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ON THE REGULARITY OF STATIONARY MEASURES

BY

YVES BENOIST

Institut de mathématique d'Orsay, UMR 8628 CNRS
Université Paris-Sud, Bât. 307, F-91405 Orsay Cedex, France
e-mail: yves.benoist@math.u-psud.fr

AND

JEAN-FRANÇOIS QUINT

CNRS—Université Bordeaux I, 33405 Talence, France
e-mail: Jean-Francois.Quint@math.u-bordeaux1.fr

ABSTRACT

Extending a construction of Bourgain for SL(2,R), we construct on any semisimple real Lie group G a symmetric probability measure whose stationary measure on the Furstenberg boundary has a smooth density and whose support is finite and generates a dense subgroup of G.

1. Introduction

1.1. NOTATIONS. Let G be a connected real semisimple Lie group and let $P \subset G$ be a parabolic subgroup. We recall that a parabolic subgroup is a subgroup P that contains a minimal parabolic subgroup $P_{\min}$ and that a minimal parabolic subgroup is a subgroup that is equal to the normalizer of a maximal unipotent subgroup of G. Equivalently, $P_{\min}$ is a maximal amenable subgroup of G such that $G/P_{\min}$ is compact. The homogeneous space $X := G/P$ is called a partial flag variety. The homogeneous space $G/P_{\min}$ is called the full flag variety or the Furstenberg boundary.

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Let μ be a (Borel) probability measure on G. In this paper, the probability measure μ will often be a finite average of Dirac masses $\mu = |F|^{-1}\sum_{f\in F}\delta_f$ where F is a finite subset of G.

1.2. Example. The main example is the following: the group G is the special linear group $G = \mathrm{SL}(d,\mathbb{R})$, the parabolic subgroup P is the stabilizer in G of a line of $\mathbb{R}^d$ and X is the real projective space $X = \mathbb{P}(\mathbb{R}^d)$.

1.3. Main result. A probability measure ν on X is said to be μ-stationary if $\nu = \mu * \nu$ where $\mu * \nu = \int_G g_*\nu \, \mathrm{d}\mu(g)$.

The following fact, which is the starting point of this note, is due to Furstenberg in [16] and to Goldsheid and Margulis in [21]. We denote by Γ_μ the subgroup of G spanned by the support of μ. We will assume that Γ_μ is Zariski dense in G. Here, this means that no finite index subgroup of Γ_μ is included in a proper connected closed subgroup of G.

Fact 1.1: *When Γ_μ is Zariski dense in G, there exists a unique μ-stationary probability measure ν on X.*

We will call this measure ν the Furstenberg measure. The importance of this measure relies on the fact that it controls the behavior of the random walk on G obtained by multiplying random elements of G chosen independently with law μ. See the articles [14], [18], [24], or the surveys [8], [9], [13], [27]. The question we address in this short note is: What is the regularity of ν? Our main result is the construction of examples where ν has regularity C^k while Γ_μ is (topologically) dense.

Theorem 1.2: *Let G be a connected semisimple real Lie group, P be a parabolic subgroup of G and let $k \geq 1$. Then, there exists a finitely supported symmetric probability measure μ on G with Γ_μ dense in G whose stationary measure ν on the flag variety $X := G/P$ of G has a C^k-smooth density.*

With no loss of generality, we can assume that G has finite center and we denote by $K \subset G$ a maximal compact subgroup. For instance when $G = \mathrm{SL}(d,\mathbb{R})$, the maximal compact subgroup K is the special orthogonal group $K = \mathrm{SO}(d,\mathbb{R})$.

The main property of G/P used in the proof will be Fact 1.1. Hence Theorem 1.2 is still true for any compact algebraic homogeneous space G/P. Indeed, by [6, Prop. 5.5], those also support a unique μ-stationary probability measure.

The conclusion of Theorem 1.2 means that one can write $\nu = \psi\,\mathrm{d}x$ where $\psi \in C^k(X)$ is a k-times continuously differentiable function on X and where $\mathrm{d}x$ is the K-invariant probability measure on X.

When $G = \mathrm{SL}(2,\mathbb{R})$, this existence theorem is due to B. Barany, M. Pollicott and K. Simon in [4, Section 9], if we do not insist on μ to be symmetric. If we insist on μ to be symmetric, the first example of such a measure μ when $G = \mathrm{SL}(2,\mathbb{R})$ is due to J. Bourgain in [10]. Moreover, the example of Bourgain is given by an explicit construction. Our proof below will assume that $G \neq \mathrm{SL}(2,\mathbb{R})$ and will also give an explicit construction of such a measure μ.

1.4. Related results. We survey now a few regularity results for the Furstenberg measure which help to put our theorem in perspective. We fix a K-invariant Riemannian metric on X.

(i) *When μ has a C^1 density, then ν has a C^∞ density.* Just because the convolution by μ is then a regularizing operator, it sends measures to measures with C^1 density and measures with C^k density to measures with C^{k+1} density.

(ii) *If Γ_μ is Zariski dense in G and μ has a finite exponential moment, then ν is Hölder regular.* Recall that **finite exponential moment** means that there exists $\varepsilon > 0$ such that $\int_G \|\mathrm{Ad}g\|^\varepsilon\,\mathrm{d}\mu(g) < \infty$. Recall also that **Hölder regular** means that there exists $\alpha > 0$ and $C > 0$ such that $\nu(B(x,r)) \leq Cr^\alpha$ for all balls $B(x,r)$ in X of radius r. This fact is due to Guivarc'h in [22]. See also the survey [8, Chap. 13]

(iii) *For any lattice Λ in G, one can find μ such that $\Gamma_\mu = \Lambda$ and $\nu = \mathrm{d}x$.* This fact is due to Furstenberg in [15] and to Lyons and Sullivan in [29]. See also [31]. Ballmann and Ledrappier have proved in [3] that one can choose μ to be symmetric. When Λ is cocompact, the construction of Lyons and Sullivan gives a probability measure μ with a finite exponential moment. Note that all these constructions provide measures with infinite support.

(iv) *If $G = \mathrm{SL}(2,\mathbb{R})$, if Γ_μ is a non-cocompact lattice in G and if μ has a finite first moment, then ν is singular with respect to $\mathrm{d}x$.* This fact is due to Guivarc'h and Le Jan in [23]. See also [11] and [19].

(v) *If $G = \mathrm{SL}(d,\mathbb{R})$, there exists a finitely supported symmetric probability measure μ on G such that Γ_μ is dense in G and ν is singular with respect to $\mathrm{d}x$.* This fact is due to Kaimanovich and Le Prince in [26] and the construction allows to obtain a Furstenberg measure ν whose Hausdorff dimension is arbitrarily small. The authors of [26] conjectured there that the Furstenberg measure ν of a finitely supported probability measure μ might always be singular. As we have already seen, the first counterexamples for $G = \mathrm{PSL}(2,\mathbb{R})$ are due to Barany, Pollicott and Simon in [4] and to Bourgain in [10] with a symmetric measure μ. The main theorem of this note is a counterexample for each semisimple Lie group G.

(vi) It is not known whether there exists a finitely supported probability measure μ on G with Γ_μ discrete and Zariski dense and whose Furstenberg measure is absolutely continuous with respect to $\mathrm{d}x$. Note that the constructions in [4] and [10] provide measures with Γ_μ dense.

(vii) We end this introduction by quoting a few results whose proofs rely on an understanding of the regularity of Furstenberg measures.

-The Margulis superrigidity theorem for a lattice Λ in G relies on the Furstenberg construction of a measure μ supported by Λ whose Furstenberg measure has a continuous density. See (iii) and [31, Section VI.4].

- The classification of stationary measures on finite volume homogeneous spaces relies on the Hölder regularity theorem due to Guivarc'h for the Furstenberg measure when μ has a finite exponential moment. See (ii) and [5, Section 4.5].

- The central limit theorem for the product of random independent elements of G when their law μ has a finite second moment, i.e., when

$$\int_G (\log \|\mathrm{Ad} g\|)^2 \, \mathrm{d}\mu(g) < \infty,$$

relies on the log-regularity of the Furstenberg measure. See [7, Prop. 4.5].

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2. Construction of the law

We begin now the proof of Theorem 1.2.

2.1. First reductions. We notice that, if Theorem 1.2 is true for two semisimple Lie groups, then it will be true for their product. We notice also that, when the group G is compact, the space X is a singleton. Since moreover Bourgain has proved Theorem 1.2 for $G = \mathrm{PSL}(2,\mathbb{R})$, we can assume with no loss of generality that

$$G \text{ is a non-compact simple Lie group, } G \neq \mathrm{PSL}(2,\mathbb{R}).$$

We will first construct in Section 2.6 probability measures μ for which the Furstenberg measure ν has an L^2 density. We will explain then in Section 2.7 that the same method allows to construct probability measures μ for which the Furstenberg measure ν has a C^k density.

2.2. Transfer operators. We introduce some notation and a few remarks that will relate Theorem 1.2 to a spectral property of the transfer operators that we will prove later. We will use the Hilbert space

$$L^2(X) := \{\varphi : X \to \mathbb{C} \mid \|\varphi\|_{L^2}^2 := \textstyle\int_X |\varphi(x)|^2 \,\mathrm{d}x < \infty\}.$$

The main tool will be the two transfer operators

$$P_\mu : L^2(X) \to L^2(X) \quad \text{and} \quad P_\mu^* : L^2(X) \to L^2(X)$$

defined for compactly supported measures μ on G by, for all φ, ψ in $L^2(X)$,

$$P_\mu\varphi\,(x) = \int_G \varphi(gx)\,\mathrm{d}\mu(g)$$

and

$$P_\mu^*\psi\,(x) = \int_G \psi(g^{-1}x)\mathrm{Jac}(g^{-1},x)\,\mathrm{d}\mu(g),$$

where $\mathrm{Jac}(g^{-1},x)$ is the Jacobian determinant of the map $x \mapsto g^{-1}x$ with respect to the volume form $\mathrm{d}x$.

Remark 2.1: (i) These operators P_μ and P_μ^* are bounded operators which are adjoint of one another, i.e., for all φ, ψ in $L^2(X)$, one has

$$\int_X P_\mu\varphi\,\psi\,\mathrm{d}x = \int_X \varphi\,P_\mu^*\psi\,\mathrm{d}x\,.$$

(ii) Their norms as operators of $L^2(X)$ are equal $\|P_\mu\|_{L^2} = \|P_\mu^*\|_{L^2}$. When μ is a Dirac mass $\mu = \delta_g$ with g in G, one has

$$\|P_{\delta_g}\|_{L^2} = \sup_{x\in X}\mathrm{Jac}(g^{-1},x)^{1/2}.$$

(iii) When μ is a symmetric probability measure $\mu = \sigma$ supported on K, one has the equalities

$$P_\sigma^* = P_\sigma \quad \text{and} \quad \|P_\sigma\|_{L^2} = 1,$$

because the measure $\mathrm{d}x$ is K-invariant and because $P_\sigma \mathbf{1} = \mathbf{1}$.

(iv) For all compactly supported measures μ_1, μ_2 on G, one has

$$P_{\mu_1 * \mu_2} = P_{\mu_2} P_{\mu_1}.$$

(v) Whenever the equation

$$P_\mu^* \psi = \psi$$

has a solution ψ in $L^2(X)$, the measure $\psi\,\mathrm{d}x$ is μ-stationary. In particular, if Γ_μ is Zariski dense, by uniqueness, the stationary measure ν must be proportional to $\psi\,\mathrm{d}x$, hence ν has an L^2 density. Moreover, whenever this solution ψ can be found in $C^k(X)$, the stationary measure ν has a C^k density.

(vi) The equation $P_\mu \varphi = \varphi$ always has a solution in $L^2(X)$: the constant function $\varphi = \mathbf{1}$. Hence we will just have to use the following general Fact 2.3 which allows us sometimes to deduce that 1 is an eigenvalue of P_μ^* from the input that 1 is an eigenvalue of P_μ.

2.3. Essential spectral radius. Let E be a Banach space and let $T \in \mathcal{L}(E)$ be a bounded operator. We denote by E^* the dual Banach space and $T^* \in \mathcal{L}(E^*)$ the adjoint operator. We recall that the **spectral radius** of T is

$$\rho(T) = \lim_{n\to\infty} \|T^n\|_E^{1/n}$$

and that the **essential spectral radius** is

$$\rho_e(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n},$$

where $\gamma(T)$ is the infimum of the radii R such that the image $T(B(0,1))$ of the ball of radius 1 is included in a finite union of translates of the ball $B(0,R)$. The operator T is said to be **quasicompact** if one has $\rho_e(T) < \rho(T)$.

The following two related facts will be useful.

Fact 2.2: *One has $\rho_e(T) < 1$ if and only if some positive power T^d of T can be written as a sum $T^d = T_0 + T_1$ of two operators with T_0 compact and $\|T_1\| < 1$.*

Fact 2.3: *Let λ be a complex number such that $|\lambda| > \rho_e(T)$. Then the following dimensions are finite and are equal:*

$$\dim \operatorname{Ker}(T^* - \lambda) = \dim \operatorname{Ker}(T - \lambda).$$

For a proof of these classical facts, see for instance [8, Prop. B.13]. For more on the essential spectral radius see [32] and [34, Section 2.4].

We will also need the following fact which tells us that the spectral radius and the essential spectral radius vary upper semicontinuously in the norm topology.

FACT 2.4: *Let $T_n \in \mathcal{L}(E)$ be a sequence of bounded operators which converges in norm toward an operator $T_\infty \in \mathcal{L}(E)$. Then, one has*

$$\limsup_{n\to\infty} \rho(T_n) \leq \rho(T_\infty)$$

and

$$\limsup_{n\to\infty} \rho_e(T_n) \leq \rho_e(T_\infty).$$

These inequalities could be strict. An example due to Kakutani is given in [36, §17.13].

Proof. The first inequality is true in any Banach algebra by [33, Thm. 10.20]. Recall that the Calkin algebra is the quotient of the Banach algebra $\mathcal{L}(E)$ by the closed ideal $\mathcal{K}(E)$ of compact operators. Since the essential spectral radius of T is the spectral radius of the image of T in the Calkin algebra (see [8, Sect. B.2.4]), the second formula follows from the same [33, Thm. 10.20] applied in the Calkin algebra. ∎

2.4. SPECTRAL GAP. We recall that G is now a non-compact simple Lie group of dimension $d > 3$ and with finite center. The following fact is well-known.

FACT 2.5: *G contains a simple 3-dimensional compact subgroup S.*

Proof. See [25, Prop. VIII.5.1]. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{e}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{e}$ its orthogonal for the Killing form. Since $\mathfrak{k}$ is a maximal Lie subalgebra, $\mathfrak{k}$ acts irreducibly on $\mathfrak{e}$. If Fact 2.5 were not true, $\mathfrak{k}$ would be abelian hence one would have $\dim\mathfrak{e} \leq 2$. Since $\mathfrak{k} = [\mathfrak{e}, \mathfrak{e}]$, one would also have $\dim\mathfrak{k} \leq 1$. Contradiction with $d > 3$. ∎

This subgroup S is locally isomorphic to the orthogonal group $\mathrm{SO}(3, \mathbb{R})$. We will say that a probability measure σ on S has a **spectral gap** if there exists $\varepsilon > 0$ such that, for every unitary representation $(\mathcal{H}, \pi)$ of S with no S-invariant non-zero vectors, one has $\|\pi(\sigma)\| \leq 1 - \varepsilon$ where $\pi(\sigma)$ is the bounded operator of $\mathcal{H}$ given by $\pi(\sigma) := \int_G \pi(s)\, \mathrm{d}\sigma(s)$. The following fact is due to Drinfeld in [12] (see also [30]).

FACT 2.6: *There exists a finitely supported symmetric probability measure σ on S which has a spectral gap.*

Here are two comments on this well-known fact.

- An explicit example of such a probability measure σ on $\mathrm{SO}(3,\mathbb{R})$ has been given by Lubotzky, Phillips and Sarnak in [28] (see also [35, Section 2.5]). One can choose σ to be

$$\sigma = \tfrac{1}{6}\sum_{i\leq 3}(\delta_{R_i} + \delta_{R_i^{-1}})$$

where the R_i's are the rotations of angle $\arccos(-3/5)$ with respect to the i^{th} coordinate axis. One has then $\|\pi(\sigma)\| = \sqrt{3}/5$.

- When a probability measure σ on S has a spectral gap, the subgroup spanned by the support of σ is dense in S. Conversely, it is conjectured that any probability measure σ on S whose support spans a dense subgroup has a spectral gap.

2.5. CONSTRUCTION OF μ. We choose now a finitely supported symmetric probability measure σ on S with a spectral gap. We choose also a finitely supported symmetric probability measure μ_0 on G of the form

$$\mu_0 = |F_0|^{-1}\sum_{f\in F_0}\delta_f$$

where

(i) F_0 is a symmetric finite subset of G with $|F_0| = 4d$,

(ii) F_0 is included in a small neighborhood B of e so that, for g in B,

$$\|P_{\delta_g}\| \leq (1+\varepsilon_0)^{1/d} \quad \text{with } \varepsilon_0 = |F_0|^{-d}/2.$$

Such a neighborhood B does exist by Remark 2.1(ii).

(iii) F_0 contains elliptic elements $g_i = e^{X_i}$ of infinite order where the elements X_i span the Lie algebra $\mathfrak{g}$ of G.

(iv) One can find a finite sequence $f_1, \ldots, f_d$ in F_0, such that the Lie algebra $\mathfrak{s}$ of S together with the images $\mathrm{Ad}(f_1\cdots f_i)(\mathfrak{s})$ with $1 \leq i \leq d$ span $\mathfrak{g}$ as a vector space.

It is elementary to construct such a finite set F_0.

- The equality $|F_0| = 4d$ is not important: it can be relaxed easily. In our construction we need d elements to check (iii), d elements to check (iv), and the inverses of these $2d$ elements so that μ be symmetric.

- The condition (iii) ensures that the subgroup Γ_{μ_0} is dense in G. Indeed the Lie algebra of the closure $\overline{\Gamma_\mu}$ contains all the elements X_i and hence is equal to $\mathfrak{g}$.
- The condition (iv) will be used to ensure that the set $Sf_1S\cdots f_dS$ has non-empty interior.

We will choose the probability measure μ to be

$$\mu = \mu_n := \sigma^{*n} * \mu_0 * \sigma^{*n},$$

for n large enough. The subgroup Γ_{μ_n} is also dense in G.

Notice that a similar construction cannot provide a discrete subgroup Γ_{μ_n} as required in Question 1.4(vi), because the group Γ_σ is dense in S.

2.6. Stationary measure with L^2 density.

Proposition 2.7: *For n large enough, the essential spectral radius of P_{μ_n} in $L^2(X)$ is strictly smaller than 1:*

$$\rho_e(P_{\mu_n}) < 1.$$

Hence the μ_n-stationary measure ν_n on X has an L^2 density.

Since 1 is an eigenvalue of P_{μ_n}, this Proposition 2.7 tells us also that the operator P_{μ_n} is quasicompact in $L^2(X)$.

The proof of Proposition 2.7 will rely on the following Lemma 2.8. Let σ_∞ be the S-invariant probability measure on S and let

$$\mu_\infty := \sigma_\infty * \mu_0 * \sigma_\infty.$$

Lemma 2.8: *The essential spectral radius of P_{μ_∞} in $L^2(X)$ is strictly smaller than 1: $\rho_e(P_{\mu_\infty}) < 1$.*

Proof of Lemma 2.8. The proof will be based on Fact 2.2. Recall that $d = \dim G$ and $\varepsilon_0 := |F_0|^{-d}/2$. We first claim that we can write

$$\mu_\infty^{*d} = \varepsilon_0\alpha_0 + (1-\varepsilon_0)\alpha_1, \tag{2.1}$$

with α_0, α_1 positive measures on G such that α_0 has a C^∞ density and

$$\|P_{\alpha_1}\| \le 1 + \varepsilon_0. \tag{2.2}$$

Indeed, by construction μ_∞^{*d} is the average of $|F_0|^d$ probability measures of the form

$$\sigma_\infty * \delta_{f_1} * \sigma_\infty * \cdots * \delta_{f_d} * \sigma_\infty,$$

with the f_i's in F_0. If one chooses $(f_1, \ldots, f_d)$ in F_0^d to be the d-tuple given by condition (iv), the map $\pi : S^{d+1} \to G$ given by

$$\pi(s_0, \ldots, s_d) = s_0 f_1 s_1 \cdots f_d s_d$$

is submersive near the point $(e, \ldots, e)$. Since this map π is algebraic, it is submersive on a non-empty Zariski open subset $U \subset S^{d+1}$. This open subset U has full $\sigma_\infty^{\otimes d+1}$-measure. Hence there exists a compactly supported function $\varphi \in C_c^\infty(U)$ with $0 \leq \varphi \leq 1$ on U such that $\int_U \varphi \, \mathrm{d}\sigma_\infty^{\otimes d+1} = 1/2$. The measure

$$\alpha_0 := \pi_*(2\varphi\sigma_\infty^{\otimes d+1})$$

is a probability measure on G with C^∞ density. Since $\varepsilon_0 = |F_0|^{-d}/2$, one can write $\mu_\infty^{*d} = \varepsilon_0\alpha_0 + (1 - \varepsilon_0)\alpha_1$ where α_1 is another probability measure on G. It remains only to check (2.2).

Notice that, by construction, the operator P_{α_1} is an average of operators of the form P_{δ_g} where $g = s_0 f_1 s_1 \cdots f_d s_d$ with the s_i's varying in S and the f_i's varying in F_0. The condition (ii) tells us that these operators P_{δ_g} have norm at most $1 + \varepsilon_0$, hence one also has $\|P_{\alpha_1}\| \leq 1 + \varepsilon_0$ as required.

Now, the operator $T := P_{\mu_\infty}^d$ of $L^2(X)$ is equal to the sum $T = T_0 + T_1$ where $T_0 := \varepsilon_0 P_{\alpha_0}$ and $T_1 := (1-\varepsilon_0)P_{\alpha_1}$. The measure α_0 has a C^∞ density, hence the convolution operator by α_0 is a continuous operator from $L^2(X)$ to $C^\infty(X)$. Because of the Ascoli Theorem, the embedding $C^\infty(X) \hookrightarrow L^2(X)$ is compact, hence the first operator T_0 is a compact operator of $L^2(X)$. The norm of the second operator T_1 is bounded by

$$\|T_1\| \leq (1 - \varepsilon_0)\|P_{\alpha_1}\| \leq 1 - \varepsilon_0^2 < 1.$$

Using Fact 2.2, this proves that $\rho_e(P_{\mu_\infty}) < 1$ in $L^2(X)$. ∎

Proof of Proposition 2.7. Since the probability measure σ has a spectral gap, and since the operator P_{σ_∞} is the orthogonal projection on the S-invariant vectors in $L^2(X)$, one has the convergences in $\mathcal{L}(L^2(X))$ for the norm topology,

$$P_{\sigma^{*n}} \xrightarrow[n\to\infty]{} P_{\sigma_\infty} \text{ and hence } P_{\mu_n} \xrightarrow[n\to\infty]{} P_{\mu_\infty}.$$

By Fact 2.4, the essential spectral radius varies in an upper semicontinuous way in the norm topology, Hence by Lemma 2.8, one has $\rho_e(P_{\mu_n}) < 1$ for n large enough.

Since 1 is always an eigenvalue of P_{μ_n} and since $\rho_e(P_{\mu_n}) < 1$, using Fact 2.3, one infers that 1 is also an eigenvalue of $P^*_{\mu_n}$. Let $\psi_n \in L^2(X)$ be the corresponding eigenvector. According to Remark 2.1(v), the μ_n-stationary probability measure ν_n on X is proportional to $\psi_n \,\mathrm{d}x$. In particular ν_n has an L^2 density. ∎

2.7. Stationary measure with C^k-density. We explain now how to modify the previous arguments to show that for n large enough the μ_n-stationary measure ν_n has a C^k-density.

The main modification is to replace the Hilbert space $L^2(X)$ by the Sobolev space $E = H^{-s}(X)$ and by its dual $E^* = H^s(X)$. We first recall the definition of Sobolev spaces. For more details, one can consult [1] for Sobolev spaces over $\mathbb{R}^n$ and [2, Chap. 2] for Sobolev spaces over Riemannian manifolds. We denote by $C^\infty(X)$ the Frechet space of C^∞-functions on X, and by $\mathcal{D}'(X)$ the Frechet space of generalized functions (or distributions) on X. By definition, $\mathcal{D}'(X)$ is the topological dual of $C^\infty(X)$. The duality on $C^\infty(X)$ given by, for all φ, ψ in $C^\infty(X)$,

$$(\varphi, \psi) := \int_X \varphi(x)\psi(x)\,\mathrm{d}x, \tag{2.3}$$

identifies the space $C^\infty(X)$ with a dense subspace of $\mathcal{D}'(X)$.

We denote by Δ the Laplacian of the K-invariant Riemannian metric on X. It is a symmetric operator on $C^\infty(X)$ that has a unique continuous extension, also denoted by Δ, as an operator of $\mathcal{D}'(X)$. The operator $1 - \Delta$ is invertible both in $C^\infty(X)$ and in $\mathcal{D}'(X)$. For s an even integer, the Sobolev spaces are given by

$$H^s(X) := \{\psi \in \mathcal{D}'(X) \mid (1-\Delta)^{s/2}\psi \in L^2(X)\}.$$

The Sobolev space $H^s(X)$ is a Hilbert space for the norm

$$\|\psi\|_{H^s} := \|(1-\Delta)^{s/2}\psi\|_{L^2}.$$

This Hilbert norm is K-invariant. When a probability measure μ on G has compact support, the operators P_μ and P^*_μ introduced in Section 2.2 have a unique continuous extension, also denoted by P_μ and P^*_μ, as operators of $\mathcal{D}'(X)$. These operators P_μ and P^*_μ preserve the Sobolev spaces. In what follows, we will assume $s > k + \frac{1}{2}\dim X$ so that, by the Sobolev embedding theorem (see [2, Thms. 2.10 & 2.20]), one has $H^s(X) \subset C^k(X)$. We will consider P_μ as a bounded operator of $H^{-s}(X)$ and P_μ^* as a bounded operator of $H^s(X)$:

$$P_\mu : H^{-s}(X) \to H^{-s}(X) \quad \text{and} \quad P_\mu^* : H^s(X) \to H^s(X).$$

We recall that the duality (2.3) on $C^\infty(X)$ extends as a duality also denoted by $(\cdot,\cdot)$ between $H^{-s}(X)$ and $H^s(X)$. This duality identifies $H^s(X)$ with the dual of $H^{-s}(X)$. The operators P_μ and P_μ^* are still adjoint to each other for this duality, i.e., one has, for all φ in $H^{-s}(X)$ and ψ in $H^s(X)$,

$$(P_\mu \varphi, \psi) = (\varphi, P_\mu^* \psi).$$

Proof of Theorem 1.2. We use the same probability measures σ and σ_∞ on K, μ_n and μ_∞ on G as in Sections 2.5 and 2.6, maybe with a smaller neighborhood B and a larger value of n. Our claim follows from the previous discussion and the following Proposition 2.9. ∎

Proposition 2.9: *Let $s \geq 0$. For n large enough, the essential spectral radius of P_{μ_n} in $H^{-s}(X)$ is strictly smaller than 1: $\rho_e(P_{\mu_n}) < 1$. Hence the μ_n-stationary measure ν_n on X has a H^s density.*

Since 1 is an eigenvalue of P_{μ_n}, this Proposition 2.9 tells us also that the operator P_{μ_n} is quasicompact in $H^{-s}(X)$.

Proof of Proposition 2.9. The proof is the same as for Proposition 2.7. We just replace Lemma 2.8 by Lemma 2.10 below. ∎

Lemma 2.10: *Let $s \geq 0$. For r small enough, the essential spectral radius of P_{μ_∞} in $H^{-s}(X)$ is strictly smaller than 1: $\rho_e(P_{\mu_\infty}) < 1$.*

Proof of Lemma 2.10. The proof is the same as for Lemma 2.8. ∎

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