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ENTROPY IN THE CUSP AND PHASE TRANSITIONS FOR GEODESIC FLOWS

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ABSTRACT

In this paper we study the geodesic flow for a particular class of Riemannian non-compact manifolds with variable pinched negative sectional curvature. For a sequence of invariant measures we are able to prove results relating the loss of mass and bounds on the measure entropies. We compute the entropy contribution of the cusps. We develop and study the corresponding thermodynamic formalism. We obtain certain regularity results for the pressure of a class of potentials. We prove that the pressure is real analytic until it undergoes a phase transition, after which it becomes constant. Our techniques are based on the one hand on symbolic methods and Markov partitions, and on the other on geometric techniques and approximation properties at the level of groups.

1. Introduction

This paper is devoted to studying ergodic and geometric properties of a class of geodesic flows defined over non-compact manifolds of variable pinched negative curvature. These flows can be coded with suspension flows defined over Markov shifts, albeit on a countable alphabet. This paper addresses problems where the non-compactness of the ambient manifold plays a fundamental role. Inspired by some recent results proved in the context of homogeneous dynamics ([ELMV12, EKP15]), we establish properties that relate the escape of mass of a sequence of invariant probability measures for the geodesic flow with its measure theoretic entropies (see Section 5). Our study combines both geometric and symbolic methods. A consequence of these results is that we can describe the thermodynamic formalism for the flow. In particular, we construct a class of potentials for which the pressure exhibits a phase transition (see Section 6).

The class of manifolds that we will be working on in the paper were introduced in [DP98]. These manifolds are obtained as the quotient of a Hadamard manifold with an extended Schottky group (see Subsection 4.2 for precise definitions). Groups in this class have maximal parabolic subgroups of rank 1, therefore the manifolds are non-compact. It was shown in [DP98] that the geodesic flow over the unit tangent bundle of those manifolds can be coded as suspension flows over countable Markov shifts. The existence of a Markov coding for the geodesic flow is essential for our results.

The idea of coding a flow in order to describe its dynamical and ergodic properties has a long history, and a great deal of interesting and important results have been obtained with these methods. Probably, some of the most relevant results using this technique are related to counting closed geodesics and also estimating the rate at which their number grow [PP90]. A landmark result is the construction of Markov partitions for Axiom A flows defined over compact manifolds done by Bowen [Bow73] and Ratner [Rat73]. They actually showed that Axiom A flows can be coded with suspension flows defined over sub-shifts of finite type on finite alphabets with regular (Hölder) roof functions. The study in the non-compact setting is far less developed. However, some interesting results have been obtained. Recently, Hamenstädt [Ham] and also Bufetov and Gurevich [BG11] have coded Teichmüller flows with suspension flows over countable alphabets, and using this representation have proved, for example, the uniqueness of the measure of maximal entropy. Another important example for which codings on countable alphabets have been constructed is a type of Sinai billiards [BS81a, BS81b].

As mentioned before, a main goal of the paper is to investigate the loss of mass of sequences of invariant measures for the geodesic flow. Recently, the loss of mass has been studied for the modular surface in [ELMV12]. Despite being a particular case, the method displayed in [ELMV12] is quite flexible and has the advantage that it can be understood purely geometrically. A more general situation is studied in [EKP15], where this type of result is shown to hold for geodesic flows on finite volume hyperbolic spaces of any dimension and type (real hyperbolic, complex hyperbolic, quaternionic, Cayley plane).

We begin by introducing the notion of entropy at infinity of a dynamical system defined over a non-compact topological space. This notion has also been considered in a similar form by Buzzi in [Buz10] for countable Markov shifts.

Definition 1.1: Let Y be a non-compact topological space and $\Psi=(\psi_t)_{t\in\mathbb{R}}:Y\to Y$ a continuous flow. We define the "**entropy at infinity**" of the dynamical system as the number

$$h_\infty(\Psi,Y)=\sup_{(\nu_n)\rightharpoonup 0}\limsup_{n\to\infty} h_{\nu_n}(\Psi),$$

where the supremum is taken over all the sequences of invariant probability measures for the flow converging in the vague topology to the zero measure. If no such sequence exists we set $h_\infty(\Psi,Y)=0$. Here $h_\nu(\Psi)$ denotes the measure-theoretic entropy of a probability Ψ-invariant measure ν.

Recall that the total mass of probability measures is not necessarily preserved under vague convergence (as opposed to weak convergence). Note that Definition 1.1 can be extended to more general group actions whenever an entropy theory has been developed for the group in consideration. Amenable groups are a classical example of such.

In this paper we are able to compute $h_\infty(g, T^1X/\Gamma)$, where X is a Hadamard manifold with pinched negative sectional curvature, Γ is an extended Schottky group generated by N_1 hyperbolic isometries and N_2 parabolic ones, and (g_t) is the geodesic flow on the unit tangent bundle T^1X/Γ (see Subsection 4.2 for precise definitions). Define

$$\delta_{p,\max} = \max\{\delta_{\mathcal{P}} : \mathcal{P} \text{ parabolic subgroup of } \Gamma\},$$

where δ_H denotes the critical exponent of $H < Iso(X)$. We prove that

$$h_\infty(g, T^1X/\Gamma) = \delta_{p,\max}.$$

It is worth mentioning that $\delta_{p,\max}$ is strictly less than the topological entropy of the geodesic flow. In our context, the non-compact pieces of dynamical interest are modeled by cusps. That is why we refer to this quantity as **entropy in the cusps**. More concretely, we prove that if a sequence of measures is dissipating through the cusps, then the entropy contribution of the sequence is at most $\delta_{p,max}$. In [EKP15] it is proven that $h_\infty(A, \Gamma\backslash G) = h_{top}/2$, where G is a connected semisimple Lie group of real rank 1 with finite center, Γ a lattice in G, and A a one-parameter subgroup of diagonalizable elements over $\mathbb{R}$ acting by right multiplication. In particular $h_\infty(g, T^1S) = 1/2$, where S is a hyperbolic surface with finite volume. We also obtain results in the case where the sequence of measures keeps some mass at the limit. Our bounds are less concrete than the analogous result in the homogeneous dynamical case though. The following is one of our main results and gives the calculation of the entropy in the cusps mentioned before.

Theorem 1.2: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every $c > \delta_{p,\max}$ there exists a constant $m = m(c) > 0$, with the following property: If (ν_n) is a sequence of (g_t)-invariant probability measures on T^1X/Γ satisfying $h_{\nu_n}(g) \geqslant c$, then for* *every vague limit* $\nu_n \rightharpoonup \nu$, *we have*

$$\nu(T^1X/\Gamma) \geqslant m.$$

In particular, if $\nu_n \rightharpoonup 0$, *then* $\limsup h_{\nu_n}(g) \leqslant \delta_{p,\max}$. *Moreover, the value* $\delta_{p,\max}$ *is optimal in the following sense: there exists a sequence* (ν_n) *of* (g_t)*-invariant probability measures on* T^1X/Γ *such that* $h_{\nu_n}(g) \to \delta_{p,\max}$ *and* $\nu_n \rightharpoonup 0$.

We also study regularity properties of the pressure function. In order to do so, we make strong use of the symbolic coding that the geodesic flow has in the manifolds we are considering. The idea of using symbolic dynamics to study thermodynamic formalism of flows of geometric nature can be traced back to the work of Bowen and Ruelle [BR75]. They studied in great detail ergodic theory and thermodynamic formalism for Axiom A flows defined on compact manifolds. The techniques they used were symbolic in nature and were based on the symbolic codings obtained by Bowen [Bow73] and Ratner [Rat73]. In this work we follow this strategy. We stress, however, that our symbolic models are non-compact. There are several difficulties related to the lack of compactness that have to be addressed, but also new phenomena are observed.

To begin with, in Subsection 2.6 we propose a definition of topological pressure, $P(\cdot)$, that satisfies not only the variational principle, but also an approximation by the compact invariant sets property. These provide symbolic proofs to results obtained by different (non-symbolic) methods in far more general settings by Paulin, Pollicott and Schapira [PPS15]. The strength of our approach is perhaps better appreciated in our regularity results for the pressure (Subsection 6.2). Note that the techniques in [PPS15] do not provide these type of results. We say that the pressure function $t \mapsto P(tf)$ has a **phase transition** at $t = t_0$ if it is not analytic at that point. It readily follows from work by Bowen and Ruelle [BR75] that the pressure for Axiom A flows and regular potentials is real analytic and hence has no phase transitions. Regularity properties of the pressure of geodesic flows defined on non-compact manifolds, as far as we know, have not been studied, with the exception of the geodesic flow defined on the modular surface (see [IJ13, Section 6]).

There is a general strategy used to study regularity properties of the pressure of maps and flows with strong hyperbolic or expanding properties in most of the phase space but not in all of it. Indeed, if there exists a subset of the phase space $B \subset X$ for which the restricted dynamics is not expansive and its entropy equal to A, then it is possible to construct potentials $f : X \to \mathbb{R}$ for which the pressure function has the form

$$P(tf) := \begin{cases} \text{real analytic, strictly decreasing and convex} & \text{if } t < t'; \\ A & \text{if } t > t'. \end{cases} \tag{1}$$

Well known examples of this phenomena include the Manneville–Pomeau map (see for example [Sar01]) in which the set B consists of a parabolic fixed point and therefore $A = 0$. The potential considered is the geometrical one: $-\log |T'|$. Similar results for multimodal maps have been obtained, for example, in [DT, IT10, PRL]. In this case the set B corresponds to the post-critical set and $A = 0$. Examples of maps in which $A > 0$ have been studied in [DGR11, IT13]. For suspension flows over countable Markov shifts, similar examples were obtained in [IJ13].

In the case of geodesic flows, roughly speaking we are considering the set B as the union of the cusps of the manifold. More interestingly, as we mentioned before we are able to compute the entropy contributions of the cusps in the geodesic flow. In Subsection 6.2 we construct a class of potentials, that we denote by $\mathcal{F}$, for which the pressure exhibits similar behaviour as in equation (1). In those examples $A = \delta_{p,\max}$. Note that it is possible for t' to be infinity and in that case the pressure is real analytic. The following is the precise statement:

Theorem 1.3: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. If $f \in \mathcal{F}$, then:*

1. *For every $t \in \mathbb{R}$ we have that $P_g(tf) \geqslant \delta_{p,\max}$.*
2. *We have that $\lim_{t\to-\infty} P_g(tf) = \delta_{p,\max}$.*
3. *Let $t' := \sup\{t \in \mathbb{R} : P_g(tf) = \delta_{p,\max}\}$. Then*

$$P_g(tf) = \begin{cases} \delta_{p,\max} & \text{if } t < t'; \\ \text{real analytic, strictly convex, strictly increasing} & \text{if } t > t'. \end{cases}$$

4. *If $t > t'$, the potential tf has a unique equilibrium measure. If $t < t'$ it has no equilibrium measure.*

In order to prove this result we need to relate symbolic quantities with geometrical ones. This is achieved in Theorem 4.10 in which a symbolic parameter of the suspension flow, the number s_∞, is proven to be equal to the geometric parameter of the group $\delta_{p,\max}$. We stress that when coding a flow a great deal of geometric information is lost. With this result we are able to recover part of it.

Remark 1.4: In [RV] the authors recently studied the escape of mass phenomena and the thermodynamical formalism for the geodesic flow on geometrically finite groups (extended Schottky groups are a particular case of such). Their approach is purely geometric, which gives more explicit bounds of the mass of limit measures. However, our symbolic approach allows one to give precise answers to questions relating the thermodynamical formalism, such as approximation results for the topological pressure and the regularity of the pressure function .

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2. Preliminaries on thermodynamic formalism and suspension flows

This section is devoted to providing the necessary background on thermodynamic formalism and on suspension flows required in the rest of the article.

2.1. Thermodynamic formalism for countable Markov shifts. Let M be an incidence matrix defined on the alphabet of natural numbers. The associated one sided countable Markov shift (Σ^+, σ) is the set

$$\Sigma^+ := \{(x_n)_{n\in\mathbb{N}} : M(x_n, x_{n+1}) = 1 \text{ for every } n \in \mathbb{N}\},$$

together with the shift map $\sigma : \Sigma^+ \to \Sigma^+$ defined by $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$. A standing assumption we will make throughout the article is that (Σ^+, σ) is

topologically mixing. We equip Σ^+ with the topology generated by the cylinder sets:

$$C_{a_1 \cdots a_n} = \{x \in \Sigma^+ : x_i = a_i \text{ for } i = 1, \ldots, n\}.$$

We stress that, in general, Σ^+ is a non-compact space. Given a function $\varphi\colon \Sigma^+ \to \mathbb{R}$ we define the n-th variations of φ by

$$V_n(\varphi) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in \Sigma^+,\ x_i = y_i \text{ for } i = 1, \ldots, n\},$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. We say that φ has **summable variation** if $\sum_{n=1}^{\infty} V_n(\varphi) < \infty$. We say that φ is **locally Hölder** if there exists $\theta \in (0, 1)$ such that for all $n \geqslant 1$, we have $V_n(\varphi) \leqslant O(\theta^n)$.

This section is devoted to recalling some of the notions and results of thermodynamic formalism in this setting. The following definition was introduced by Sarig [Sar99] based on work by Gurevich [Gur69].

Definition 2.1: Let $\varphi\colon \Sigma^+ \to \mathbb{R}$ be a function of summable variation. The **Gurevich pressure** of φ is defined by

$$P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{x : \sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x) \right) \chi_{C_{i_1}}(x),$$

where $\chi_{C_{i_1}}(x)$ is the characteristic function of the cylinder $C_{i_1} \subset \Sigma^+$.

It is possible to show (see [Sar99, Theorem 1]) that the limit always exists and that it does not depend on i_1. The following two properties of the pressure will be relevant for our purposes (see [Sar99, Theorems 2 and 3] and [IJT15, Theorem 2.10]). If $\varphi\colon \Sigma^+ \to \mathbb{R}$ is a function of summable variations, then:

(1) (Approximation property)

$$P(\varphi) = \sup\{P_K(\varphi) : K \in \mathcal{K}\},$$

where

$$\mathcal{K} := \{K \subset \Sigma^+ : K \neq \varnothing \text{ compact and } \sigma\text{-invariant}\}$$

and $P_K(\varphi)$ is the classical topological pressure on K (see [Wal82, Chapter 9]).

(2) (Variational Principle) Denote by $\mathcal{M}_\sigma$ the space of σ-invariant probability measures and by $h_\mu(\sigma)$ the entropy of the measure μ

(see [Wal82, Chapter 4]). If $\varphi\colon \Sigma^+ \to \mathbb{R}$ is a function of summable variation, then

$$P_\sigma(\varphi) = \sup\left\{h_\mu(\sigma) + \int \varphi \mathrm{d}\mu : \mu \in \mathcal{M}_\sigma \text{ and } -\int \varphi \mathrm{d}\mu < \infty\right\}.$$

A measure $\mu \in \mathcal{M}_\sigma$ attaining the supremum, that is, $P_\sigma(\varphi) = h_\mu(\sigma) + \int \varphi \mathrm{d}\mu$, is called an **equilibrium measure** for φ. A potential of summable variations has at most one equilibrium measure (see [BS03, Theorem 1.1]).

It turns out that under a combinatorial assumption on the incidence matrix M, which roughly means to be similar to a full-shift, the thermodynamic formalism is well behaved.

Definition 2.2: We say that a countable Markov shift (Σ^+, σ), defined by the transition matrix $M(i,j)$ with $(i,j) \in \mathbb{N} \times \mathbb{N}$, satisfies the **BIP (Big Images and Preimages) condition** if and only if there exists $\{b_1, \ldots, b_n\} \subset \mathbb{N}$ such that for every $a \in \mathbb{N}$ there exists $i, j \in \mathbb{N}$ with $M(b_i, a)M(a, b_j) = 1$.

The following theorem summarises results proven by Sarig in [Sar99, Sar01, Sar03] and by Mauldin and Urbański, [MU03], where they show that thermodynamic formalism in this setting is similar to that observed for sub-shifts of finite type on finite alphabets. For precise statements see [MU03, Theorem 2.6.12] and [Sar03, Section 3].

THEOREM 2.3: *Let (Σ^+, σ) be a countable Markov shift satisfying the BIP condition and $\varphi : \Sigma^+ \to \mathbb{R}$ a non-positive locally Hölder potential. Then, there exists $s_\infty \geqslant 0$ such that pressure function $t \to P_\sigma(t\varphi)$ has the following properties:*

$$P_\sigma(t\varphi) = \begin{cases} \infty & \text{if } t < s_\infty; \\ \text{real analytic} & \text{if } t > s_\infty. \end{cases}$$

Moreover, if $t > s_\infty$, there exists a unique equilibrium measure for $t\varphi$.

2.2. SUSPENSION FLOWS. Let (Σ^+, σ) be a topologically mixing countable Markov shift and $\tau : \Sigma^+ \to \mathbb{R}^+$ a function of summable variations bounded away from zero. Consider the space

$$Y = \{(x,t) \in \Sigma^+ \times \mathbb{R} \colon 0 \leqslant t \leqslant \tau(x)\}, \tag{2}$$

with the points $(x, \tau(x))$ and $(\sigma(x), 0)$ identified for each $x \in \Sigma^+$. The **suspension semi-flow** over σ with **roof function** τ is the semi-flow $\Phi = (\varphi_t)_{t \geqslant 0}$ on

Y defined by

$$\varphi_t(x,s) = (x, s+t) \quad \text{whenever } s+t \in [0, \tau(x)].$$

In particular,

$$\varphi_{\tau(x)}(x,0) = (\sigma(x), 0).$$

2.3. Invariant measures. Let (Y, Φ) be a suspension semi-flow defined over a countable Markov shift (Σ^+, σ) with roof function $\tau : \Sigma^+ \to \mathbb{R}^+$ bounded away from zero. Denote by $\mathcal{M}_\Phi$ the space of invariant probability measures for the flow. It follows from a classical result by Ambrose and Kakutani [AK42] that every measure $\nu \in \mathcal{M}_\Phi$ can be written as

$$\nu = \frac{(\mu \times m)|_Y}{(\mu \times m)(Y)}, \tag{3}$$

where $\mu \in \mathcal{M}_\sigma$ and m denotes the one dimensional Lebesgue measure. When (Σ^+, σ) is a sub-shift of finite type defined on a finite alphabet the relation in equation (3) is actually a bijection between $\mathcal{M}_\sigma$ and $\mathcal{M}_\Phi$. If (Σ^+, σ) is a countable Markov shift with roof function bounded away from zero, the map defined by

$$\nu \mapsto \frac{(\mu \times m)|_Y}{(\mu \times m)(Y)}$$

is surjective. However, it can happen that $(\mu \times m)(Y) = \infty$. In this case the image can be understood as an infinite invariant measure.

The case which is more subtle is when the roof function is only assumed to be positive. We will not be interested in that case here, but we refer to [IJT15] for a discussion on the pathologies that might occur.

2.4. Of flows and semi-flows. In 1972 Sinai [Sin72, Section 3] observed that in order to study thermodynamic formalism for suspension flows, it suffices to study thermodynamic formalism for semi-flows. Denote by (Σ, σ) a two-sided countable Markov shift. Recall that two continuous functions $\varphi, \gamma \in C(\Sigma)$ are said to be **cohomologous** if there exists a continuous function $\psi \in C(\Sigma)$, called a transfer function, such that $\varphi = \gamma + \psi \circ \sigma - \psi$. The relevant remark is that thermodynamic formalism for two cohomologous functions is exactly the same. Thus, if every continuous function $\varphi \in C(\Sigma)$ is cohomologous to a continuous function $\gamma \in C(\Sigma)$ which only depends on future coordinates, then thermodynamic formalism for the flow can be studied in the corresponding semi-flow. The next result formalises this discussion.

PROPOSITION 2.4: *If $\varphi \in C(\Sigma)$ has summable variation, then there exists $\gamma \in C(\Sigma)$ of summable variation cohomologous to φ via a bounded transfer function, such that $\gamma(x) = \gamma(y)$ whenever $x_i = y_i$ for all $i \geqslant 0$ (that is, γ depends only on the future coordinates).*

Proposition 2.4 has been proved with different regularity assumptions in the compact setting and in the non-compact case in [Dao13, Theorem 7.1].

2.5. ABRAMOV AND KAC. The entropy of a flow with respect to an invariant measure can be defined as the entropy of the corresponding time one map. The following result was proved by Abramov [Abr59].

PROPOSITION 2.5 (Abramov): *Let $\nu \in \mathcal{M}_\Phi$ be such that*

$$\nu = (\mu \times m)|_Y/(\mu \times m)(Y), \quad \textit{where } \mu \in \mathcal{M}_\sigma.$$

Then the entropy of ν with respect to the flow, that we denote $h_\nu(\Phi)$, satisfies

$$h_\nu(\Phi) = \frac{h_\mu(\sigma)}{\int \tau d\mu}. \tag{4}$$

In Proposition 2.5 a relation between the entropy of a measure for the flow and a corresponding measure for the base dynamics was established. We now prove a relation between the integral of a function on the flow with the integral of a related function on the base. Let $f \colon Y \to \mathbb{R}$ be a continuous function. Define $\Delta_f \colon \Sigma^+ \to \mathbb{R}$ by

$$\Delta_f(x) := \int_0^{\tau(x)} f(x,t)\ \mathrm{d}t.$$

PROPOSITION 2.6 (Kac's Lemma): *Let $f \colon Y \to \mathbb{R}$ be a continuous function and $\nu \in \mathcal{M}_\Phi$ an invariant measure that can be written as*

$$\nu = \frac{\mu \times m}{(\mu \times m)(Y)},$$

where $\mu \in \mathcal{M}_\sigma$. Then

$$\int_Y f\ \mathrm{d}\nu = \frac{\int_\Sigma \Delta_f\ \mathrm{d}\mu}{\int_\Sigma \tau\ \mathrm{d}\mu}.$$

Propositions 2.5 and 2.6 together with the relation between the spaces of invariant measures for the flow and for the shift established by Ambrose and Kakutani (see Subsection 2.3) allow us to study thermodynamic formalism for the flow by means of the corresponding one on the base.

2.6. Thermodynamic formalism for suspension flows. Let (Σ^+, σ) be a topologically mixing countable Markov shift and $\tau : \Sigma^+ \to \mathbb{R}$ a positive function bounded away from zero of summable variations. Denote by (Y, Φ) the suspension semi-flow over (Σ^+, σ) with roof function τ. Thermodynamic formalism has been studied in this context by several people with different degrees of generality: Savchenko [Sav98], Barreira and Iommi [BI06], Kempton [Kem11] and Jaerisch, Kesseböhmer and Lamei [JKL14]. Thermodynamic formalism for suspension flows where the base (Σ^+, σ) is a sub-shift of finite type defined on a finite alphabet has been studied, for example, in [BR75, PP90]. The next result provides equivalent definitions for the pressure, $P_\Phi(\cdot)$, on the flow.

Theorem 2.7: *Let $f : Y \to \mathbb{R}$ be a function such that $\Delta_f : \Sigma^+ \to \mathbb{R}$ is of summable variations. Then the following equalities hold:*

$$\begin{aligned} P_\Phi(f) &:= \lim_{t\to\infty} \frac{1}{t} \log \Bigg(\sum_{\varphi_s(x,0)=(x,0),0<s\leqslant t} \exp\left(\int_0^s f(\varphi_k(x,0))\, \mathrm{d}k \right) \chi_{C_{i_0}}(x) \Bigg) \\ &= \inf\{t \in \mathbb{R} : P_\sigma(\Delta_f - t\tau) \leqslant 0\} = \sup\{t \in \mathbb{R} : P_\sigma(\Delta_f - t\tau) \geqslant 0\} \\ &= \sup\{P_{\Phi|K}(f) : K \in \mathcal{K}(\Phi)\}, \end{aligned}$$

where $\mathcal{K}(\Phi)$ denotes the space of compact Φ-invariant sets.

In particular, the topological entropy of the flow is the unique number $h_{top}(\Phi)$ satisfying

$$h_{top}(\Phi) = \inf\{t \in \mathbb{R} : P(-t\tau) \leqslant 0\}. \tag{5}$$

Note that in this setting the Variational Principle also holds (see [BI06, JKL14, Kem11, Sav98]).

Theorem 2.8 (Variational Principle): *Let $f : Y \to \mathbb{R}$ be a function such that $\Delta_f : \Sigma^+ \to \mathbb{R}$ is of summable variations. Then*

$$P_\Phi(f) = \sup\left\{ h_\nu(\Phi) + \int_Y f\, \mathrm{d}\nu : \nu \in \mathcal{M}_\Phi \text{ and } -\int_Y f\, \mathrm{d}\nu < \infty \right\}.$$

A measure $\nu \in \mathcal{M}_\Phi$ is called an **equilibrium measure** for f if

$$P_\Phi(f) = h_\nu(\Phi) + \int f\, \mathrm{d}\nu.$$

It was proved in [IJT15, Theorem 3.5] that potentials f for which Δ_f is locally Hölder have at most one equilibrium measure. Moreover, the following result (see [BI06, Theorem 4]) characterises functions having equilibrium measures.

THEOREM 2.9: *Let $f\colon Y \to \mathbb{R}$ be a continuous function such that Δ_f is of summable variations. Then there is an equilibrium measure $\nu_f \in \mathcal{M}_\Phi$ for f if and only if $P_\sigma(\Delta_f - P_\Phi(f)\tau) = 0$ and there exists an equilibrium measure $\mu_f \in \mathcal{M}_\sigma$ for $\Delta_f - P_\Phi(f)\tau$ such that $\int \tau d\mu_f < \infty$.*

Remark 2.10: We stress that the situation is more complicated when τ is not assumed to be bounded away from zero. For results in that setting see [IJT15].

3. Entropy and escape of mass

Over the last few years there has been interest, partially motivated for its connections with number theory, in studying the relation between entropy and the escape of mass of sequences of invariant measures for diagonal flows on homogenous spaces (see [EKP15, ELMV12, KKLM17]). Some remarkable results have been obtained bounding the amount of mass that an invariant measure can give to an unbounded part of the domain (a cusp) in terms of the entropy of the measure (see for example [EKP15, Theorem A] or [KKLM17, Theorem 1.3]). The purpose of this section is to prove similar results in the context of suspension flows defined over countable Markov shifts. As we will see, the proofs in this setting suggest a geometrical interpretation that we pursue in Section 5.

Let (Σ^+, σ) be a topologically mixing countable Markov shift of infinite topological entropy and $\tau : \Sigma^+ \to \mathbb{R}^+$ a potential of summable variations bounded away from zero. Denote by (Y, Φ) the associated suspension flow, which we assume to have finite topological entropy. Note that since (Σ^+, σ) has infinite entropy and τ is non-negative, the entropy $h_{top}(\Phi)$ of the flow satisfies $P_\sigma(-h_{top}(\Phi)\tau) \leqslant 0$ (see equation (5)). Therefore, there exists a real number $s_\infty \in (0, h_{top}(\Phi)]$ such that

$$P_\sigma(-t\tau) = \begin{cases} \text{infinite} & \text{if } t < s_\infty; \\ \text{finite} & \text{if } t > s_\infty. \end{cases}$$

As it turns out the number s_∞ will play a crucial role in our work.

For geodesic flows defined in non-compact manifolds there are vectors that escape through the cusps; they do not exhibit any recurrence property. That phenomenon is impossible in the symbolic setting; every point will return to the base after some time. The following definition describes the set of points that escape on average (compare with an analogous definition given in [KKLM17]).

Definition 3.1: We say that a point $(x,t) \in Y$ **escapes on average** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \tau(\sigma^i x) = \infty.$$

We denote the set of all points which escape on average by $\mathcal{E}_{\mathcal{A}}(\tau)$.

Remark 3.2: Note that if $\nu \in \mathcal{M}_\Phi$ is ergodic and $\nu = (\mu \times m)/(\mu \times m)(Y)$ with $\mu \in \mathcal{M}_\sigma$ then Birkhoff's theorem implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \tau(\sigma^i x) = \int \tau d\mu.$$

Thus, no measure in $\mathcal{M}_\Phi$ is supported on $\mathcal{E}_{\mathcal{A}}(\tau)$. We can, however, describe the dynamics of the set $\mathcal{E}_{\mathcal{A}}(\tau)$ by studying sequences of measures $\nu_n \in \mathcal{M}_\Phi$ such that the associated measures $\mu_n \in \mathcal{M}_\sigma$ satisfy

$$\lim_{n\to\infty} \int \tau d\mu_n = \infty.$$

In our first result we show that a measure of sufficiently large entropy cannot give too much weight to the set of points for which the return time to the base is very high. More precisely,

THEOREM 3.3: *Let (Y,Φ) be a finite entropy suspension flow defined over an infinite entropy countable Markov shift with roof function bounded away from zero. Assume that $s_\infty < h_{top}(\Phi)$ and let $c \in (s_\infty, h_{top}(\Phi))$. Then there exists a constant $C > 0$ such that for every $\nu \in \mathcal{M}_\Phi$ with $h_\nu(\Phi) \geqslant c$, we have that*

$$\int \tau d\mu \leqslant C.$$

Proof. Let $\nu \in \mathcal{M}_\Phi$ with $h_\nu(\Phi) = c$ and let $\mu \in \mathcal{M}_\sigma$ be the invariant measure satisfying $\nu = (\mu \times m)/((\mu \times m)(Y))$. By the Abramov formula we have

$$h_\mu(\sigma) - c \int \tau d\mu = 0.$$

We will consider the straight line $L(t) := h_\mu(\sigma) - t \int \tau d\mu$. Note that $L(c) = 0$ and $L(0) = h_\mu(\sigma)$. Let $s \in (s_\infty, c)$. Note that $P_\sigma(-s\tau) < \infty$ and by the variational principle $L(s) \leqslant P_\sigma(-s\tau)$. This remark readily implies a bound on the slope of $L(t)$. Indeed,

$$\int \tau d\mu \leqslant \frac{P_\sigma(-s\tau)}{c-s}.$$

Thus the constant $C = P_\sigma(-s\tau)/(c-s)$ satisfies the theorem. In order to obtain the best possible constant we have to compute the infimum of the function defined for $s \in (s_\infty, c)$ by

$$s \mapsto \frac{P_\sigma(-s\tau)}{c-s}. \qquad \blacksquare$$

Remark 3.4: We stress that the constant C in Theorem 3.3 depends only on the entropy bound c and not on the measure ν.

Remark 3.5: In Section 4 we will see that in the geometrical context of geodesic flows the assumption $s_\infty < h_{top}(\Phi)$ in Theorem 3.3 has a very natural interpretation. Indeed, it will be shown to be equivalent to the parabolic gap property (see [DP98, Section III] or Definition 4.12 for precise definitions).

COROLLARY 3.6: *If (Σ^+, σ) is a Markov shift defined countable alphabet satisfying the BIP condition, then the best possible constant $C \in \mathbb{R}$ in Theorem 3.3 is given by*

$$C = \frac{P_\sigma(-s_m\tau)}{c - s_m} = \int \tau d\mu_{s_m},$$

where $s_m \in \mathbb{R}$ is such that the equilibrium measure μ_{s_m} for $-s_m\tau$ satisfies

$$c = \frac{h_{\mu_{s_m}}(\sigma)}{\int \tau d\mu_{s_m}}.$$

Proof. Since the system satisfies the BIP condition, the function $P_\sigma(-s\tau)$, when finite, is differentiable (see Theorem 2.3). Moreover, its derivative is given by (see [Sar15, Theorem 6.5])

$$\frac{d}{ds}P_\sigma(-s\tau)\Big|_{s=s_m} = -\int \tau d\mu_{s_m},$$

where μ_{s_m} is the (unique) equilibrium measure for $-s_m\tau$. The critical points of the function $s \mapsto \frac{P_\sigma(-s\tau)}{c-s}$ are those which satisfy

$$(c-s)P'_\sigma(-s\tau) + P_\sigma(-s\tau) = 0. \tag{6}$$

Equivalently,

$$-(c-s)\int \tau d\mu_s + h_{\mu_s}(\sigma) - s\int \tau d\mu_s = 0.$$

Therefore, equation (6) is equivalent to

$$c = \frac{h_{\mu_s}(\sigma)}{\int \tau d\mu_s}. \qquad \blacksquare$$

In the next Theorem we prove that the entropy of the flow on $\mathcal{E}_{\mathcal{A}}(\tau)$ is bounded above by s_∞ and that, under some additional assumptions, it is actually equal to it. This result could be thought of as a symbolic estimation for the entropy of a flow in a cusp. Theorem 3.7 describes a phenomenon first observed in [FJLR15, Lemma 2.5] in a dimension theory context and used in the setting of suspension flows in [IJ13].

THEOREM 3.7: *Let (Y, Φ) be a finite entropy suspension flow defined over an infinite entropy countable Markov shift and with roof function bounded away from zero. Assume that $s_\infty < h_{top}(\Phi)$. Let $(\nu_n)_n \subset \mathcal{M}_\Phi$ be a sequence of invariant probability measures for the flow of the form*

$$\nu_n = \frac{\mu_n \times m}{(\mu_n \times m)(Y)},$$

where $\mu_n \in \mathcal{M}_\sigma$. If $\lim_{n\to\infty} \int \tau d\mu_n = \infty$, then

$$\limsup_{n\to\infty} h_{\nu_n}(\Phi) \leqslant s_\infty.$$

Moreover, there exists a sequence $(\nu_n)_n \in \mathcal{M}_\Phi$ such that $\lim_{n\to\infty} \int \tau d\mu_n = \infty$ and

$$\lim_{n\to\infty} h_{\nu_n}(\Phi) = s_\infty.$$

Proof. Observe that the first claim is a direct consequence of Theorem 3.3. Let us construct now a sequence $(\nu_n)_n \in \mathcal{M}_\Phi$ with $\lim_{n\to\infty} \int \tau d\mu_n = \infty$ such that $\lim_{n\to\infty} h_{\nu_n}(\Phi) = s_\infty$. First note that it is a consequence of the approximation property of the pressure, that there exists a sequence of compact invariant sets $(K_N)_N \subset \Sigma$ such that $\lim_{N\to\infty} P_{K_N}(-t\tau) = P_\sigma(-t\tau)$. In particular, for every $n \in \mathbb{N}$ we have that

$$\lim_{N\to\infty} P_{K_N}\left(-\left(s_\infty - 1/n\right)\tau\right) = \infty. \tag{7}$$

For the same reason, for any $n \in \mathbb{N}$ and $N \in \mathbb{N}$ we have that

$$P_{K_N}\left(-\left(s_\infty + 1/n\right)\tau\right) \leqslant P_\sigma\left(-\left(s_\infty + 1/n\right)\tau\right) < \infty. \tag{8}$$

Thus, given $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that

$$n^2 < \frac{P_{K_N}\left(-\left(s_\infty - 1/n\right)\tau\right) - P_{K_N}\left(-\left(s_\infty + 1/n\right)\tau\right)}{2/n}.$$

Since the function $t \mapsto P_{K_N}(-t\tau)$ is real analytic, by the mean value theorem there exists $t_n \in [s_\infty - 1/n, s_\infty + 1/n]$ such that $P'_{K_N}(-t_n\tau) > n^2$. Denote by

μ_n the equilibrium measure for $-t_n\tau$ in K_N. We have that

$$n^2 < P'_{K_N}(-t_n\tau) = \int \tau d\mu_n.$$

In particular, the sequence $(\mu_n)_n$ satisfies

$$\lim_{n\to\infty} \int \tau d\mu_n = \infty.$$

Since $s_\infty < h_{top}(\Phi)$, we have that for $n \in \mathbb{N}$ large enough

$$h_{\mu_n}(\sigma) - t_n \int \tau d\mu_n > 0.$$

In particular,

$$t_n < \frac{h_{\mu_n}(\sigma)}{\int \tau d\mu_n}.$$

Since $t_n \in (s_\infty - 1/n, s_\infty + 1/n)$, we have that

$$s_\infty = \lim_{n\to\infty} t_n \leqslant \lim_{n\to\infty} \frac{h_{\mu_n}(\sigma)}{\int \tau d\mu_n} = \lim_{n\to\infty} h_{\nu_n}(\Phi). \tag{9}$$

But we already proved that the limit cannot be larger than s_∞, thus the result follows. ∎

4. The geodesic flow on extended Schottky groups

4.1. Some preliminaries in negative curvature. Let X be a Hadamard manifold with pinched negative sectional curvature, that is a complete simply connected Riemannian manifold whose sectional curvature K satisfies $-b^2 \leqslant K \leqslant -1$ (for some fixed $b \geqslant 1$). Denote by ∂X the boundary at infinity of X. Finally, denote by d the Riemannian distance on X. A crucial object in the study of the dynamics of the geodesic flow is the Busemann function. Let $\xi \in \partial X$ and $x, y \in X$. For every geodesic ray $t \mapsto \xi_t$ pointing to ξ, the limit

$$B_\xi(x,y) := \lim_{t\to\infty} [d(x,\xi_t) - d(y,\xi_t)]$$

always exists, and is independent of the geodesic ray ξ_t since X has negative sectional curvature. The **Busemann function** $B : \partial X \times X^2 \to \mathbb{R}$ is the continuous function defined as $B(\xi, x, y) \mapsto B_\xi(x,y)$. A (open) **horoball** based in ξ and passing through x is the set of $y \in X$ such that $B_\xi(x,y) > 0$. In the hyperbolic case, when $X = \mathbb{D}$, a horoball based in $\xi \in \mathbb{S}^1$ and passing through $x \in \mathbb{D}$ is the interior of a euclidean circle containing x and tangent to $\mathbb{S}^1$ at ξ.

Recall that every isometry of X can be extended to a homeomorphism of $X \cup \partial X$. A very important property of the Busemann function is the fact that it is invariant by isometries. More precisely, if $\varphi : X \to X$ is an isometry of X, then for every $x, y \in X$ we have

$$B_{\varphi\xi}(\varphi x, \varphi y) = B_\xi(x, y). \tag{10}$$

Let $o \in X$ be a reference point, which is often called the **origin** of X. The unit tangent bundle T^1X of X can be identified with $\partial^2 X \times \mathbb{R}$ via Hopf's coordinates, where $\partial^2 X = (\partial X \times \partial X)\backslash$diagonal. A vector $v \in T^1X$ is identified with $(v^-, v^+, B_{v^+}(o, \pi(v)))$, where v^- (resp. v^+) is the negative (resp. positive) endpoint of the geodesic defined by v. Here $\pi : T^1X \to X$ is the natural projection of a vector to its base point. Observe that the geodesic flow $(g_t) : T^1X \to T^1X$ acts by translation in the third coordinate of this identification. Another crucial object in this setting is the Poincaré series.

Definition 4.1: Let G be a discrete subgroup of isometries of X and let $x \in X$. The **Poincaré series** $P_G(s, x)$ associated with G is defined by

$$P_G(s, x) := \sum_{g \in G} e^{-sd(x, gx)}.$$

The **critical exponent** δ_G of G is the number

$$\delta_G := \inf \{s \in \mathbb{R} : P_G(s, x) < \infty\}.$$

The group G is said to be of **divergence type** (resp. **convergence type**) if $P_G(\delta_G, x) = \infty$ (resp. $P_G(\delta_G, x) < \infty$).

Remark 4.2: By the triangle inequality, the critical exponent of a discrete group of isometries of X is independent of $x \in X$. Moreover, as the sectional curvature of X is bounded from below, it is finite.

The isometries of X are categorized in three types: those fixing a unique point in X called **elliptic isometries**, those fixing a unique point in ∂X called **parabolic isometries**, and finally, those fixing uniquely two points in ∂X called **hyperbolic isometries**. For g a non-elliptic isometry of X, denote by δ_g the critical exponent of the group $< g >$. If g is a hyperbolic isometry, it is fairly straightforward to see that $\delta_g = 0$ and that the group $< g >$ is of divergence type. If g is parabolic, it was shown in [DP98, Theorem III.1] that $\delta_g \geqslant \frac{1}{2}$.

Let Γ be a discrete subgroup of isometries of X. Denote by Λ the limit set of Γ, that is, the set

$$\Lambda = \overline{\Gamma \cdot o} \backslash \Gamma \cdot o.$$

The group Γ is **non-elementary** if Λ contains infinitely many elements. We recall the following fact proved in [DOP00, Proposition 2].

Theorem 4.3: *Let Γ be a non-elementary discrete subgroup of isometries of a Hadamard manifold X. If G is a divergence type subgroup of Γ and its limit set is strictly contained in the limit set of Γ, then $\delta_\Gamma > \delta_G$.*

In particular, if Γ is a non-elementary discrete group of isometries and there is an element $g \in \Gamma$ such that $< g >$ is of divergence type, then $\delta_\Gamma > \delta_{<g>}$ (see also [DP98, Theorem III.1]). Note that a non-elementary group always contains a hyperbolic isometry (in fact, infinitely many non-conjugate of them), hence a non-elementary group Γ always satisfies $\delta_\Gamma > 0$.

We end this subsection by giving an important relation between the critical exponent of a group and the topological entropy of the geodesic flow on the associated quotient manifold. Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be a non-elementary torsion free discrete subgroup of isometries of X. Denote by

$$(g_t) : T^1X/\Gamma \to T^1X/\Gamma$$

the geodesic flow on the unit tangent bundle of the quotient manifold X/Γ. Otal and Peigné [OP04, Theorem 1] proved that, if the derivatives of the sectional curvature are uniformly bounded, then the topological entropy $h_{top}(g)$ of the geodesic flow equals the critical exponent of the Poincaré series of the group Γ, that is

$$h_{top}(g) = \delta_\Gamma. \tag{11}$$

We stress the fact that the assumption on the derivatives of the sectional curvature is crucial in order to compute the topological entropy of the geodesic flow. This assumption implies the Hölder regularity of the strong unstable and stable foliations (see for instance [PPS15, Theorem 7.3]), which is used in the proof of [OP04, Theorem 1].

4.2. The symbolic model for extended Schottky groups. In this section we recall the definition of an extended Schottky group. To the best of our knowledge this definition was introduced in [DP98] by Dal'bo and Peigné. The basic idea is to extend the classical notion of Schottky groups to the context of manifolds where the non-wandering set of the geodesic flow is non-compact.

Let X be a Hadamard manifold as in Subsection 4.1. Let N_1, N_2 be two non-negative integers such that $N_1 + N_2 \geqslant 2$ and $N_2 \geqslant 1$. Consider N_1 hyperbolic isometries $h_1, \dots, h_{N_1}$ and N_2 parabolic ones $p_1, \dots, p_{N_2}$ satisfying the following conditions:

(C1) For $1 \leqslant i \leqslant N_1$ there exist in ∂X a compact neighbourhood C_{h_i} of the attracting point ξ_{h_i} of h_i and a compact neighbourhood $C_{h_i^{-1}}$ of the repelling point $\xi_{h_i^-}$ of h_i, such that

$$h_i(\partial X \backslash C_{h_i^{-1}}) \subset C_{h_i}.$$

(C2) For $1 \leqslant i \leqslant N_2$ there exists in ∂X a compact neighbourhood C_{p_i} of the unique fixed point ξ_{p_i} of p_i, such that

$$\forall n \in \mathbb{Z}^* \quad p_i^n(\partial X \backslash C_{p_i}) \subset C_{p_i}.$$

(C3) The $2N_1 + N_2$ neighbourhoods introduced in (1) and (2) are pairwise disjoint.

(C4) The elementary parabolic groups $< p_i >$, for $1 \leqslant i \leqslant N_2$, are of divergence type.

The group

$$\Gamma = < h_1, \dots, h_{N_1}, p_1, \dots, p_{N_2} >$$

is a non-elementary free group which acts properly discontinuously and freely on X (see [DP98, Corollary II.2]). Such a group Γ is called an **extended Schottky group**. Note that if $N_2 = 0$, that is the group Γ only contains hyperbolic elements, then Γ is a classical Schottky group and its geometric and dynamical properties are well understood. Indeed, in that case, the non-wandering set $\Omega \subset T^1X/\Gamma$ of the geodesic flow is compact, which implies that $(g_t)|_\Omega$ is an Axiom A flow. If $N_2 \geqslant 1$, then X/Γ is a non-compact manifold and Ω is a non-compact subset of T^1X/Γ. Figure 1 is an example of a Schottky group acting on the hyperbolic disk $\mathbb{D}$. It has two generators, one hyperbolic and the other parabolic.

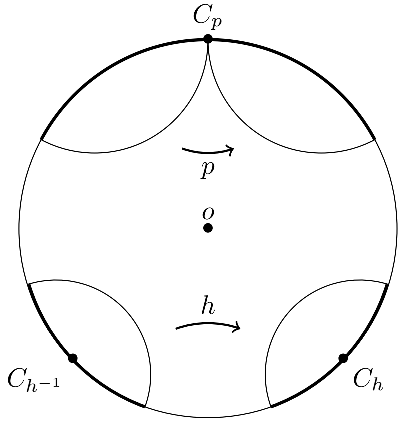

Figure 1. Schottky group $\Gamma =< h, p >$.

In [DP98] the authors proved that there exists a (g_t)-invariant subset Ω_0 of T^1X/Γ, contained in the non-wandering set of (g_t), such that $(g_t)|_{\Omega_0}$ is topologically conjugated to a suspension flow over a countable Markov shift (Σ, σ). The Theorem below summarizes their construction together with some dynamical properties.

Theorem 4.4: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group. Then there exists a (g_t)-invariant subset Ω_0 of T^1X/Γ, a countable Markov shift (Σ, σ) and a function $\tau : \Sigma \to \mathbb{R}$, such that:*

(1) *the function τ is locally Hölder and bounded away from zero,*
(2) *the geodesic flow $(g_t)|_{\Omega_0}$ over Ω_0 is topologically conjugated to the suspension flow over Σ with roof function τ,*
(3) *the Markov shift (Σ, σ) satisfies the BIP condition,*
(4) *if $N_1 + N_2 \geqslant 3$, then (Σ, σ) is topologically mixing.*

Proof. Let $\mathcal{A} = \{h_1, \ldots, h_{N_1}, p_1, \ldots, p_{N_2}\}$ and consider the symbolic space Σ defined by

$$\Sigma = \{(a_i^{m_i})_{i\in\mathbb{Z}} : a_i \in \mathcal{A}, m_i \in \mathbb{Z} \text{ and } a_{i+1} \neq a_i \forall i \in \mathbb{Z}\}.$$

Note that the space Σ is a sequence space defined on the countable alphabet $\{a_i^m : a_i \in \mathcal{A}, m \in \mathbb{Z}\}$. Let Λ^0 be the limit set Λ minus the Γ-orbit of the fixed points of the elements of $\mathcal{A}$. We denote by $\tilde{\Omega}_0$ the set of vectors in T^1X

identified with $(\Lambda^0 \times \Lambda^0 \backslash \text{diagonal}) \times \mathbb{R}$ via Hopf's coordinates. Finally, define $\Omega_0 := \tilde{\Omega}_0/\Gamma$, where the action of Γ is given by

$$\gamma \cdot (\xi^-, \xi^+, s) = (\gamma(\xi^-), \gamma(\xi^+), s - B_{\xi^+}(o, \gamma^{-1}0)).$$

Observe that Ω_0 is invariant by the geodesic flow.

Fix now $\xi_0 \in \partial X \backslash \bigcup_{a \in \mathcal{A}} C_{a^\pm}$, where $C_{a^\pm} = C_a \cup C_{a^{-1}}$. Dal'bo and Peigné [DP98, Property II.5] established the following coding property: for every $\xi \in \Lambda^0$ there exists a unique sequence $\omega(\xi) = (a_i^{m_i})_{i \geqslant 1}$ with $a_i \in \mathcal{A}$, $m_i \in \mathbb{Z}^*$ and $a_{i+1} \neq a_i$ such that

$$\lim_{k \to \infty} a_1^{m_1} \cdots a_k^{m_k} \xi_0 = \xi.$$

For each $a \in \mathcal{A}$ define $\Lambda^0_{a^\pm} = \Lambda^0 \cap C_{a^\pm}$ and set $\partial^2 \Lambda^0 = \bigcup_{\substack{\alpha, \beta \in \mathcal{A} \\ \alpha \neq \beta}} \Lambda^0_{\alpha^\pm} \times \Lambda^0_{\beta^\pm}$. For any pair $(\xi^-, \xi^+) \in \partial^2 \Lambda^0$, if a^m is the first term of the sequence $\omega(\xi^+)$, define $\tilde{\tau}_0(\xi^+) = B_{\xi^+}(o, a^m o)$ and $\overline{T}(\xi^-, \xi^+) = (a^{-m}\xi^-, a^{-m}\xi^+)$. Define also $\overline{T}_{\tilde{\tau}_0}$ by the formula

$$\overline{T}_{\tilde{\tau}_0}(\xi^-, \xi^+, s) = (\overline{T}(\xi^-, \xi^+), s - \tilde{\tau}_0(\xi^+)).$$

Observe that $\overline{T}_{\tilde{\tau}_0}$ maps $\partial^2 \Lambda^0 \times \mathbb{R}$ to itself.

Lemma 4.5: *The set Ω_0 can be identified with the quotient $\partial^2 \Lambda^0 \times \mathbb{R} / < \overline{T}_{\tilde{\tau}_0} >$.*

Proof. Let $(\xi^-, \xi^+) \in \partial^2 \Lambda^0$. The geodesic determined by (ξ^-, ξ^+) in X intersects the horosphere based in ξ^+ and passing through o in only one point x^0_{o,ξ^-,ξ^+}. Denote by $v^0_{o,\xi^-,\xi^+} \in T^1 X$ the unit vector based in x^0_{o,ξ^-,ξ^+} and pointing to ξ^+. Finally, set

$$S = \{v^0_{o,\xi^-,\xi^+} : (\xi^-, \xi^+) \in \partial^2 \Lambda^0\} \subset T^1 X/\Gamma. \tag{12}$$

To prove this lemma we first observe that the set S is a cross-section in $T^1 X/\Gamma$, so for any $v \in \Omega_0$ there exists a minimal time $t \geqslant 0$ such that $g_{-t} v \in S$. If v^0_{o,ξ^-,ξ^+} denotes the vector $g_{-t} v$, then (ξ^-, ξ^+, t) corresponds to v. Observe that $g_s v \notin S$ for every $0 \leqslant s < \tilde{\tau}_0(\xi^+) - t$, so $\tilde{\tau}_0(\xi^+)$ is the first return time of v^0_{o,ξ^-,ξ^+} into S. This give us the identification. ∎

The coding property implies that the set $\partial^2 \Lambda^0$ is identified with Σ by considering (ξ^-, ξ^+) as a bilateral sequence $(\omega^*(\xi^-), \omega(\xi^+))$. If $\omega(\xi^-) = (b_i^{n_i})_{i \geqslant 1}$, we define $\omega^*(\xi^-)$ as the sequence $(\ldots, b_2^{-n_2}, b_1^{-n_1})$; then $(\omega^*(\xi^-), \omega(\xi^+))$ represent the concatenated sequence. Let Σ^+ be the one-sided symbolic space obtained from Σ by forgetting the negative time coordinates. We define the function $\tau_0 : \Sigma^+ \to \mathbb{R}$ as

$$\tau_0(x) = \tau(\omega^{-1}(x)) = B_{\omega^{-1}(x)}(o, a^m o),$$

where $w : \Lambda^0 \to \Sigma$ is the coding function and a^m the first symbol in $w^{-1}(x)$. We extend τ to Σ by making it independent of the negative time coordinates.

Lemma 4.6: *The function $\tau_0 : \Sigma \to \mathbb{R}$ is cohomologous to a Hölder-continuous positive function $\tau : \Sigma \to \mathbb{R}$ bounded away from zero and depending only on future coordinates.*

Proof. By [DP98, Proposition V.1], there exists $N \geqslant 1$ such that for every $n \geqslant N$ and every $x \in \Sigma$, we have

$$\sum_{i=0}^{n} \tau_0(\sigma^i(x)) \geqslant c > 0.$$

Let $\epsilon = \frac{1}{N}$ and $m_i = 1 - i\epsilon$ for every $i = 0, \ldots, N$. Define the function $f : \Sigma \to \mathbb{R}$ as

$$f(x) = \sum_{i=0}^{N-1} m_i \tau_0(\sigma^i(x)).$$

Then

$$\begin{aligned} f(x) - f(\sigma(x)) &= \sum_{i=0}^{N-1} m_i \tau_0(\sigma^i(x)) - \sum_{i=0}^{N-1} m_i \tau_0(\sigma^{i+1}(x)) \\ &= m_0 \tau_0(x) - m_N \tau_0(\sigma^N(x)) + \sum_{i=1}^{N} m_i \tau_0(\sigma^i(x)) - \sum_{i=1}^{N} m_{i-1} \tau_0(\sigma^i(x)). \end{aligned}$$

Since $m_0 = 1$, $m_N = 0$ and $m_i - m_{i-1} = -\epsilon$ for every $i = 1, \ldots, N$, we get

$$f(x) - f(\sigma(x)) = \tau_0(x) - \epsilon \sum_{i=1}^{N} \tau_0(\sigma^i(x)). \tag{13}$$

Define the function $\tau : \Sigma \to \mathbb{R}$ as $\tau(x) = \epsilon \sum_{i=1}^{N} \tau_0(\sigma^i(x))$. It is positive bounded away from zero since $\tau(x) \geqslant \frac{c}{N}$ and depends only on future coordinates by construction. By equation (13) it is cohomologous to τ_0. Finally, the Hölder-regularity of τ follows from the Hölder-regularity of τ_0, which is proved in [DP98, Lemma VII] together with a standard recoding argument. All points above conclude the proof of this lemma. ∎

Lemma 4.6 implies (1) in the conclusion of Theorem 4.4. Using the identification given by Lemma 4.5 together with the previous construction, we deduce that the geodesic flow (g_t) on Ω_0 can be coded as the suspension flow on

$$Y = \{(x,t) \in \Sigma \times \mathbb{R} : 0 \leqslant t \leqslant \tau(x)\}/\sim,$$

which gives us (2).

Lemma 4.7: *Under the hypothesis of Theorem 4.4 the countable Markov shift* (Σ, σ) *satisfies the BIP condition. Moreover, if* $N_1 + N_2 \geqslant 3$, *then the countable Markov shift* (Σ, σ) *is topologically mixing.*

Proof. It is not hard to see that the set $\mathcal{A}$ satisfies the required conditions in order for (Σ, σ) to be BIP (see Definition 2.2). Suppose now that $N_1 + N_2 \geqslant 3$. Recall that the Markov shift (Σ, σ) is topologically mixing if for every $a, b \in \{a_i^m : a_i \in \mathcal{A}, m \in \mathbb{Z}\}$ there exists $N(a,b) \in \mathbb{N}$, such that for every $n > N(a,b)$ there exists an admissible word of length n of the form $ai_1i_2 \cdots i_{n-1}b$. The set of allowable sequences is given by

$$\{(a_i^{m_i})_{i \in \mathbb{Z}} : a_i \in \mathcal{A}, m_i \in \mathbb{Z} \text{ and } a_{i+1} \neq a_i \forall i \in \mathbb{N}\}.$$

Since $N_1 + N_2 \geqslant 3$, then given any pair of symbols in $\{a_i^m : a_i \in \mathcal{A}, m \in \mathbb{Z}\}$, say $a_1^{m_1}, a_2^{m_2}$, we can consider the symbol $a_3 \notin \{a_1, a_2\}$. Hence the following words are admissible:

$$a_1^{m_1} a_3 a_1 a_3 \cdots a_1 a_2^{m_2}, \quad a_1^{m_1} a_3 a_1 a_3 a_1 \cdots a_3 a_2^{m_2}.$$

Thus the system is topologically mixing. ∎

Since Lemma 4.7 above shows the points (3) and (4), we have concluded the proof of Theorem 4.4. ∎

Remark 4.8: Under the condition $N_1 + N_2 \geqslant 3$ we have proved that (Σ, σ) is a topologically mixing countable Markov shift satisfying the BIP condition (Lemma 4.7) and that the roof function τ is locally Hölder and bounded away from zero (Lemma 4.6). Therefore, the associated suspension semi-flow (Y, Φ) can be studied with the techniques presented in Section 2.

So far we have proved that the geodesic flow restricted to the set Ω_0 can be coded by a suspension flow over a countable Markov shift. We now describe, from the ergodic point of view, the geodesic flow in the complement

$\left(T^1X/\Gamma\right)\backslash\Omega_0$ of Ω_0. We denote by $\mathcal{M}_{\Omega_0}$ the space of (g_t)-invariant probability measures supported in the set where we have coding, in other words in $\partial^2\Lambda_0 \times \mathbb{R}/ < \overline{T}_\tau >$. We describe the difference between the space $\mathcal{M}_{\Omega_0}$ and the space $\mathcal{M}_g$ of all (g_t)-invariant probability measures. Recall that in Γ there are hyperbolic isometries $h_1, \ldots, h_{N_1}$, each of which fixes a pair of points in ∂X. The geodesic connecting the fixed points of h_i will descend to a closed geodesic in the quotient by Γ. We denote ν^{h_i} the probability measure equidistributed along such a geodesic.

Proposition 4.9: *The set of ergodic measures in $\mathcal{M}_g\backslash\mathcal{M}_{\Omega_0}$ is finite, those are exactly the set $\{\nu^{h_i} : 1 \leqslant i \leqslant N_1\}$. Moreover, for every $\nu \in \mathcal{M}_g$ with support in $X\backslash\Omega_0$, we have $h_\nu(g) = 0$.*

Proof. Let $\nu \in \mathcal{M}_g\backslash\mathcal{M}_{\Omega_0}$ be an ergodic measure. Take v a generic vector for ν. Since a generic vector is recurrent, the orbit $g_t v$ does not go to infinity, therefore v^+ is not parabolic. Now consider the case when v points toward a hyperbolic fixed point z. Let $\gamma : \mathbb{R} \to X$ be a geodesic flowing at positive time to z with initial condition $\gamma'(0) = v$ and let γ_i be the geodesic connecting z with the associated hyperbolic fixed point. By reparametrization we can assume

$$\gamma_i(+\infty) = z$$

and that $\gamma_i(0)$ lies in the same horosphere centered at z rather than v. By estimates in [HIH77] we have $d(\gamma_i(t), \gamma(t)) \to 0$ exponentially fast (here d stands for hyperbolic distance, actually in [HIH77] the stronger exponential decay in the horospherical distance is obtained). Since the vectors along the geodesics are perpendicular to the horospheres centered at z, we have the desired geometric convergence in TX. Observe that γ_i descends to a periodic orbit in TX/Γ. This gives the convergence of γ to the periodic orbit and the Birkhoff ergodic theorem gives that the measure generated by such a geodesic is exactly one of ν^{h_i}.

The fact that $h_\nu(g) = 0$ for every ergodic $\nu \in \mathcal{M}_g\backslash\mathcal{M}_{\Omega_0}$ is a classical result for measures supported on periodic orbits. ∎

4.3. Geometric meaning of s_∞. One of our main technical results is the following. Let τ be the roof function constructed in subsection 4.2. In the next Theorem we give a geometrical characterisation of the value s_∞ defined as the unique real number satisfying

$$P_\sigma(-t\tau) = \begin{cases} \text{infinite} & \text{if } t < s_\infty; \\ \text{finite} & \text{if } t > s_\infty. \end{cases}$$

One of the key ingredients in this paper, and the important result of this section, is the relation between s_∞ and the largest parabolic critical exponent. This relation will allow us to translate several results at the symbolic level into the geometrical one.

THEOREM 4.10: *Let Γ be an extended Schottky group satisfying $N_1 + N_2 \geqslant 3$. Let (Σ, σ) and $\tau : \Sigma \to \mathbb{R}$ be the base space and the roof function of the symbolic representation of the geodesic flow (g_t) on Ω_0. Then $s_\infty = \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$.*

Proof. Since τ is cohomologous to τ_0, we have $P_\sigma(-t\tau) = P_\sigma(-t\tau_0)$. We first show that $s_\infty \leqslant \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$. Now

$$\begin{aligned} P_\sigma(-t\tau_0) &= \lim_{n\to\infty} \frac{1}{n+1} \log \sum_{x:\sigma^{n+1}x=x} \exp\left(\sum_{i=0}^{n} -t\tau_0(\sigma^i x)\right) \chi_{C_{h_1}}(x) \\ &= \lim_{n\to\infty} \frac{1}{n+1} \log \sum_{\xi=\overline{h_1 x_2 \dots x_n x_{n+1}}\xi_0} \exp\left(\sum_{i=0}^{n} -tB_{\omega^{-1}(\sigma^i x)}(o, x_{i+1}o)\right) \\ &\geqslant \lim_{n\to\infty} \frac{1}{n+1} \log \sum_{\xi=\overline{h_1 x_2 \dots x_{n+1}}\xi_0} \exp\left(\sum_{i=0}^{n} -td(o, x_{i+1}o)\right). \end{aligned}$$

The last inequality follows from $d(x,y) \geqslant B_\xi(x,y)$. By removing words having h_1^m (some m) in more places than just the first coordinate, we conclude that the argument of the function log in the limit above is greater than

$$e^{-td(o,h_1o)} \sum_{(c_1,\dots,c_n)\in(\mathcal{A}\backslash h_1)^n_*} \sum_{(m_1,\dots,m_n)\in\mathbb{Z}^n} \exp\left(\sum_{i=1}^{n} -td(o, c_i^{m_i} o)\right),$$

where $(\mathcal{A}\backslash h_1)^n_*$ represent the set of admissible words of length n for the code, i.e. $c_i \neq c_{i+1}^{\pm 1}$. Let $k \geqslant 1$. For all $0 \leqslant j \leqslant k-1$ and $1 \leqslant i \leqslant N_1 + N_2 - 1$, define

$$b_{i+j(N_1+N_2-1)} = \begin{cases} h_{i+1}, & \text{if } 1 \leqslant i \leqslant N_1 - 1, \\ p_{i+1-N_1}, & \text{if } N_1 \leqslant i \leqslant N_1 + N_2 - 1. \end{cases}$$

Consider $n+1 = k(N_1 + N_2 - 1)$. By restricting the above sum to words with $c_i = b_i$ for every $i = 1, \dots, n$, we can continue the sequence of inequalities above to get

$$P_\sigma(-t\tau) \geqslant \sum_{m_1,\ldots,m_n\in\mathbb{Z}} \exp\left(\sum_{i=1}^{n} -td(o, b_i^{m_i} o)\right),$$

where the right-hand side is equal to

$$\prod_{i=1}^{n}\sum_{m\in\mathbb{Z}} \exp(-td(o, b_i^{m} o)).$$

By definition of the b_i's, the last term is equal to

$$\left(\prod_{i=2}^{N_1}\sum_{m\in\mathbb{Z}} \exp(-td(o, h_i^{m} o))\right)^k \left(\prod_{i=1}^{N_2}\sum_{m\in\mathbb{Z}} \exp(-td(o, p_i^{m} o))\right)^k.$$

Hence, it follows that

$$\begin{aligned} &P_\sigma(-t\tau)\\ &\quad\geqslant \frac{1}{N_1+N_2}\log\left(\prod_{i=2}^{N_1}\sum_{m\in\mathbb{Z}} \exp(-td(o, h_i^{m} o))\right)\left(\prod_{i=1}^{N_2}\sum_{m\in\mathbb{Z}} \exp(-td(o, p_i^{m} o))\right)\\ &\quad= \frac{1}{N_1+N_2}\log\prod_{a\in\mathcal{A}\backslash h_1} P_{<a>}(t,o). \end{aligned}$$

In particular, if $t < \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$ then $P_\sigma(-t\tau) = +\infty$. This shows that $s_\infty \geqslant \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$.

Before proving the other inequality we need to prove first a technical lemma. Let $\mathcal{A}^\pm = \{h_1^{\pm1}, \ldots, h_{N_1}^{\pm1}, p_1, \ldots, p_{N_2}\}$ and consider for every $a \in \mathcal{A}^\pm$ the convex hull U_a in $X \cup \partial X$ of the set C_a.

Lemma 4.11: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group. Fix $o \in X$. Then there exists a universal constant $C > 0$ (depending only on the generators of Γ and the fixed point o) such that for every $a_1, a_2 \in \mathcal{A}^\pm$ satisfying $a_1 \neq a_2^{\pm1}$, and for every $x \in U_{a_1}$ and $y \in U_{a_2}$, we have*

$$d(x,y) \geqslant d(x,o) + d(y,o) - C. \tag{14}$$

Proof. Since C_{a_1} and C_{a_2} are disjoint, for every $a_1, a_2 \in \mathcal{A}^\pm$ satisfying $a_1 \neq a_2^{\pm1}$, the same happens for the sets U_{a_1} and U_{a_2}. Let $x \in U_{a_1}$ and $y \in U_{a_2}$. The geodesic segments $[o,x]$ and $[o,y]$ form an angle uniformly bounded below, hence $d(x,y) \geqslant d(x,o) + d(y,o) - C$ for a universal constant $C > 0$. ∎

Let (ξ_t^i) be the geodesic ray $[o, \omega^{-1}(\sigma^{i+1}x))$. Using (14), we have

$$\begin{aligned}\tau_0(\sigma^i x) &= B_{\omega^{-1}(\sigma^i x)}(o, x_i o)\\ &= B_{\omega^{-1}(\sigma^{i+1}x)}(x_i^{-1}o, o)\\ &= \lim_{t\to\infty} d(\xi_t^i, x_i o) - d(\xi_t^i, o)\\ &\geqslant [d(\xi_t^i, o) + d(o, x_i o) - C] - d(\xi_t^i, o)\\ &= d(o, x_i o) - C.\end{aligned}$$

Thus

$$\exp(-t\tau_0(\sigma^i x)) \leqslant \exp(tC)\exp(-td(o, x_i o)).$$

Therefore

$$\begin{aligned}P_\sigma(-t\tau_0) &\leqslant \lim_{n\to\infty}\frac{1}{n}\log\sum_{a_1,\dots,a_n}\sum_{m_1,\dots,m_n}\prod_{i=1}^{n}\exp(tC)\exp(-td(o, a_i^{m_i}o))\\ &= \log\left(C^t\prod_{a\in\mathcal{A}}P_{<a>}(t,o)\right).\end{aligned}$$

In particular,
the pressure $P_\sigma(-t\tau)$ is finite for every $t > \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$. ∎

Denote $\delta_{p,\max} := \max\{\delta_{p_i}, 1 \leqslant i \leqslant N_2\}$. The simplest example to consider is a real hyperbolic space X. In this case $\delta_{<p_i>} = 1/2$ for any $i \in \{1, \dots, N_2\}$. In particular, $\delta_{p,\max} = 1/2$. More generally, if we replace hyperbolic space by a manifold of constant negative curvature equal to $-b^2$, then $\delta_{<p>} = b/2$. In the case of non-constant curvature some bounds are known, indeed if the curvature is bounded above by $-a^2$ then $\delta_{<p>} \geqslant a/2$ (see [DOP00]).

Recall that at a symbolic level we have $h_{top}(\Phi) = \inf\{t : P_\sigma(-t\tau) \leqslant 0\}$. In particular, if the derivatives of the sectional curvature are uniformly bounded, then Theorem 4.4, Proposition 4.9 and equality (11) imply

$$h_{top}(g) = \delta_\Gamma = h_{top}(\Phi). \tag{15}$$

The existence of a measure of maximal entropy for the flow (g_t) is related to convergence properties of the Poincaré series at the critical exponent. Indeed, using the construction of Patterson and Sullivan ([Pat76], [Sul84]) of a Γ-invariant measure on $\partial^2 X$, it is possible to construct a measure on T^1X which is invariant under the action of Γ and the geodesic flow. This measure induces a (g_t)-invariant measure on T^1X/Γ called the Bowen–Margulis measure. It turns out that, if the group Γ is of convergence type, then the Bowen–Margulis measure is infinite and dissipative. Hence the geodesic flow does not have a measure of maximal entropy. On the other hand, if the group Γ is of divergence type, then the Bowen–Margulis measure is ergodic and conservative. If finite, it is the measure of maximal entropy.

It is therefore of interest to determine conditions that will ensure that the group is of divergence type and that the Bowen–Margulis measure is finite. It is along these lines that Dal'bo, Otal and Peigné [DOP00] introduced the following:

Definition 4.12: A geometrically finite group Γ satisfies the **parabolic gap condition (PGC)** if its critical exponent δ_Γ is strictly greater than the one of each of its parabolic subgroups.

It was shown in [DOP00, Theorem A] that if a group satisfies the PGC-condition, then the group is divergent and the measure of Bowen–Margulis is finite [DOP00, Theorem B]. In particular, it has a measure of maximal entropy. Note that a divergent group in the case of constant negative curvature satisfies the PGC-property.

In our context, an extended Schottky group is a geometrically finite group such that all the parabolic subgroups have rank 1. Moreover, by Condition (C4) and Theorem 4.3, it satisfies the PGC-condition. Thus, the following property is a direct consequence of Theorem 4.10, Theorem 4.3 and (15).

PROPOSITION 4.13: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. If (Y, Φ) is the symbolic representation of the geodesic flow on T^1X/Γ, then $s_\infty < h_{top}(\Phi)$.*

5. Escape of mass for geodesic flows

This section contains our main results relating the escape of mass of a sequence of invariant probability measures for a class of geodesic flows defined over noncompact manifolds. We prove that there is a uniform bound, depending only on the entropy of a measure, for the amount of mass a measure can give to the cusps. We also characterise the amount of entropy that the cusp can have. These results are similar in spirit to those obtained in [EKP15, ELMV12, KKLM17] for other types of flows.

THEOREM 5.1: *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every $c > \delta_{p,\max}$ there exists a constant $M = M(c) > 0$ such that for every $\nu \in \mathcal{M}_{\Omega_0}$ with $h_\nu(g) \geqslant c$, we have*

$$\int \tau d\mu \leqslant M,$$

where ν has a symbolic representation as $(\mu \times m)/((\mu \times m)(Y))$. Moreover, the value $\delta_{p,\max}$ is optimal in the following sense: there exists a sequence $(\nu_n) \subset \mathcal{M}_{\Omega_0}$ of g-invariant probability measures such that $\lim_{n\to\infty} \int \tau d\mu_n = \infty$ and

$$\lim_{n\to\infty} h_{\nu_n}(g) = \delta_{p,\max}.$$

Proof. This is a direct consequence of Theorems 3.3 and 3.7 using the symbolic model for the geodesic flow on T^1X/Γ. ∎

The following corollary is just an equivalence of the first conclusion in Theorem 5.1 (see also Theorem 3.7).

COROLLARY 5.2: *Assume X and Γ as in Theorem 5.1. If $(\nu_n) \subset \mathcal{M}_{\Omega_0}$ is a sequence of (g_t)-invariant probability measures such that $\lim_{n\to\infty} \int \tau d\mu_n = \infty$, then*

$$\limsup_{n\to\infty} h_{\nu_n}(g) \leqslant \delta_{p,\max}.$$

We are now in position to prove the main result about escape of mass.

THEOREM (1.2): *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every $c > \delta_{p,\max}$ there exists a constant $m = m(c) > 0$, with the following property: If (ν_n) is a sequence of ergodic (g_t)-invariant probability measures on T^1X/Γ satisfying $h_{\nu_n}(g) \geqslant c$, then for every vague limit $\nu_n \rightharpoonup \nu$, we have*

$$\nu(T^1X/\Gamma) \geqslant m.$$

In particular, if $\nu_n \rightharpoonup 0$*, then* $\limsup h_{\nu_n}(g) \leqslant \delta_{p,\max}$*. Moreover, the value* $\delta_{p,\max}$ *is optimal in the following sense: there exists a sequence* (ν_n) *of* (g_t)*-invariant probability measures on* T^1X/Γ *such that* $h_{\nu_n}(g) \to \delta_{p,\max}$ *and* $\nu_n \rightharpoonup 0$*.*

Proof. Since every ergodic measure in $\mathcal{M}_g$ with support in $X\backslash\Omega_0$ has zero entropy, we can suppose that ν_n belongs to $\mathcal{M}_{\Omega_0}$ for every $n \in \mathbb{N}$. Observe now that the cross-section $S \subset T^1X/\Gamma$ defined in (12) is bounded. Hence, using the identification $\Psi : \Omega_0 \to Y$ given by Theorem 4.4 and fixing $0 < r \leqslant \inf_{x\in\Sigma} \tau(x)$, there exists a compact set $K_r \subset T^1X/\Gamma$ such that

$$\Sigma \times [0, r]/ \sim\, \subset \Psi(K_r).$$

Let μ_n be the probability measure on Σ associated to the symbolic representation of ν_n. By Theorem 5.1, we have

$$\int \tau d\mu_n \leqslant M.$$

Hence

$$\begin{aligned}\nu_n(K_r) &= \Psi_*\nu_n(\Psi(K_r)) \geqslant \Psi_*\nu_n(\Sigma \times [0, r]/ \sim)\\ &= \frac{\int_\Sigma \int_0^r dtd\mu_n}{\int \tau d\mu_n}\\ &\geqslant \frac{r}{M}.\end{aligned}$$

In other words, every vague limit ν of the sequence of ergodic probability measures $(\nu_n)_n$ satisfies $\nu(K_r) \geqslant r/M$. In particular, we obtain $\nu(T^1X/\Gamma) \geqslant r/M$. By setting $m = \inf_{x\in\Sigma} \tau(x)/M$, the conclusion follows.

Before giving the proof of the optimality of $\delta_{p,\max}$, we need the following result.

Proposition 5.3: *Let* Γ *be an extended Schottky group of isometries of* X*. Let* $p \in \mathcal{A}$ *be a parabolic isometry. We can choose a hyperbolic isometry* $h \in \Gamma$ *for which the groups* $\Gamma_n =< p, h^n >$ *satisfy the following conditions:*

1. *The group* Γ_n *is of divergence type for every* $n \geqslant 1$*.*
2. *The sequence* $(\delta_{\Gamma_n})_n$ *of critical exponents satisfy* $\delta_{\Gamma_n} \to \delta_{\mathcal{P}}$ *as* n *goes to* ∞*.*

(3) *The following limit holds:*

$$\lim_{n\to\infty} \frac{n}{\sum_{\gamma\in\mathcal{P}} e^{-\delta_{\Gamma_n} d(x,\gamma x)}} = 0.$$

Proof. The proof is based on that of [DOP00, Theorem C]. Let G be a group; we will use the notation G^* for $G\backslash\{id\}$. Define $\mathcal{P} =< p >$ and take $U_{\mathcal{P}} \subset X \cup \partial X$ a connected compact neighbourhood of the fixed point ξ_p of p such that for every $m \in \mathbb{Z}^*$ we have $p^m(\partial X\backslash U_{\mathcal{P}}) \subset U_{\mathcal{P}}$. We could take $U_{\mathcal{P}}$ so that $U_{\mathcal{P}} \cap \partial X$ is a fundamental domain for the action of $\mathcal{P}$ in $\partial X\backslash\{\xi_p\}$. Because Γ is non-elementary and Λ_Γ is not contained in $U_{\mathcal{P}}$, it is possible to choose $h \in \Gamma$ a hyperbolic isometry of X such that its two fixed points ξ_{h^-}, ξ_h do not lie in $U_{\mathcal{P}}$. We have used the fact that a pair of points fixed by a hyperbolic isometry is dense in $\Lambda \times \Lambda$. Fix $x \in X$ over the axis of h. Since Γ is an extended Schottky group, for every $k \in \mathbb{N}$ the elements p and h^k are in Schottky position. In particular, for $H_k =< h^k >$ we can find a compact subset $U_{H_k} \subset X \cup \partial X$ satisfying the following three conditions:

(1) $H^*(\partial X\backslash U_{H_k}) \subset U_{H_k}$.
(2) $U_{H_k} \cap U_{\mathcal{P}} = \varnothing$.
(3) $x \notin U_{H_k} \cup U_{\mathcal{P}}$.

Since $\mathcal{P}$ and U_{H_k} are in Schottky position it is a consequence of the Ping Pong Lemma that the group generated by them is a free product. By the same argument as that of Lemma 4.11, there is a positive constant $C \in \mathbb{R}$ such that for every $y \in U_{H_k}$ and $z \in U_{\mathcal{P}}$, we have

$$d(y,z) \geqslant d(x,y) + d(x,z) - C. \tag{16}$$

Applying inequality (16) and the inclusion properties described above we obtain

$$d(x, p^{m_1}h^{kn_1}\cdots p^{m_j}h^{kn_j}x) \geqslant \sum_i d(x,p^{m_i}x) + \sum_i d(x,h^{kn_i}x) - 2kC, \tag{17}$$

where $m_i \in \mathbb{Z}^*$. As remarked in [DOP00] the sum

$$\tilde{P}(s) = \sum_{j\geqslant 1}\sum_{n_i,m_i\in\mathbb{Z}*} \exp(-sd(x,p^{m_1}h^{kn_1}\cdots p^{m_j}h^{kn_j}x))$$

is comparable with the Poincaré series of Γ_k. Indeed, since h is hyperbolic both have the same critical exponent. Using the inequality (17) we obtain

$$\tilde{P}(s) \leqslant \sum_{j\geqslant 1}\left(e^{2sC}\sum_{n\in\mathbb{Z}*} e^{-sd(x,h^{kn}x)}\sum_{m\in\mathbb{Z}*} e^{-sd(x,p^m x)}\right)^j.$$

Because of our choice of x, if $l := d(x, hx)$ then $d(x, h^N x) = |N|l$ for all $N \in \mathbb{Z}$. Thus

$$\sum_{n\in\mathbb{Z}^*} e^{-sd(x,h^{kn}x)} \leqslant 2\frac{e^{-slk}}{1-e^{-slk}}.$$

Let $s_\epsilon := \delta_{\mathcal{P}} + \epsilon > \delta_{\mathcal{P}}$ and denote $P_s = \sum_{m\in\mathbb{Z}} e^{-(\delta_{\mathcal{P}}+s)d(x,p^m x)}$; then the sum P_ϵ is finite. Assuming ϵ small, we get a constant D such that

$$e^{2s_\epsilon C} 2\frac{e^{-s_\epsilon kl}}{1-e^{-s_\epsilon kl}} \sum_{m\in\mathbb{Z}^*} e^{-s_\epsilon d(x,p^m x)} < De^{-s_\epsilon kl} P_\epsilon.$$

Hence, if $\log(DP_\epsilon)/s_\epsilon l < k$, then $De^{-s_\epsilon kl}P_\epsilon < 1$ and therefore $\delta_{\Gamma_k} \leqslant s_\epsilon$. Observe that the function $t \mapsto \log(DP_t)/s_t$ is continuous, decreasing and unbounded in the interval $(0, \eta)$, for any $0 < \eta \leqslant 1$. We can then solve the equation $\log(DP_t)/s_t l = k - 1$, where $t \in (\delta_p, \delta_p + \eta)$ and k is large enough. We call this solution ϵ_k. By construction $\delta_{\Gamma_k} \leqslant s_{\epsilon_k}$. It follows from the definition of ϵ_k that $\lim_{n\to\infty} P_{\epsilon_k} = \infty$. Observe that

$$\frac{k}{\sum_{\gamma\in\mathcal{P}} e^{-\delta_{\Gamma_k} d(x,\gamma x)}} \leqslant \frac{k}{P_{\epsilon_k}} = \frac{\log(DP_{\epsilon_k})/(s_{\epsilon_k} l) + 1}{P_{\epsilon_k}},$$

but the RHS goes to 0 as $k \to \infty$. Since p is of divergence type, it follows from [DOP00, Theorem A] that Γ_n is of divergence type. ∎

We proceed to show an explicit family of measures satisfying the property claimed in the second part of Theorem 1.2. We remark that the measures constructed in Theorem 5.1 cannot be used at this point, since a compact set in T^1X/Γ is not necessarily a compact set in the topology of Y. Hence, the fact that $\int \tau d\mu_n \to \infty$ does not imply that $\nu_n \rightharpoonup 0$. Despite this difficulty, we can use the geometry to construct the desired family.

Denote by p a parabolic isometry in the generator set $\mathcal{A}$ with maximal critical exponent, that is $\delta_{p,\max} = \delta_p$. Take $\Gamma_n =< p, h^n >$ as in Proposition 5.3. Let m_n^{BM} be the Bowen–Margulis measure on T^1X/Γ_n. Since an extended Schottky group is a geometrically finite group, the measure m_n^{BM} is finite [DOP00, Theorem B]. Moreover, it maximises the entropy of the geodesic flow on T^1X/H_n [OP04, Theorem 2]. In other words $h_{m_n^{BM}}(g)$ equals δ_{Γ_n}. Recall that the critical exponent δ_{Γ_n} converges to $\delta_{p,\max}$ as n goes to infinity, therefore

$$h_{m_n^{BM}}(g) \to \delta_{p,\max}. \tag{18}$$

Using the coding property, we know that T^1X/Γ_n (except vectors defining geodesics pointing to the Γ_n-orbit of the fixed points of h and p) is identified with $Y_n = \{(x,t) \in \Sigma_n \times \mathbb{R} : 0 \leqslant t \leqslant \tau(x)\}/\sim$, where

$$\Sigma_n = \{(a_i^{m_i})_{i\in\mathbb{Z}} : a_i \in \{p, h^n\}, m_i \in \mathbb{Z}\},$$

and the geodesic flow is conjugated to the suspension flow on (Y_n, τ) (same τ as before, but for this coding). It is convenient to think of (Σ_n, σ) as a sub-shift of (Σ, σ). Since the Bowen–Margulis measure m_n^{BM} is ergodic and has positive entropy, it needs to be supported in Y_n under the corresponding identification, i.e. in the space of geodesics modeled by the suspension flow. In particular, we can consider m_n^{BM} as supported in some invariant subset of Y. Let us call ν_n^{BM} the image measure of m_n^{BM} induced by the inclusion $Y_n \hookrightarrow Y$ and normalized so that ν_n^{BM} is a probability measure. Observe that (18) implies that

$$\lim_{n\to\infty} h_{\nu_n}(g) = \delta_{p,\max}.$$

We just need to prove that $\nu_n^{BM} \rightharpoonup 0$ to end the proof of Theorem 1.2. This sequence actually dissipates through the cusp associated to the parabolic element p. Recall that ξ_p denotes the fixed point of p at infinity. Define

$$N_{\xi_p}(s) := \{x \in X : B_{\xi_p}(o,x) > s\},$$

where $o \in X$ is a reference point. Since Γ is geometrically finite, for s large enough $N_{\xi_p}(s)/ < p >$ embeds isometrically into T^1X/Γ, i.e. it is a standard model for the cusp at ξ_p. By definition, the group $< p >$ acts co-compactly on $\Lambda_\Gamma \backslash \{\xi_p\}$. In other words, if we consider a fundamental domain for the action of $\mathcal{P}$ on $\Lambda_\Gamma \backslash \{\xi_p\}$, say D, then $\Lambda_\Gamma \bigcap D$ is relatively compact in D. Clearly the other fundamental domains are given by γD where $\gamma \in \mathcal{P}$.

In [DOP00] it is proven that for any geometrically finite group Γ the Bowen–Margulis measure in the cusp C satisfies a bound of the type

$$\begin{aligned}\frac{1}{A_{\Gamma,C}} \sum_{p\in\mathcal{P}} d(x,px)e^{-\delta_\Gamma d(x,px)} &\leqslant m_{BM}^{\Gamma}(T^1C) \\ &\leqslant A_{\Gamma,C} \sum_{p\in\mathcal{P}} d(x,px)e^{-\delta_\Gamma d(x,px)}.\end{aligned} \tag{19}$$

Here the point x is chosen inside C and the constant $A_{\Gamma,C}$ basically depend on the size of C and the minimal distance between $\Lambda_\Gamma \cap D$ and ∂D.

Define $\mathcal{Q}_i = N_{\xi_p}(s_i)/ < p >$, where the sequence $(s_i)_{i\geqslant 1}$ is increasing with $\lim_i s_i = \infty$. We assume Q_1 provides a standard cusp neighborhood. Denote by p_n the projection

$$p_n : T^1X/\Gamma_n \to T^1X/\Gamma,$$

induced by the inclusion at the level of groups. By definition

$$\nu_n^{BM} = \frac{1}{m_n^{BM}(T^1X/\Gamma_n)}(p_n)_* m_n^{BM}.$$

We will prove that $\lim_{n\to\infty} \nu_n^{BM}(T^1((X/\Gamma)\backslash\mathcal{Q}_i)) = 0$ for any i. For this it is enough to prove the limit

$$\lim_{n\to\infty} \frac{m_n^{BM}(p_n^{-1}T^1((X/\Gamma)\backslash\mathcal{Q}_i))}{m_n^{BM}(p_n^{-1}T^1\mathcal{Q}_i)} = 0.$$

Observe that, if $\pi_n : X/\Gamma_n \to X/\Gamma$ is the natural projection, then the sets $\pi_n^{-1}\mathcal{Q}_i$ are represented by the same one in the universal covering. We denote by S_i this cusp neighborhood.

Lemma 5.4: *The measure $m_n^{BM}(p_n^{-1}T^1((X/\Gamma)\backslash\mathcal{Q}_i)))$ grows at most linearly in k_n, that is, for a certain positive constant C_i we have*

$$m_n^{BM}(p_n^{-1}T^1((X/\Gamma)\backslash\mathcal{Q}_i)) \leqslant C_i n. \tag{20}$$

Proof. Let D_0 (resp. D_n) be the fundamental domain of Γ (resp. Γ_n) containing $o \in X$. By the definition of fundamental domain, there exists a set $T_n \subset \Gamma$ such that

(1) for any $\gamma_1, \gamma_2 \in T_n$ and $\gamma_1 \neq \gamma_2$, we have $\gamma_1\mathrm{int}(D_0) \cap \gamma_2\mathrm{int}(D_0) = \varnothing$, and
(2) $\bigcup_{\gamma\in T_n} \gamma D_0 = D_n$.

Denote by K_i the compact $K_i = (X/\Gamma)\backslash\mathcal{Q}_i$ and let $\widetilde{K}_i$ be the lift of K_i into X intersecting D_0. By definition, any lift of K_i into X intersecting D_n is a translation of $\widetilde{K}_i$ by an element in T_n. Since m_n^{BM} is supported in Λ_{Γ_n}, the Bowen–Margulis measure $\tilde{m}_n^{BM}$ on X satisfies

$$m_n^{BM}(p_n^{-1}T^1(K_i)) \leqslant \sum_{\substack{\gamma\in T_n \\ \gamma\widetilde{K}_i\cap C(\Gamma_n)\neq\varnothing}} \tilde{m}_n^{BM}(T^1(\gamma\widetilde{K}_i)),$$

where $C(\Gamma_n)$ is the convex hull of $L(\Gamma_n) \times L(\Gamma_n)$ in $X \cup \partial X$. By construction and convexity of the domains C_γ, we obtain

$$\#\{\gamma \in T_n : \gamma\widetilde{K}_i \cap C(\Gamma_n) \neq \varnothing\} \leqslant 2n - 1.$$

In particular, we have

$$m_n^{BM}(p_n^{-1}T^1(K_i)) \leqslant (2n-1)\tilde{m}_n^{BM}(T^1(\widetilde{K}_i)).$$

But again, by estimates given in [DOP00], the measure $\tilde{m}_n^{BM}(T^1(\widetilde{K}_i))$ satisfies

$$m_n^{BM}(T^1(\widetilde{K}_i)) \leqslant L_i,$$

where L_i is a constant depending on the diameter of $\widetilde{K}_i$. By setting $C_i = 2L_i$, the conclusion follows. ∎

Using the comments just below equation (19) we know that the constants $A_{H_n,\mathcal{Q}_i}$ can all be considered equal to $A_{H_1,\mathcal{Q}_i}$. We have then

$$m_n^{BM}(T^1 S_i) \asymp_{A_{H_1,\mathcal{Q}_i}} \sum_{p\in\mathcal{P}} d(x,px)e^{-\delta_{\Gamma_n} d(x,px)}. \tag{21}$$

Hence, from (20) and (21), we get

$$\begin{aligned}\frac{m_n^{BM}(p_n^{-1}T^1(X\backslash\mathcal{Q}_i))}{m_n^{BM}(p_n^{-1}T^1\mathcal{Q}_i)} &\leqslant \frac{A_{H_1,\mathcal{Q}_i}C_i n}{\sum_{p\in\mathcal{P}} d(x,px)e^{-\delta_{\Gamma_n} d(x,px)}}\\ &\leqslant \frac{C_i' n}{\sum_{p\in\mathcal{P}} e^{-\delta_{\Gamma_n} d(x,px)}}.\end{aligned}$$

Finally, property (3) in Proposition 5.3 implies that the last term above converges to 0. Therefore

$$\lim_{n\to\infty} \frac{m_n^{BM}(p_n^{-1}T^1(X\backslash\mathcal{Q}_i))}{m_n^{BM}(p_n^{-1}T^1\mathcal{Q}_i)} = 0,$$

which concludes the proof of Theorem 1.2. ∎

Corollary 5.5: *Let X and Γ be as in Theorem 1.2. Then the entropy at infinity of the geodesic flow is equal to the maximal parabolic critical exponent, that is*

$$h_\infty(g, T^1X/\Gamma) = \delta_{p,\max}.$$

6. Thermodynamic formalism

In this section we always consider X a Hadamard manifold with pinched negative sectional curvature and Γ an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. We also assume that the derivatives of the sectional curvature are uniformly bounded. Our goal is to obtain several results on thermodynamic formalism for the geodesic flow over X/Γ. Some of these results were already obtained, without symbolic methods, by Coudène (see [Cou03]) and Paulin, Pollicott and Schapira (see [PPS15]). However, the strength of our symbolic approach will be clear in the study of regularity properties of the pressure (subsection 6.2).

Here we keep the notation of subsection 4.2. Thus, the geodesic flow (g_t) in the set Ω_0 can be coded by a suspension semi-flow (Y, Φ) with base (Σ, σ) and roof function $\tau : \Sigma \to \mathbb{R}$.

6.1. Equilibrium measures. We will consider the following class of potentials.

Definition 6.1: A continuous function $f : T^1X/\Gamma \to \mathbb{R}$ belongs to the class of **regular** functions, that we denote by $\mathcal{R}$, if the symbolic representation $\Delta_f : \Sigma \to \mathbb{R}$ of $f|_{\Omega_0}$ has summable variations.

We begin studying thermodynamic formalism for the geodesic flow restricted to the set Ω_0. The following results can be deduced from the general theory of suspension flows over countable Markov shifts and from the symbolic model for the geodesic flow. With a slight abuse of notation, using the identification explained before, we still denote by $f : Y \to \mathbb{R}$ the given map $f : \Omega_0 \to \mathbb{R}$.

Definition 6.2: Let $f \in \mathcal{R}$. Then the pressure of f with respect to the geodesic flow $g := (g_t)$ restricted to the set Ω_0 is defined by

$$P_{\Omega_0}(f) := \lim_{t\to\infty} \frac{1}{t} \log \left(\sum_{\varphi_s(x,0)=(x,0), 0<s\leqslant t} \exp\left(\int_0^s f(\varphi_k(x,0))\, \mathrm{d}k \right) \chi_{C_{i_0}}(x) \right).$$

This pressure satisfies the following properties:

Proposition 6.3 (Variational Principle): *Let $f \in \mathcal{R}$. Then*

$$P_{\Omega_0}(f) = \sup\left\{ h_\nu(g) + \int_{\Omega_0} f\, \mathrm{d}\nu : \nu \in \mathcal{M}_{\Omega_0} \text{ and } -\int_{\Omega_0} f\, \mathrm{d}\nu < \infty \right\},$$

where $\mathcal{M}_{\Omega_0}$ denotes the set of (g_t)-invariant probability measures supported in Ω_0.

Proposition 6.4: *Let $f \in \mathcal{R}$. Then*

$$P_{\Omega_0}(f) = \sup\{P_{g|K}(f) : K \in \mathcal{K}_{\Omega_0}(g)\},$$

where $\mathcal{K}_{\Omega_0}(g)$ denotes the space of compact g-invariant sets in Ω_0.

Remark 6.5 (Convexity): It is well known that for any $K \in \mathcal{K}_{\Omega_0}(g)$ the pressure function $P_{g|K}(\cdot)$ is convex. Since the supremum of convex functions is a convex function, it readily follows that $P_{\Omega_0}(\cdot)$ is convex.

Proposition 6.6: *Let $f \in \mathcal{R}$. Then there is an equilibrium measure $\nu_f \in \mathcal{M}_{\Omega_0}$, that is,*

$$P_{\Omega_0}(f) = h_{\nu_f}(g) + \int_{\Omega_0} f \, \mathrm{d}\nu_f,$$

for f if and only if we have that $P_\sigma(\Delta_f - P_\Phi(f)\tau) = 0$ and there exists an equilibrium measure $\mu_f \in \mathcal{M}_\sigma$ for $\Delta_f - P_\Phi(f)\tau$ such that $\int \tau d\mu_f < \infty$. Moreover, if such an equilibrium measure exists, then it is unique.

In order to extend these results to the geodesic flow in T^1X/Γ we use the second conclusion of Proposition 4.9.

Definition 6.7: Let $f \in \mathcal{R}$. Then the pressure of f with respect to the geodesic flow $g := (g_t)$ in T^1X/Γ is defined by

$$P_g(f) := \max\left\{P_{\Omega_0}(f), \int f \, \mathrm{d}\nu^{h_1}, \ldots, \int f \, \mathrm{d}\nu^{h_{N_1}}\right\}.$$

Proposition 6.8 (Variational Principle): *Let $f \in \mathcal{R}$. Then*

$$P_g(f) = \sup\left\{h_\nu(g) + \int f \, \mathrm{d}\nu : \nu \in \mathcal{M}_g \text{ and } -\int f \, \mathrm{d}\nu < \infty\right\}.$$

Proposition 6.9: *Let $f \in \mathcal{R}$. Then*

$$P_g(f) = \sup\{P_{g|K}(f) : K \in \mathcal{K}(g)\},$$

where $\mathcal{K}(g)$ denotes the space of compact g-invariant sets.

Proposition 6.10: *Let $f \in \mathcal{R}$ be such that $\sup f < P_g(f)$. Then there is an equilibrium measure $\nu_f \in \mathcal{M}_g$ for f if and only if we have that $P_{\Omega_0}(\Delta_f - P_\Phi(f)\tau) = 0$ and there exists an equilibrium measure $\mu_f \in \mathcal{M}_\sigma$ for $\Delta_f - P_\Phi(f)\tau$ such that $\int \tau d\mu_f < \infty$. Moreover, if such an equilibrium measure exists, then it is unique.*

Proof. Note that if $\sup f < P_g(f)$, then an equilibrium measure for f, if it exists, must have positive entropy, since the measures ν^{h_i}, with $i \in \{1, \ldots, N_1\}$, have zero entropy (see Proposition 4.9). The result follows from Proposition 6.6. ∎

The next result shows that potentials with small oscillation do have equilibrium measures; this result can also be deduced from [Cou03, PPS15]. Our proof is short and uses the symbolic structure.

THEOREM 6.11: *Let $f \in \mathcal{R}$. If*

$$\sup f - \inf f < h_{top}(g) - \delta_{p,\max},$$

then f has an equilibrium measure.

Proof. Assume that the measures ν^{h_i} are not equilibrium measures for f, otherwise the theorem is proved. Therefore, we have that $P_g(f) = P_{\Omega_0}(f)$. We first show that $P_\sigma(\Delta_f - P_g(f)\tau) = 0$. Note that for every $x \in \Sigma$,

$$\tau(x) \inf f \leqslant \Delta_f(x) \leqslant \tau(x) \sup f.$$

By monotonicity of the pressure we obtain

$$P_\sigma((\inf f - t)\tau) \leqslant P_\sigma(\Delta_f - t\tau) \leqslant P_\sigma((\sup f - t)\tau).$$

Let $t \in (s_\infty + \sup f, h_{top}(g) + \inf f)$ and recall that $s_\infty = \delta_{p,\max}$. Then

$$0 < P_\sigma((\inf f - t)\tau) \leqslant P_\sigma(\Delta_f - t\tau) \leqslant P_\sigma((\sup f - t)\tau) < \infty.$$

Since $P_g(f) < \infty$ and the function $t \to P_\sigma(\Delta_f - t\tau)$ is continuous with $\lim_{t\to\infty} P_\sigma(\Delta_f - t\tau) = -\infty$, we obtain that $P_\sigma(\Delta_f - P_g(f)\tau) = 0$. Since the system Σ has the BIP condition and the potential $\Delta_f - P_g(f)\tau$ is of summable variations, it has an equilibrium measure μ. It remains to prove the integrability condition. Recall that

$$\frac{\partial}{\partial t} P_\sigma(\Delta_f - t\tau)\Big|_{t=P_g(f)} = -\int \tau d\mu.$$

But we have proved that the function $t \to P_\sigma(\Delta_f - t\tau)$ is finite (at least) in an interval of the form $[P_g(f) - \epsilon, P_g(f) + \epsilon]$. The result now follows, because when finite the function is real analytic. ∎

6.2. PHASE TRANSITIONS. This subsection is devoted to studying the regularity properties of pressure functions $t \mapsto P_g(tf)$ for a certain class of functions f. We say that the pressure function $t \mapsto P_g(tf)$ has a **phase transition** at $t = t_0$ if the pressure function is not real analytic at $t = t_0$. The set of points at which the pressure function exhibits phase transitions might be a very large set. However, since the pressure is a convex function it can only have a countable set of points where it is not differentiable. Regularity properties of the pressure are related to important dynamical properties, for example exponential decay of correlations of equilibrium measures. In the Axiom A case the pressure is real analytic. Indeed, this can be proved noting that, in that setting, the function $t \mapsto P_\sigma(\Delta_f - t\tau)$ is real analytic and that $P_\sigma(\Delta_f - P_\Phi \tau) = 0$. The result then follows from the implicit function theorem noticing that the non-degeneracy condition is fulfilled:

$$\frac{\partial}{\partial t} P_\sigma(\Delta_f - t\tau) = -\int \tau d\mu < 0,$$

where μ is the equilibrium measure corresponding to $\Delta_f - t\tau$. The inequality above, together with the coding properties established in [Bow73, BR75, Rat73], allow us to establish that the pressure is real analytic for regular potentials in the Axiom A setting. In the non-compact case the situation can be different. However, the only results involving the regularity properties of the pressure function for geodesic flows defined on non-compact manifolds, that we are aware of, are those concerning the modular surface (see [IJ13, Section 6]). In this section we establish regularity results for pressure functions of geodesic flows defined on extended Schottky groups. We begin by defining conditions (F1) and (F2) on the potentials.

Definition 6.12: Consider a non-negative continuous function $f : T^1X/\Gamma \to \mathbb{R}$. We will say f satisfies Condition (F1) or (F2) if the corresponding property below holds.

- (F1) The symbolic representation $\Delta_f : \Sigma^+ \to \mathbb{R}$ is locally Hölder and bounded away from zero in every cylinder $C_{a^m} \subset \Sigma^+$, where $a \in \mathcal{A}$, $m \in \mathbb{Z}$.
- (F2) Consider any indexation $(C_n)_{n\in\mathbb{N}}$ of the cylinders of the form C_{a^m}. Then

$$\lim_{n\to\infty} \frac{\sup\{\Delta_f(x) : x \in C_n\}}{\inf\{\tau(x) : x \in C_n\}} = 0.$$

We say f belongs to the class $\mathcal{F}$ if it satisfies (F1) and (F2).

In the following Lemma we establish two properties of potentials in $\mathcal{F}$ that will be used in the sequel.

Lemma 6.13: *Let f be a potential satisfying (F1) and (ν_n) a sequence of measures in $\mathcal{M}_{\Omega_0}$ such that $\nu_n = \frac{\mu_n \times m|_Y}{(\mu_n \times m)(Y)}$. Then:*

1. *if $\lim_{n\to\infty} \int_{\Omega_0} f d\nu_n = 0$, then $\lim_{n\to\infty} \int \tau d\mu_n = \infty$;*
2. *if f satisfies (F2) and $\lim_{n\to\infty} \int \tau d\mu_n = \infty$, then $\lim_{n\to\infty} \int_{\Omega_0} f d\nu_n = 0$.*

Proof. To prove (1) we will argue by contradiction. Assume, passing to a subsequence if necessary, that

$$\lim_{n\to\infty}\int \tau d\mu_n = C.$$

Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n > N$ we have that

$$\left|\int \tau d\mu_n - C\right| < \epsilon.$$

Lemma 6.14: *Let $r > 1$. Then for every $n > N$ we have that*

$$\mu_n(\{x : \tau(x) \leqslant r\}) > 1 - \frac{C+\epsilon}{r}.$$

Proof of Lemma 6.14. Since the function τ is positive we have

$$\int \tau d\mu_n \geqslant r\mu_n(\{x : \tau(x) \geqslant r\}) + \int_{\{x:\tau(x)\leqslant r\}} \tau d\mu_n.$$

Thus

$$C+\epsilon \geqslant r\mu_n(\{x : \tau(x) \geqslant r\}),$$
$$\frac{C+\epsilon}{r} \geqslant \mu_n(\{x : \tau(x) \geqslant r\}).$$

Finally

$$\mu_n(\{x : \tau(x) \leqslant r\}) > 1 - \frac{C+\epsilon}{r}. \qquad \blacksquare$$

Note that the set $\{x : \tau(x) \leqslant r\}$ is contained in a finite union of cylinders on Σ. This follows from the inequality $d(o, a^m o) - C \leqslant \tau(x)$, which is a consequence of Lemma 4.11, and the fact that $\mathcal{A}$ is finite. Since Δ_f is bounded away from zero in every one of them, there exists a constant $G(r) > 0$ such that

$$\Delta_f(x) > G(r),$$

on $\{x : \tau(x) \leqslant r\}$. Thus

$$\int_{\Omega_0} f d\nu_n = \frac{\int_\Sigma \Delta_f(x) d\mu_n}{\int_\Sigma \tau d\mu_n} \geqslant \frac{\int_{\{x:\tau(x)\leqslant r\}} \Delta_f(x) d\mu_n}{\int_\Sigma \tau d\mu_n} \geqslant \frac{G(r)\left(1 - \frac{C+\epsilon}{r}\right)}{C-\epsilon}.$$

If we choose r large enough so that $1 - \frac{C+\epsilon}{r} > 0$ we obtain the desired contradiction.

To prove (2), observe that for every $\varepsilon > 0$ there exists $N \geqslant 1$ such that for every $k \geqslant N$ we have

$$\frac{\sup\{\Delta_f(x) : x \in C_k\}}{\inf\{\tau(x) : x \in C_k\}} < \varepsilon.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty}\int f d\nu_n &= \lim_{n\to\infty}\frac{1}{\int\tau d\mu_n}\sum_{k\geqslant 1}\int_{C_k}\Delta_f d\mu_n\\
&= \lim_{n\to\infty}\frac{1}{\int\tau d\mu_n}\sum_{k\geqslant N}\int_{C_k}\Delta_f d\mu_n\\
&\leqslant \lim_{n\to\infty}\frac{1}{\int\tau d\mu_n}\sum_{k\geqslant N}\int_{C_k}\frac{\sup\{\Delta_f(x):x\in C_k\}}{\inf\{\tau(x):x\in C_k\}}\inf\{\tau(x):x\in C_k\}d\mu_n\\
&\leqslant \lim_{n\to\infty}\frac{1}{\int\tau d\mu_n}\sum_{k\geqslant N}\int_{C_k}\varepsilon\inf\{\tau(x):x\in C_k\}d\mu_n\\
&\leqslant \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion of the second claim follows. ∎

Combining Theorem 3.7 and Lemma 6.13, we obtain the following

Lemma 6.15: *Let Γ be an extended Schottky group satisfying $N_1 + N_2 \geqslant 3$ and let f be a function satisfying property (F1). If $(\nu_n)_n \subset \mathcal{M}_g$ is a sequence of invariant probability measures for the geodesic flow such that*

$$\lim_{n\to\infty}\int_{T^1X/\Gamma} f d\nu_n = 0,$$

then

$$\limsup_{n\to\infty} h_{\nu_n}(g) \leqslant \delta_{p,\max}.$$

Proof. If $\nu_n \in \mathcal{M}_{\Omega_0}$, the Lemma follows directly from (1) in Lemma 6.13 and Theorem 3.7. If $\nu_n \in \mathcal{M}_g$ with support in $X\backslash\Omega_0$, then we can consider the measure $\tilde{\nu}_n := \nu_n - \sum_{i=1}^{N_1} c_i^n \nu^{h_i}$ where the constants $c_i^n \geqslant 0$ are chosen so that $\tilde{\nu}_n$ (periodic orbit associated to hyperbolic generator h_i) $= 0$. Let $C_n = \tilde{\nu}_n(X/\Gamma)$. If $C_n = 0$, then $h_{\nu_n}(g) = 0$ and there is no contribution to the desired $\limsup h_{\nu_n}(g)$. Otherwise define $u_n = C_n^{-1}\tilde{\nu}_n$. By definition u_n is a probability measure in $\mathcal{M}_{\Omega_0}$; we claim $\lim_{n\to\infty}\int f du_n = 0$. Observe that $\int f d\nu_n = \int f d\tilde{\nu}_n + \sum_{i=1}^{N_1} c_i^n \nu^{h_i}(f)$ has non-negative summands and it is converging to zero, it follows that $\int f d\tilde{\nu}_n \to 0$, $c_i^n \to 0$ as $n \to \infty$. By definition $C_n = 1 - \sum_{i=1}^{N_1} c_i^n$, therefore $C_n \to 1$. Recalling $u_n = C_n^{-1}\tilde{\nu}_n$ we get $\lim \int f du_n = 0$.

Because $\nu_n = C_n u_n + (1-C_n)(1-C_n)^{-1}\sum_{i=1}^{N_1} c_i^n \nu^{h_i}$, we have $h_{\nu_n}(g) = C_n h_{u_n}(g)$. Finally, since $C_n \to 1$ and $\limsup_{n\to\infty} h_{u_n}(g) \leqslant \delta_{p,\max}$ (because $u_n \in \mathcal{M}_{\Omega_0}$), we get $\limsup_{n\to\infty} h_{\nu_n}(g) \leqslant \delta_{p,\max}$. ∎

The next Theorem is the main result of this subsection and it is an adaptation of results obtained at a symbolic level in [IJ13]. It is possible to translate those symbolic results into this geometric setting thanks to Theorem 4.10.

Theorem (1.3): *Let X be a Hadamard manifold with pinched negative sectional curvature and let Γ be an extended Schottky group of isometries of X satisfying $N_1 + N_2 \geqslant 3$. Assume that the derivatives of the sectional curvature are uniformly bounded. If $f \in \mathcal{F}$, then*

1. *For every $t \in \mathbb{R}$ we have that $P_g(tf) \geqslant \delta_{p,\max}$.*
2. *We have that $\lim_{t\to-\infty} P_g(tf) = \delta_{p,\max}$.*
3. *Let $t' := \sup\{t \in \mathbb{R} : P_g(tf) = \delta_{p,\max}\}$. Then*

$$P_g(tf) = \begin{cases} \delta_{p,\max} & \text{if } t < t'; \\ \text{real analytic, strictly convex, strictly increasing} & \text{if } t > t'. \end{cases}$$

4. *If $t > t'$, the potential tf has a unique equilibrium measure. If $t < t'$ it has no equilibrium measure.*

Note that Theorem 1.3 shows that when t' is finite, then the pressure function exhibits a phase transition at $t = t'$, whereas when $t' = -\infty$ the pressure function is real analytic where defined (see Figure 2). Recall that $\delta_{p,\max} = s_\infty$.

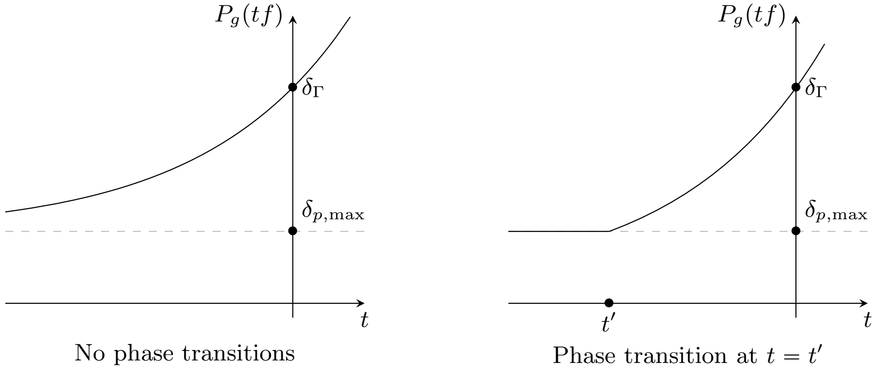

Figure 2.

Proof of (1). The first claim follows from the variational principle. By Theorem 5.1 there exists a sequence $(\nu_n) \subset \mathcal{M}_g$ such that $\lim_{n\to\infty} h_{\nu_n}(g) = s_\infty$ and their corresponding probability σ-invariant measures (μ_n) in Σ satisfy $\lim_{n\to\infty} \int \tau d\mu_n = \infty$. Therefore, by (2) in Lemma 6.13, we also have that

$\lim_{n\to\infty}\int f d\nu_n = 0$. Hence, for every $t \in \mathbb{R}$, we have

$$s_\infty = \delta_{p,\max} = \lim_{n\to\infty}\left(h_{\nu_n}(g) + t\int_{\Omega_0} f d\nu_n\right)$$
$$\leqslant \sup\left\{h_\nu(g) + t\int f d\nu : \nu \in \mathcal{M}_g\right\} = P_g(tf). \qquad \blacksquare$$

Proof of (2). Since $t \mapsto P_g(tf)$ is non-decreasing and bounded below, the following limit $\lim_{t\to-\infty} P_g(tf)$ exists. Define $A \in \mathbb{R}$ as the limit $\lim_{t\to-\infty} P_g(tf) := A$. Using the Variational Principle, we can choose a sequence of measures $(\nu_n)_n$ in $\mathcal{M}_g$ for which

$$\lim_{n\to\infty} h_{\nu_n}(g) - n\int f d\nu_n = A.$$

Since A is finite it follows that $\lim_{n\to\infty}\int f d\nu_n = 0$. Hence, from Lemma 6.15, we obtain $\limsup_{n\to\infty} h_{\nu_n}(g) \leqslant s_\infty$. In particular,

$$\begin{aligned} s_\infty &\leqslant \lim_{t\to-\infty} P_g(tf) \\ &= \lim_{n\to\infty} h_{\nu_n}(g) - n\int f d\nu_n \\ &\leqslant \lim_{n\to\infty} h_{\nu_n}(g) \leqslant s_\infty. \end{aligned}$$

Therefore, we have that $A = \delta_{p,\max}$. $\blacksquare$

Proof of (3). Real analyticity. We first prove $P_g(tf) = P_{\Omega_0}(tf)$. After this is done we can proceed with standard regularity arguments in the symbolic picture. Observe that for $t < 0$ the pressure $P_{\Omega_0}(tf)$ is always positive while the contribution of the pressure on $(T^1X/\Gamma)\backslash\Omega_0$ is negative, so $P_g(tf) = P_{\Omega_0}(tf)$ for every $t \leqslant 0$. Consider now $t > 0$. Pick ν^{h_i} as in Proposition 4.9 (see also Definition 6.7). Denote by x_- (resp. x_+), the repulsor (attractor) of h_i and γ_{h_i}, the geodesic defined by those points. Consider p a parabolic element in $\mathcal{A}$ and let γ_n be the geodesic connecting the points ξ^- and ξ^+ where $\omega(\xi^-) = \overline{p^{-1}h^{-n}}$ and $\omega(\xi^+) = \overline{h^n p}$. Denote γ_∞ the geodesic connecting $p^{-1}x_-$ and x_+. Observe that γ_n descends to a closed geodesic in T^1X/Γ. By comparing γ_n and γ_∞ we see that for any $\epsilon > 0$, the amount of time γ_n leaves a ϵ-neighborhood of γ_{h_i} is uniformly bounded for big enough n. Let ν_n be the invariant probability measure defined by the closed geodesic γ_n; then we get the weak convergence $\nu_n \to \nu^{h_i}$. Then

$$t\int f d\nu^{h_i} = \lim_{n\to\infty} t\int f d\nu_n \leqslant \lim_{n\to\infty}(h_{\nu_n}(g) + t\int f d\nu_n) \leqslant P_{\Omega_0}(tf).$$

This give us $P_g(tf) = P_{\Omega_0}(tf)$.

The pressure function $t \mapsto P_g(tf)$ is convex, non-decreasing and bounded from below by s_∞. We now prove that for $t > t'$ it is real analytic. Note that since $t' < t$ we have that

$$P_\sigma(t\Delta_f - s_\infty \tau) > 0,$$

possibly infinity, and that there exists $p > s_\infty$ such that $0 < P_\sigma(t\Delta_f - p\tau) < \infty$ (see [IJ13, Lemma 4.2]). Moreover, Condition $(F2)$ implies that $P_g(tf) < \infty$ for every $t > t'$, hence

$$P_\sigma(\Delta_{tf} - P_g(tf)\tau) \leqslant 0.$$

Since τ is positive, the function $s \mapsto P_\sigma(t\Delta_f - s\tau)$ is decreasing and

$$\lim_{s \to +\infty} P_\sigma(t\Delta_f - s\tau) = -\infty.$$

Moreover, since the base map of the symbolic model satisfies the BIP condition, the function $(s,t) \mapsto P_\sigma(t\Delta_f - s\tau)$ is real analytic in both variables. Hence, there exists a unique real number $s_f > s_\infty$ such that $P_\sigma(t\Delta_f - s_f\tau) = 0$ and

$$\frac{\partial}{\partial s} P_\sigma(t\Delta_f - s\tau)\Big|_{s=s_f} < 0.$$

Therefore, $P_g(tf) = s_f$ and, by the Implicit Function Theorem, the function $t \mapsto P_g(tf)$ is real analytic in $(t', t^\star)$. ∎

Proof of (4). First note that the previous claims imply that no zero entropy measure can be an equilibrium measure. Moreover, in the proof of (3) we obtained that for $t \in (t', \infty)$ we have that $P_\sigma(t\Delta_f - P_g(tf)\tau) = 0$. Since the system satisfies the BIP condition, there exists an equilibrium measure $\mu_f \in \mathcal{M}_\sigma$ for $t\Delta_f - P_g(tf)\tau$ such that $\int \tau d\mu_f < \infty$ (see Theorem 2.9). Therefore it follows from Proposition 6.6 that tf has an equilibrium measure.

In order to prove the last claim, assume by contradiction that for some $t_1 < t'$ the potential $t_1 f$ has an equilibrium measure ν_{t_1}. Then

$$s_\infty = P_g(t_1 f) = h_{\nu_{t_1}}(g) + t_1 \int_{\Omega_0} f d\nu_{t_1}.$$

Since $f > 0$ on Ω_0, we have that $\int_{\Omega_0} f d\nu_{t_1} := B > 0$. Thus the straight line $r \to h_{\nu_{t_1}}(g) + r\int_{\Omega_0} f d\nu_{t_1}$ is increasing with r, therefore for $t \in (t_1, t')$ we have that

$$h_{\nu_{t_1}}(g) + t\int_{\Omega_0} f d\nu_{t_1} > s_\infty = P_g(tf).$$

This contradiction proves the statement. ∎

6.3. Examples. We will use the following criterion, first introduced in [IJ13], to construct phase transitions.

Proposition 6.16: *Let $f \in \mathcal{F}$. Then:*

(1) *If there exists $t_0 \in \mathbb{R}$ such that $P_\sigma(t_0\Delta_f - s_\infty\tau) < \infty$, then there exists $t' < t_0$ such that for every $t < t'$ we have*

$$P_g(tf) = s_\infty.$$

(2) *Suppose that there exists an interval I such that $P_\sigma(t\Delta_f - s_\infty\tau) = \infty$ for every $t \in I$. Then $t \mapsto P_g(tf)$ is real analytic on I. In particular, if for every $t \in \mathbb{R}$ we have $P_\sigma(t\Delta_f - s_\infty\tau) = \infty$, then $t \mapsto P_g(tf)$ is real analytic in $\mathbb{R}$.*

The proof of this Lemma follows as in [IJ13, Lemma 4.5, Theorem 4,1]. We now present an example of a phase transition (Example 6.18) and another one with pressure real analytic everywhere (Example 6.19). A useful lemma in order to construct an example of a phase transition is the following

Lemma 6.17: *Let $(a_n)_n$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n^t$ converges for every $t > t^*$ and diverges at $t = t^*$. Then there exists a sequence $(\varepsilon_n)_n$ of positive numbers such that*

$$\lim_{n\to\infty} \varepsilon_n = 0$$

and

$$\sum_{n=1}^{\infty} a_n^{t^*+\varepsilon_n} < \infty.$$

Proof of Lemma 6.17. Let $(\alpha_m)_m$ be any sequence of real numbers in $(0, 1]$ converging to zero. Note that for every $m \geqslant 1$ we have

$$\sum_{n=1}^{\infty} a_n^{t^*+\alpha_m} < \infty.$$

Then there exists an integer $N_m \geqslant 1$ such that

$$\sum_{n=N_m}^{\infty} a_n^{t^*+\alpha_m} < 1/m^2.$$

We can suppose without loss of generality that $N_m < N_{m+1}$. Define ε_n for every $N_m \leqslant n < N_{m+1}$ as $\varepsilon_n = \alpha_m$, and $\varepsilon_n = 1$ for $1 \leqslant n < N_1$. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^{t^*+\varepsilon_n} &= \sum_{n=1}^{N_1-1} a_n^{t^*+\varepsilon_n} + \sum_{m=1}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} a_n^{t^*+\varepsilon_n} \\ &= \sum_{n=1}^{N_1-1} a_n^{t^*+1} + \sum_{m=1}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} a_n^{t^*+\alpha_m} \\ &\leqslant \sum_{n=1}^{N_1-1} a_n^{t^*+1} + \sum_{m=1}^{\infty} \sum_{n=N_m}^{\infty} a_n^{t^*+\alpha_m} \\ &\leqslant \sum_{n=1}^{N_1-1} a_n^{t^*+1} + \sum_{m=1}^{\infty} 1/m^2 \\ &<\infty. \qquad \blacksquare\end{aligned}$$

Example 6.18 (Phase transition): Let Γ be a Schottky group satisfying $N_1 + N_2 \geqslant 3$ and assume that there are at least 2 different cusps, i.e. $N_2 \geqslant 2$. Moreover, assume there exists a unique parabolic generator p with $\delta_p = \delta_{p,\max}$. Recall that the series $\sum_{m\in\mathbb{Z}} e^{-\delta_p d(o,p^m o)}$ diverges since p is a parabolic isometry of divergence type. Take a decreasing sequence of real numbers $\varepsilon_m > 0$, as in Lemma 6.17, such that $\lim_{m\to\infty} \varepsilon_m = 0$ and $\sum_{m\in\mathbb{Z}} e^{-(\delta_p+\varepsilon_m)d(o,p^m o)} < \infty$. Define a function $f^0 : \Sigma \to \mathbb{R}^+$ by

(1) $f^0(x) = \varepsilon_m \tau(x)$ if the first symbol of x is p^m for some $m \in \mathbb{Z}$,
(2) $f^0(x) = 1$ otherwise.

Observe that since τ is locally Hölder, the function f^0 is also locally Hölder. We first see that $f^0 \in \mathcal{F}$; for this, it is enough to check that Condition (F2) holds. There exists a constant C independent of m such that $d(o, p^m o) - C \leqslant \tau(x)$ whenever $x \in C_{p^m}$. Then, if $x, y \in C_{p^m}$ we have

$$\tau(x)/\tau(y) \leqslant d(o,p^m o)/(d(o,p^m o) - C),$$

i.e. $\sup_{x\in C_{p^m}} \tau(x)/\inf_{x\in C_{p^m}} \tau(x)$ is uniformly bounded in m; this implies Condition (F2). As shown in [BRW04, Section 2], we can construct a continuous function $f : Y \to \mathbb{R}$ with $\Delta_f = f^0$. We define $t : \Sigma \to \mathbb{R}$ by $t(x) = (s_\infty + \varepsilon_m)$ if the first symbol of x is p^m, and s_∞ otherwise. By simplicity we will denote $s(a^m) = t(x)$ if the first symbol of x is a^m. Following the notation and ideas of the second part of the proof of Theorem 4.10, we obtain

$$\begin{aligned}
&P_\sigma(-\Delta_f - s_\infty\tau)\\
&\qquad= \lim_{n\to\infty}\frac{1}{n}\log\sum_{x:\sigma^n x=x}\exp\left(\sum_{i=0}^{n-1}-(\Delta_f(\sigma^i x)+s_\infty\tau(\sigma^i x))\right)\chi_{C_{h_1}}(x)\\
&\qquad\leqslant \lim_{n\to\infty}\frac{1}{n}\log\sum_{x:\sigma^n x=x}\exp\left(\sum_{i=0}^{n-1}-\tau(\sigma^i x)t(\sigma^i x)\right)\chi_{C_{h_1}}(x)\\
&\qquad\leqslant \lim_{n\to\infty}\frac{1}{n}\log\sum_{a_1,\dots,a_n}\sum_{m_1,\dots,m_n}\prod_{i=1}^{n}C^{s(a_i^{m_i})}e^{-s(a_i^{m_i})d(o,a_i^{m_i}o)}\\
&\qquad\leqslant \lim_{n\to\infty}\frac{1}{n}\log C^{n(s_\infty+1)}\sum_{a_1,\dots,a_n}\sum_{m_1,\dots,m_n}\prod_{i=1}^{n}e^{-s(a_i^{m_i})d(o,a_i^{m_i}o)}\\
&\qquad= \lim_{n\to\infty}\frac{1}{n}\log C^{n(s_\infty+1)}\left(\sum_{a\in\mathcal{A}}\sum_{m\in\mathbb{Z}}e^{-s(a^m)d(o,a^m o)}\right)^n\\
&\qquad= \log C^{s_\infty+1}\left(\sum_{a\in\mathcal{A}}\sum_{m\in\mathbb{Z}}e^{-s(a^m)d(o,a^m o)}\right).
\end{aligned}$$

Observe that $\sum_m e^{-s(a^m)d(o,a^m o)}$ converges for every $s>\delta_a$ and every $a\neq p$. On the other hand, the series $\sum_{m\in\mathbb{Z}} e^{-(\delta_p+\varepsilon_m)d(o,p^m o)}$ is finite by construction. In particular $P_\sigma(-\Delta_f - s_\infty\tau)$ is finite. Observe that f is a potential belonging to the family $\mathcal{F}$. Then from Proposition 6.16 it follows that $t\mapsto P_g(tf)$ exhibits a phase transition.

Example 6.19 (No phase transition): Let Γ be a Schottky group satisfying $N_1+N_2\geqslant 3$. Define $f^0:\Sigma\to\mathbb{R}^+$ to be constant of value 1 and construct a continuous function $f:Y\to\mathbb{R}$ with $\Delta_f=f^0$. Observe that

$$P_\sigma(t-s_\infty\tau)=t+P_\sigma(-s_\infty\tau)=\infty.$$

Recall that $P_\sigma(-s_\infty\tau)=\infty$, because the maximal parabolic generator is of divergence type (see the first part of Theorem 4.10). Since τ is unbounded and f^0 is constant, we can apply Proposition 6.16 to show that $t\mapsto P_g(tf)$ is real analytic in $\mathbb{R}$. In particular, it never attains the lower bound s_∞.

Remark 6.20: In both examples above, the potential f is defined (a priori) only on the set Ω_0. To extend it continuously to the entire manifold T^1X/Γ, it is enough to define it to be equal to 0 on the complement $(T^1X/\Gamma)\backslash\Omega_0$.

References

[Abr59] L. M. Abramov, *On the entropy of a flow*, Doklady Akademii Nauk SSSR **128** (1959), 873–875.

[AK42] W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Mathematical Journal **9** (1942), 25–42.

[BG11] A. I. Bufetov and B. M. Gurevich, *Existence and uniqueness of a measure with maximal entropy for the Teichmüller flow on the moduli space of abelian differentials*, Rossiĭskaya Akademiya Nauk. Matematicheskiĭ Sbornik **202** (2011), 3–42.

[BI06] L. Barreira and G. Iommi, *Suspension flows over countable Markov shifts*, Journal of Statistical Physics **124** (2006), 207–230.

[Bow73] R. Bowen, *Symbolic dynamics for hyperbolic flows*, American Journal of Mathematics **95** (1973), 429–460.

[BR75] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Inventiones Mathematicae **29** (1975), 181–202.

[BRW04] L. Barreira, L. Radu and C. Wolf, *Dimension of measures for suspension flows*, Dynamical Systems **19** (2004), 89–107.

[BS03] J. Buzzi and O. Sarig, *Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps*, Ergodic Theory and Dynamical Systems **23** (2003), 1383–1400.

[BS81a] L. A. Bunimovich and Y. G. Sinaĭ, *Markov partitions for dispersed billiards*, Communications in Mathematical Physics **78** (1980/81), 247–280.

[BS81b] L. A. Bunimovich and Y. G. Sinaĭ, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Communications in Mathematical Physics **78** (1980/81), 479–497.

[Buz10] J. Buzzi, *Puzzles of quasi-finite type, zeta functions and symbolic dynamics for multi-dimensional maps*, Université de Grenoble. Annales de l'Institut Fourier **60** (2010), 801–852.

[Cou03] Y. Coudene, *Gibbs measures on negatively curved manifolds*, Journal of Dynamical and Control Systems **9** (2003), 89–101.

[Dao13] Y. Daon, *Bernoullicity of equilibrium measures on countable Markov shifts*, Discrete and Continuous Dynamical Systems **33** (2013), 4003–4015.

[DGR11] L. J. Díaz, K. Gelfert and M. Rams, *Rich phase transitions in step skew products*, Nonlinearity **24** (2011), 3391–3412.

[DOP00] F. Dal'bo, J.-P. Otal and M. Peigné, *Séries de Poincaré des groupes géométriquement finis*, Israel Journal of Mathematics **118** (2000), 109–124.

[DP98] F. Dal'bo and M. Peigné, *Some negatively curved manifolds with cusps, mixing and counting*, Journal für die Reine und Angewandte Mathematik **497** (1998), 141–169.

[DT] N. Dobbs and M. Todd, *Free energy jumps up*, preprint, arXiv:1512.09245..

[EKP15] M. Einsiedler, S. Kadyrov and A. Pohl, *Escape of mass and entropy for diagonal flows in real rank one situations*, Israel Journal of Mathematics **210** (2015), 245–295.

[ELMV12] M. Einsiedler, E. Lindenstrauss, P. Michel and A. Venkatesh, *The distribution of closed geodesics on the modular surface, and Duke's theorem*, Enseignement des Mathématiques **58** (2012), 249–313.

[FJLR15] A.-H. Fan, T. Jordan, L. Liao and M. Rams, *Multifractal analysis for expanding interval maps with infinitely many branches*, Transactions of the American Mathematical Society **367** (2015), 1847–1870.

[Gur69] B. M. Gurevič, *Topological entropy of a countable Markov chain*, Soviet Mathematics. Doklady **10** (1969), 911–915.

[Ham] U. Hamenstädt, *Symbolic dynamics for the teichmüller flow*, preprint, arXiv:1112.6107.

[HIH77] E. Heintze and H.-C. Im Hof, *Geometry of horospheres*, Journal of Differential Geometry **12** (1977), 481–491 (1978).

[IJ13] G. Iommi and T. Jordan, *Phase transitions for suspension flows*, Communications in Mathematical Physics **320** (2013), 475–498.

[IJT15] G. Iommi, T. Jordan and M. Todd, *Recurrence and transience for suspension flows*, Israel Journal of Mathematics **209** (2015), 547–592.

[IT10] G. Iommi and M. Todd, *Natural equilibrium states for multimodal maps*, Communications in Mathematical Physics **300** (2010), 65–94.

[IT13] G. Iommi and M. Todd, *Transience in dynamical systems*, Ergodic Theory and Dynamical Systems **33** (2013), 1450–1476.

[JKL14] J. Jaerisch, M. Kesseböhmer and S. Lamei, *Induced topological pressure for countable state Markov shifts*, Stochastics and Dynamics **14** (2014), 1350016, 31 pp.

[Kem11] T. Kempton, *Thermodynamic formalism for suspension flows over countable Markov shifts*, Nonlinearity **24** (2011), 2763–2775.

[KKLM17] S. Kadyrov, D. Kleinbock, E. Lindenstrauss and G. A. Margulis, *Singular systems of linear forms and non-escape of mass in the space of lattices*, Journal d'Analayse Mathématique **133** (2017), 253–277.

[MU03] R. D. Mauldin and M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics, Vol. 148, Cambridge University Press, Cambridge, 2003.

[OP04] J.-P. Otal and M. Peigné, *Principe variationnel et groupes kleiniens*, Duke Mathematical Journal **125** (2004), 15–44.

[Pat76] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica **136** (1976), 241–273.

[PP90] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187/188** (1990).

[PPS15] F. Paulin, M. Pollicott and B. Schapira, *Equilibrium states in negative curvature*, Astérisque **373** (2015).

[PRL] F. Przytycki and J. Rivera-Letelier, *Geometric pressure for multimodal maps of the interval*, Memoirs of the American Mathematical Society, to appear.

[Rat73] M. Ratner, *Markov partitions for Anosov flows on n-dimensional manifolds*, Israel Journal of Mathematics **15** (1973), 92–114.

[RV] F. Riquelme and A. Velozo, *Escape of mass and entropy for geodesic flows*, Ergodic Theory and Dynamical Systems, to appear, arXiv:1610.04683.

[Sar99] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory and Dynamical Systems **19** (1999), 1565–1593.

[Sar01] O. Sarig, *Phase transitions for countable Markov shifts*, Communications in Mathematical Physics **217** (2001), 555–577.

[Sar03] O. Sarig, *Existence of Gibbs measures for countable Markov shifts*, Proceedings of the American Mathematical Society **131** (2003), 1751–1758.

[Sar15] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, in *Hyperbolic Dynamics, Fluctuations and Large Deviations*, Proceedings of Symposia in Pure Mathematics, Vol. 89, American Mathematical Society, Providence, RI, 2015, pp. 81–117.

[Sav98] S. V. Savchenko, *Special flows constructed from countable topological Markov chains*, Funktsional'nyĭ Analiz i ego Prilozheniya **32** (1998), 40–53, 96.

[Sin72] J. G. Sinaĭ, *Gibbs measures in ergodic theory*, Akademija Nauk SSSR i Moskovskoe Matematičeskoe Obščestvo. Uspehi Matematičeskih Nauk **27** (1972), 21–64.

[Sul84] D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Mathematica **153** (1984), 259–277.

[Wal82] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Vol. 79, Springer-Verlag, New York–Berlin, 1982.