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POLYNOMIAL IDENTITIES FOR MATRICES OVER THE GRASSMANN ALGEBRA

BY

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ABSTRACT

We determine minimal Cayley–Hamilton and Capelli identities for matrices over a Grassmann algebra of finite rank. For minimal standard identities, we give lower and upper bounds on the degree. These results improve on upper bounds given by L. Márki, J. Meyer, J. Szigeti and L. van Wyk in a recent paper.

1. Notations

Let R be a commutative ring with 1. For $m \geq 0$, consider the Grassmann algebra

$$E^m = R\langle v_1, \ldots, v_m\rangle/(v_k^2, v_iv_j + v_jv_i \mid 1 \leq k \leq m, 1 \leq i < j \leq m)$$

of rank m. It is a graded R-algebra (each v_i has degree 1). We write E_i^m for the degree i component, so

$$E^m = \bigoplus_{i=0}^{m} E_i^m, \qquad E_0^m = R, \quad E_m^m = Rv_1 \cdots v_m.$$

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We write

$$E^m_{\geq r} = \bigoplus_{i=r}^{m} E^m_i.$$

Let M_nX be the set of n-square matrices with entries in the set X.

2. Cayley–Hamilton identity

L. Márki, J. Meyer, J. Szigeti and L. van Wyk have shown [4, Theorem 3.4] that if the base ring R is a field of characteristic zero and $m \geq 2$, then any element of M_nE^m satisfies a monic polynomial of degree $n \cdot 2^{m-1}$ over R. This was achieved by constructing a CT-embedding of E^m into a 2^{m-1}-square matrix algebra over a suitable commutative R-algebra — an interesting result in its own right. In the present paper, we allow R to be any commutative ring with 1 and reduce the degree of the monic polynomial from $n \cdot 2^{m-1}$ to $n \cdot (\lceil m/2 \rceil + 1)$, which turns out to be least possible in general. Moreover, a suitable polynomial of this degree is given explicitly. For the proof, we do not use CT-embeddings. Instead, we directly exploit the nilpotency and supercommutativity properties of the Grassmann algebra.

We now set up notation that will be used throughout this section. Let $A \in M_nE^m$. We decompose A into its homogeneous components:

$$A = \sum_{i=0}^{m} A_i, \qquad A_i \in M_nE^m_i.$$

Let

$$f(x) = \det(xI - A_0) \in R[x]$$

be the (monic) characteristic polynomial of $A_0 \in M_nR$. The main result of this paper is

THEOREM 1: *For any $A \in M_nE^m$, we have*

$$f(A)^{\lceil m/2 \rceil + 1} = 0.$$

For $m = 0$, this recovers the Cayley–Hamilton Theorem.

As a first step towards the proof, we decompose $B = f(A) \in M_nE^m$ into its homogeneous components:

$$B = \sum_{i=0}^{m} B_i, \qquad B_i \in M_nE^m_i.$$

Before attacking Theorem 1 in its full generality, we treat a special case.

LEMMA 2: *If the degree zero component A_0 of $A \in M_nE^m$ is a diagonal matrix $A_0 = \operatorname{diag}(\lambda_i)_{i=1}^n$ with distinct diagonal elements $\lambda_i \in R$, and $A_1 = (v_{ij})$ with $v_{ij} \in E_1^m$, then*

(0) $B_0 = 0$,

(1) $B_1 = \operatorname{diag}(f'(\lambda_i)v_{ii})_{i=1}^n$,

(2) $B_1^2 = 0$,

(3) $(B_2)_{rs} = \frac{1}{\lambda_r - \lambda_s}\left(f'(\lambda_r)v_{rr} + f'(\lambda_s)v_{ss}\right)v_{rs}$ *for* $r \neq s$.

(4) *If B_2^+ and B_2^- denote the diagonal and off-diagonal part of B_2 respectively, then B_1 commutes with B_2^+ but anticommutes with B_2^-.*

Proof. We have

$$f(x) = \prod_{i=1}^n (x - \lambda_i),$$

whence

$$B = f(A) = \prod_{i=1}^n (A - \lambda_i I). \tag{2.1}$$

(0) We have $B_0 = f(A_0)$, which is zero by the Cayley–Hamilton Theorem, or, if you prefer, by the trivial computation

$$f(A_0) = \prod_{i=1}^n (A_0 - \lambda_i I) = \operatorname{diag}\left(\prod_{i=1}^n (\lambda_j - \lambda_i)\right)_{j=1}^n = 0.$$

(1) The factors in (2.1) commute, so, for any indices $r \neq s$, we have

$$B = (A - \lambda_r I)C(A - \lambda_s I) \tag{2.2}$$

for some $C \in M_nE^m$. In the first factor, the r-th row has no degree zero component. In the last factor, the s-th column has no degree zero component. Thus, the (r, s) entry in B has no degree 1 component.

The degree 1 component of the (r, r) entry in B arises by taking a degree 1 component from the r-th row of $A - \lambda_r I$ and degree zero components from all other $A - \lambda_i I$. But the degree zero components of these matrices are diagonal, so the only possibility is to use the (r, r) entry from each factor. The result is

$$v_{rr} \prod_{i \neq r} (\lambda_r - \lambda_i),$$

as claimed.

(2) This is clear because B_1 is diagonal and homogeneous of degree 1.

(3) Using formulas (2.1) and (2.2), we obtain

$$(B_2)_{rs} = \sum_{j=1}^{n} v_{rj}v_{js} \prod_{i\neq r,s}(\lambda_j - \lambda_i) = v_{rr}v_{rs}\frac{f'(\lambda_r)}{\lambda_r - \lambda_s} + v_{rs}v_{ss}\frac{f'(\lambda_s)}{\lambda_s - \lambda_r},$$

which yields the result.

(4) The first statement is clear because B_1 and B_2^+ are diagonal and B_2^+ is homogeneous of degree 2 — note that in the Grassmann algebra, homogeneous elements of even degree are central.

For the second statement, observe that B_1 is diagonal and B_2^- is off-diagonal, so their product, in either order, is off-diagonal. Moreover, for $r \neq s$, the (r, s) entry in the product is

$$(B_1B_2^-)_{rs} = (B_1)_{rr}(B_2)_{rs} = f'(\lambda_r)v_{rr}\frac{1}{\lambda_r - \lambda_s}f'(\lambda_s)v_{ss}v_{rs}$$

for one order and is

$$(B_2^-B_1)_{rs} = (B_2)_{rs}(B_1)_{ss} = \frac{1}{\lambda_r - \lambda_s}f'(\lambda_r)v_{rr}v_{rs}f'(\lambda_s)v_{ss}$$

for the other order. These add up to zero as claimed. ∎

Proof of Theorem 1. The coefficients of the polynomial $f(x)$ are polynomials with integer coefficients in the entries of A_0. Hence, the coordinates in the natural R-basis

$$\{v_{i_1} \cdots v_{i_k} | i_1 < \cdots < i_k\} \tag{2.3}$$

of the entries of the matrix $f(A)^{\lceil m/2\rceil+1}$ are polynomials with integer coefficients in the coordinates of the entries of A. The theorem is equivalent to the statement that these $n^2 2^m$ polynomials are all identically zero. Thus, we may assume that $R = \mathbb{C}$. We may assume that A_0 has n distinct eigenvalues, since such matrices are dense in $M_n(\mathbb{C})$. Then A_0 is diagonalizable by an invertible complex matrix P. Since the conjugation by P is an automorphism of M_nE^m as a graded algebra over $\mathbb{C}$, we may assume that $P = I$, i.e., A_0 is diagonal with distinct diagonal entries. Then, by Lemma 2, we have

$$B = \sum_{i=1}^{m} B_i,$$

where $B_i \in M_n E_i^m$, $B_1^2 = 0$, $B_2 = B_2^+ + B_2^-$, and $B_1 B_2^\pm = \pm B_2^\pm B_1$. If $B^k \neq 0$ for an exponent k, then there is a nonzero product

$$B_{i_1}^\bullet \cdots B_{i_k}^\bullet,$$

where $i_1, \ldots, i_k \in \{1, \ldots, m\}$, and $B_i^\bullet$ means $B_2^\pm$ if $i = 2$ and means B_i otherwise. But then, in the sequence $i_1, \ldots, i_k$, any two 1's are separated by at least one $i \geq 3$, whence

$$m \geq i_1 + \cdots + i_k \geq 2k - 1,$$

so $k \leq \lfloor (m+1)/2 \rfloor = \lceil m/2 \rceil$. ∎

We now show that the degree of the polynomial in Theorem 1 cannot be reduced.

Proposition 3: *Let R be a field of characteristic either 0 or a prime $p > \lceil m/2 \rceil$. Let $\lambda_1, \ldots, \lambda_n \in R$ be distinct elements and*

$$v = v_1 v_2 + v_3 v_4 + \cdots + v_{2\lfloor m/2 \rfloor - 1} v_{2\lfloor m/2 \rfloor} \{+ v_m\} \in E^m$$

(the last term appears only if m is odd).

Let

$$A = \operatorname{diag}(\lambda_i + v)_{i=1}^n \in M_n E^m,$$

so that $A_0 = \operatorname{diag}(\lambda_i)_{i=1}^n$. Then the characteristic polynomial of A_0 is

$$f(x) = \prod_{i=1}^n (x - \lambda_i)$$

and the minimal polynomial of A over R is

$$f(x)^{\lceil m/2 \rceil + 1}.$$

Proof. Observe that

$$v^{\lceil m/2 \rceil} = \lceil m/2 \rceil! v_1 \cdots v_m \neq 0.$$

Thus, the polynomial

$$g_i(x) = \frac{f(x)^{\lceil m/2 \rceil + 1}}{x - \lambda_i} = (x - \lambda_i)^{\lceil m/2 \rceil} \prod_{j \neq i} (x - \lambda_j)^{\lceil m/2 \rceil + 1}$$

does not vanish at A. Indeed, the (i,i)-entry of $g_i(A)$ is

$$v^{\lceil m/2 \rceil} \prod_{j \neq i} (\lambda_i - \lambda_j + v)^{\lceil m/2 \rceil + 1} = v^{\lceil m/2 \rceil} f'(\lambda_i)^{\lceil m/2 \rceil + 1} \neq 0. \quad ∎$$

3. Capelli identity

Recall [3, Definition 1.5.3] that the Capelli polynomial d_k is defined by the formula

$$d_k(x_1,\dots,x_k;y_0,\dots,y_k)=\sum_{\pi\in\mathfrak{S}_k}(-1)^\pi y_0x_{\pi(1)}y_1x_{\pi(2)}\cdots y_{k-1}x_{\pi(k)}y_k.$$

We say that the Capelli identity of x-degree k holds in a ring $\mathfrak{A}$ if the above expression is 0 for all $x_1,\dots,x_k,y_0,\dots,y_k\in\mathfrak{A}$. It is trivial that the Capelli identity of x-degree k implies the Capelli identity of x-degree $k+1$.

It is well known that the ring of n-square matrices over a commutative ring satisfies the Capelli identity of x-degree n^2+1 (because the Capelli polynomial is alternating in the variables x_i), but does not satisfy the Capelli identity of x-degree n^2 if the base ring has $1\neq 0$ (because we may choose the x_i to be the usual matrix units in some order, and choose the y_i to be suitable matrix units such that exactly one term in d_{n^2} is nonzero). We now wish to generalize this to matrices over the Grassmann algebra E^m. We shall need the following lemma.

Lemma 4: *Let $a_1,\dots,a_k$ be elements of a ring. Suppose that $[k]=\{1,\dots,k\}=M\cup N$ is a disjoint union. Suppose that a_i and a_j anticommute for distinct $i,j\in M$, but commute otherwise.*

Let $\mathcal{P}$ be a partition of $[k]$ into $|N|$ classes, each class containing exactly one element of N. Let $\mathfrak{S}\subseteq\mathfrak{S}_k$ be the Young subgroup corresponding to $\mathcal{P}$ (i.e., $\mathfrak{S}$ is the group of permutations leaving each class invariant).

(a) *If $|M|$ is odd, then*

$$\sum_{\pi\in\mathfrak{S}}(-1)^\pi a_{\pi(1)}\cdots a_{\pi(k)}=0. \tag{3.1}$$

(b) *If $\mathcal{P}$ consists of intervals of odd cardinalities $m_1+1,\ \dots,\ m_{|N|}+1$ respectively, and N consists of the leftmost elements of these intervals, then*

$$\sum_{\pi\in\mathfrak{S}}(-1)^\pi a_{\pi(1)}\cdots a_{\pi(k)}=m_1!\cdots m_{|N|}!a_1\cdots a_k. \tag{3.2}$$

Proof. (a) We use induction on $m=|M|$. For $m=1$, the group $\mathfrak{S}$ has two elements of distinct sign, and all a_i commute, so (3.1) holds.

Let $m \geq 3$ be odd. Suppose that the claim is true for $m-2$. Let us prove it for m.

Consider the special case when $\mathcal{P}$ is a partition into intervals. Then the left-hand side of (3.1) can be written as a product. For each interval $I \in \mathcal{P}$, we get a factor of the form

$$\sum_{\pi \in \mathfrak{S}_I} (-1)^\pi \prod_{i \in I} a_{\pi(i)}. \tag{3.3}$$

Since m is odd, we can choose an interval $I \in \mathcal{P}$ that has an even number of elements. There is a unique $i \in I \cap N$. The terms in (3.3) where $\pi^{-1}(i)$ is even can be paired off with those where $\pi^{-1}(i)$ is odd. This can be done so that in each pair π has different signs but $\prod_{i \in I} a_{\pi(i)}$ is the same, so the sum within each pair is zero. This proves the special case.

To finish the proof, it suffices to prove the following. If the lemma is true for a sequence $a_1, \ldots, a_k$ and a partition $\mathcal{P}$, and $i-1$ and i are in distinct classes S and T of $\mathcal{P}$ respectively, then the lemma remains true for $a_i' = a_{i-1}$, $a_{i-1}' = a_i$, $M' = M\Delta\{i-1, i\}$, $N' = N\Delta\{i-1, i\}$, $S' = (S - \{i-1\}) \cup \{i\}$, $T' = (T - \{i\}) \cup \{i-1\}$ (all other data remain unchanged).

To prove this, we examine the change made in the left-hand side of (3.1). The terms remain the same up to order and sign. The terms that change sign are exactly those where $\pi(i-1)$ and $\pi(i)$ both come from the set M. It suffices to prove that these terms sum to zero. This is true even if $\pi(i-1)$ and $\pi(i)$ are fixed elements of M, due to the induction hypothesis.

(b) We may assume that $|N| = 1$. Then the claim is trivial. ∎

Theorem 5: *The ring $M_n E^m$ satisfies the Capelli identity of x-degree $k = n^2 + 2\lfloor m/2 \rfloor + 1$.*

Proof. Let $A_1, \ldots, A_k, B_0, \ldots, B_k \in M_n E^m$. We prove that

$$d_k(A_1, \ldots, A_k; B_0, \ldots, B_k) = 0. \tag{3.4}$$

By multilinearity, we may assume that each A_i and each B_i has only one nonzero entry, which is an element of the standard R-basis (2.3) of E^m. Moreover, we may assume that the degrees of these $2k+1$ basis elements sum to at most m. Then at most m of these degrees are nonzero, i.e., at least $2k+1-m$ of these $2k+1$ basis elements are 1. At least $k-m$ of these 1's come from the matrices A_i.

If m is even, then $k-m=n^2+1$, so, by the pigeonhole principle, there exist indices $i \neq i'$ such that $A_i = A_{i'}$, whence (3.4) holds.

If m is odd, then $k-m=n^2$. We may assume that

$$A_1, \ldots, A_{n^2}$$

is the standard R-basis of M_nR, while

$$A_{n^2+i} = v_i A_{j_i},$$

where $1 \leq j_i \leq n^2$ for $i = 1, \ldots, m$. We may also assume that each B_i comes from the standard R-basis of M_nR. The claim now follows from Lemma 4(a), applied to the nonzero entries of the matrices A_i. ∎

Proposition 6: *The ring M_nE^m does not satisfy the Capelli identity of x-degree $k = n^2 + 2\lfloor m/2 \rfloor$ if the base ring R is a field of characteristic either zero or a prime $p > 2\left\lceil \lfloor m/2 \rfloor / n^2 \right\rceil$.*

Proof. Let us write $2\lfloor m/2 \rfloor$ as a sum of n^2 even numbers that are smaller than p if the characteristic is $p > 0$. Let these even numbers be $m_1, \ldots, m_{n^2}$. Let $A_1, \ldots, A_{n^2}$ be the standard basis of M_nR. For each r, consider m_r matrices of the form v_iA_r, chosen so that each index $i = 1, \ldots, 2\lfloor m/2 \rfloor$ is used exactly once. Let us insert the chosen m_r multiples of A_r immediately after A_r into the sequence $A_1, \ldots, A_{n^2}$. This gives us a sequence $C_1, \ldots, C_k$. Now let $B_0, \ldots, B_k$ be elements from the standard basis of M_nR with the property that $B_0C_1B_1 \cdots C_kB_k \neq 0$. Then

$$d_k(C_1, \ldots, C_k; B_0, \ldots, B_k) = B_0C_1B_1 \cdots C_kB_k \prod_{r=1}^{n^2} m_r! \tag{3.5}$$

by Lemma 4(b), applied to the case where a_i is the unique nonzero entry of the matrix C_i $(i = 1, \ldots, k)$. The right-hand side of (3.5) is nonzero because $m_r < p$ if the characteristic is $p > 0$. ∎

4. Standard identity

The standard polynomial s_k is defined by the formula

$$s_k(x_1, \ldots, x_k) = \sum_{\pi \in \mathfrak{S}_k} (-1)^\pi x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)}.$$

We say that the standard identity of degree k holds in a ring $\mathfrak{A}$ if the above expression is 0 for all $x_1, \ldots, x_k \in \mathfrak{A}$. It is trivial that the standard identity of degree k implies the standard identity of degree $k+1$. When $\mathfrak{A} \ni 1$ and k is even, the converse implication holds as well because

$$s_k(x_1, ..., x_k) = s_{k+1}(1, x_1, ..., x_k).$$

Also, the Capelli identity of x-degree k implies the standard identity of degree $2\lfloor k/2 \rfloor$ if $\mathfrak{A} \ni 1$. Indeed, we may substitute 1 for each y_i in the Capelli identity to get the standard identity of degree k, and then we can use the previous remark.

The celebrated Amitsur–Levitzki theorem [1], see e.g. also [3], states that the ring of n-square matrices over a commutative ring satisfies the standard identity of degree $2n$. An easy example shows that it does not satisfy the standard identity of degree $2n-1$ if the base ring has $1 \neq 0$. We now wish to generalize this to matrices over the Grassmann algebra E^m.

L. Márki, J. Meyer, J. Szigeti and L. van Wyk [4, 3.7 Theorem] used an embedding into a matrix algebra over a commutative ring and invoked the Amitsur–Levitzki Theorem to show that for $m \geq 1$, the standard identity of degree $2^m n$ holds in $M_n E^m$. They also invoked a very general theorem of M. Domokos [2, Theorem 5.5] to show that the standard identity of degree $(m+1)n^2+1$ holds in $M_n E^m$ [4, 3.8 Remark]. We now show that these degree bounds can be substantially reduced. For the latter one, this is already clear from Theorem 5, which yields

Corollary 7: *The standard identity of degree*

$$2\left(\left\lfloor \frac{n^2+1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor\right)$$

holds in $M_n E^m$.

An improvement of the degree bound $2^m n$ is given by

Proposition 8: *The standard identity of degree* $k = 2n(\lfloor m/2 \rfloor + 1)$ *holds in* $M_n E^m$.

Proof. We prove the stronger identity

$$s_{2n}(A_1, \ldots, A_{2n}) s_{2n}(A_{2n+1}, \ldots, A_{4n}) \cdots s_{2n}(A_{k-2n+1}, \ldots, A_k) = 0$$

for all $A_1, \ldots, A_k \in M_n E^m$. It suffices to prove that each of the $\lfloor m/2 \rfloor + 1$ factors is contained in $M_n E^m_{\geq 2}$. In fact, it suffices to prove this for the first factor. Observe that the ring

$$E^m/(v_2, \ldots, v_m) \simeq R[v_1]/(v_1^2)$$

is commutative. Thus, by the Amitsur–Levitzki Theorem, n-square matrices over this ring satisfy the standard identity of degree $2n$. Thus, each entry in the matrix $s_{2n}(A_1, \ldots, A_{2n})$ is contained in the ideal $(v_2, \ldots, v_m)$; moreover, by the same argument, it is contained in

$$\bigcap_{i=1}^{m} (v_j | j \neq i) = E^m_{\geq 2},$$

as claimed. ∎

Note that for $m = 0$ or $m = 1$, the ring $M_n E^m$ is commutative and Proposition 8 reduces to the Amitsur–Levitzki Theorem ($k = 2n$) and therefore is sharp.

Proposition 8 is sharp for $n = 1$, and Corollary 7 is sharp for $n = 1$ or $n = 2$. More generally, we have

PROPOSITION 9: *The standard identity of degree $k = 2(n + \lfloor m/2 \rfloor) - 1$ does not hold in $M_n E^m$ if the base ring R is a field of characteristic either zero or a prime $p > 2\lfloor m/2 \rfloor$.*

Proof. Consider the $2n - 1$ matrices e_{12}, e_{23}, $\ldots$, $e_{n-1,n}$, e_{nn}, $e_{n,n-1}$, $e_{n-1,n-2}$, $\ldots$, e_{21}, together with the $2\lfloor m/2 \rfloor$ further matrices $v_i e_{11}$, where $i = 1, \ldots, 2\lfloor m/2 \rfloor$. The standard polynomial s_k evaluated at these k matrices is the same as

$$s_{2\lfloor m/2 \rfloor + 1}(e_{11}, v_1 e_{11}, \ldots, v_{2\lfloor m/2 \rfloor} e_{11}).$$

By Lemma 4(b), applied to the trivial partition, this is

$$(2\lfloor m/2 \rfloor)! v_1 \cdots v_{2\lfloor m/2 \rfloor} e_{11} \neq 0. \qquad ∎$$

PROBLEM 10: *Does the standard identity of degree $2(n + \lfloor m/2 \rfloor)$ hold in $M_n E^m$?*

For $m = 0$, or $m = 1$, or $n = 1$, or $n = 2$, the answer is clearly affirmative.

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