}}$ (cf. the stable reduction of $\mathbf{X}(\mathfrak{p}^2)_{\mathbf{C}}$ in [T1]). To give a more precise description of the stable reduction, we consider these $\overline{\mathbf{Y}}_{n,i}$ $(i < n)$ above.

Remark 5.4: For $n = 1$, this is a special case of [Y]. For general n, a family of affinoids is studied in [T2].

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