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UNDER-RECURRENCE IN THE KHINTCHINE RECURRENCE THEOREM

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ABSTRACT

The Khintchine recurrence theorem asserts that in a measure preserving system, for every set A and $\varepsilon > 0$, we have $\mu(A \cap T^{-n}A) \geq \mu(A)^2 - \varepsilon$ for infinitely many $n \in \mathbb{N}$. We show that there are systems having under-recurrent sets A, in the sense that the inequality $\mu(A \cap T^{-n}A) < \mu(A)^2$ holds for every $n \in \mathbb{N}$. In particular, all ergodic systems of positive entropy have under-recurrent sets. On the other hand, answering a question of V. Bergelson, we show that not all mixing systems have under-recurrent sets. We also study variants of these problems where the previous strict inequality is reversed, and deduce that under-recurrence is a much more rare phenomenon than over-recurrence. Finally, we study related problems pertaining to multiple recurrence and derive some interesting combinatorial consequences.

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1. Introduction and main results

1.1. Introduction. One of the most classic results in ergodic theory is the Khintchine recurrence theorem which provides a quantitative refinement of the celebrated recurrence theorem of Poincaré:

Khintchine Recurrence Theorem ([11]): *Let* $(X, \mathcal{X}, \mu, T)$ *be a measure preserving system and* $A \in \mathcal{X}$ *be a set. Then for every* $\varepsilon > 0$ *we have*

$$\mu(A \cap T^{-n}A) \geq \mu(A)^2 - \varepsilon$$

for infinitely many $n \in \mathbb{N}$.

By considering mixing systems it is easy to see that the lower bound $\mu(A)^2$ cannot be in general improved. It is less clear whether the ε that appears on the right-hand side of Khintchine's estimate is a necessity or can be removed. This raises the following question:

Question 1: *Is there a measure preserving system* $(X, \mathcal{X}, \mu, T)$ *and a set* $A \in \mathcal{X}$ *such that* $\mu(A \cap T^{-n}A) < \mu(A)^2$ *holds for every* $n \in \mathbb{N}$ *? Can we take this system to be mixing?*

We show that the answer to both questions is affirmative (see Theorem 2.1). Moreover, we construct examples that answer affirmatively analogous questions pertaining to multiple recurrence (see Theorem 2.3).

Another natural question, first raised by V. Bergelson in [4, Problem 1], is whether such constructions can be carried out on every mixing system:

Question 2: *Is it true that for every mixing measure preserving system* $(X, \mathcal{X}, \mu, T)$ *there exists a set* $A \in \mathcal{X}$ *such that* $\mu(A \cap T^{-n}A) \leq \mu(A)^2$ *holds for every* $n \in \mathbb{N}$*?*

Rather surprisingly, the answer to this question is negative. In fact, we show (see the remark after Theorem 2.2) that if a system has an under-recurrent set, then it necessarily has a Lebesgue component (these notions are defined in the next section). Hence, if a system has singular maximal spectral type, then for every set $A \in \mathcal{X}$ with $0 < \mu(A) < 1$ we have $\mu(A \cap T^{-n}A) > \mu(A)^2$ for infinitely many $n \in \mathbb{N}$. Systems with singular maximal spectral type include all rigid systems and several (potentially all, as conjectured in [12]) rank one transformations. So, in a sense, systems that have under-recurrent sets are rather rare.

Another interesting fact is that although there are examples of over-recurrent sets for which the sequence $\mu(A \cap T^{-n}A) - \mu(A)^2$ converges to 0 arbitrarily slowly (see Theorem 2.4), for under-recurrent sets some stringent conditions apply which force this sequence to always be (absolutely) summable (see the remark after Theorem 2.5). In a sense, sets do not like to be under-recurrent.

In the next section, we give the precise statements of the results alluded to in the previous discussion and also give several relevant refinements and combinatorial consequences.

2. Main results

To facilitate our discussion we first introduce some notation.

A **measure preserving system**, or simply a **system**, is a quadruple $(X, \mathcal{X}, \mu, T)$ where $(X, \mathcal{X}, \mu)$ is a probability space and $T\colon X \to X$ is a measure preserving transformation. Throughout, all functions are assumed to be real valued and with Tf we denote the composition $f \circ T$.

Definition: Let $(X, \mathcal{X}, \mu, T)$ be a system. We say that:

(i) The set $A \in \mathcal{X}$ is **under-recurrent** if $0 < \mu(A) < 1$ and
$$\mu(A \cap T^{-n}A) \leq \mu(A)^2 \quad \text{for every } n \in \mathbb{N}.$$

(ii) The set $A \in \mathcal{X}$ is **over-recurrent** if $0 < \mu(A) < 1$ and
$$\mu(A \cap T^{-n}A) \geq \mu(A)^2 \quad \text{for every } n \in \mathbb{N}.$$

(iii) The function $f \in L^2(\mu)$ is **under-recurrent** if it is non-constant and
$$\int f \cdot T^n f \, d\mu \leq \left(\int f \, d\mu \right)^2 \quad \text{for every } n \in \mathbb{N}.$$

(iv) The function $f \in L^2(\mu)$ is **over-recurrent** if it is non-constant and
$$\int f \cdot T^n f \, d\mu \geq \left(\int f \, d\mu \right)^2 \quad \text{for every } n \in \mathbb{N}.$$

(v) If we have strict inequality, we say that the set or the function is **strictly** under-recurrent and over-recurrent respectively.

It is not hard to see using the von Neumann ergodic theorem that for every system $(X, \mathcal{X}, \mu, T)$ and function $f \in L^2(\mu)$ we have

$$\limsup_{n\to\infty} \int f \cdot T^n f \, d\mu \geq \left(\int f \, d\mu \right)^2.$$

If in addition the system is ergodic, then

$$\liminf_{n\to\infty} \int f \cdot T^n f \, d\mu \leq \left(\int f \, d\mu \right)^2.$$

This explains why we use the constants $\mu(A)^2$ and $(\int f \, d\mu)^2$ in the above definitions. Furthermore, we note that over-recurrent functions are not hard to come by; for instance, on a cartesian product system for each zero mean $f \in L^2(\mu)$ the function $f \otimes f$ is over-recurrent.

2.1. Under- and over-recurrent sets. We start by stating some results related to under- and over-recurrent sets.

Our first result gives an affirmative answer to Question 1 in the introduction.

Theorem 2.1: *There exists a mixing system that has strictly under-recurrent and strictly over-recurrent sets.*

Remark: We give two proofs of this result. One uses an explicit construction on Bernoulli systems and implies that every positive entropy system has under- and over-recurrent sets (see Theorem 7.1). The other is an indirect construction that is somewhat more versatile (see Section 4.2); for a large class of systems we establish the existence of under- and over-recurrent functions with values on $[0, 1]$ (see Proposition 3.4) and we then deduce using Proposition 4.1 the existence of under- and over-recurrent sets on different systems. Using this second method we can prove more delicate results like the following: For any partition $\mathbb{N} = S_+ \cup S_-$, there exist a mixing system and a set A, such that $\mu(A \cap T^{-n}A) > \mu(A)^2$ for every $n \in S_+$ and $\mu(A \cap T^{-n}A) < \mu(A)^2$ for every $n \in S_-$.

V. Bergelson asked in [4, Problem 1] whether every mixing system has a strictly under-recurrent set and whether it has a strictly over-recurrent set. We show that the answer to the first question (and thus to Question 2 in the introduction) is negative. The second question remains open; see Problem 1 in Section 2.4.

THEOREM 2.2: *There exists a mixing system with no under-recurrent sets.*

Remark: We show something stronger: Any system with singular maximal spectral type (with respect to the Lebesgue measure on $\mathbb{T}$) has no under-recurrent functions and in fact, for every non-constant $f \in L^2(\mu)$ we have $\int f \cdot T^n f \, d\mu > (\int f \, \mu)^2$ for infinitely many $n \in \mathbb{N}$ (see the remark after Theorem 2.5). Examples of mixing systems with singular maximal spectral type include the Chacon map and several other rank one transformations [5, 12] as well as certain Gaussian systems.

It is natural to inquire whether variants of Theorem 2.1 hold that deal with the concept of multiple under-recurrence. It is not hard to prove the following extension of the Khintchine recurrence theorem: For every system $(X, \mathcal{X}, \mu, T)$, set $A \in \mathcal{X}$, and every $\varepsilon > 0$, we have for every $\ell \in \mathbb{N}$ that

$$\mu(A \cap T^{-n_1} A \cap \cdots \cap T^{-n_\ell} A) \geq \mu(A)^{\ell+1} - \varepsilon$$

for infinitely many distinct $n_1, \ldots, n_\ell \in \mathbb{N}$. The next result shows that the ε in the above estimate cannot be removed.

THEOREM 2.3: *For every $d \in \mathbb{N}$ there exist a multiple mixing system*[1] *$(X, \mathcal{X}, \mu, T)$ and a set $A \in \mathcal{X}$ with $0 < \mu(A) < 1$ and such that for $\ell = 1, \ldots, d$ we have*

$$\mu(A \cap T^{-n_1} A \cap \cdots \cap T^{-n_\ell} A) < \mu(A)^{\ell+1}$$

for all distinct $n_1, \ldots, n_\ell \in \mathbb{N}$.

A similar result also holds with the strict inequality reversed.

Finally, we remark that if we do not impose any ergodicity assumptions on the system, we can prove a variant of Theorem 2.1 which gives more information about the possible values of the difference

$$d_A(n) := \mu(A \cap T^{-n} A) - \mu(A)^2.$$

We say that the sequence $(a_n)_{n \in \mathbb{N}}$ **decreases convexly to** 0 if it converges to 0 and satisfies $a_{n-1} + a_{n+1} - 2a_n \geq 0$ for every $n \geq 2$. Note that then $(a_n)_{n \in \mathbb{N}}$ is necessarily non-negative and decreasing.

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[1] A system $(X, \mathcal{X}, \mu, T)$ is multiple mixing if for every $\ell \in \mathbb{N}$ and $f_0, f_1, \ldots, f_\ell \in L^\infty(\mu)$ one has $\lim_{n_1, \ldots, n_\ell} \int f_0 \cdot T^{n_1} f_1 \cdots T^{n_\ell} f_\ell \, d\mu = \prod_{i=1}^{\ell} \int f_i \, d\mu$ where the limit is taken when $\min\{n_1, n_2 - n_1, \ldots, n_\ell - n_{\ell-1}\} \to \infty$.

Theorem 2.4: *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence. If either $\sum_{n=1}^{\infty}|a_n|<1/16$, or $(a_n)_{n\in\mathbb{N}}$ decreases convexly to 0 and satisfies $a_1 \leq 1/8$, then there exist a system $(X,\mathcal{B},\mu,T)$ and a set $A\in\mathcal{X}$ such that $d_A(n)=a_n$ for every $n\in\mathbb{N}$.*

Remarks:

- We show something stronger: If σ is a symmetric probability measure on $\mathbb{T}$, then there exist a system $(X,\mathcal{B},\mu,T)$ and a set $A\in\mathcal{X}$ such that $d_A(n)=\frac{1}{8}\,\widehat{\sigma}(n)$ for every $n\in\mathbb{N}$.
- When it comes to under-recurrence, the first hypothesis is not so severe; if A is an under-recurrent set, then the proof of Proposition 3.1 gives $\sum_{n=1}^{\infty}|\,d_A(n)|<1/2$.

2.2. Under- and over-recurrent functions. Next, we state results related to under- and over-recurrence properties of functions. For a particular class of systems, which we define next, it is possible to guarantee the existence of under- and over-recurrent functions.

Definition: We say that the system $(X,\mathcal{X},\mu,T)$ has a **Lebesgue component** if there exists a function $f\in L^2(\mu)$ with spectral measure equal to the Lebesgue measure on $\mathbb{T}$, that is, satisfies $\|f\|_{L^2(\mu)}=1$ and $\int f\cdot T^n f\,d\mu=0$ for every $n\in\mathbb{N}$.

Systems having a Lebesgue component include ergodic nilsystems that are not rotations [2, Theorem 4.2], [9, Proposition 2.1], and positive entropy systems [15].

The next result includes a convenient characterization of systems that have under-recurrent functions (it is the equivalence (i)$\Longleftrightarrow$(iii)).

Theorem 2.5: *Let $(X,\mathcal{X},\mu,T)$ be a system. Then the following are equivalent:*

(i) *The system has an under-recurrent function.*
(ii) *The system has a strictly under-recurrent function.*
(iii) *The system has a Lebesgue component.*
(iv) *For every non-negative $\phi\in L^1(m_{\mathbb{T}})$ there exists $g\in L^2(\mu)$ such that*

$$\int g\,d\mu=0 \quad\text{and}\quad \int g\cdot T^n g\,d\mu=\widehat{\phi}(n)\quad \text{for every } n\in\mathbb{N}.$$

Remark: Our argument shows that if $f\in L^2(\mu)$ is under-recurrent, then the spectral measure of f is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$ and

$$\sum_{n=1}^{\infty}\left|\int f\cdot T^n f\,d\mu-\left(\int f\,d\mu\right)^2\right|<\infty.$$

We deduce from the previous result the following:

COROLLARY 2.6: *Suppose that the system $(X, \mathcal{X}, \mu, T)$ has an under-recurrent function. Then it also has a strictly over-recurrent function. Furthermore, if the real valued sequence $(a_n)_{n\in\mathbb{N}}$ is either absolutely summable or decreases convexly to 0, then there exists $f \in L^2(\mu)$ such that $d_f(n) = a_n$ for every $n \in \mathbb{N}$, where*

$$d_f(n) := \int f \cdot T^n f \, d\mu - \Big(\int f \, d\mu \Big)^2 .$$

Finally, we record a variant of Theorem 2.5 which can be used (via Proposition 4.1 below) in order to deduce the existence of under- or over-recurrent sets for certain classes of systems.

THEOREM 2.7: *If the system $(X, \mathcal{X}, \mu, T)$ has a Lebesgue (spectral) measure realized by a bounded function, then it has a strictly under-recurrent function with values in $[0, 1]$ and a strictly over-recurrent function with values in $[0, 1]$.*

Remark: For a stronger statement see Proposition 3.4. Note also that by a theorem of V. M. Alexeyev [1] our assumption is satisfied for every system that has Lebesgue maximal spectral type.

2.3. COMBINATORIAL CONSEQUENCES. Finally, we use under- and over-recurrence properties of ergodic measure preserving systems in order to deduce some combinatorial consequences. In what follows, with $d(E)$ and $\bar{d}(E)$ we denote the density and the upper density of a set $E \subset \mathbb{N}$ respectively. Whenever we write $d(E)$ we implicitly assume that the density of the set E exists. Using the remark made immediately after Theorem 2.1 we deduce the following:

THEOREM 2.8: *For any partition $\mathbb{N} = S_+ \cup S_-$ there exists $E \subset \mathbb{N}$, such that*

(i) $d(E \cap (E - n)) > d(E)^2$ *for every* $n \in S_+$;
(ii) $d(E \cap (E - n)) < d(E)^2$ *for every* $n \in S_-$;
(iii) $\lim_{n\to\infty} d(E \cap (E - n)) = d(E)^2$.

Note that even the existence of a set of positive integers E that satisfies Property (ii) for $S_- = \mathbb{N}$ seems non-trivial to establish.

It can be shown (one way is to deduce this from the corresponding ergodic statement via the Furstenberg correspondence principle) that for every set $E \subset \mathbb{N}$

and every $\ell \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\bar{d}(E \cap (E - n_1) \cap \cdots \cap (E - n_\ell)) \geq \bar{d}(E)^{\ell+1} - \varepsilon$$

for infinitely many distinct $n_1, \ldots, n_\ell \in \mathbb{N}$. Using Theorem 2.8 we show that for every $\ell \in \mathbb{N}$ the ε in the above statement cannot in general be removed.

Theorem 2.9: *For every $r \in \mathbb{N}$ there exists a set of positive integers E such that for $\ell = 1, \ldots, r$ we have*

(i) $d(E \cap (E - n_1) \cap \cdots \cap (E - n_\ell)) < d(E)^{\ell+1}$ *for all distinct* $n_1, \ldots, n_\ell \in \mathbb{N}$*;*

(ii) $\lim_{n_1, \ldots, n_\ell} d(E \cap (E - n_1) \cap \cdots \cap (E - n_\ell)) = d(E)^{\ell+1}$,

where in (ii) *the limit is taken when* $\min\{n_1, n_2 - n_1, \ldots, n_\ell - n_{\ell-1}\} \to \infty$.

Moreover, there exists $E \subset \mathbb{N}$ that satisfies Property (i) *with the strict inequality reversed, and Property* (ii).

2.4. Open problems. Theorem 2.2 asserts that there exist mixing systems with no under-recurrent functions and the key to our construction was the fact that every under-recurrent function has spectral measure absolutely continuous with respect to the Lebesgue measure. This property is not shared by over-recurrent functions (see the example in Section 8), and in fact, constructing weakly mixing systems that have no over-recurrent functions (or sets) turns out to be much harder, perhaps even impossible.[2] This leads to the following question (a variant of it was also asked by V. Bergelson in [4, Problem 1]):

Problem 1: Does every (weakly) mixing system have an over-recurrent set or a bounded over-recurrent function?

C. Badea and V. Müller showed in [3] that every mixing system has a strictly over-recurrent function in $L^2(\mu)$. In fact, on every mixing system, for every sequence $(a_n)_{n \in \mathbb{N}}$ of positive reals that converges to 0, and every $\varepsilon > 0$, there exists a function $f \in L^2(\mu)$ with

$$\|f\|_{L^2(\mu)} \leq \sup_n a_n + \varepsilon$$

and such that

$$\int f \cdot T^n f \, d\mu > a_n$$

for every $n \in \mathbb{N}$.

[2] On the other hand, it is not hard to verify that ergodic Kronecker systems do not have over-recurrent functions.

It is possible (via Proposition 4.1 below) to transfer recurrence properties of a function with values in $[0, 1]$ in a given system, to recurrence properties of a set in a different system. It is not clear whether a similar construction can take place without changing the system.

Problem 2: If a system has an under-recurrent function with values in $[0, 1]$ does it always have an under-recurrent set?

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3. Under- and over-recurrent functions

In this section, we give the proofs of the results pertaining to under- and over-recurrence properties of functions; in the next section, we use some of these results in order to deduce analogous properties for sets.

3.1. Proof of Theorem 2.5 and Corollary 2.6. In this subsection, it is convenient to think of a correlation sequence $(\int f \cdot T^n f \, d\mu)_{n\in\mathbb{N}}$ as the sequence of Fourier coefficients of the spectral measure σ_f of the function f. This way, if $\int f \, d\mu = 0$, then under- or over-recurrence properties of a function $f \in L^2(\mu)$ correspond to statements about the sign of the sequence $(\widehat{\sigma_f}(n))_{n\in\mathbb{N}}$. Keeping this in mind, the key to the proof of Theorem 2.5 is the following simple Fourier analysis result:

Proposition 3.1: *Let σ be a probability measure on $\mathbb{T}$ such that $Re(\widehat{\sigma}(n)) \le 0$ for every $n \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} |Re(\widehat{\sigma}(n))| \le 1/2.$$

Remark: Our argument shows that if σ is a symmetric probability measure on $\mathbb{T}$ with a convergent sum of positive Fourier coefficients, then

$$-\sum_{n\in F_-} \widehat{\sigma}(n) \le \sum_{n\in F_+} \widehat{\sigma}(n) + \frac{1}{2},$$

where

$$F_+ := \{n \in \mathbb{N} \colon \widehat{\sigma}(n) > 0\} \quad \text{and} \quad F_- := \{n \in \mathbb{N} \colon \widehat{\sigma}(n) < 0\}.$$

Proof of Proposition 3.1. For every $N \in \mathbb{N}$ we have

$$0 \le \int \Big| \sum_{n=1}^{N} e(nt) \Big|^2 \, d\sigma(t) = N + 2\,\mathrm{Re} \sum_{1 \le m < n \le N} \int e((n-m)t)\, d\sigma(t)$$

$$= N + 2 \sum_{k=1}^{N} (N-k)\,\mathrm{Re}(\widehat{\sigma}(k))$$

$$= N + 2 \sum_{k=1}^{N-1} S_k,$$

where $S_k = \sum_{n=1}^{k} \mathrm{Re}(\widehat{\sigma}(n))$ and $e(t) := e^{2\pi i t}$. We conclude that

$$\frac{1}{2} + \frac{1}{2N} + \frac{1}{N} \sum_{n=1}^{N} S_n \ge 0 \quad \text{for every } N \in \mathbb{N}. \tag{1}$$

Since $(-S_n)$ is a non-decreasing sequence in $[0, \infty)$, it has a limit

$$L = \sum_{n=1}^{\infty} |\mathrm{Re}(\widehat{\sigma}(n))| \in [0, \infty].$$

Taking $N \to \infty$ in (1) results in the inequality $\frac{1}{2} - L \ge 0$, completing the proof. ∎

Corollary 3.2: *Let σ be a symmetric probability measure on $\mathbb{T}$ such that $\widehat{\sigma}(n) \le 0$ for every $n \in \mathbb{N}$. Then σ is equivalent to the Lebesgue measure on $\mathbb{T}$.*

Proof. Using Proposition 3.1 we get that

$$\sum_{n=1}^{\infty} |c_n| \le 1/2$$

where $c_n := \widehat{\sigma}(n)$, $n \in \mathbb{N}$. Then $d\sigma = \phi\, dm_{\mathbb{T}}$ where

$$\phi(t) := 1 + 2 \sum_{n=1}^{\infty} c_n \cos(2\pi n t).$$

We have that $\phi(t) \ge 0$ for all $t \in \mathbb{R}$, with equality only if

$$\sum_{n=1}^{\infty} |c_n \cos(2\pi n t)| = \sum_{n=1}^{\infty} |c_n| = 1/2.$$

This can happen only for finitely many $t \in \mathbb{T}$. It follows that σ is equivalent to the measure $m_{\mathbb{T}}$. ∎

We also need the following classic result from the spectral theory of unitary operators:

Proposition 3.3: *Let $(X, \mathcal{X}, \mu, T)$ be a system, $f \in L^2(\mu)$ be a function, and let ρ be a finite measure that is absolutely continuous with respect to σ_f. Then there exists $g \in L^2(\mu)$ with $\sigma_g = \rho$.*

Proof. We have that $d\rho = \phi\, d\sigma_f$ for some non-negative function $\phi \in L^1(\sigma_f)$. Then $\phi = \psi^2$ for some real valued $\psi \in L^2(\sigma_f)$. Let $g := \psi(T)f$ (see [14, Corollary 2.15] for the definition of the operator $\psi(T)$). Then g is real valued, $g \in L^2(\mu)$, and $d\sigma_g = \psi^2\, d\sigma_f = d\rho$, that is, $\sigma_g = \rho$. ∎

Proof of Theorem 2.5. We show that (i)$\Longrightarrow$(iii). Suppose that f is an under-recurrent function. Then the spectral measure of the real valued function $g := f - \int f\, d\mu$ is symmetric, not identically 0 (since our standing assumption is that f is non-constant), and satisfies

$$\widehat{\sigma_g}(n) = \int f \cdot T^n f\, d\mu - \left(\int f\, d\mu\right)^2 \leq 0 \quad \text{for every } n \in \mathbb{N}.$$

Hence, by Corollary 3.2, the measure σ_g is equivalent to $m_{\mathbb{T}}$. We deduce from this and Proposition 3.3 that there exists a function $h \in L^2(\mu)$ with $\sigma_h = m_{\mathbb{T}}$. Hence, the system has a Lebesgue component.

We show that (iii)$\Longrightarrow$(iv). Let $f \in L^2(\mu)$ have spectral measure $\sigma_f = m_{\mathbb{T}}$ and let $\phi \in L^1(m_{\mathbb{T}})$ be non-negative. By Proposition 3.3 there exists a zero mean $g \in L^2(\mu)$ such that $d\sigma_g = \phi\, dm_{\mathbb{T}}$. Then

$$\int g \cdot T^n g\, d\mu = \widehat{\sigma}_g(n) = \widehat{\phi}(n) \quad \text{for every } n \in \mathbb{N}.$$

Furthermore, since $\int g \cdot T^n g\, d\mu = \widehat{\phi}(n) \to 0$ as $n \to \infty$, we deduce from the ergodic theorem that

$$\left(\int g\, d\mu\right)^2 \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int g \cdot T^n g\, d\mu = 0.$$

Hence, $\int g\, d\mu = 0$.

We show that (iv)$\Longrightarrow$(ii). We apply (iv) for the non-negative (real valued) function $\phi \in L^\infty(\mathbb{T})$ defined by

$$\phi(t) := 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} (e(nt) + e(-nt)).$$

We get that there exists $g \in L^2(\mu)$ such that $\sigma_g = \phi\, dm_{\mathbb{T}}$. Then

$$\int g \cdot T^n g\, d\mu = -\frac{1}{2^{n+2}} < 0 = \left(\int g\, d\mu\right)^2 \quad \text{for every } n \in \mathbb{N}.$$

Hence, the function g is strictly under-recurrent.

Finally, the implication (ii)$\Longrightarrow$(i) is obvious. ∎

Proof of Corollary2.6. The proof follows by combining the implication (i)$\Longrightarrow$(iv) of Theorem 2.5 and the fact that under the stated assumptions there exists a non-negative even function $\phi \in L^1(m_{\mathbb{T}})$ such that $\widehat{\phi}(n) = a_n$ for every $n \in \mathbb{N}$. Indeed, if $(a_n)_{n\in\mathbb{N}}$ decreases convexly to 0, then this is a classic result (see for example [10, Theorem 4.1]); on the other hand, if $\sum_{n=1}^{\infty} |a_n| = A$, we let $\phi(t) := 2A + \sum_{n=1}^{\infty} a_n(e(nt) + e(-nt))$. ∎

3.2. Proof of Theorem 2.7. We prove a more general result:

Proposition 3.4: *Let $(X, \mathcal{X}, \mu, T)$ be a system that has a Lebesgue component defined by an $L^\infty(\mu)$ function. Then for any partition $\mathbb{N} = S_+ \cup S_-$ there exists $f \in L^\infty(\mu)$, with values in $[0, 1]$, such that*

(i) $\int f \cdot T^n f\, d\mu > (\int f\, d\mu)^2$ *for every* $n \in S_+$;
(ii) $\int f \cdot T^n f\, d\mu < (\int f\, d\mu)^2$ *for every* $n \in S_-$.

Proof. First note that it suffices to find $f \in L^\infty(\mu)$ that satisfies Properties (i) and (ii) without imposing any other restriction on its range. Indeed, then $\tilde{f} := \frac{1}{2\|f\|_\infty}(\|f\|_\infty + f)$ still satisfies Properties (i) and (ii) and takes values in $[0, 1]$.

Our assumptions imply that there exists a function $g \in L^\infty(\mu)$ such that

$$\int g \cdot T^n g\, d\mu = 0 \quad \text{for every } n \in \mathbb{N} \text{ and } \|g\|_{L^2(\mu)} = 1.$$

Note that then $\int g\, d\mu = 0$; indeed, the ergodic theorem implies that

$$\left(\int g\, d\mu\right)^2 \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int g \cdot T^n g\, d\mu = 0.$$

Now let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

(i) $(|a_n|)_{n\in\mathbb{N}}$ is decreasing;
(ii) $a_k > 0$ for $k \in S_+$ and $a_k < 0$ for $k \in S_-$;
(iii) $\sum_{k=1}^{\infty} |a_k| < 1$.

Consider the function

$$f := g + \sum_{k=1}^{\infty} a_k T^k g.$$

Then $f \in L^\infty(\mu)$ and $\int f\, d\mu = 0$. We set $a_0 := 1$. Then

$$\int f \cdot T^n f\, d\mu = \sum_{k,l \geq 0} a_k \cdot a_l \int T^l g \cdot T^{k+n} g\, d\mu \quad \text{for every } n \in \mathbb{N}.$$

Since, by assumption, $\{g, Tg, T^2g, \ldots\}$ is an orthonormal set, we deduce that

$$\int f \cdot T^n f\, d\mu = a_n + \sum_{k=1}^{\infty} a_k \cdot a_{k+n} \quad \text{for every } n \in \mathbb{N}.$$

Hence, using Properties (i)–(iii), we get for $n \in S_+$ that

$$\int f \cdot T^n f\, d\mu \geq a_n - \sum_{k=1}^{\infty} |a_k| \cdot |a_{k+n}| \geq a_n - a_n \sum_{k=1}^{\infty} |a_k| > 0,$$

and for $n \in S_-$ that

$$\int f \cdot T^n f\, d\mu \leq a_n + \sum_{k=1}^{\infty} |a_k| \cdot |a_{k+n}| \leq a_n - a_n \sum_{k=1}^{\infty} |a_k| < 0.$$

This completes the proof. ∎

4. Under- and over-recurrent sets

4.1. From functions to sets. We will use the following “correspondence principle” in order to translate statements about correlation sequences of functions with values on the interval $[0,1]$ to statements about correlation sequences of sets.

Proposition 4.1: *Let $(X, \mathcal{X}, \mu, T)$ be a system and $f \in L^\infty(\mu)$ be a function that takes values in $[0,1]$. Then there exist an invertible system $(Y, \mathcal{Y}, \nu, S)$ and a set $A \in \mathcal{Y}$ such that*

$$\nu(S^{-n_1}A \cap \cdots \cap S^{-n_\ell}A) = \int T^{n_1} f \cdots T^{n_\ell} f\, d\mu$$

holds for every $\ell \in \mathbb{N}$ and all distinct non-negative integers $n_1, \ldots, n_\ell$.

Moreover, if the system $(X, \mathcal{X}, \mu, T)$ is ergodic, weak-mixing, mixing, or multiple-mixing, then so is the system $(Y, \mathcal{Y}, \nu, S)$.

Proof. In the sequence space $Y := \{0,1\}^{\mathbb{Z}}$ we denote the cylinder sets by

$$[\epsilon_m\epsilon_{m+1}\cdots\epsilon_n] := \{x = (x_m)_{m\in\mathbb{Z}} \colon x_m = \epsilon_m, x_{m+1} = \epsilon_{m+1}, \ldots, x_n = \epsilon_n\},$$

where $m, n \in \mathbb{Z}$, $m \leq n$, and $\epsilon_i \in \{0,1\}$ for $i \in \mathbb{Z}$. We let

$$f_0 := f, \quad f_1 := 1 - f,$$

and define the measure ν on cylinder sets by

$$\nu([\epsilon_0\epsilon_1\cdots\epsilon_n]) := \int f_{\epsilon_0}\cdot Tf_{\epsilon_1}\cdots T^n f_{\epsilon_n}\, d\mu. \tag{2}$$

We extend ν to all cylinder sets in a stationary way. The consistency conditions of Kolmogorov's extension theorem are satisfied, thus ν extends to a stationary measure on the Borel σ-algebra of the sequence space Y. If

$$A := \{x \in Y \colon x_0 = 1\},$$

and S is the shift transformation on Y, then

$$\nu(S^{-n_1}A \cap \cdots \cap S^{-n_\ell}A) = \int T^{n_1}f \cdot \ldots \cdot T^{n_\ell} f\, d\mu$$

holds for every $\ell \in \mathbb{N}$ and all distinct non-negative integers $n_1, \ldots, n_\ell$.

Finally, suppose that the system $(X, \mathcal{X}, \mu, T)$ is mixing (ergodicity, weak-mixing, and multiple-mixing can be treated similarly). Let

$$A := [\epsilon_0\epsilon_1\cdots\epsilon_k], \quad B := [\tilde{\epsilon}_0\tilde{\epsilon}_1\cdots\tilde{\epsilon}_l]$$

be two cylinder sets, and let

$$g := f_{\epsilon_0}\cdot Tf_{\epsilon_1}\cdot\ldots\cdot T^k f_{\epsilon_k}, \quad h := f_{\tilde{\epsilon}_0}\cdot Tf_{\tilde{\epsilon}_1}\cdot\ldots\cdot T^l f_{\tilde{\epsilon}_l}.$$

Then for $n > \max\{k, l\}$, by the defining property of ν (see (2)), we have

$$\nu(A\cap S^{-n}B) = \int g\cdot T^n h\, d\mu \to \int g\, d\mu \cdot \int h\, d\mu = \nu(A)\cdot\nu(B).$$

By stationarity, we get a similar statement for any two cylinder sets, and by density for all Borel subsets of Y. This proves the asserted mixing property for the system $(Y, \mathcal{Y}, \nu, S)$. ∎

4.2. Proof of Theorems 2.1 and 2.2. We are now ready to prove results about under- and over-recurrent sets.

Proof of Theorem 2.1. Let $(X, \mathcal{X}, \mu, T)$ be a mixing system with a Lebesgue component defined by a bounded function (for example a Bernoulli system). Then by Theorem 2.7 there exists a strictly under-recurrent function f with values in $[0, 1]$. By Proposition 4.1 there exist a mixing system $(Y, \mathcal{Y}, \nu, S)$ and a set $A \in \mathcal{Y}$ with $\nu(A) = \int f \, d\mu$ (then $0 < \nu(A) < 1$ since f is non-constant), and such that for every $n \in \mathbb{N}$ we have

$$\nu(A \cap S^{-n}A) = \int f \cdot T^n f \, d\mu < \Big(\int f \, d\mu \Big)^2 = \nu(A)^2.$$

Hence, the set A is strictly under-recurrent.

A similar argument proves the existence of a strictly over-recurrent set B in some other mixing system $(Y, \mathcal{Y}, \nu, S)$.

To get a single system with a strictly under-recurrent and a strictly over-recurrent set, we consider the direct product $(X \times Y, \mathcal{X} \times \mathcal{Y}, \mu \times \nu, T \times S)$ of the two systems. This system is still mixing, the set $A \times Y$ is strictly under-recurrent, and the set $X \times B$ is strictly over-recurrent. ∎

To prove the statement in the remark following Theorem 2.1 we repeat the previous argument replacing Theorem 2.7 with Proposition 3.4.

Proof of Theorem 2.2. It is known that there exist mixing systems with singular maximal spectral type (see the remark after Theorem 2.2). Then by Theorem 2.5 any such system does not have under-recurrent functions, and as a consequence, does not have under-recurrent sets. ∎

The remark after Theorem 2.2 follows in a similar fashion since if $f \in L^2(\mu)$ satisfies $\int f \cdot T^n f \, d\mu \leq (\int f \, d\mu)^2$ for all large enough $n \in \mathbb{N}$, then the argument used in the proof of Corollary 3.2 shows that the spectral measure of f is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.

4.3. Proof of Theorem 2.4. Recall that for $f \in L^2(\mu)$ and $n = 0, 1, 2, \dots$ we define

$$d_f(n) := \int f \cdot T^n f \, d\mu - \Big(\int f \, d\mu \Big)^2.$$

The proof of Theorem 2.4 is based on the following result:

Proposition 4.2: *Let σ be a symmetric probability measure on $\mathbb{T}$. Then there exist a system $(X, \mathcal{B}, \mu, T)$ and a function $f \in L^\infty(\mu)$ that takes values in $[0,1]$ and such that*

$$d_f(n) = \frac{1}{8} \cdot \widehat{\sigma}(n) \quad \text{for } n = 0, 1, 2 \dots .$$

Proof. Consider the system $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \mu, T)$ where

$$T(x,y) := (x, y+x) \pmod 1$$

and $\mu = \sigma \times m_{\mathbb{T}}$ ($m_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$). Note that T preserves the measure μ. Let

$$f(x,y) := \frac{1}{2}(1 + \cos(2\pi y)) = \frac{1}{2}\Big(1 + \frac{e(y) + e(-y)}{2}\Big).$$

Then f takes values in $[0,1]$, $\int f \, d\mu = \frac{1}{2}$, and for $n = 0, 1, 2, \dots$ we have

$$\int f \cdot T^n f \, d\mu = \frac{1}{4} + \frac{\widehat{\sigma}(n) + \widehat{\sigma}(-n)}{16} = \Big(\int f \, d\mu\Big)^2 + \frac{1}{8}\widehat{\sigma}(n). \qquad \blacksquare$$

Proof of Theorem 2.4. Combining Propositions 4.1 and 4.2, we get that for any given symmetric probability measure σ on $\mathbb{T}$, there exist a system $(Y, \mathcal{Y}, \nu, S)$ and a set $A \in \mathcal{Y}$, with $\nu(A) = \int f \, d\mu$, and such that

$$\nu(A \cap S^{-n}A) = \nu(A)^2 + \frac{1}{8} \cdot \widehat{\sigma}(n) \quad \text{for } n = 0, 1, 2 \dots .$$

In order to complete the proof, we choose the measure σ appropriately, as in the proof of Corollary 2.6, taking this time into account that it is a probability measure. $\blacksquare$

5. Multiple under- and over-recurrence

In this subsection, we prove Theorem 2.3. The next definition gives a substitute for the notion of a Lebesgue component that is better suited for our multiple recurrence setup.

Definition: Let $(X, \mathcal{X}, \mu)$ be a probability space and S be a subset of $\mathbb{Z}$. We say that the sequence $(f_n)_{n \in S}$, of real valued functions in $L^\infty(\mu)$, is **an orthogonal sequence of order** $\ell \in \mathbb{N}$, if $\int f_{n_1} \cdot f_{n_2} \cdots f_{n_\ell} \, d\mu = 0$ whenever among the indices $n_1, \dots, n_\ell \in S$ there is at least one index not equal to each of the others.

Example: Consider a Bernoulli $(1/2, 1/2)$-system on the sequence space $\{-1,1\}^{\mathbb{Z}}$. Then the sequence of functions $(T^k f)_{k\in\mathbb{Z}}$, where $f(x) = x(0)$ and T is the shift transformation (defined by $(Tx)(k) = x(k+1)$, $k \in \mathbb{Z}$), is an orthogonal sequence of order ℓ for every $\ell \in \mathbb{N}$.

Proposition 5.1: *Let $(X, \mathcal{X}, \mu, T)$ be a system and let $\ell \in \mathbb{N}$. Suppose that there exists $g \in L^\infty(\mu)$ with values in $\{-1,1\}$ such that $(T^n g)_{n\geq 0}$ is an orthogonal sequence of order 2ℓ. Then there exists $f \in L^\infty(\mu)$ with values in $[0,1]$ such that for $d = 2, \ldots, \ell+1$ we have that*

$$\int T^{n_1} f \cdots T^{n_d} f \, d\mu < \left(\int f \, d\mu \right)^d$$

for all distinct $n_1, \ldots, n_d \in \mathbb{N}$. Furthermore, a similar statement holds with the strict inequality reversed.

Proof. Let $\ell \in \mathbb{N}$ and g satisfies the asserted hypothesis. We can assume that $\|g\|_{L^\infty(\mu)} \leq 1$. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that

(i) $a_0 = 0$ and $0 < a_n = -a_{-n} < \frac{1}{2^{d+1}\ell!}$ for every $n \in \mathbb{N}$;
(ii) $\sum_{n=1}^\infty a_n \leq \frac{1}{2}$.

Consider the function

$$h := g \cdot \sum_{k\in\mathbb{Z}}^\infty a_k T^k g.$$

Then

$$\|h\|_\infty \leq \sum_{k\in\mathbb{Z}} |a_k| = 2\sum_{k=1}^\infty a_k \leq 1.$$

Note that

$$\int h \, d\mu = \sum_{k\in\mathbb{Z}} a_k \int g \cdot T^k g \, d\mu = 0,$$

where the last equality follows from the order 2 orthogonality of the sequence $(T^n g)_{n\geq 0}$ and the fact that $a_0 = 0$. Let

$$f := \frac{1+h}{2}.$$

Then f takes values in $[0,1]$ and $\int f \, d\mu = \frac{1}{2}$. We claim that f satisfies the asserted under-recurrence property. For the reader's convenience, we first explain how the argument works for $d = 2, 3$; the general case is similar but the notation is more cumbersome.

Proof for $d = 2, 3$. A simple computation that uses the order 4 orthogonality of the sequence $(T^n g)_{n\geq 0}$, that $g^2 = 1$, and the properties of the sequence $(a_k)_{k\in\mathbb{Z}}$, shows that

$$\int T^{n_1}h \cdot T^{n_2}h \, d\mu = a_{n_1-n_2}a_{n_2-n_1} = -a_{n_1-n_2}^2$$

for all distinct $n_1, n_2 \in \mathbb{N}$. We deduce that for distinct $n_1, n_2 \in \mathbb{N}$ we have

$$\begin{aligned} 4\int T^{n_1}f \cdot T^{n_2}f \, d\mu =& 1 + 2\int h \, d\mu + \int T^{n_1}h \cdot T^{n_2}h \, d\mu \\ =& 1 - a_{n_1-n_2}^2 < 1, \end{aligned}$$

where we used that $\int h \, d\mu = 0$. Hence, for all distinct $n_1, n_2 \in \mathbb{N}$ we have

$$\int T^{n_1}f \cdot T^{n_2}f \, d\mu < \frac{1}{4} = \left(\int f \, d\mu\right)^2.$$

A similar computation, this time using the order 6 orthogonality of the sequence $(T^n g)_{n\geq 0}$, shows that

$$\int T^{n_1}h \cdot T^{n_2}h \cdot T^{n_3}h \, d\mu = a_{n_1-n_2}a_{n_2-n_3}a_{n_3-n_1} + a_{n_1-n_3}a_{n_2-n_1}a_{n_3-n_2} = 0$$

for all distinct $n_1, n_2, n_3 \in \mathbb{N}$. Furthermore, for distinct $n_1, n_2, n_3 \in \mathbb{N}$ we have

$$\begin{aligned} 8\int T^{n_1}f \cdot T^{n_2}f \cdot T^{n_3}f \, d\mu =& 1 + 3\int h \, d\mu + \int T^{n_1}h \cdot T^{n_2}h \, d\mu \\ &+ \int T^{n_1}h \cdot T^{n_3}h \, d\mu + \int T^{n_2}h \cdot T^{n_3}h \, d\mu \\ &+ \int T^{n_1}h \cdot T^{n_2}h \cdot T^{n_3}h \, d\mu \end{aligned}$$

which is equal to

$$1 - a_{n_1-n_2}^2 - a_{n_1-n_3}^2 - a_{n_2-n_3}^2 < 1.$$

Hence, for all distinct $n_1, n_2, n_3 \in \mathbb{N}$ we have

$$\int T^{n_1}f \cdot T^{n_2}f \cdot T^{n_3}f \, d\mu < \frac{1}{8} = \left(\int f \, d\mu\right)^3. \qquad \blacksquare$$

Proof for $d \geq 4$. First note that since $\int h \, d\mu = 0$, we have that

$$2^d \int T^{n_1}f \cdot \ldots \cdot T^{n_d}f \, d\mu = 1 + A + B, \tag{3}$$

where

$$A := \sum_{1\leq i<j\leq d} \int T^{n_i}h \cdot T^{n_j}h \, d\mu = -\sum_{1\leq i<j\leq d} a_{n_i-n_j}^2, \tag{4}$$

and

$$B := \text{ sum of at most } 2^d \text{ terms of the form } \int T^{m_1}h\cdots T^{m_{d'}}h\,d\mu \tag{5}$$

where $d' \in \{3,\ldots,d\}$ and $m_1,\ldots,m_{d'} \in \{n_1,\ldots,n_d\}$ are distinct integers. Let

$$\alpha := \max_{1\le i\neq j\le d}\{|a_{n_i-n_j}|\}. \tag{6}$$

Generalizing the computation done in the case $d = 2, 3$, this time using the order $2d$ orthogonality of the sequence $(T^n g)_{n\ge 0}$, we get for every $d \ge 2$ and distinct $n_1,\ldots,n_d \in \mathbb{N}$ that

$$\int T^{n_1}h\cdots T^{n_d}h\,d\mu = \sum_{\pi\in\Sigma[d]} a_{n_1-\pi(n_1)}\cdots a_{n_d-\pi(n_d)} \tag{7}$$

where $\Sigma[d]$ denotes the set of all permutations of the set $\{1,\ldots,d\}$ that have no fixed points. Combining (5), (6), (7), and using that $|a_n| \le \frac{1}{2^{d+1}d!}$ for all $n \in \mathbb{N}$, we get that

$$|B| \le 2^d d!\,\alpha^3 \le 2^d d!\,\frac{1}{2^{d+1}d!}\,\alpha^2 = \frac{\alpha^2}{2}.$$

Combining this with (4), we deduce that

$$1 + A + B \le 1 - \alpha^2 + \frac{\alpha^2}{2} < 1.$$

Hence, (3) gives that

$$\int T^{n_1}f\cdots T^{n_d}f\,d\mu < \frac{1}{2^d} = \Big(\int f\,d\mu\Big)^d$$

for all distinct $n_1,\ldots,n_d \in \mathbb{N}$, as required.

A similar (and simpler) argument proves the asserted over-recurrence property. The only change needed is that in the definition of the sequence $(a_k)_{k\in\mathbb{Z}}$ we impose that $a_{-n} = a_n$ for every $n \in \mathbb{N}$. ∎

Proof of Theorem 2.3. Consider a multiple mixing system that satisfies the assumptions of Proposition 5.1 (for example, any Bernoulli system). Using Proposition 4.1 we get a multiple mixing system and a set satisfying the asserted properties. ∎

6. Combinatorial consequences

In this short section, we deduce Theorems 2.8 and 2.9 from their ergodic counterparts.

Proof of Theorem 2.8. Let $(X, \mathcal{X}, \mu, T)$ be the mixing system and let A be the set given by the remark following Theorem 2.1. The ergodic theorem guarantees that for some $x_0 \in X$ and for every non-negative integer n we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_n(T^k x_0) = \int f_n \, d\mu$$

where $f_n := \mathbf{1}_{A\cap T^{-n}A}$. Let $E := \{m \in \mathbb{N}\colon T^m x_0 \in A\}$. Then

$$d(E) = \mu(A) \quad \text{and} \quad d(E\cap(E-n)) = \mu(A\cap T^{-n}A) \quad \text{for every } n \in \mathbb{N}.$$

Hence,

$$d(E\cap(E-n)) = \mu(A\cap T^{-n}A) > \mu(A)^2 = d(E)^2 \quad \text{for every } n \in S_+$$

and, similarly,

$$d(E\cap(E-n)) = \mu(A\cap T^{-n}A) < \mu(A)^2 = d(E)^2 \quad \text{for every } n \in S_-.$$

Moreover, since the system is mixing, we have that

$$d(E\cap(E-n)) = \mu(A\cap T^{-n}A) \to \mu(A)^2 = d(E)^2$$

as $n \to \infty$. ∎

In a similar fashion, we deduce Theorem 2.9 from Theorem 2.3. We include the details for the reader's convenience.

Proof of Theorem 2.9. For $r \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T)$ be the multiple mixing system and A be the set given by Theorem 2.3. The ergodic theorem guarantees that for some $x_0 \in X$ and for all non-negative integers $n_1, \ldots, n_r$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_{n_1,\ldots,n_d}(T^k x_0) = \int f_{n_1,\ldots,n_r} \, d\mu$$

where $f_{n_1,\ldots,n_r} := \mathbf{1}_{A\cap T^{-n_1}A\cap\cdots\cap T^{-n_r}A}$. Let $E := \{m \in \mathbb{N}\colon T^m x_0 \in A\}$. One concludes the proof of Property (i) exactly as in the proof of Theorem 2.8. Property (ii) follows in a similar way using the fact that the system is assumed to be multiple mixing.

The existence of a set E that satisfies Property (ii) with the strict inequality reversed and also satisfies Property (ii), follows in a similar fashion from Theorem 2.3. ∎

7. Under- and over-recurrent sets in positive entropy systems

In this section we give explicit constructions of under- and over-recurrent sets in Bernoulli systems and deduce that every system with positive entropy has under- and over-recurrent sets.

THEOREM 7.1: *Every ergodic system with positive entropy has a strictly over-recurrent and a strictly under-recurrent set.*

Proof. Suppose that the system $(X, \mathcal{X}, \mu, T)$ has entropy $h > 0$. It is known [16] that any Bernoulli shift with entropy smaller than h is a factor of the system $(X, \mathcal{X}, \mu, T)$. Hence, it suffices to show that there exist Bernoulli shifts with arbitrarily small entropy that have strictly under- and over-recurrent sets.

Thus, henceforth, we work with Bernoulli systems on the space $X := \{0,1,2\}^{\mathbb{N}}$ and for $i = 0, 1, 2$ we let

$$p_i := \mu([i]) \in (0,1),$$

where with $[x_1 \cdots x_k]$ we denote the cylinder set consisting of those $x \in X$ whose first k entries are $x_1, \ldots, x_k \in \{0,1,2\}$.

We first deal with over-recurrence. Let

$$A := \{x \in X : \text{the first non-zero entry of } x \text{ is } 1\}.$$

Then

$$A = \bigcup_{n=1}^{\infty} A_n$$

where

$$A_n := \{x \in X : \text{the first non-zero entry of } x \text{ is } 1 \text{ and it is at place } n\}.$$

Since $\mu(A_n) = p_0^{n-1} p_1$, we have

$$\mu(A) = \sum_{n=1}^{\infty} p_0^{n-1} p_1 = \frac{p_1}{1 - p_0} = \frac{p_1}{p_1 + p_2} =: a.$$

Moreover, we have

$$A \cap T^{-n} A = A'_n \cap T^{-n} A$$

where

$$A'_n := \left(\bigcup_{k=1}^{n} [(0)_{k-1}1] \right) \cup [(0)_n]$$

and $(0)_i$ is used to denote i-consecutive zero entries. Since the set A'_n depends on the first n entries of elements of X only, we have

$$\mu(A \cap T^{-n}A) = \mu(A'_n) \cdot \mu(T^{-n}A) = a \cdot \mu(A'_n)$$

where

$$\mu(A'_n) = p_0^n + \sum_{k=1}^{n} p_0^k p_1 = p_0^n + p_1 \cdot \frac{1-p_0^n}{1-p_0} = p_0^n + a(1-p_0^n) = a + p_0^n(1-a).$$

It follows that

$$\mu(A \cap T^{-n}A) = a^2 + p_0^n(1-a)a > a^2 = \mu(A)^2 \quad \text{for every } n \in \mathbb{N}.$$

Hence, the set A is strictly over-recurrent. Note also that by choosing p_1 sufficiently close to 1 (then p_0, p_2 will be close to 0) we can make the Bernoulli shift have arbitrarily small entropy.

Next we deal with under-recurrence. We let

$$A := \{x \in X\colon \text{the first two non-zero entries of } x \text{ are 1 and 2 in this order}\}.$$

Then

$$A = \bigcup_{k,l \geq 0} A_{k,l} \quad \text{where } A_{k,l} := [(0)_k 1 (0)_l 2].$$

Hence,

$$\mu(A) = \sum_{k,l \geq 0} p_1 p_2 p_0^{k+l} = p_1 p_2 \left(\sum_{k \geq 0} p_0^k \right)^2 = \frac{p_1 p_2}{(p_1+p_2)^2} =: a.$$

Next we fix $n \in \mathbb{N}$ and compute the measure of the set $A \cap T^{-n}A$. We partition the set A into three sets. The first, call it A_1, consists of those $x \in A$ whose first two non-zero entries (which are 1 and 2) occur at the first n places. Then

$$\begin{aligned}\mu(A_1 \cap T^{-n}A) =& a \cdot \sum_{0 \leq k+l \leq n-2} \mu([(0)_k 1 (0)_l 2]) = a p_1 p_2 \sum_{k=0}^{n-2} (k+1) p_0^k \\ =& a^2(1 + n p_0^n - n p_0^{n-1} - p_0^n).\end{aligned}$$

The second, call it A_2, consists of those $x \in A$ whose first two non-zero entries occur after the first n places. Then

$$\mu(A_2 \cap T^{-n}A) = a \cdot \mu([0]_n) = ap_0^n.$$

The third, call it A_3, consists of those $x \in A$ whose first non-zero entry (which is 1) occurs at the first n places and the second (which is 2) occurs after the first n places. Then clearly $A_3 \cap T^{-n}A = \emptyset$, hence

$$\mu(A_3 \cap T^{-n}A) = 0.$$

Combining the above, we deduce that

$$\mu(A \cap T^{-n}A) = a^2(1 + np_0^n - np_0^{n-1} - p_0^n) + ap_0^n, \quad n \in \mathbb{N}. \tag{8}$$

Then

$$\mu(A \cap T^{-n}A) < \mu(A)^2 = a^2 \iff n > \frac{p_0(1-a)}{a(1-p_0)}.$$

So it remains to choose p_0, p_1, p_2 so that $p_0 < a$; then the last estimate will be satisfied for all $n \in \mathbb{N}$ and the set A will be strictly under-recurrent. We let $p_1 = 1 - s$ and $p_2 = ts$ with $s, t \in (0,1)$. Then the estimate $p_0 < a = \frac{p_1 p_2}{(p_1+p_2)^2}$ leads to the equivalent estimate $1 - t < \frac{t(1-s)}{(1-s+ts)^2}$ which is satisfied, for example, if $t = \frac{3}{4}$ and $s < \frac{1}{2}$.

Summarizing, taking $p_0 = \frac{1}{4}s$, $p_1 = 1 - s$, $p_2 = \frac{3}{4}s$, we have that for all $s < \frac{1}{2}$ the set A, defined above, is strictly under-recurrent. Taking s close to 0 we deduce the existence of Bernoulli shifts with arbitrarily small entropy that have strictly under-recurrent sets. This finishes the proof. ∎

8. Singular over-recurrent functions on a mixing system

In Theorem 2.2 we showed that there exist mixing systems with no under-recurrent sets, and the key to our construction was that a function with singular spectral measure cannot be under-recurrent. In this section we show that a similar approach cannot be used in order to construct mixing systems with no over-recurrent functions. We will show that there exists a mixing system that has a strictly over-recurrent function with singular spectral measure.

First, we briefly review some basic facts regarding Riesz-products; their proofs can be found in [14, pages 5–7] and [6, 13, 17]. If

$$P_N(t) = \prod_{j=0}^{N-1} (1 + a_j \cos(3^j t)), \quad N \in \mathbb{N},$$

where $(a_j)_{j \geq 0}$ are real numbers with $|a_j| \leq 1$, then the sequence of probability measures $(\sigma_N)_{N\in\mathbb{N}}$, defined by

$$d\sigma_N := P_N(t)\, dt, \quad N \in \mathbb{N},$$

converges w^* to a symmetric probability measure σ on $[0,1]$ with Fourier coefficients $\widehat{\sigma}(0) = 1$ and

$$\widehat{\sigma}(n) = \prod_j (\frac{a_j}{2}), \quad \text{if } n = \sum_{j=0}^{k} \epsilon_j 3^j,\ \epsilon_j = -1, 0, 1, \tag{9}$$

where the product is taken over those $j \in \{0, \ldots, k\}$ for which $\epsilon_j \neq 0$.

The measure σ is equivalent to the Lebesgue measure if $\sum_{j=0}^{\infty} |a_j|^2 < \infty$ and is continuous and singular if $\sum_{j=0}^{\infty} |a_j|^2 = \infty$.

We also review some basic facts regarding Gaussian systems; their proofs can be found in [7, pages 369–371] and [8, pages 90–92]. If σ is a symmetric probability measure on the circle, then there exist a Gaussian system $(X, \mathcal{X}, \mu, T)$ and a function $f \in L^2(\mu)$ (f is a real Gaussian variable, so it is not bounded) with spectral measure σ, meaning it satisfies

$$\int f \cdot T^n f\, d\mu = \widehat{\sigma}(n) \quad \text{for } n = 0, 1, 2, \ldots. \tag{10}$$

The Gaussian system is mixing if and only if the measure σ is Rajchman, meaning it satisfies $\widehat{\sigma}(n) \to 0$ as $n \to \infty$. Note that in this case we have $\int f\, d\mu = 0$.

We proceed now to the construction (shown to us by B. Host). We take σ to be the w^*-limit of the sequence of measures $(\sigma_N)_{N\in\mathbb{N}}$ defined by

$$d\sigma_N = \prod_{j=0}^{N} \Big(1 + \frac{\cos(3^j t)}{\sqrt{j+1}}\Big)\, dt, \quad N \in \mathbb{N}.$$

Since

$$\sum_{j=0}^{\infty} \Big(\frac{1}{\sqrt{j+1}}\Big)^2 = \infty,$$

as remarked above, the measure σ is singular. Moreover, it follows from (9) that $\widehat{\sigma}(n) > 0$ for every $n \in \mathbb{N}$ and $\widehat{\sigma}(n) \to 0$ as $n \to \infty$.

Next, we consider a Gaussian system and a function $f \in L^2(\mu)$ that satisfies (10). As remarked above, this system is mixing. Moreover, the function f has singular spectral measure by construction, and satisfies

$$\int f \cdot T^n f \, d\mu = \widehat{\sigma}(n) > 0 \quad \text{for every } n \in \mathbb{N}.$$

Hence, the function f is strictly over-recurrent, as required.

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