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CENTRAL POLYNOMIALS AND GROWTH FUNCTIONS

BY

Antonio Giambruno*

Dipartimento di Matematica e Informatica, Università di Palermo
Via Archirafi 34, 90123 Palermo, Italy
e-mail: antonio.giambruno@unipa.it

AND

Mikhail Zaicev**

Department of Algebra, Faculty of Mathematics and Mechanics
Moscow State University, Moscow, 119992 Russia
e-mail: zaicevmv@mail.ru

ABSTRACT

The growth of central polynomials for the algebra of $n \times n$ matrices in characterstic zero was studied by Regev in [13]. Here we study the growth of central polynomials for any finite-dimensional algebra over a field of characteristic zero. For such an algebra A we prove the existence of two limits called the central exponent and the proper central exponent of A. They give a measure of the exponential growth of the central polynomials and the proper central polynomials of A. We study the range of such limits and we compare them with the PI-exponent of the algebra.

1. Introduction

Throughout this paper all algebras will be associative and over an algebraically closed field F of characteristic zero. Let $F\langle X\rangle$ be the free associative algebra on a countable set X over F and let A be an F-algebra.

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Recall that a polynomial $f \in F\langle X\rangle$ is a central polynomial of A if for any $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) \in Z(A)$, the center of A. In case f takes only the zero value, f is a polynomial identity (PI) of A whereas if f takes a non-zero value in $Z(A)$, we say that f is a proper central polynomial.

Even if an algebra has a non-zero center, the existence of proper central polynomials is not granted. Nevertheless a famous conjecture of Kaplansky (see [8]) asserting that the algebra of $n \times n$ matrices has proper central polynomials was proved in the early 70's independently by Formanek and Razmyslov ([2], [10]).

Here we want to compare the growth of the spaces of central polynomials, proper central polynomials and polynomial identities of an algebra in the following sense.

Let $Id(A)$ be the T-ideal of polynomial identities of A and, following [13], we let $Id^z(A)$ be the space of central polynomials of A. Notice that $Id^z(A)$ is a T-space, i.e., a vector space invariant under all endomorphisms of $F\langle X\rangle$. Clearly $Id(A) \subseteq Id^z(A)$, and the proper central polynomials correspond to the quotient space $Id^z(A)/Id(A)$.

Regev in [13] introduced the notion of central codimensions as follows. Let P_n be the space of multilinear polynomials in $x_1, \ldots, x_n$ and set

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}, \quad P_n^z(A) = \frac{P_n}{P_n \cap Id^z(A)}.$$

The quotient space

$$\Delta_n(A) = \frac{P_n \cap Id^z(A)}{P_n \cap Id(A)}$$

corresponds to the space of proper central polynomials.

We write $c_n(A) = \dim P_n(A)$, $c_n^z(A) = \dim P_n^z(A)$ and $\delta_n(A) = \dim \Delta_n(A)$, respectively and it is easily seen that

$$c_n(A) = \delta_n(A) + c_n^z(A). \tag{1}$$

We call $c_n(A), c_n^z(A)$ and $\delta_n(A)$, $n = 1, 2, \ldots$, the sequences of codimensions, central codimensions and proper central codimensions, respectively.

It is well known that for any PI-algebra A the sequence $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded ([11]). Moreover, the limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

always exists and is a non-negative integer called the PI-exponent of A ([5]). Clearly from (1) it follows that if A is a PI-algebra, the sequences $c_n^z(A)$ and

$\delta_n(A)$, $n = 1, 2, \ldots$, are also exponentially bounded and it is worth asking if the corresponding limits

$$\exp^z(A) = \lim_{n\to\infty} \sqrt[n]{c_n^z(A)}, \quad \exp^\delta(A) = \lim_{n\to\infty} \sqrt[n]{\delta_n(A)} \tag{2}$$

exist.

Here we shall prove that for any finite-dimensional algebra A, the central exponent $\exp^z(A)$ and the proper central exponent $\exp^\delta(A)$ exist and are non-negative integers. Moreover, they are both the dimension of suitable subalgebras of A.

What about searching for all allowed values of the limits in (2)? It is well-known that $\exp(A)$ can be any positive integer. Here we show that for any finite-dimensional algebra A such that $\exp(A) \geq 2$, the central exponent $\exp^z(A)$ and the PI-exponent $\exp(A)$ coincide. Concerning the proper central exponent we prove that for any integer $N \geq 1$ there exists a finite-dimensional algebra A such that $\exp^\delta(A) \neq 0$ and $\exp(A) - \exp^\delta(A) > N$.

2. A general setting

Throughout, A is a finite-dimensional algebra over an algebraically closed field of characteristic zero. The spaces $P_n(A)$, $P_n^z(A)$ and $\Delta_n(A)$ become S_n-modules via the usual permutation action of the symmetric group S_n: if $f(x_1, \ldots, x_n) \in P_n$ and $\sigma \in S_n$, then $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The corresponding characters are denoted $\chi_n(A)$, $\chi_n^z(A)$ and $\chi_n(\Delta(A))$, respectively.

We decompose such characters into a sum of irreducibles:

$$\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda, \quad \chi_n^z(A) = \sum_{\lambda\vdash n} m'_\lambda \chi_\lambda, \quad \chi_n(\Delta(A)) = \sum_{\lambda\vdash n} m''_\lambda \chi_\lambda,$$

where χ_λ is the irreducible character of S_n corresponding to the partition λ of n and $m_\lambda, m'_\lambda, m''_\lambda$ are the multiplicities. Clearly $m_\lambda = m'_\lambda + m''_\lambda$, for all $\lambda \vdash n$ and we write

$$\chi_n(A) = \chi_n(\Delta(A)) + \chi_n^z(A). \tag{3}$$

A special algebra, whose properties we shall use in this paper, is the algebra of upper block triangular matrices $UT(d_1, \ldots, d_k)$. Recall that this is the following subalgebra of $M_{d_1+\cdots+d_k}(F)$

$$UT(d_1,\ldots,d_k)=\begin{pmatrix} A_1 & A_{12} & \cdots & A_{1k} \\ & A_2 & & \vdots \\ & & \ddots & A_{k-1k} \\ 0 & & & A_k \end{pmatrix},$$

where $A_i \cong M_{d_i}(F)$, $1 \le i \le k$, and $A_{ij} \cong M_{d_i \times d_j}(F)$, the space of $d_i \times d_j$ matrices over F, $1 \le i < j \le k$.

In the next lemma we shall prove that such an algebra has no proper central polynomials.

Lemma 1: *If $k > 1$, the algebra $A = UT(d_1,\ldots,d_k)$ has no proper central polynomials.*

Proof. Let f be a central polynomial of A and assume, as we may, that $f = f(x_1,\ldots,x_n)$ is a multilinear polynomial. Notice that the center of A is the set of scalar matrices.

Write

$$f = f_1x_1 + \cdots + f_nx_n,$$

where f_i is a polynomial in $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$, $1 \le i \le n$, and consider any evaluation $\varphi : F\langle X\rangle \to A$ such that $\varphi(x_1) \in A_{12}$ and $\varphi(x_i) \in A_1$, $2 \le i \le n$. Since $A_{12}A_1 = 0$, we get that $\varphi(f_2x_2) = \cdots = \varphi(f_nx_n) = 0$. Recalling that $\varphi(f_1x_1) \in A_{12}$ and f is a central polynomial of A, then we deduce that $\varphi(f_1x_1) = 0$. Since f_1 is evaluated in A_1 and x_1 in A_{12}, by making a suitable evaluation, we deduce that f_1 is an identity of A_1. It is clear that the above argument applied to the polynomials $f_2,\ldots,f_n$ says that they are all polynomial identities of A_1. Hence $f \in Id(A_1)$.

In a similar fashion we can prove that $f \in \bigcap_{i=1}^{k-1} Id(A_i)$. Now rewrite f as $f = x_1g_1 + \cdots + x_ng_n$, where for $i = 1,\ldots,n$, the variable x_i does not appear in g_i. By making an evaluation similar to the one above, we deduce that f is an identity of A_k.

We have proved that f is an identity of $A_1 \oplus \cdots \oplus A_k$. But then any non-zero evaluation of f takes values in $J(A)$, the Jacobson radical of A. Since f is a central polynomial we deduce that f is an identity of A and we are done. ∎

3. Computing the exponential growth

Let A be a finite-dimensional algebra over an algebraically closed field of characteristic zero. By the Wedderburn–Malcev theorem we can write $A = \bar{A} + J$, where $\bar{A} = A_1 \oplus \cdots \oplus A_m$ is a semisimple algebra with the A_i's simple algebras and $J = J(A)$ is the Jacobson radical of A.

It is clear that if $A = J$ is a nilpotent algebra, then $\delta_n(A) = 0$, for n large. Then we may assume that $m \geq 1$. In this case we make the following definition.

Definition 1: A semisimple subalgebra B of A is centrally admissible if $B = A_{i_1} \oplus \cdots \oplus A_{i_k}$ where $A_{i_1}, \ldots, A_{i_k} \in \{A_1, \ldots, A_m\}$ are distinct and there exists a central polynomial $f = f(x_1, \ldots, x_n)$ of A such that $\varphi(f) \neq 0$ for some evaluation φ with $\varphi(x_j) \in A_{i_j}$, $1 \leq j \leq k$.

Notice that even if A is not nilpotent and has proper central polynomials, centrally admissible subalgebras do not necessarily exist.

For instance, let $A = B \oplus R$ where B is the subalgebra of $M_3(F)$ consisting of all matrices whose third row is zero and $R = \frac{F\langle x_1, \ldots, x_N\rangle}{F\langle x_1, \ldots, x_N\rangle^{N+1}}$. Here $F\langle x_1, \ldots, x_N\rangle$ is the free associative algebra with N generators.

Notice that R is a free nilpotent algebra of index $N+1$ ($R^{N+1} = 0, R^N \neq 0$). Hence $ann_R(R) = \{a \in R \mid aR = Ra = 0\} = R^N = Z(R)$. It follows that any multilinear polynomial of degree N is not a polynomial identity of R and lies in $ann_R(R)$.

Since $Z(B) = 0$, then $Z(A) = Z(R) = R^N$. It follows that any multilinear polynomial identity of B of degree N is not a polynomial identity of R and lies in $R^N = Z(A)$. Thus A has proper central polynomials.

On the other hand, let $f(x_1, \ldots, x_k)$ be a multilinear central polynomial of A and at least one variable is evaluated in $M_2(F) \subset B$. If all the other variables are evaluated in B, then f vanishes in A since $Z(B) = 0$. If also at least one variable is evaluated in R, then f vanishes in A since $A = B \oplus R$. Thus A has no centrally admissible subalgebras.

Recall that for a partition $\lambda \vdash n$ we write $\chi_\lambda(1) = d_\lambda$ for the degree of the irreducible S_n-character corresponding to λ. Also, if $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ are two partitions, we write $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$, for all $i \geq 1$.

In the following lemma we assume that A has centrally admissible subalgebras.

LEMMA 2: *Let d be the maximal dimension of a centrally admissible subalgebra of A. Then $\delta_n(A) \le Cn^t d^n$, for some constants C, t.*

Proof. Consider the S_n-character $\chi_n(\Delta(A))$ and its decomposition into irreducibles

$$\chi_n(\Delta(A)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{4}$$

We claim that if $\lambda \vdash n$ is such that $m_\lambda \neq 0$, then the diagram of λ contains at most s boxes out of the first d rows, where d is the integer defined above, and s is such that $J^s \neq 0$ and $J^{s+1} = 0$.

In fact, let $M_\lambda \subseteq P_n \cap Id^z(A)$ be an irreducible S_n-module corresponding to λ such that $M_\lambda \not\subseteq P_n \cap Id(A)$. This says that there exists a proper central polynomial f and a tableau T_λ such that $M_\lambda = FS_n e_{T_\lambda} f \subseteq P_n \cap Id^z(A)$. The reader can find in [7] all the basic properties of the representation theory of the symmetric group needed here.

Recall that $e_{T_\lambda} = (\sum_{\sigma \in R_{T_\lambda}} \sigma)(\sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau)$ is an essential idempotent of the group algebra FS_n, where R_{T_λ} and C_{T_λ} are the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. Hence M_λ is also generated by $(\sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau) e_{T_\lambda} f$ and this implies that

$$g = \left(\sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau \right) e_{T_\lambda} f$$

is also a proper central polynomial of A. Notice that the polynomial g is alternating on distinct sets of variables corresponding to the columns of T_λ.

Let φ be a non-zero evaluation of g in A. Since g is multilinear, we can restrict to evaluations of g on a basis of A that is the union of bases of the simple components and the Jacobson radical. Since $\varphi(g)$ takes a non-zero central value in A, by the definition of d, every alternating set of variables of g can be evaluated in at most d basis elements of $\bar{A}$ and the remaining elements in the basis of J. Since the sets on which g is alternating correspond to the columns of T_λ, this says that the number of variables evaluated in J is greater than or equal to the number of boxes of λ out of the first d rows. Since $J^{s+1} = 0$, this says that there are at most s boxes out of the first d rows of λ. This proves the claim.

By (3) we have that $\chi_n(\Delta(A)) \le \chi_n(A)$, i.e., the multiplicity of every irreducible χ_λ appearing in $\chi_n(\Delta(A))$ is less than or equal to the multiplicity of χ_λ in $\chi_n(A)$. Since these last multiplicities are polynomially bounded ([1, Theorem 16]), we get that also the m_λ's in (4) are polynomially bounded and let $m_\lambda \le Cn^q$.

Recall that if $\lambda \subseteq \mu$ are two partitions such that $\lambda \vdash n$ and $\mu \vdash n+s$, then $d_\mu \le n^s d_\lambda$ (see [7, Lemma 6.2.4]).

Let us write $H(n,d) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{d+1} = 0\}$ for the set of partitions of n in at most d parts and recall that $\sum_{\lambda \in H(n,d)} d_\lambda \le C' n^t d^n$, for some constants C', t ([7, Lemma 6.2.5]).

We have

$$\begin{aligned}\delta_n(A) = &\sum_{\lambda \vdash n} m_\lambda d_\lambda \le Cn^q \sum_{\lambda \in H(n,d) \cup S} d_\lambda \\ \le &C'' n^{q+s} \sum_{\mu \in H(n,d)} d_\mu \\ \le &C_1 n^l d^n,\end{aligned}$$

for some constants C_1, l, where $H(n,d) \cup S$ is the set of partitions containing at most s boxes out of the first d rows. This proves the lemma. ∎

Now we can prove the main result of this section.

THEOREM 1: *Let A be a finite-dimensional algebra over an algebraically closed field of characteristic zero.*

1. *If A has no proper central polynomials or has no centrally admissible subalgebras, then $\delta_n(A) = 0$, for all n large enough.*
2. *If A has centrally admissible subalgebras, then*

$$C_1 n^{t_1} d^n \le \delta_n(A) \le C_2 n^{t_2} d^n,$$

for some constants $C_1 > 0, C_2, t_1, t_2$, where d is the maximal dimension of a centrally admissible subalgebra of A.

Proof. (1) If A has no proper central polynomials, then clearly $\delta_n(A) = 0$, for all $n \ge 1$.

Suppose that A has proper central polynomials but no centrally admissible subalgebras. Let s be such that $J^s = 0$, where J is the Jacobson radical of A. Clearly any proper central polynomial has non-zero evaluations only on elements of J. This implies that its degree is less than s and $\delta_n(A) = 0$ whenever $n \ge s$.

(2) Now suppose that A has centrally admissible subalgebras. In this case an upper bound for $\delta_n(A)$ is given in Lemma 2. Let B be a centrally admissible subalgebra of A of maximal dimension. We may clearly assume that $B = A_1 \oplus \cdots \oplus A_k$ and let $f = f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m})$ be a central polynomial of A such that $0 \neq f(a_1, \ldots, a_k, b_1, \ldots, b_m) \in Z(A)$, with $a_1 \in A_1, \ldots, a_k \in A_k$. Let $A_i = M_{d_i}(F)$ so that $\dim A_i = d_i^2$.

Let $G_t(w_1, \ldots, w_{t^2}, w'_1, \ldots, w'_{t^2})$ denote the Regev central polynomial for $M_t(F)$ which is alternating on $w_1, \ldots, w_{t^2}$ and on $w'_1, \ldots, w'_{t^2}$ ([3, Theorem 16]).

Starting with f we construct a sequence of polynomials f_t, $t = 1, 2, \ldots$, as follows. Set $f_1 = f$ and define

$$f_2 = f_2(z_{1,2}, \ldots, z_{k,2}, y_1, \ldots, y_m) = \mathrm{Alt}_1 \mathrm{Alt}_2 f_1(z_{1,2}, \ldots, z_{k,2}, y_1, \ldots, y_m),$$

where

$$\begin{aligned} z_{1,2} =& x_1 G_{d_1}(u_{1,1}, \ldots, u_{d_1^2,1}, v_{1,1}, \ldots, v_{d_1^2,1}) \\ &\vdots \\ z_{k,2} =& x_k G_{d_k}(u_{1,k}, \ldots, u_{d_k^2,k}, v_{1,k}, \ldots, v_{d_k^2,k}). \end{aligned}$$

Here Alt_1 is alternation on the set $\{u_{1,1}, \ldots, u_{d_1^2,1}, \ldots, u_{1,k}, \ldots, u_{d_k^2,k}\}$ and Alt_2 is alternation on the set $\{v_{1,1}, \ldots, v_{d_1^2,1}, \ldots, v_{1,k}, \ldots, v_{d_k^2,k}\}$.

Since f_1 is a central polynomial of A, also f_2 is central. Now, every G_{d_i} has an evaluation φ_i in $A_i = M_{d_i}(F)$ such that $\varphi_i(G_{d_i})$ is a non-zero scalar matrix. Moreover, if we exchange two variables belonging to distinct polynomials, say, G_{d_i} and G_{d_j}, the corresponding evaluation is zero since $A_i A_j = 0$ for $i \neq j$. It follows that we can extend the evaluations φ_i to an evaluation φ of f_2 in A, $\varphi(x_j) = a_j$, $1 \leq j \leq k$, $\varphi(y_l) = b_l$, $1 \leq l \leq m$, such that $\varphi(f_2) \neq 0$. Hence f_2 is a proper central polynomial of A. Notice that

$$\deg f_2 = 2 \sum_{i=1}^{k} d_i^2 + k + m = 2d + k + m.$$

By repeatedly applying this procedure we can construct a sequence of polynomial $f_2, f_3, \ldots$ with the following properties: for any $t \geq 2$, f_t is a proper central polynomial of A of degree $2(t-1)d + k + m$ depending on $x_1, \ldots, x_k, y_1, \ldots, y_m$ and $2(t-1)$ alternating sets of variables each of order d. Moreover, there exists a non-zero evaluation of f_t in A such that all variables of the $2(t-1)$ alternating sets take values in $A_1 \oplus \cdots \oplus A_k$.

Next we shall compute a lower bound of $\delta_n(A)$ when $n = 2td + k + m$, any $t \geq 1$. Now, since f_{t+1} is a proper central polynomial of A,

$$f_{t+1} \in P_n \cap Id^z(A) \setminus P_n \cap Id(A).$$

Let the symmetric group S_{2td} act on the variables of P_n belonging to the $2t$ sets on which f_{t+1} is alternating. We consider the S_{2td}-module generated by f_{t+1}. As in the proof of Lemma 2, it turns out that in the decomposition of the S_{2td}-character of $\Delta_n(A)$ into irreducibles, the characters $\chi_{((2t)^d)}$ appears with non-zero multiplicity. Hence $\delta_n(A) = \dim \Delta_n(A) \geq \deg \chi_{((2t)^d)}$.

An asymptotic estimate of $\deg \chi_{((2t)^d)}$ was done in [12]. For our purpose here we make the following computation.

Denote $n_0 = 2td$. Then by the hook formula giving the dimension of an irreducible representation of the symmetric group we have

$$\deg \chi_{((2t)^d)} > \frac{(2td)!}{((2t)!)^d} \cdot \frac{1}{n_0^{d^2}}.$$

Noticing that $\frac{(2td)!}{((2t)!)^d}$ is a generalized binomial coefficient and $n = 2td + k + m$, we get

$$\frac{(2td)!}{((2t)!)^d} > \frac{d^{n_0}}{(n_0+1)^d} > \frac{1}{d^{m+k}} \cdot \frac{1}{n^d} d^n.$$

Hence

$$\delta_n(A) \geq \alpha_0 n^{q_0} d^n, \tag{5}$$

where $\alpha_0 = d^{-m-k}, q_0 = -d^2 - d$, for all $n = 2td + k + m$, $t \geq 1$.

Now suppose that n is such that $k+m+2td < n < k+m+2(t+1)d$, for some t. Clearly, if we replace x_1 with $x_1 x_{n+1}$ in f_{t+1}, then the resulting polynomial f'_{t+1} will also be a proper central polynomial since A_1 is an unitary algebra. Hence $\delta_{N+1}(A) \geq \delta_N(A)$ for all $N \geq \deg f$. Denote $p = n - (k + m + 2td)$. Then $p < 2d$ and by (5) we have

$$\begin{aligned} \delta_n(A) \geq & \alpha_0 (n-p)^{q_0} d^{n-p} \\ > & \alpha_0 (n-2d)^{q_0} d^{n-2d} \\ = & \alpha_0 d^{-2d} (n-2d)^{-d-d^2} d^n > \alpha_1 n^{q_1} d^n \end{aligned}$$

for all n large enough where $\alpha_1 = \alpha_0 d^{-2d}$ and $q_1 = -d - d^2 - 1$. ∎

As a consequence of Theorem 1 we get the following two corollaries.

Corollary 1: *If A is a finite-dimensional algebra, then the proper central exponent* $\exp^\delta(A)$ *exists and is a non-negative integer.*

Corollary 2: *Let A be a finite-dimensional algebra. Then the sequence $\delta_n(A)$, $n = 1, 2, \ldots$, is either polynomially bounded or grows as an exponential function a^n with $a \geq 2$.*

4. A few examples

In this section we shall give some examples of non semisimple algebras A with proper central polynomials and we shall compare the three sequences $c_n(A)$, $c_n^z(A)$, $\delta_n(A)$, $n = 1, 2, \ldots$.

Let $F[z]$ be the algebra of polynomials over F in the variable z and let $F[z]_0$ be its subalgebra of polynomials with zero constant term. Define the quotient algebras

$$T = F[z]/(z^{2t^2}) \quad \text{and} \quad Q = F[z]_0/(z^{2t^2}),$$

where (z^{2t^2}) is the ideal generated by z^{2t^2}.

In $M_t(T)$, the algebra of $t \times t$ matrices over T, $t > 1$, we consider the subalgebra

$$A = B + M_t(Q)$$

where B is the semisimple algebra of diagonal matrices $\mathrm{diag}(\theta_1, \ldots, \theta_t)$ with $\theta_1, \ldots, \theta_t \in F$, and $M_t(Q) = J(A)$ is the Jacobson radical of A. Recalling that $G_t(x_1, \ldots, x_{t^2}, y_1, \ldots, y_{t^2})$ denotes the Regev central polynomial of $M_t(F)$, we have the following.

Lemma 3: *The polynomial G_t is a proper central polynomial of A.*

Proof. Since G_t is a central polynomial of $M_t(F)$ and $M_t(T) \simeq M_t(F) \otimes T$, then also G_t is a central polynomial of A. Let $v_1, \ldots, v_t$ denote the diagonal matrix units $e_{11}, \ldots, e_{tt}$ and let $v_{t+1}, \ldots, v_{t^2}$ denote the matrices $ze_{ij}, 1 \leq i, j \leq t, i \neq j$. Then, since the polynomial G_t evaluated on any basis of $M_t(F)$ gives a non-zero scalar value, we get

$$G_t(v_1, \ldots, v_{t^2}; v_1, \ldots, v_{t^2}) = \alpha z^{2t^2 - 2t} E,$$

for some non-zero scalar $\alpha \in F$, where E is the identity matrix. Thus G_t is a proper central polynomial of A. ∎

Corollary 3: $\exp^{\delta}(A) = \exp(A) = t$. *Hence, for any integer* $t \geq 2$ *there exists a finite-dimensional algebra* A *with* $\exp^{\delta}(A) = t$.

Proof. Recalling the Wedderburn–Malcev decomposition of $A = B + M_t(Q)$, we see that B is a maximal centrally admissible subalgebra of A, hence by Theorem 1, $\exp^{\delta}(A) = t$. Also $\exp(A) = t$ since A is a reduced algebra (see [7, Definition 9.4.2]). ∎

We know by (1) that for any PI-algebra A, $\exp(A) \geq \exp^{\delta}(A)$. Next we want to show that the difference between the two exponents can be any positive integer. To this end we modify the previous construction, recalling that $T = F[z]/(z^{2t^2})$ and $Q = F[z]_0/(z^{2t^2})$ we set

$$R = F + M_t(Q)$$

a subalgebra of $M_t(T)$, where F is identified with the subalgebra of $M_t(T)$ of scalar matrices isomorphic to F.

Then define

$$A_1 = UT(p,q) \oplus R,$$

where $UT(p,q)$ is the algebra of upper block triangular matrices with diagonal blocks of size p and q respectively, and we require that $p + q \leq t$.

The algebra A_1 has proper central polynomials. In fact we prove the following.

Lemma 4: *The polynomial* G_t *is a proper central polynomial of* A_1 *whose values lie in* R.

Proof. Since $R \subseteq M_t(T)$, then $G_t = G_t(x_1, \ldots, x_{t^2}, y_1, \ldots, y_{t^2})$ is a central polynomial for both $UT(p,q)$ and R.

Let φ be an evaluation of G_t in a basis of A_1. If for all $1 \leq i \leq t^2$, $\varphi(x_i), \varphi(y_i) \in UT(p,q)$, then by Lemma 1 we have that $\varphi(G_t) = 0$. If there are two variables of G_t that are evaluated, one in $UT(p,q)$ and the other in R, then also in this case we get $\varphi(G_t) = 0$ since $UT(p,q)R = RUT(p,q) = 0$. Hence we must have $\varphi(G_t) \in R$.

Now consider an evaluation φ such that

$$\varphi(x_1) = \varphi(y_1) = E \in F, \quad \varphi(x_i) = \varphi(y_i) = ze_{ii} \in M_t(Q), \quad 2 \leq i \leq t,$$

and the remaining $\varphi(x_k) = \varphi(y_k)$ are equal to distinct ze_{ij}, with $i \neq j$, $t+1 \leq k \leq t^2$. Since the elements $E, e_{22}, \ldots, e_{tt}, e_{ij}$, $1 \leq i,j \leq t, i \neq j$ form a basis of $M_t(F)$, we have that $0 \neq \varphi(G_t) = \lambda z^{2t^2-1} E \in M_t(Q)$, and we are done. ∎

Corollary 4: $\exp^\delta(A_1) = 1$ *whereas* $\exp(A_1) = p^2+q^2$. *Hence, for any integer* $N \geq 0$ *there exists a finite-dimensional algebra* C *such that* $\exp^\delta(C) \neq 0$ *and* $\exp(C) - \exp^\delta(C) > N$.

Proof. By the basic properties of the PI-exponent,

$$\exp(A_1) = \exp(UT(p,q)) = p^2 + q^2$$

(see [7, Section 6.2]).

In order to compute $\exp^\delta(A_1)$, we notice that $M_p(F) \oplus M_q(F)$ is isomorphic to a maximal semisimple subalgebra of $UT(p,q)$. Hence a maximal semisimple subalgebra of R is isomorphic to $M_p(F) \oplus M_q(F) \oplus B$ where $B = F$.

Let f be a proper central polynomial of A_1. If φ is an evaluation in A_1, then as in Lemma 4, either $\varphi(f) \in UT(p,q)$ or $\varphi(f) \in R$.

If the first case occurs, then $\varphi(f)$=0 by Lemma 1. In the second case we notice that only B=F can be a centrally admissible subalgebra of A_1. By Lemma 4, B is centrally admissible and by Theorem 1 the proof is complete. ∎

5. Relations among codimensions

In this section we shall compare for any finite-dimensional algebra the sequence of codimensions and the sequence of central codimensions. We have the following.

Theorem 2: *For any finite-dimensional algebra* A *with* $\exp(A) \geq 2$, *the central exponent* $\exp^z(A)$ *exists and is a non-negative integer. Moreover,*

$$\exp^z(A) = \exp(A).$$

Proof. Suppose $\exp(A) = d \geq 2$. Then by [5], there are constants $C_1 > 0$, C_2, t_1, t_2 such that

$$C_1 n^{t_1} d^n \leq c_n(A) \leq C_2 n^{t_2} d^n,$$

holds for all n. By (1) we then have that

$$c_n^z(A) \leq C_2 n^{t_2} d^n. \tag{6}$$

On the other hand, by [6, Lemma 2], A contains a subalgebra B isomorphic to $UT(d_1,\ldots,d_m)$, for some $d_1,\ldots,d_m$ with $d_1^2+\cdots+d_m^2 = d = exp(B)$. Moreover $c_n(B) \geq C_0 n^{t_0} d^n$, for some constants $C_0 > 0, t_0$ (see [4, Theorem 3]).

Suppose first that $m>1$ and let $N=c_n(B)$. Let $f_1,\ldots,f_N$ be multilinear polynomials in $x_1,\ldots,x_n$ linearly independent modulo the T-ideal $Id(B)\supseteq Id(A)$. If $f = \lambda_1 f_1 + \cdots + \lambda_N f_N + g \in Id^z(A)$, with $g \in Id(B)$, for some scalars $\lambda_1,\ldots,\lambda_N$, then $\bar f = \lambda_1 f_1 + \cdots + \lambda_N f_N$ is a central polynomial of B. But then by Lemma 1, $\bar f$ is an identity of B and this says that $\lambda_1 = \cdots = \lambda_N = 0$. Thus

$$c_n^z(A) \geq N = c_n(B) \geq C_0 n^{t_0} d^n. \tag{7}$$

Now let $m = 1$. This says that $B = M_{d_1}(F)$, $d_1 \times d_1$ matrices over F and $d = d_1^2$. Consider m copies of Regev central polynomial $f_1,\ldots,f_m$ in distinct sets of variables and let $f = f_1\cdots f_m$. Then f generates an irreducible S_n-module M where $n = 2md$ with character $\chi(M) = \chi_\lambda$, $\lambda = ((2m)^d)$. If y is another variable, $f' = fy$ is not a central polynomial and generates an irreducible S_{n+1}-module whose character is χ_μ, where $\mu = (2m+1,(2m)^{d-1})$. It is not difficult to check that $d_\mu \geq C_0(n+1)^t d^{n+1}$. Hence $c_{n+1}^z(A)$ satisfies the same inequality as in (7).

The relations (6) and (7) imply that $\exp^z(A)$ exists and $\exp^z(A)=\exp(A)$. ∎

When $\exp(A) = 0$, then A is nilpotent and $\exp^z(A) = 0$. In case $\exp(A) = 1$, then either $\exp^z(A) = 1$ or $\exp(A) = 0$. If $\exp(A) = 1$, then A is not nilpotent and the sequence of codimensions is polynomially bounded. Clearly the same holds for the sequence of central codimensions. Thus $\exp^z(A) = 1$ provided $c_n^z(A) \neq 0$ for all n.

The case when $c_n^z(A) = 0$ can be characterized as follows.

Proposition 1: *Let A be a finite-dimensional algebra such that* $\exp^z(A) = 0$. *Then $A = A_1 \oplus A_2$ where A_1 is a nilpotent algebra and A_2 is a commutative algebra.*

Proof. If $c_n^z(A) = 0$ for some $n \geq 2$, then any monomial of degree n is a central polynomial of A. In particular, $x_1\cdots x_n$ is central.

Consider the Wedderburn–Malcev decomposition $A = B + J$ where

$$B = B_1 \oplus \cdots \oplus B_m$$

is the sum of simple components and J is the Jacobson radical. Since $x_1\cdots x_n$ is central, all $B_1,\ldots,B_m$ are one dimensional and central subalgebras. Denote by e the unity of B_1. Then $J = J_0 \oplus J_1$ where $xe = ex = 0$ if $x \in J_0$ and $ye = ey = y$ if $y \in J_1$. Also $B_iJ_1 = 0$ for all $i = 2,\ldots,m$ since $eB_i = 0$. Hence $A = (B_1 + J_1) \oplus (B_2 + \cdots + B_m + J_0)$. Moreover, $B_1 + J_1$ is commutative since $y = e^{n-1}y$ lies in the center of A for any $y \in J_1$. Repeating this procedure we get $A = C_1 \oplus \cdots \oplus C_m \oplus I$ where all $C_1, \ldots, C_m$ are commutative and $I \subset J$ is nilpotent. ∎

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