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NON-CENTRAL SECTIONS OF CONVEX BODIES

BY

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ABSTRACT

Let K be a convex body in $\mathbb{R}^n$. Is K uniquely determined by the areas of its sections? There are classical results that explain what happens in the case of sections passing through the origin. However, much less is known about sections that do not contain the origin. We discuss several problems of this type and establish the corresponding uniqueness results.

1. Introduction

Geometric Tomography is an area of Mathematics that deals with the study of properties of objects (such as convex bodies or star bodies) based on information about the size of their sections, projections, etc. It is a well-known result, which goes back to Minkowski and Funk (see [4]), that an origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by the areas of its central sections. More precisely, if K and L are origin-symmetric star bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$$

for every hyperplane H passing through the origin, then $K = L$. On the other hand, in the class of general (not necessarily symmetric) star bodies the latter result is not true.

In view of this, it is natural to ask what information is needed to determine non-symmetric bodies. Falconer [2] and Gardner [3] have shown that if K and

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L are convex bodies in $\mathbb{R}^n$ that contain two points p and q in their interiors and such that $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that passes through either p or q, then $K = L$. In this context, let us also mention the problem of Klee about the inner section function of convex bodies, which is given by $m_K(u) = \max_{t\in\mathbb{R}} \mathrm{vol}_{n-1}(K \cap \{u^\perp + tu\})$. In 1969 Klee asked whether the knowledge of m_K is sufficient to determine the body K uniquely. In [5] the problem was solved in the negative, and a little later a nonspherical body with a constant inner section function was constructed in [12].

Recently, a lot of attention has been attracted to the following problem, posed by Barker and Larman in [1]. Note that a similar question on the sphere was considered earlier by Santaló [13].

Problem 1.1: *Let K and L be convex bodies in $\mathbb{R}^n$ $(n \geq 2)$ that contain a Euclidean ball B in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports B, does it follow that $K = L$?*

The problem is open even in $\mathbb{R}^2$. Some particular cases are known to be true. In particular, a body K in $\mathbb{R}^2$ all of whose sections by lines supporting a disk have the same length, must itself be a disk; see [1]. The problem also has a positive answer in the class of convex polytopes in $\mathbb{R}^n$; see [15].

Barker and Larman also suggested a more general version of Problem 1.1.

Problem 1.2: *Let K and L be convex bodies in $\mathbb{R}^n$ $(n \geq 2)$ that contain a convex body D in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports D, does it follow that $K = L$?*

Closely related is the following open problem.

Problem 1.3: *Let K and L be convex bodies in $\mathbb{R}^n$ and let D be a convex body in the interior of $K \cap L$. If $\mathrm{vol}_n(K \cap H^+) = \mathrm{vol}_n(L \cap H^+)$ for every hyperplane H supporting D, does it follow that $K = L$? Here, H^+ is the half-space bounded by the hyperplane H that does not intersect the interior of D.*

It is interesting to see what happens if the hypotheses of Problems 1.2 and 1.3 hold for two distinct bodies D_1 and D_2 simultaneously (i.e. if we double the amount of information). We show that in this case the answer in $\mathbb{R}^2$ is affirmative under some mild assumptions on D_1 and D_2. After this paper was written, it was brought to our attention that Theorem 3.1 (see Section 3 below) was known to Barker and Larman, as they mention it can be found in Barker's thesis. We include it here anyway, because it is a simple consequence of Theorem 3.4.

In Section 4 we discuss some higher-dimensional analogues. In particular, Groemer [7] has shown that convex bodies are uniquely determined by the areas of "half-sections". More precisely, consider half-planes of the form $H(u, w) = \{x \in \mathbb{R}^n : x \in u^\perp, \langle x, w\rangle \geq 0\}$, where $u \in S^{n-1}$ and $w \in S^{n-1} \cap u^\perp$. Then the equality $\mathrm{vol}_{n-1}(K \cap H(u, w)) = \mathrm{vol}_{n-1}(L \cap H(u, w))$ for all such half-planes implies that $K = L$. We give a version of this result for half-planes that do not pass through the origin. Some other types of sections are also discussed.

2. Definitions and preliminaries

In this section we collect some basic concepts and definitions that we use in the paper. For further facts in Convex Geometry and Geometric Tomography the reader is referred to the books by Schneider [14] and Gardner [4].

A set in $\mathbb{R}^n$ is called *convex* if it contains the closed line segment joining any two of its points. A convex set is a *convex body* if it is compact and has non-empty interior. A convex body is *strictly convex* if its boundary contains no line segments.

A hyperplane H *supports* a set E at a point x if $x \in E \cap H$ and E is contained in one of the two closed half-spaces bounded by H. We say H is a *supporting hyperplane* of E if H supports E at some point.

The *support function* of K is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\},$$

for $x \in \mathbb{R}^n$. If h_K is of class C^k on $\mathbb{R}^n \backslash \{O\}$, we will simply say that K has a C^k support function. For a convex body $K \subset \mathbb{R}^2$ it is often convenient to write h_K as a function of the polar angle θ. So, abusing notation, we will use $h_K(\theta)$ to denote $h_K((\cos\theta, \sin\theta))$. If H is the supporting line to $K \subset \mathbb{R}^2$ with the outer normal vector $(\cos\theta, \sin\theta)$, and K has a C^1 support function, then K has a unique point of contact with H, and $|h'_K(\theta)|$ is the distance from this point to the foot of the perpendicular from the origin O to H; see [4, p. 24].

A compact set L is called a *star body* if the origin O is an interior point of L, every line through O meets L in a line segment, and its *Minkowski functional* defined by

$$\|x\|_L = \min\{a \geq 0 : x \in aL\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of L is given by $\rho_L(x) = \|x\|_L^{-1}$, for $x \in \mathbb{R}^n\backslash\{O\}$. If $x \in S^{n-1}$, then $\rho_L(x)$ is just the radius of L in the direction of x. If p is a point in the interior of L, and $L - p$ is a star body, then we will use $\rho_{L,p}$ to denote ρ_{L-p}.

Let K be a convex body in $\mathbb{R}^n$, and D be a strictly convex body in the interior of K. Let H be a supporting plane to D with outer unit normal vector ξ, and $p = D \cap H$ be the corresponding point of contact. If $u \in S^{n-1} \cap \xi^\perp$, we denote by $\rho_{K,D}(u,\xi) = \rho_{K,p}(u)$ the radial function of $K \cap H$ with respect to p.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$. Functions from this space are called test functions. For a function $\psi \in \mathcal{S}(\mathbb{R}^n)$, its *Fourier transform* is defined by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-i\langle x,\xi\rangle}\, dx,\ \ \xi \in \mathbb{R}^n.$$

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. Elements of this space are referred to as distributions. By $\langle f, \psi\rangle$ we denote the action of the distribution f on the test function ψ. Note that $\hat{\psi}$ is also a test function, which allows to introduce the following definition. We say that the distribution $\hat{f}$ is *the Fourier transform of the distribution* f if

$$\langle \hat{f}, \psi\rangle = \langle f, \hat{\psi}\rangle,$$

for every test function ψ. The reader is referred to the book [8] for applications of Fourier transforms to the study of convex bodies.

3. Main results: 2-dimensional cases

We will start with the following definition. We say that convex bodies D_1 and D_2 in $\mathbb{R}^2$ are *admissible* if they have C^2 support functions, $D_1 \cup D_2$ is not convex, and there are only two lines that support both D_1 and D_2 and do not separate D_1 and D_2. The last condition is satisfied, when, for example, the bodies D_1 and D_2 are disjoint, or they touch each other, or they overlap, but their boundaries have only two common points.

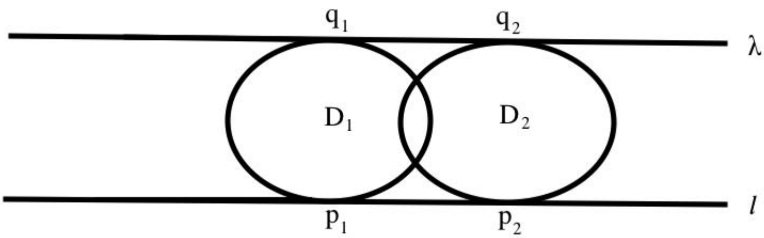

Figure 1

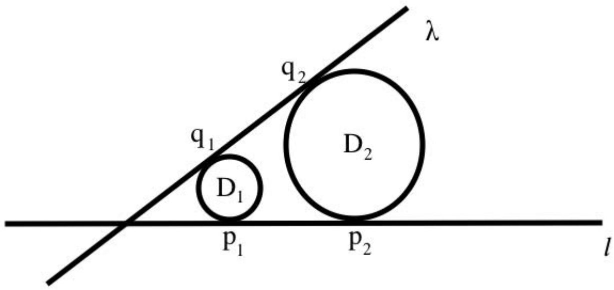

Figure 2

Figures 1 and 2 show two examples of admissible convex bodies. For simplicity, the reader could just think of two disks (not necessarily of the same radius) such that none of them is contained in the other.

We will now prove the following two results.

Theorem 3.1: *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting either D_1 or D_2, then $K = L$.*

If H is a supporting line to a body $D \subset \mathbb{R}^2$, we will denote by H^+ the half-plane bounded by H and disjoint from the interior of D.

Theorem 3.2: *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. If $\mathrm{vol}_2(K \cap H^+) = \mathrm{vol}_2(L \cap H^+)$ for every H supporting D_1 or D_2, then $K = L$.*

We will obtain these theorems as particular cases of a more general statement, Theorem 3.4 below. First, we will need the following lemma.

Lemma 3.3: *Let $D \subset \mathbb{R}^2$ be a convex body with a C^2 support function. Let $Q \in \partial D$ and l be the supporting line to D at Q. Suppose the origin O is located on the line perpendicular to l and passing through Q, and $O \neq Q$. Consider a polar coordinate system centered at O with the polar axis $\overrightarrow{OQ}$. Then, for θ small enough, we have*

$$h'_D(\theta)\sin\theta + h_D(0) - h_D(\theta)\cos\theta \approx \sin^2\theta, \tag{1}$$

where $f \approx g$ means that $C_1 g \leq f \leq C_2 g$, for some constants C_1 and C_2, which depend on f and g.

Proof. Since Q is both the point where l supports D and the foot of the perpendicular from O to l, it follows that $h'_D(0) = 0$. Thus,

$$h_D(\theta) = h_D(0) + \frac{h''_D(0)}{2}\theta^2 + o(\theta^2).$$

Therefore, for θ small enough, we have

$$\begin{aligned} h_D(0) - h_D(\theta)\cos\theta &= h_D(0) - \left(h_D(0) + \frac{h''_D(0)}{2}\theta^2 + o(\theta^2)\right)\left(1 - \frac{1}{2}\theta^2 + o(\theta^2)\right) \\ &= \frac{h_D(0) - h''_D(0)}{2}\theta^2 + o(\theta^2) \\ &\approx \theta^2 \approx \sin^2\theta. \end{aligned}$$

Also, $h'_D(\theta) = h''_D(0)\theta + o(\theta) \approx \theta \approx \sin\theta$, and thus $h'_D(\theta)\sin\theta \approx \sin^2\theta$. ∎

Theorem 3.4: *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. Assume that for some $i > 0$ one of the following two conditions holds:*

(I) $\rho^i_{K,D_j}(u,\xi) + \rho^i_{K,D_j}(-u,\xi) = \rho^i_{L,D_j}(u,\xi) + \rho^i_{L,D_j}(-u,\xi)$, *for* $j = 1, 2$,

(II) $\partial K \cap \partial L \neq \emptyset$ *and* $\rho^i_{K,D_j}(u,\xi) - \rho^i_{K,D_j}(-u,\xi) = \rho^i_{L,D_j}(u,\xi) - \rho^i_{L,D_j}(-u,\xi)$, *for* $j = 1, 2$,

for all $\xi, u \in S^1$ such that $u \perp \xi$.

Then $K = L$.

Proof. We will present the proof of the theorem only using condition (I). The other case is similar and we will just make a brief comment on how the proof should be adjusted.

For the reader's convenience let us first outline the idea of the proof. The proof consists of four steps. In Step 1 we fix a common supporting line to D_1 and D_2 that has a certain property. Denoting this line by l, in Step 2 we show that $K \cap l = L \cap l$. In Step 3 we prove that the boundaries of K and L coincide in some neighborhood of the line l. This allows to conclude in Step 4 that the boundaries of K and L coincide everywhere.

Step 1. Since there are two common supporting lines to D_1 and D_2 (that do not separate D_1 and D_2), we will denote them by l and λ, and let $p_1 = D_1 \cap l$, $q_1 = D_1 \cap \lambda$, $p_2 = D_2 \cap l$, $q_2 = D_2 \cap \lambda$; see Figures 1 and 2. We claim that at least one of the (possibly degenerate) segments $[p_1, p_2]$ or $[q_1, q_2]$ is not entirely contained in $D_1 \cup D_2$. We will prove this claim in a slightly more general setting, i.e. without the assumption that D_1 and D_2 are strictly convex. In that case, instead of single points of contact we may have intervals, and $[p_1, p_2]$ or $[q_1, q_2]$ will just stand for the convex hulls of the corresponding support sets. To prove the claim, we will argue by contradiction. Assume that $[p_1, p_2]$ and $[q_1, q_2]$ are contained in $D_1 \cup D_2$. Then there are points $p \in [p_1, p_2]$ and $q \in [q_1, q_2]$ that both belong to $D_1 \cap D_2$. We can assume that the origin is an interior point of the interval $[p, q]$. Since there are only two common supporting lines to D_1 and D_2, we have exactly two directions u_1 and u_2, such that $h_{D_1}(u_1) = h_{D_2}(u_1)$ and $h_{D_1}(u_2) = h_{D_2}(u_2)$. These directions divide the circle S^1 into two open arcs U_1 and U_2, satisfying $h_{D_1}(u) > h_{D_2}(u)$ for all $u \in U_1$, and $h_{D_1}(u) < h_{D_2}(u)$ for all $u \in U_2$. Thus the line $l(p, q)$ through the points p and q cuts each of the bodies D_1 and D_2 into two convex parts: $D_1 = D_{11} \cup D_{12}$ and $D_2 = D_{21} \cup D_{22}$, such that $D_{11} \supset D_{21}$ and $D_{12} \subset D_{22}$. In other words, $D_1 \cup D_2 = D_{11} \cup D_{22}$, where D_{11} and D_{22} are separated by $l(p, q)$. Now, if we take two points X, $Y \in D_1 \cup D_2$, then we have two cases: either they lie on one side of $l(p, q)$, or on different sides. In the first case, either X, $Y \in D_{11}$, or X, $Y \in D_{22}$, which means that $[X, Y] \subset D_1 \cup D_2$. In the second case, the segment $[X, Y]$ intersects $[p, q]$ (since p and q belong to the supporting lines l and λ correspondingly), and thus one part of $[X, Y]$ lies in D_{11}, and the other in D_{22}, which again implies that $[X, Y] \subset D_1 \cup D_2$, meaning that $D_1 \cup D_2$ is convex. Contradiction. Thus, we have proved that at least one of the segments $[p_1, p_2]$ or $[q_1, q_2]$ is not entirely contained in $D_1 \cup D_2$. We will assume it is the segment $[p_1, p_2]$ and will fix the corresponding supporting line l.

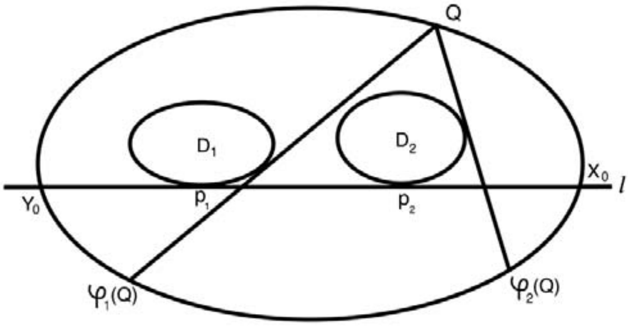

Figure 3

Step 2. Here we will show that $\partial K \cap l = \partial L \cap l$. To this end, we define two mappings φ_1 and φ_2 (see Figure 3). We will start with φ_1; the other is similar. Let Q be a point outside of D_1. There are two unique supporting lines to D_1 passing through Q. Choose the one that lies on the left of the body D_1, when viewing from the point Q. Let T be the point of contact of the chosen supporting line and the body D_1. On this line we take a point $\varphi_1(Q)$, such that T is inside the segment $[Q, \varphi_1(Q)]$ and

$$|QT|^i + |\varphi_1(Q)T|^i = \rho^i_{K,D_1}(u,\xi) + \rho^i_{K,D_1}(-u,\xi),$$

where u is a unit vector parallel to $\overrightarrow{TQ}$ and ξ is the outward unit normal vector to D_1 at T (which is perpendicular to u). The definition for φ_2 is similar; one only needs to replace D_1 by D_2. Note that the domains of φ_1 and φ_2 include the symmetric difference $K \triangle L$. An important observation is that if Q is on the boundary of K (resp. L), then $\varphi_1(Q)$, $\varphi_1^{-1}(Q)$, $\varphi_2(Q)$, and $\varphi_2^{-1}(Q)$ are also on the boundary of K (resp. L).

Note that there exists at least one point $Q \in \partial K \cap \partial L$. Otherwise, one of ∂K or ∂L would be strictly contained inside the other, thus violating condition (1) of the proposition. The line l divides the plane into two closed half-planes l^+ and l^-, where l^+ is the one that contains D_1 and D_2. If $Q \in l^+$, then applying φ_1 finitely many times, we will get a point in l^- (since φ_1 cannot miss the whole half-plane), which is also a common point of the boundaries of K and L. Thus from now on we will assume that $Q \in l^-$. If $Q \in l$, then the proof of Step 2 is finished. If Q is strictly below l, we will apply the following procedure.

Without loss of generality, we can assume that, if the line λ intersects l, then the point of intersection lies to the left of the point p_1, as in Figure 2. Let us also denote by X_0 and Y_0 the points of intersection of the boundary of K with the line l, as in Figure 3. Let $Q_0 = \varphi_2^{-1}(Q)$. The line $l(Q, Q_0)$ through Q and Q_0 is tangent to D_2 and therefore cannot have common points with D_1 (otherwise rolling this line along the boundary of D_2 we would find a third common supporting line to both D_1 and D_2). Now consider $\varphi_1(Q_0)$ and the line $l(\varphi_1(Q_0), Q_0)$ through $\varphi_1(Q_0)$ and Q_0. Note that $\varphi_1(Q_0)$ is below l. Since $l(Q, Q_0)$ and $l(\varphi_1(Q_0), Q_0)$ are different, the points Q and $\varphi_1(Q_0)$ are also different. Moreover, we have $\angle(\overrightarrow{\varphi_1(Q_0)Q_0}, \overrightarrow{p_1X_0}) < \angle(\overrightarrow{QQ_0}, \overrightarrow{p_1X_0})$. Repeating this procedure, we construct $Q_1 = \varphi_2^{-1}(\varphi_1(Q_0))$ and observe that $\angle(\overrightarrow{\varphi_1(Q_0)Q_1}, \overrightarrow{p_1X_0}) < \angle(\overrightarrow{\varphi_1(Q_0)Q_0}, \overrightarrow{p_1X_0})$; see Figure 4.

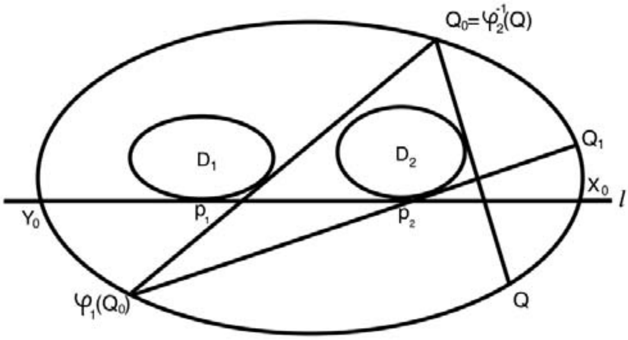

Figure 4

Continuing in this manner, we obtain a sequence of points $\{Q_j\}_{j=0}^{\infty}$ and a corresponding sequence of angles $\{\theta_j\}_{j=0}^{\infty}$, defined by $Q_{j+1} = \varphi_2^{-1}(\varphi_1(Q_j))$ and $\theta_j = \angle(\overrightarrow{\varphi_1(Q_j)Q_j}, \overrightarrow{p_1X_0})$. We note that $Q_j \in l^+ \cap \partial K \cap \partial L$, and $\theta_j > \theta_{j+1}$, for all j. Thus, the sequence $\{\theta_j\}$ is strictly decreasing and positive, and therefore convergent. To reach a contradiction, let us assume that the limit is not zero. Then there is a point $\tilde{Q} = \lim_{j\to\infty} Q_j$ that lies above the line l and satisfies $\varphi_1(\tilde{Q}) = \varphi_2(\tilde{Q})$. Thus, we have a third line that supports both D_1 and D_2. Contradiction. Hence, $\lim_{j\to\infty}\theta_j = 0$, and we conclude that $\partial K \cap l = \partial L \cap l = \{X_0, Y_0\}$.

Step 3. We will prove that ∂K and ∂L coincide in some one-sided neighborhood of the point X_0. Since

$$\frac{|Y_0p_1||X_0p_2|}{|X_0p_1||Y_0p_2|} < 1,$$

we can choose positive numbers a, b, c, d such that

$$0 < a < |X_0p_1|,\ |Y_0p_1| < b,\ 0 < c < |Y_0p_2|,\ |X_0p_2| < d,\ \text{and}\ \frac{bd}{ac} < 1.$$

By the continuity of the boundaries of K, L, D_1, and D_2, there exist neighborhoods, $\mathcal{N}(X_0)$, $\mathcal{N}(Y_0)$, of X_0 and Y_0 respectively, such that

$$\begin{cases} |XT_1| > a \text{ and } |XT_2| < d, & \text{if } X \in \mathcal{N}(X_0), \\ |YT_3| > c \text{ and } |YT_4| < b, & \text{if } Y \in \mathcal{N}(Y_0), \end{cases} \tag{2}$$

where T_1 is the point of intersection of l and the line through X supporting D_1 (if X is itself on the line l, then we let $T_1 = p_1$). Similarly, T_2 is the point of intersection of l and the line through X supporting D_2 (again, if X is on the line l, then we let $T_2 = p_2$). Here and below, by the supporting lines we mean those that are closest to l. There is no ambiguity, since X is sufficiently close to l. (The points T_3 and T_4 are defined similarly, if we replace X by Y.)

Next we claim that there are points of $\partial K \cap \partial L$ in the set $\mathcal{N}(X_0) \cap l^+$. Indeed, if in Step 2 there was a point $Q \in \partial K \cap \partial L$ strictly below the line l, then the points from the corresponding sequence $\{Q_i\}$ all lie in $\partial K \cap \partial L \cap \mathcal{N}(X_0) \cap l^+$ for i large enough. If in Step 2 the point Q was on the line l, then we can take $\varphi_1(\varphi_2^{-1}(X_0))$, which will be strictly below l, and repeat the same procedure.

Our goal is to show that ∂K and ∂L coincide in $\mathcal{N}(X_0) \cap l^+$. Taking a smaller neighborhood $\mathcal{N}(X_0)$ if needed, we can assume that $\varphi_1(\mathcal{N}(X_0) \cap l^+) \subset \mathcal{N}(Y_0)$. Discarding finitely many terms of the sequence $\{Q_j\}$, we can also assume that $Q_j \in \mathcal{N}(X_0) \cap l^+$ for all $j \geq 0$. Now consider the segments of the boundaries of ∂K and ∂L between the points Q_0 and Q_1. If they coincide, then we are done, since the boundaries of ∂K and ∂L would have to coincide between Q_j and Q_{j+1} for all j. So, we will next assume that ∂K and ∂L are not identically the same between Q_0 and Q_1. Let E_0 be the component of $K \triangle L$ with endpoints Q_0 and Q_1, i.e. E_0 is the subset of $(K \triangle L) \cap l^+$ located between the lines $l(Q_0, \varphi_1(Q_0))$ and $l(Q_1, \varphi_1(Q_0))$. We will define a sequence of sets $\{E_j\}_{j=0}^{\infty}$, where $E_{j+1} = \varphi_2^{-1}(\varphi_1(E_j))$. Each E_j is a component of $K \triangle L$ with endpoints Q_j and Q_{j+1}.

Now consider a Cartesian coordinate system with l being the x-axis, and the y-axis perpendicular to l. We will be using ideas similar to those in [4, Section 5.2]. For a measurable set E define

$$\nu_i(E) = \iint_E |y|^{i-2}\, dx\, dy. \tag{3}$$

Note that $\nu_i(E)$ is invariant under shifts parallel to the x-axis. This allows us to associate with each D_1 and D_2 their own Cartesian systems. In both systems l is the x-axis, but in the coordinate system associated with D_1 the origin is at p_1, while in the system associated with D_2 the origin is at p_2.

Our goal is to estimate $\nu_i(E_j)$. Fix the Cartesian system associated with D_1, with p_1 being the origin. For a point $(x, y) \in \mathcal{N}(X_0) \cup \mathcal{N}(Y_0)$ we will introduce new coordinates (r, θ) as follows. Let $\theta = \angle(l_{\theta,1}, l)$, where $l_{\theta,1}$ is the line passing through (x, y) and supporting D_1. Define r to be the signed distance between (x, y) and the foot of the perpendicular from the point $(0, 1)$ to the line $l_{\theta,1}$. (The word "signed" means that $r > 0$ in the neighborhood of X_0 and $r < 0$ in the neighborhood of Y_0.) Let $h_{D_1}(\theta)$ be the support function of D_1 measured from the point $(0, 1)$ in the direction of $(\sin\theta, -\cos\theta)$. Using that

$$(x, y) = h_{D_1}(0) \cdot (0, 1) + r(\cos\theta, \sin\theta) + h_{D_1}(\theta) \cdot (\sin\theta, -\cos\theta),$$

we will write the integral (3) in the (r, θ)-coordinates associated with D_1. Since the Jacobian is $|r - h'_{D_1}(\theta)|$, and $r = h'_{D_1}(\theta)$ corresponds to the point of contact of $l_{\theta,1}$ and D_1, we get

$$\begin{aligned}
\nu_i(E_j) &= \iint_{E_j} |y|^{i-2}\, dx\, dy \\
&= \int_{\theta_{j+1}}^{\theta_j} \left| \int_{\rho_{K,D_1}(u,\xi) - h'_{D_1}(\theta)}^{\rho_{L,D_1}(u,\xi) - h'_{D_1}(\theta)} |r\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2} |r - h'_{D_1}(\theta)|\, dr \right| d\theta \\
&= \int_{\theta_{j+1}}^{\theta_j} \left| \int_{\rho_{K,D_1}(u,\xi)}^{\rho_{L,D_1}(u,\xi)} |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2} r\, dr \right| d\theta,
\end{aligned}$$

where $u = (\cos\theta, \sin\theta)$, and $\xi = (\sin\theta, -\cos\theta)$. Here the absolute value of the integral with respect to r is needed, since we do not know which of ρ_K or ρ_L is greater.

For small θ, Lemma 3.3 yields that

$$h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta \approx \sin^2\theta.$$

Since E_j is inside $\mathcal{N}(X_0)$, there exists a constant $C > 0$ such that

$$\begin{aligned}(1 - C\sin\theta)r\sin\theta &\le r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta\\ &\le (1 + C\sin\theta)r\sin\theta,\end{aligned}$$

where we assume that θ is small enough so that $1 - C\sin\theta > 0$.

If $i \ge 2$, for small $\theta > 0$ we have

$$\begin{aligned}\left(\frac{1 - C\sin\theta}{1 + C\sin\theta}\right)^{i-2}(r\sin\theta)^{i-2} &\le (1 - C\sin\theta)^{i-2}(r\sin\theta)^{i-2}\\ &\le |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2}\\ &\le (1 + C\sin\theta)^{i-2}(r\sin\theta)^{i-2} \le \left(\frac{1 + C\sin\theta}{1 - C\sin\theta}\right)^{i-2}(r\sin\theta)^{i-2}.\end{aligned}$$

On the other hand, for $i < 2$,

$$\begin{aligned}\left(\frac{1 + C\sin\theta}{1 - C\sin\theta}\right)^{i-2}(r\sin\theta)^{i-2} &\le (1 + C\sin\theta)^{i-2}(r\sin\theta)^{i-2}\\ &\le |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2}\\ &\le (1 - C\sin\theta)^{i-2}(r\sin\theta)^{i-2} \le \left(\frac{1 - C\sin\theta}{1 + C\sin\theta}\right)^{i-2}(r\sin\theta)^{i-2}.\end{aligned}$$

Thus, for both $i \ge 2$ and $i < 2$, we have

$$\begin{aligned}&\frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1 - C\sin\theta}{1 + C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi) - \rho^i_{L,D_1}(u,\xi)\right| d\theta \le \nu_i(E_j)\\ &\qquad\le \frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1 + C\sin\theta}{1 - C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi) - \rho^i_{L,D_1}(u,\xi)\right| d\theta. \qquad (4)\end{aligned}$$

Now apply the same estimates to $\nu_i(\varphi_1(E_j))$. Since $\varphi_1(E_j) \subset \mathcal{N}(Y_0)$, and assuming that the constant C chosen above works for both $\mathcal{N}(X_0)$ and $\mathcal{N}(Y_0)$, we get

$$\nu_i(\varphi_1(E_j)) \geq \frac{1}{i}\int_{\theta_{j+1}}^{\theta_j} \left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{|i-2|} \times (\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(-u,\xi)-\rho^i_{L,D_1}(-u,\xi)\right| d\theta$$

$$= \frac{1}{i}\int_{\theta_{j+1}}^{\theta_j} \left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{|i-2|} \times (\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right| d\theta$$

$$= \frac{1}{i}\int_{\theta_{j+1}}^{\theta_j} \left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{2|i-2|}\left(\frac{1+C\sin\theta}{1-C\sin\theta}\right)^{|i-2|} \times (\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right| d\theta$$

$$\geq \left(\frac{1-C\sin\theta_j}{1+C\sin\theta_j}\right)^{2|i-2|}\nu_i(E_j),$$

since $\dfrac{1-C\sin\theta}{1+C\sin\theta}$ is decreasing.

Define another sequence of angles $\eta_j = \angle(\overrightarrow{\varphi_1(Q_j)Q_{j+1}}, \overrightarrow{p_1X_0})$. Then calculations similar to those above give

$$\nu_i(E_{j+1}) \geq \left(\frac{1-C\sin\eta_j}{1+C\sin\eta_j}\right)^{2|i-2|}\nu_i(\varphi_1(E_j)).$$

Thus,

$$\nu_i(E_{j+1}) \geq \left(\frac{1-C\sin\eta_j}{1+C\sin\eta_j}\right)^{2|i-2|}\left(\frac{1-C\sin\theta_j}{1+C\sin\theta_j}\right)^{2|i-2|}\nu_i(E_j). \tag{5}$$

Observe that (2) implies, for all j,

$$\frac{\sin\theta_{j+1}}{\sin\theta_j} = \frac{\sin\theta_{j+1}}{\sin\eta_j}\frac{\sin\eta_j}{\sin\theta_j} \leq \frac{db}{ac} < 1,$$

and, similarly,

$$\frac{\sin\eta_{j+1}}{\sin\eta_j} \leq \frac{db}{ac}.$$

Set $\sigma = \dfrac{db}{ac}$, where $\sigma \in (0,1)$. Then $\sin\theta_j \leq \sigma^j \sin\theta_0 \leq \sigma^j$ and $\sin\eta_j \leq \sigma^j \sin\eta_0 \leq \sigma^j$.

For sufficiently small $x > 0$, we have the following inequalities: $1 + x \leq e^x$ and $1 - x \geq e^{-2x}$. Let $N > 0$ be large enough so that $x = C\sigma^j$ satisfies the latter two inequalities for all $j \geq N$. Then for all $j \geq N$, we have

$$\begin{aligned}\nu_i(E_{j+1}) &\geq \left(\frac{1 - C\sigma^j}{1 + C\sigma^j}\right)^{4|i-2|} \nu_i(E_j) \\ &\geq \left(\frac{e^{-2C\sigma^j}}{e^{C\sigma^j}}\right)^{4|i-2|} \nu_i(E_j) = e^{-12C|i-2|\sigma^j}\nu_i(E_j).\end{aligned}$$

Using the latter estimate inductively, we get

$$\begin{aligned}\nu_i(E_{j+1}) &\geq \prod_{m=N}^{j} e^{-12C|i-2|\sigma^m}\nu_i(E_N) = \exp\left\{-12C|i-2|\sum_{m=N}^{j}\sigma^m\right\}\nu_i(E_N) \\ &\geq \gamma\nu_i(E_N),\end{aligned}$$

where

$$\gamma = \exp\left\{-12C|i-2|\sum_{m=N}^{\infty}\sigma^m\right\} > 0.$$

Since all E_j are disjoint, and since $\nu_i(E_N) \geq \tilde{C}\nu_i(E_0) > 0$, for some constant $\tilde{C}$ (by virtue of (5)), we conclude that

$$\nu_i\left(\bigcup_{j=N+1}^{\infty} E_j\right) = \sum_{j=N+1}^{\infty}\nu_i(E_j) \geq \gamma\sum_{j=N+1}^{\infty}\nu_i(E_N) = \infty.$$

Since $l \cap (K \triangle L) = \{X_0, Y_0\}$, there exists a triangle T with one vertex at X_0 satisfying $T \cap l = X_0$ and $\bigcup_{j=N+1}^{\infty} E_j \subset T$, implying

$$\nu_i(T) \geq \nu_i\left(\bigcup_{j=N+1}^{\infty} E_j\right) = \infty.$$

However, by [4, Lemma 5.2.4], any triangle of the form

$$T = \{(x,y) : a|x - x_0| \leq y \leq b\},$$

for $a, b > 0$, has finite ν_i-measure. We get a contradiction. Thus, $\partial K = \partial L$ in $\mathcal{N}(X_0) \cap l^+$.

Step 4. To finish the proof, we take any point $A \in \partial K$. Applying φ_1 to A finitely many times, we can get a point A' in $l^- \cap \partial K$. As in Step 2, produce a sequence of points $A_{j+1} = \varphi_2^{-1}(\varphi_1(A_j))$ with $A_0 = \varphi_2^{-1}(A')$. As we have seen above, there is a number M large enough such that $A_M \in \mathcal{N}(X_0) \cap l^+$. Applying the conclusion of Step 3, we get $A_M \in \partial K \cap \partial L$. Tracing the sequence $\{A_i\}$ backwards, we conclude that $A \in \partial K \cap \partial L$. Therefore, $K = L$.

We now briefly comment on how to proceed if we use condition (II) of the theorem. Note that here we require that there is a point $Q \in \partial K \cap \partial L$. We define φ_1 and φ_2 in a similar way as above, with the only difference that

$$|QT|^i - |\varphi_j(Q)T|^i = \rho^i_{K,D_j}(u,\xi) - \rho^i_{K,D_j}(-u,\xi),$$

for $j = 1, 2$. The rest of the proof goes without any changes. ∎

Remark 3.5: The C^2-smoothness assumption for the support functions of the bodies D_1 and D_2 can be relaxed. As we saw above, we only need the C^2 condition in some neighborhoods of the points p_1 and p_2 correspondingly. Moreover, D_1 or D_2 can also be polygons. In the latter case, ρ_{K,D_j} is not well defined for finitely many supporting lines, but this is not an issue. Step 1 of the proof does not need any changes, since it was proved for bodies that are not necessarily strictly convex. In Step 2, we consider small one-sided neighborhoods of X_0 and Y_0, where ρ_{K,D_j} is well-defined. As for Step 3, the proof will be similar to [4, Section 5.2], since all supporting lines to a polygon D_j passing through points $X \in \mathcal{N}(X_0) \cap l^+$ will contain the same vertex of D_j. Thus, as in [4], the measure ν_i would be invariant under φ_j. So, whenever we speak about admissible bodies, one can consider a larger class of admissible bodies by including the bodies described in this remark.

Theorem 3.1 (with admissible bodies as in the above remark) is now a consequence of Theorem 3.4 (use part (I) with $i = 1$). The following is an immediate corollary of Theorem 3.1.

Corollary 3.6: *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^2$ and let D be a convex body in the interior of $K \cap L$, such that D and $-D$ are admissible bodies. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting D, then $K = L$. In particular, D can be a disk not centered at the origin.*

Using the same ideas, one can prove the following.

Corollary 3.7: *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^2$ and let D be a convex body outside of $K \cup L$ (either a polygon or a body with a C^2 support function). If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting D, then $K = L$.*

We will now prove Theorem 3.2 using the class of admissible bodies described in Remark 3.5.

Proof. First we will prove the following claim. Let K and L be convex bodies in $\mathbb{R}^2$, D be a convex body in the interior of $K \cap L$, where D is either a body with C^2 support function or a polygon. If $\mathrm{vol}_2(K \cap H^+) = \mathrm{vol}_2(L \cap H^+)$ for every H supporting D, then

$$\rho^2_{K,D}(u, \xi) - \rho^2_{K,D}(-u, \xi) = \rho^2_{L,D}(u, \xi) - \rho^2_{L,D}(-u, \xi),$$

for every $\xi \in S^1$ and $u \in S^1 \cap \xi^\perp$, whenever well-defined. (Note that in the case when D is a polygon, the radial functions above are not well-defined for finitely many directions ξ that are orthogonal to the edges of D.)

We will treat simultaneously both the case of smooth bodies and polygons. To prove the claim, let ξ be any unit vector (and ξ is not orthogonal to an edge of D, if D is a polygon). Let H_ξ be the supporting line orthogonal to ξ. Let $\zeta \in S^1 \cap \xi^\perp$. For a small angle $\phi > 0$ let $\eta = \cos\phi\, \xi + \sin\phi\, \zeta$, and denote by H_η the supporting line orthogonal to η. Define the following sets: $E_1 = H^+_\xi \backslash H^+_\eta$, $E_2 = H^+_\xi \cap H^+_\eta$, $E_3 = H^+_\eta \backslash H^+_\xi$, and E_4 is the curvilinear triangle enclosed by H_ξ, H_η, and the boundary of D; see Figure 5.

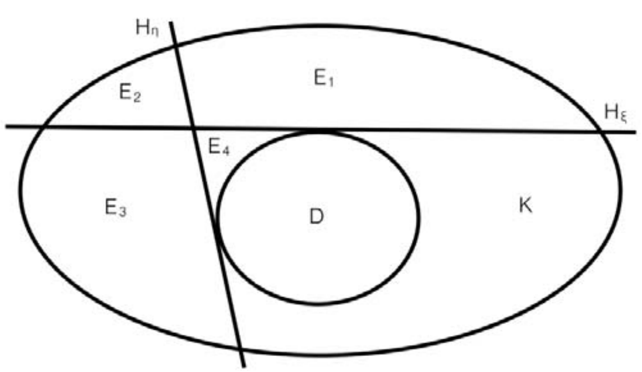

Figure 5

Note that when η and ξ are close enough, we have $E_4 \subset K \cap L$, and E_4 is empty if D is a polygon. By the assumption of the theorem,

$$\begin{aligned}\mathrm{vol}_2((E_1 \cup E_2) \cap K) - \mathrm{vol}_2((E_3 \cup E_2) \cap K)\\ = \mathrm{vol}_2((E_1 \cup E_2) \cap L) - \mathrm{vol}_2((E_3 \cup E_2) \cap L),\end{aligned}$$

implying

$$\begin{aligned}\mathrm{vol}_2((E_1 \cup E_4) \cap K) - \mathrm{vol}_2((E_3 \cup E_4) \cap K)\\ = \mathrm{vol}_2((E_1 \cup E_4) \cap L) - \mathrm{vol}_2((E_3 \cup E_4) \cap L).\end{aligned} \tag{6}$$

Now we will consider the following coordinate system (r, θ) associated with D. For a point (x, y) outside of D, we let $(x, y) = h_D(\theta)(\cos\theta\,\xi + \sin\theta\,\zeta) + r(\sin\theta\,\xi - \cos\theta\,\zeta)$, where $h_D(\theta)$ is the support function of D in the direction of $v = \cos\theta\,\xi + \sin\theta\,\zeta$. Setting $w = \sin\theta\,\xi - \cos\theta\,\zeta$, and observing that the Jacobian is $|r + h'_D(\theta)|$, we get

$$\begin{aligned}&\int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{K,D}(w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta - \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{K,D}(-w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta\\ &= \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{L,D}(w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta - \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{L,D}(-w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta,\end{aligned}$$

which after a variable change becomes

$$\begin{aligned}\int_0^{\phi}\int_0^{\rho_{K,D}(w,v)} r\,dr\,d\theta - \int_0^{\phi}\int_0^{\rho_{K,D}(-w,v)} r\,dr\,d\theta\\ = \int_0^{\phi}\int_0^{\rho_{L,D}(w,v)} r\,dr\,d\theta - \int_0^{\phi}\int_0^{\rho_{L,D}(-w,v)} r\,dr\,d\theta.\end{aligned}$$

Differentiating both sides with respect to ϕ, and setting $\phi = 0$, we get

$$\rho^2_{K,D}(u,\xi) - \rho^2_{K,D}(-u,\xi) = \rho^2_{L,D}(u,\xi) - \rho^2_{L,D}(-u,\xi),$$

as claimed.

To finish the proof of the theorem, note that $\partial K \cap \partial L \cap l^- \neq \emptyset$, where l is the common supporting line to D_1 and D_2 as in Theorem 3.4; otherwise we would have $\mathrm{vol}_2(K \cap l^-) < \mathrm{vol}_2(L \cap l^-)$ or $\mathrm{vol}_2(K \cap l^-) > \mathrm{vol}_2(L \cap l^-)$, which contradicts the hypotheses.

Now the conclusion follows from Theorem 3.4. ∎

Corollary 3.8: *Let K be a convex body in $\mathbb{R}^2$ and let D be a disk in the interior of K. If $\mathrm{vol}_2(K \cap H^+) =$ const for every H supporting D, then K is also a disk.*

After this paper was written, it was kindly pointed out to us by A. Kurusa that the latter result has also been obtained in his paper with T. Ódor; see [10]. Thus we omit the proof and refer the reader to their paper.

4. Main results: Higher-dimensional cases

Theorem 4.1: *Let K and L be convex bodies in $\mathbb{R}^n$ (where n is even) and let D be a cube in the interior of $K \cap L$. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for any hyperplane passing through a vertex of D and an interior point of D, then $K = L$.*

For $\epsilon > 0$ and $\xi \in S^{n-1}$, denote by

$$U_\epsilon(\xi) = \{\eta \in S^{n-1} : \langle \eta, \xi \rangle > \sqrt{1-\epsilon^2}\}$$

the spherical cap centered at ξ, and by

$$E_\epsilon(\xi) = \{\eta \in S^{n-1} : |\langle \eta, \xi \rangle| < \epsilon\}$$

the neighborhood of the equator $S^{n-1} \cap \xi^\perp$.

Lemma 4.2: *Let K and L be convex bodies in $\mathbb{R}^n$ (where n is even) containing the origin in their interiors. Let $\xi \in S^{n-1}$ and $\epsilon > 0$. If $\mathrm{vol}_{n-1}(K \cap u^\perp) = \mathrm{vol}_{n-1}(L \cap u^\perp)$ for every $u \in E_\epsilon(\xi)$, then*

$$\rho_K^{n-1}(\eta) + \rho_K^{n-1}(-\eta) = \rho_L^{n-1}(\eta) + \rho_L^{n-1}(-\eta)$$

for every $\eta \in U_\epsilon(\xi)$.

Proof. For every even function $\psi \in C^\infty(S^{n-1})$ with support in $U_\epsilon(\xi) \cup U_\epsilon(-\xi)$, we have

$$\int_{S^{n-1}} (\|x\|_K^{-n+1} + \|-x\|_K^{-n+1}) \psi(x)\, dx$$
$$= (2\pi)^{-n} \int_{S^{n-1}} (\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) (\psi(x/|x|)|x|^{-1})^\wedge(u)\, du,$$

where we used Parseval's formula on the sphere; see [8, Section 3.4].

Since $(\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) = 2\pi(n-1)\mathrm{vol}_{n-1}(K \cap u^\perp)$ by [8, Lemma 3.7], the assumption of the lemma yields

$$(\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) = (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1})^\wedge(u)$$

for every $u \in E_\epsilon(\xi)$. On the other hand, by formula (3.6) from [6] or [11, Lemma 5.1], we see that $(\psi(x/|x|)|x|^{-1})^\wedge\Big|_{S^{n-1}}$ is supported in $E_\epsilon(\xi)$.

Therefore,

$$\begin{aligned}\int_{S^{n-1}} &(\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})\psi(x)\,dx \\ &= (2\pi)^{-n}\int_{S^{n-1}} (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1})^\wedge(u)(\psi(x/|x|)|x|^{-1})^\wedge(u)\,du \\ &= \int_{S^{n-1}} (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1})\psi(x)\,dx.\end{aligned}$$

Since this true for any $\psi \in C^\infty(S^{n-1})$ with support in $U_\epsilon(\xi) \cup U_\epsilon(-\xi)$, the conclusion follows. ∎

Definition 4.3: Let D be a convex polytope and v_k one of its vertices. Define $C_D(v_k)$ to be the double cone centered at v_k with the property that every point in $C_D(v_k)$ lies on a line through v_k that has non-empty intersection with $D \setminus \{v_k\}$.

Note that when D is a cube, $\cup_k C_D(v_k) = \mathbb{R}^n$.

Remark 4.4: For simplicity, we stated Theorem 4.1 only in the case when D is a cube, but, in fact, it remains valid for a larger class of polytopes. In particular, any centrally symmetric polytope D satisfying the following condition will work: $\bigcup_k C_D(v_k) = \mathbb{R}^n$.

Let us say a few words about the condition $\bigcup_k C_D(v_k) = \mathbb{R}^n$. It is easy to see that it is true for any polygon in $\mathbb{R}^2$. However, the situation in higher dimensions is different. There is an origin-symmetric convex polytope $D \subset \mathbb{R}^n$ for which this condition does not hold. Consider the cube in $\{x_n = 0\}$ with the vertices $(\pm1, \pm1, \dots, \pm1, 0)$. Define D to be the convex hull of this cube and the points $(2, 0, \dots, 0, 1)$, $(-2, 0, \dots, 0, -1)$. Then $(0, \dots, 0, a) \notin \bigcup_k C_D(v_k)$ for sufficiently large a.

Proof of Theorem 4.1. We will prove the theorem for the class of polytopes described in Remark 4.4. Assume that D is such a polytope and its center of symmetry is at the origin O.

By Lemma 4.2, if v_i is a vertex of D, then

$$\rho^{n-1}_{K,v_i}(\xi) + \rho^{n-1}_{K,v_i}(-\xi) = \rho^{n-1}_{L,v_i}(\xi) + \rho^{n-1}_{L,v_i}(-\xi),$$

for every $\xi \in S^{n-1} \cap (C_D(v_i) - v_i)$. Here, ρ_{K,v_i} and ρ_{L,v_i} are the radial functions of K and L with respect to the point v_i.

For a point $Q \in C_D(v_i)$ define a mapping φ_i as follows. Let $\varphi_i(Q)$ be the point on the line through Q and v_i, such that v_i lies between Q and $\varphi_i(Q)$, and

$$|Qv_i|^{n-1} + |\varphi_i(Q)v_i|^{n-1} = \rho^{n-1}_{K,v_i}(\xi) + \rho^{n-1}_{K,v_i}(-\xi) = \rho^{n-1}_{L,v_i}(\xi) + \rho^{n-1}_{L,v_i}(-\xi),$$

where ξ is the unit vector in the direction of $\overrightarrow{v_iQ}$. Note that the domain of φ_i is not the entire set $C_D(v_i)$, but it will be enough that φ_i is defined in some neighborhood of $(K \triangle L) \cap C_D(v_i)$.

Note that $\partial K \cap \partial L \neq \emptyset$. Otherwise one of the bodies K or L would be strictly contained inside the other body, thus violating the condition $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ from the statement of the theorem. Consider a point $Q \in \partial K \cap\ \partial L$. There exists a vertex v_i of D, such that $Q \in C_D(v_i)$. Since D is origin-symmetric, there is a vertex $v_j = -v_i$. Our first goal is to show that $l(v_i, v_j) \cap \partial K = l(v_i, v_j) \cap\ \partial L$, where $l(v_i, v_j)$ is the line through v_i and v_j. If Q belongs to this line, we are done. If not, we will argue as follows.

Since $Q \in C_D(v_i) \cap \partial K \cap \partial L$, then $\varphi_i(Q)$ is also in $C_D(v_i) \cap \partial K \cap \partial L$. Let $\{F_m\}$ be the collection of the facets of D that contain the vertex v_i, and let $\{n_m\}$ be the collection of the corresponding outward unit normal vectors. Note that the condition $Q \in C_D(v_i)$ means that either $\langle \overrightarrow{v_iQ}, n_m\rangle \geq 0$ for all m, or $\langle \overrightarrow{v_iQ}, n_m\rangle \leq 0$ for all m. Without loss of generality we can assume that $\langle \overrightarrow{v_iQ}, n_m\rangle \geq 0$ for all m (otherwise, take $\varphi_i(Q)$ instead of Q).

We claim that $Q \in C_D(v_i) \cap C_D(v_j)$. Indeed, the outward unit normal vectors to the facets that contain v_j are $\{-n_m\}$. Thus,

$$\langle \overrightarrow{v_jQ}, n_m\rangle = \langle \overrightarrow{v_iQ}, n_m\rangle + \langle \overrightarrow{v_jv_i}, n_m\rangle = \langle \overrightarrow{v_iQ}, n_m\rangle + 2\langle \overrightarrow{Ov_i}, n_m\rangle \geq 0.$$

Next we claim that $\varphi_j(Q) \in C_D(v_i) \cap C_D(v_j)$. It is clear that $\varphi_j(Q) \in C_D(v_j)$. Thus, it is enough to show that $\langle \overrightarrow{v_i\varphi_j(Q)}, n_m\rangle \leq 0$ for all m. We have

$$\overrightarrow{v_i\varphi_j(Q)} = \overrightarrow{OQ} + \overrightarrow{Q\varphi_j(Q)} - \overrightarrow{Ov_i} = \overrightarrow{OQ} + \alpha\overrightarrow{Qv_j} - \overrightarrow{Ov_i},$$

where $\alpha = \dfrac{|Q\varphi_j(Q)|}{|Qv_j|} > 1$. So

$$\begin{aligned}\overrightarrow{v_i\varphi_j(Q)} = \overrightarrow{OQ} + \alpha\overrightarrow{Ov_j} - \alpha\overrightarrow{OQ} - \overrightarrow{Ov_i} &= (1-\alpha)\overrightarrow{OQ} - (1+\alpha)\overrightarrow{Ov_i} \\ &= (1-\alpha)\overrightarrow{v_iQ} - 2\alpha\overrightarrow{Ov_i}.\end{aligned}$$

Thus, for every m,

$$\langle\overrightarrow{v_i\varphi_j(Q)}, n_m\rangle = (1-\alpha)\langle\overrightarrow{v_iQ}, n_m\rangle - 2\alpha\langle\overrightarrow{Ov_i}, n_m\rangle \le 0.$$

In a similar fashion one can show that $\varphi_i(\varphi_j(Q)) \in C_D(v_i) \cap C_D(v_j)$. Thus we can produce a sequence of points $\{Q_k\}_{k=0}^{\infty}$, where $Q_0 = Q$ and $Q_k = \varphi_i(\varphi_j(Q_{k-1}))$, and such that $Q_k \in C_D(v_i) \cap C_D(v_j) \cap \partial K \cap \partial L$ for all $k \ge 0$. Moreover, all these points belong to the 2-dimensional plane spanned by the points Q, v_i, and v_j. As in Proposition 3.4 we have the corresponding sequence of angles $\theta_k = \angle(\overrightarrow{v_iQ_k}, \overrightarrow{v_iv_j})$, with $\theta_k < \theta_{k-1}$. One can see that $\lim_{k\to\infty}\theta_k = 0$. Since $Q_k \in \partial K \cap \partial L$ for all k, we have proved that $l(v_i, v_j) \cap \partial K = l(v_i, v_j) \cap \partial L$.

Denote the points of intersection of the latter line with the boundaries of K and L by X_0 and Y_0, and consider any 2-dimensional plane H through X_0 and Y_0. Using [2, Lemma 7], we see that there are neighborhoods $\mathcal{N}(X_0)$ and $\mathcal{N}(Y_0)$ of X_0 and Y_0 correspondingly, such that

$$H \cap \mathcal{N}(X_0) \cap \partial K = H \cap \mathcal{N}(X_0) \cap \partial L \text{ and } H \cap \mathcal{N}(Y_0) \cap \partial K = H \cap \mathcal{N}(Y_0) \cap \partial L.$$

If P is a point in $C_D(v_i) \cap H$ that does not belong to $\mathcal{N}(X_0)$ or $\mathcal{N}(Y_0)$, then we apply φ_j and φ_i to produce a sequence of points P_k, which after finitely many steps will belong to $\mathcal{N}(X_0)$ or $\mathcal{N}(Y_0)$. Thus, $P_N \in \partial K \cap \partial L$ for some large N. Applying inverse maps φ_i^{-1} and φ_j^{-1}, we conclude that $P \in \partial K \cap \partial L$. Thus, we have shown that

$$H \cap C_D(v_i) \cap \partial K = H \cap C_D(v_i) \cap \partial L.$$

Since this is true for every H, we have $C_D(v_i) \cap \partial K = C_D(v_i) \cap \partial L$.

Now consider any other vertex of D, say v_m, that is connected to v_i by an edge. One can see that

$$C_D(v_i) \cap C_D(v_m) \cap \partial K \cap \partial L \neq \emptyset.$$

Repeating the same process as above, we get

$$C_D(v_m) \cap \partial K = C_D(v_m) \cap \partial L.$$

Since we can do this for every vertex, it follows that $C_D(v_k)\cap\partial K = C_D(v_k)\cap\partial L$ for every k, and thus $K = L$. ∎

Problem 4.5: Does the result of Theorem 4.1 hold in odd dimensions?

Since some of the methods in the proof above come from the paper [11], the answer may be different for even and odd dimensions. One can also ask if there is a different condition that guarantees a positive answer in odd dimensions? If we replace the equality of sections by the equality of derivatives of the parallel section functions, then, for example, in $\mathbb{R}^3$ first derivatives are not enough; cf. [9, Remark 1].

The next theorem is an analogue of Groemer's result for half-sections. The difference is that we look at half-sections that do not pass through the origin. We will adopt the following notation. For a point $p \in \mathbb{R}^n$ and a vector $v \in S^{n-1}$, define $v_p^\perp = \{x \in \mathbb{R}^n : \langle x-p, v\rangle = 0\}$ and $v_p^+ = \{x \in \mathbb{R}^n : \langle x-p, v\rangle \geq 0\}$.

Theorem 4.6: *Let K and L be convex bodies in $\mathbb{R}^n$, $n \geq 3$, that contain a strictly convex body D in their interiors. Assume that*

$$\mathrm{vol}_{n-1}(K\cap H\cap v_p^+) = \mathrm{vol}_{n-1}(L\cap H\cap v_p^+),$$

for every hyperplane H supporting D and every unit vector $v \in H-p$, where $p = D\cap H$. Then $K = L$.

Proof. Let us fix a supporting plane H and consider the equality

$$\mathrm{vol}_{n-1}(K\cap H\cap v_p^+) = \mathrm{vol}_{n-1}(L\cap H\cap v_p^+),$$

for every unit vector $v \in H-p$. Then [7] implies that

$$\rho_{K,p}^{n-1}(u) - \rho_{K,p}^{n-1}(-u) = \rho_{L,p}^{n-1}(u) - \rho_{L,p}^{n-1}(-u),$$

for every vector $u \in S^{n-1}\cap(H-p)$, where $p = D\cap H$.

Now observe that $\partial K\cap\partial L \neq \emptyset$; otherwise the condition $\mathrm{vol}_{n-1}(K\cap H\cap v_p^+) = \mathrm{vol}_{n-1}(L\cap H\cap v_p^+)$ would be violated. Moreover, if $Q \in \partial K\cap\partial L$, then by [1, Lemma 3] there exists a neighborhood $\mathcal{N}(Q)$ of Q, such that $\mathcal{N}(Q)\cap\partial K \subset \partial K\cap\partial L$. Hence, $\partial K\cap\partial L$ is open in ∂K. On the other hand, by the continuity of the boundaries of K and L, $\partial K\cap\partial L$ is closed in ∂K. Therefore,

$$\partial K\cap\partial L = \partial K = \partial L. \qquad ∎$$

Corollary 4.7: *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, that contains a ball D of radius t in its interior. If*

$$\mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\} \cap v^+) = const,$$

for every $\xi \in S^{n-1}$ and every vector $v \in S^{n-1} \cap \xi^\perp$, then K is a Euclidean ball.

In the next theorem we will consider a different type of half-sections.

Theorem 4.8: *Let K and L be convex bodies in $\mathbb{R}^n$, $n \geq 3$, that contain a ball D in their interiors. Assume that*

$$\mathrm{vol}_{n-1}(K \cap H^+ \cap v^\perp) = \mathrm{vol}_{n-1}(L \cap H^+ \cap v^\perp)$$

for every hyperplane H supporting D and every unit vector $v \in H - p$, where $p = D \cap H$. Then $K = L$.

Proof. Let us fix a unit vector v, and consider $\xi, \zeta \in S^{n-1} \cap v^\perp$ such that $\xi \perp \zeta$. For a small ϕ let $\eta = \cos\phi\,\xi + \sin\phi\,\zeta$. Without loss of generality we will assume that D has radius 1 and is centered at the origin. Consider the affine hyperplanes $H_\xi = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = 1\}$ and $H_\eta = \{x \in \mathbb{R}^n : \langle x, \eta\rangle = 1\}$. Let the $(n-3)$-dimensional subspace W be the orthogonal compliment of $\mathrm{span}\{\xi, \zeta\}$ in $v^\perp$. Consider the orthogonal projection of the convex body $K \cap v^\perp$ onto the 2-dimensional subspace spanned by ξ and ζ. The picture is identical to Figure 5, with E_1, E_2, E_3, and E_4 defined similarly. If $n = 3$, we repeat the argument from the proof of Theorem 3.2. If $n \geq 4$, we will use the following modification of this argument.

Let $\bar{E}_i = E_i \times W$, for $i = 1, 2, 3, 4$. Then the equality

$$\begin{aligned}\mathrm{vol}_{n-1}(K \cap v^\perp \cap H_\xi^+) - \mathrm{vol}_{n-1}(K \cap v^\perp \cap H_\eta^+)\\ = \mathrm{vol}_{n-1}(L \cap v^\perp \cap H_\xi^+) - \mathrm{vol}_{n-1}(L \cap v^\perp \cap H_\eta^+)\end{aligned}$$

implies

$$\begin{aligned}\mathrm{vol}_{n-1}(K \cap v^\perp \cap (\bar{E}_1 \cup \bar{E}_4)) - \mathrm{vol}_{n-1}(K \cap v^\perp \cap (\bar{E}_3 \cup \bar{E}_4))\\ = \mathrm{vol}_{n-1}(L \cap v^\perp \cap (\bar{E}_1 \cup \bar{E}_4)) - \mathrm{vol}_{n-1}(L \cap v^\perp \cap (\bar{E}_3 \cup \bar{E}_4)).\end{aligned} \tag{7}$$

For $x \in \mathrm{span}\{\xi, \zeta\}$, consider the following parallel section function:

$$A_{K\cap v^\perp, W}(x) = \mathrm{vol}_{n-3}(K \cap v^\perp \cap \{W + x\}).$$

Then equation (7) and the Fubini theorem imply

$$\int_{E_1\cup E_4} A_{K\cap v^\perp,W}(x)dx - \int_{E_3\cup E_4} A_{K\cap v^\perp,W}(x)dx$$
$$= \int_{E_1\cup E_4} A_{L\cap v^\perp,W}(x)dx - \int_{E_3\cup E_4} A_{L\cap v^\perp,W}(x)dx.$$

Now we will pass to new coordinates (r,θ) as in the proof of Theorem 3.2, by letting $x(r,\theta) = \cos\theta\,\xi + \sin\theta\,\zeta + r(\sin\theta\,\xi - \cos\theta\,\zeta)$. Then

$$\int_0^\phi \int_0^\infty |r| A_{K\cap v^\perp,W}(x(r,\theta))drd\theta - \int_0^\phi \int_{-\infty}^0 |r| A_{K\cap v^\perp,W}(x(r,\theta))drd\theta$$
$$= \int_0^\phi \int_0^\infty |r| A_{L\cap v^\perp,W}(x(r,\theta))drd\theta - \int_0^\phi \int_{-\infty}^0 |r| A_{L\cap v^\perp,W}(x(r,\theta))drd\theta.$$

Differentiating with respect to ϕ and letting $\phi = 0$, we get

(8) $$\int_{-\infty}^\infty r A_{K\cap v^\perp,W}(x(r,0))dr = \int_{-\infty}^\infty r A_{L\cap v^\perp,W}(x(r,0))dr.$$

Note that

$$A_{K\cap v^\perp,W}(x(r,0)) = A_{K\cap v^\perp,W}(\xi - r\zeta) = A_{(K-\xi)\cap v^\perp,W}(-r\zeta)$$
$$= \int_{x\in\xi^\perp\cap v^\perp\cap\{\langle x,\zeta\rangle=-r\}} \chi(\|x\|_{K-\xi})dx.$$

Therefore, (8) and the Fubini theorem give

$$\int_{\xi^\perp\cap v^\perp} \langle x,\zeta\rangle \chi(\|x\|_{K-\xi})dx = \int_{\xi^\perp\cap v^\perp} \langle x,\zeta\rangle \chi(\|x\|_{L-\xi})dx.$$

Passing to polar coordinates in $\xi^\perp \cap v^\perp$, we get

$$\int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\zeta\rangle \|w\|_{K-\xi}^{-n+1} dw = \int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\zeta\rangle \|w\|_{L-\xi}^{-n+1} dw.$$

Observe that this is true for any $\zeta \in \xi^\perp \cap v^\perp$. Furthermore, for any vector $\vartheta \in \xi^\perp$ there is a vector $\zeta \in \xi^\perp \cap v^\perp$ and a number β such that $\vartheta = \zeta + \beta v$. Therefore, for every $\vartheta \in \xi^\perp$ we have

$$\int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\vartheta\rangle \|w\|_{K-\xi}^{-n+1} dw = \int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\vartheta\rangle \|w\|_{L-\xi}^{-n+1} dw.$$

Fixing ξ and ϑ, and looking at all $v \in S^{n-1} \cap \xi^\perp$, we can consider the latter equality as the equality of the spherical Radon transforms on $S^{n-1} \cap \xi^\perp$. Since the spherical Radon transform only allows to reconstruct even parts, we get

$$\langle w,\vartheta\rangle\|w\|_{K-\xi}^{-n+1}+\langle -w,\vartheta\rangle\|-w\|_{K-\xi}^{-n+1}=\langle w,\vartheta\rangle\|w\|_{L-\xi}^{-n+1}+\langle -w,\vartheta\rangle\|-w\|_{L-\xi}^{-n+1},$$

for all $w,\vartheta\in S^{n-1}\cap\xi^\perp$. That is,

$$\|w\|_{K-\xi}^{-n+1}-\|-w\|_{K-\xi}^{-n+1}=\|w\|_{L-\xi}^{-n+1}-\|-w\|_{L-\xi}^{-n+1},\quad \text{for all } w\in S^{n-1}\cap\xi^\perp.$$

We finish the proof as in Theorem 4.6. ∎

Below we will prove an analogue of the result of Falconer [2] and Gardner [4] for halfspaces. We will need the following lemma.

LEMMA 4.9: *Suppose $i>0$. Let K and L be convex bodies in $\mathbb{R}^n$, p_1 and p_2 be distinct points in the interior of $K\cap L$. If for all $\xi\in S^{n-1}$,*

$$\rho^i_{K,p_j}(\xi)-\rho^i_{K,p_j}(-\xi)=\rho^i_{L,p_j}(\xi)-\rho^i_{L,p_j}(-\xi),\quad \text{for } j=1,2, \tag{9}$$

and $\partial K\cap\partial L\neq\emptyset$, then $K=L$.

Proof. Let l be the line passing through p_1 and p_2. Our first goal is to prove that $\partial K\cap l=\partial L\cap l$. Let $Q_0\in\partial K\cap\partial L$. If $Q_0\in l$, we are done. Otherwise, we define two maps φ_1,φ_2 as follows. If Q is a point distinct from p_1, then $\varphi_1(Q)$ is defined to be the point on the line passing through Q and p_1, such that p_1 lies between Q and $\varphi_1(Q)$ and

$$|Qp_1|^i-|p_1\varphi_1(Q)|^i=\rho^i_{K,p_1}(\xi)-\rho^i_{K,p_1}(-\xi),$$

where $\xi=\dfrac{\overrightarrow{p_1Q}}{|p_1Q|}$.

Note that the domain of φ_1 contains the set $K\triangle L$. The map φ_2 is defined similarly with p_1 replaced by p_2.

For the chosen point $Q_0\in\partial K\cap\partial L$ consider the 2-dimensional plane H passing through Q_0, p_1, and p_2. Construct a sequence of points $\{Q_j\}\subset\partial K\cap\partial L\cap H$, satisfying $Q_{j+1}=\varphi_2^{-1}(\varphi_1(Q_j))$, and a sequence of angles $\{\theta_j\}=\{\angle(\overrightarrow{Q_j\varphi_1(Q_j)},l)\}$. One can see that $\lim_{j\to\infty}\theta_j=0$, and therefore the limit

$$X_0=\lim_{j\to\infty}Q_j$$

is a point on $l\cap\partial K\cap\partial L$. The claim that $\partial K\cap l=\partial L\cap l$ is now proved.

Let V be any 2-dimensional affine subspace of $\mathbb{R}^n$ that contains the line l. Consider the bodies $K\cap V$ and $L\cap V$ in V. The line l cuts both these bodies in two parts, $K\cap V=K_1\cup K_2$ and $L\cap V=L_1\cup L_2$, so that K_1 and L_1 are on the same side of l. Since $K \cap l = L \cap l$, the following star bodies are well-defined: $\tilde{K} = K_1 \cup L_2$ and $\tilde{L} = K_2 \cup L_1$. Condition (9) now implies

$$\rho^i_{\tilde{K},p_j}(\xi) + \rho^i_{\tilde{K},p_j}(-\xi) = \rho^i_{\tilde{L},p_j}(\xi) + \rho^i_{\tilde{L},p_j}(-\xi), \quad \text{for } j = 1, 2.$$

Now we can use [4, Theorem 6.2.3] to show that $\tilde{K} = \tilde{L}$, implying that $K \cap V = L \cap V$. Since V was an arbitrary affine subspace containing l, it follows that $K = L$. ∎

Remark 4.10: A version of this lemma for a smaller set of values of i (but without the assumption $\partial K \cap \partial L \neq \emptyset$) was proved by Koldobsky and Shane, [9, Lemma 6]. They also showed (see [9, Remark 1]) that one can take two balls that satisfy condition (9) with $i = 1$, but whose boundaries do not intersect.

With the help of Lemma 4.9 we obtain the following result.

Theorem 4.11: *Let K and L be convex bodies in $\mathbb{R}^n$ containing two distinct points p_1 and p_2 in their interiors. If, for every $v \in S^{n-1}$, we have*

$$\mathrm{vol}_n(K \cap v^+_{p_j}) = \mathrm{vol}_n(L \cap v^+_{p_j}) \quad \textit{for } j = 1, 2,$$

then $K = L$.

Proof. By [7], we have $\rho^n_{K,p_j}(\xi) - \rho^n_{K,p_j}(-\xi) = \rho^n_{L,p_j}(\xi) - \rho^n_{L,p_j}(-\xi)$, for $j = 1, 2$, and every $\xi \in S^{n-1}$. Also observe that $\partial K \cap \partial L \neq \emptyset$. Otherwise one of K or L would be strictly contained inside the other, which would contradict the hypothesis of the theorem. Now the result follows from Lemma 4.9. ∎

Note that Problem 1.1 is open even in the case of bodies of revolution when the center of the ball lies on the axis of revolution. However, if we consider a ball that does not intersect the axis of revolution, then the problem has a positive answer.

Theorem 4.12: *Let K and L be convex bodies of revolution in $\mathbb{R}^n$ with the same axis of revolution. Let D be a convex body in the interior of both K and L such that D does not intersect the axis of revolution. If for every hyperplane H supporting D we have*

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H),$$

then $K = L$.

Proof. Consider the two supporting hyperplanes of D that are perpendicular to the axis of revolution. Let p and q be the points where these hyperplanes intersect the axis of revolution.

Note that every plane passing through p (or q) can be rotated around the axis of revolution until it touches the body D. Due to the rotational symmetry of the bodies K and L we obtain that

$$\mathrm{vol}_{n-1}(K \cap (p + \xi^{\perp})) = \mathrm{vol}_{n-1}(L \cap (p + \xi^{\perp}))$$

and

$$\mathrm{vol}_{n-1}(K \cap (q + \xi^{\perp})) = \mathrm{vol}_{n-1}(L \cap (q + \xi^{\perp})),$$

for every $\xi \in S^{n-1}$.

The conclusion now follows from the corresponding result of Falconer [2] and Gardner [4], described in the introduction. ∎

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