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LOW DISTORTION EMBEDDINGS OF SOME METRIC GRAPHS INTO BANACH SPACES

BY

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ABSTRACT

We give a simple example of a countable metric graph M such that M Lipschitz embeds with distortion strictly less than 2 into a Banach space X only if X contains an isomorphic copy of ℓ_1. Further we show that, for each ordinal $\alpha < \omega_1$, the space $C([0,\omega^\alpha])$ does not Lipschitz embed into $C(K)$ with distortion strictly less than 2 unless $K^{(\alpha)} \neq \emptyset$. Also $C([0,\omega^{\omega^\alpha}])$ does not Lipschitz embed into a Banach space X with distortion strictly less than 2 unless $\mathrm{Sz}(X) \geq \omega^{\alpha+1}$.

1. Introduction

In 1974, Aharoni [1] proved that there exists $D \leq 6$ such that every separable metric space M Lipschitz embeds into c_0 with distortion at most D (which we

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denote by $M \hookrightarrow_D c_0$, see Section 2 for the definition). He also proved that $D \geq 2$ by showing that $\ell_1 \hookrightarrow_D c_0$ only if $D \geq 2$. The upper estimate on D was improved non-trivially several times until in 2008 Kalton and Lancien [17] had the final word $D = 2$. In the recent preprint [4], Baudier raised the following question: given an infinite Hausdorff compact space K, what is the least constant D such that $M \hookrightarrow_D C(K)$ for every separable metric space M?

In Section 3, we answer this question completely ($D = 1$ if K is not scattered, $D = 2$ otherwise, see Corollary 3.2) by constructing a countable metric space (a metric graph in fact) M with the property that $M \hookrightarrow_D X$, $D < 2$, only if the Banach space X contains an isomorphic copy of ℓ_1 (Theorem 3.1). It is not without interest that, conversely, if X contains an isomorphic copy of ℓ_1 then $M \hookrightarrow_1 (X, |\cdot|)$ for some equivalent norm $|\cdot|$ on X (see Proposition 3.5). In this section we also study bad embeddability properties of M into spaces with non-trivial generalized roundness.[1]

Section 4 is dedicated to the second main result of this article (Theorem 4.1) and its proof. It implies in particular that, for every countable ordinal α and every $\beta < \omega^\alpha$, the space $C([0, \omega^\alpha])$ does not embed into the space $C([0, \beta])$ with distortion strictly less than 2.

Section 5 deals with various consequences of Theorem 4.1. In particular, we give a partial improvement of non-linear Amir–Cambern type theorems studied by Jarosz [16], Dutrieux and Kalton [12] and Górak [14] (see Proposition 5.1 and Corollaries 5.2 and 5.3).

Finally, we show in Theorem 5.4 that if the Szlenk index of a Banach space X satisfies $\mathrm{Sz}(X) \leq \omega^\alpha$, then $C([0, \omega^{\omega^\alpha}])$ does not embed into X with distortion strictly less than 2 (Theorem 5.4). This can be understood as a refinement of the fact that $C([0, 1])$ does not embed into any Asplund space with distortion strictly less than 2 which is an immediate consequence of results of Section 3 (see also the original preprint [22] where this weaker result is proved without using Rosenthal's theorem).

Many open questions are scattered across the paper. In the next section, we introduce our notation and some facts which we will use without further reference.

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[1] *Added in proof:* a "non-reflexive" analogue of M as well as local version of Theorem 3.1 are studied in [21].

2. Preliminaries

The notation we use is standard. In particular, B_X stands for the closed unit ball of a Banach space X, c_0 is the Banach space of all real sequences converging to 0 and ℓ_1 is the Banach space of all absolutely summable real sequences. If K is a Hausdorff compact, $C(K)$ is the space of continuous functions on K. If A is a set, we denote by $|A|$ its cardinality. The symbol ω, resp. ω_1 stand for the first infinte, resp. uncountable, ordinal. The symbol $\bigsqcup$ denotes the disjoint union of not necessarily disjoint sets.

The following definitions and facts can be found, e.g., in [11]. A Banach space X is called Asplund if every closed separable subspace $Y \subset X$ has separable dual. A Hausdorff compact K is called scattered if there exists an ordinal α such that the Cantor–Bendixson derivative $K^{(\alpha)}$ is empty. A countable Hausdorff compact is necessarily scattered. If K is a Hausdorff compact, then $C(K)$ is Asplund iff K is scattered.

If α is an ordinal, then the interval $[0,\alpha] = \{\beta : 0 \leq \beta \leq \alpha\}$ becomes a Hausdorff compact space when equipped with the order topology. It is a well known theorem of Mazurkiewicz and Sierpiński (see [15, Theorem 2.56]) that every countable Hausdorff compact is homeomorphic to the interval $[0, \omega^\alpha \cdot n]$ for some $\alpha < \omega_1$ and $1 \leq n < \omega$. Thus the corresponding spaces of continuous functions are isometrically isomorphic. One also has

$$[0, \omega^\alpha \cdot n]^{(\alpha)} = \{\omega^\alpha \cdot 1, \dots, \omega^\alpha \cdot n\}.$$

A mapping $f : M \to N$ between metric spaces (M, d) and (N, ρ) is called **Lipschitz embedding** if there are constants $C_1, C_2 > 0$ such that

$$C_1 d(x,y) \leq \rho(f(x), f(y)) \leq C_2 d(x,y)$$

for all $x, y \in M$. The distortion $\mathrm{dist}(f)$ of f is defined as $\inf \frac{C_2}{C_1}$ where the infimum is taken over all constants C_1, C_2 which satisfy the above inequality. In other words $\mathrm{dist}(f) = \mathrm{Lip}(f)\,\mathrm{Lip}(f^{-1})^{-1}$ where $\mathrm{Lip}(f) = \sup_{x \neq y} \frac{\rho(f(x),f(y))}{d(x,y)}$. We say that M **Lipschitz embeds** (embeds for brevity) into N with distortion D (in short $M \hookrightarrow_D N$) if there exists a Lipschitz embedding $f : M \to N$ with $\mathrm{dist}(f) \leq D$ (in short $f : M \hookrightarrow_D N$). In this case, if the target space N is a Banach space, we may always assume (by taking $C_1^{-1} f$) that $C_1 = 1$. The N-distortion of M is defined as $c_N(M) := \inf\{D : M \hookrightarrow_D N\}$.

Metric spaces M and N are called **Lipschitz homeomorphic** if there is a surjective Lipschitz embedding from M onto N. Such embedding is then called **Lipschitz homeomorphism**. The **Lipschitz distance** of M and N is $d_L(M,N) = \inf \operatorname{dist}(f)$ where the infimum is taken over all Lipschitz homeomorphisms $f : M \to N$.

Recall that if X and Y are Banach spaces and $u : X \to Y$ is uniformly continuous, the following **Lipschitz constant of u at infinity**,

$$l_\infty(u) = \inf_{\eta>0} \sup_{\|x-x'\|\geq\eta} \frac{\|u(x)-u(x')\|}{\|x-x'\|},$$

is finite (sometimes called the Corson–Klee lemma, see [6, Proposition 1.11]). The **uniform distance** between X and Y is

$$d_U(X,Y) = \inf l_\infty(u) \cdot l_\infty(u^{-1})$$

with u ranging over all uniform homeomorphisms between X and Y.

A **net** in a Banach space X is a subset $\mathcal{N}$ of X such that there exist $a, b > 0$ which satisfy

- for any $x, x' \in \mathcal{N}$ with $x \neq x'$, we have $\|x - x'\| \geq a$, and
- for any $x \in X$, there exists $y \in \mathcal{N}$ with $\|x - y\| \leq b$.

We say that two Banach spaces are **net-equivalent** when they have Lipschitz homeomorphic nets. The **net distance** between X and Y is the number $d_N(X,Y) = \inf \operatorname{dist}(f)$ where the infimum is taken over all mappings $f : \mathcal{N} \to \mathcal{M}$ with $\mathcal{N} \subset X$ and $\mathcal{M} \subset Y$ being nets.

Finally the **Banach–Mazur distance** of X and Y is

$$d_{BM}(X,Y) = \inf \operatorname{dist}(f)$$

with f ranging over all linear isomorphisms from X to Y. It is well known and easy to see that for any couple of Banach spaces X and Y we have

$$d_N(X,Y) \leq d_U(X,Y) \leq d_L(X,Y) \leq d_{BM}(X,Y).$$

3. An unwieldy metric graph

Let $M = \{\mathbf{0}\} \cup \mathbb{N} \cup F$ where

$$F = \{A \subset \mathbb{N} : 1 \leq |A| < \infty\}$$

is the set of all finite nonempty subsets of $\mathbb{N}$. Notice that, for every $n \in \mathbb{N}$, the number n and the set $\{n\}$ are distinct elements of M. We put an edge between two points a, b of M iff $a = \mathbf{0}$ and $b \in \mathbb{N}$, or $a \in \mathbb{N}$, $b \in F$ and $a \in b$ thus introducing a graph structure on M. Let d be the shortest path metric on M. For $n \neq m \in \mathbb{N} \subset M$ and $A \neq B \in F$, it has the following values:

$$\begin{aligned} d(\mathbf{0}, n) =&1, \qquad d(n, A) = 1 \text{ if } n \in A, \qquad d(n, A) = 3 \text{ if } n \notin A, \\ d(\mathbf{0}, A) =&2, \qquad d(A, B) = 2 \text{ if } A \cap B \neq \emptyset, \qquad d(A, B) = 4 \text{ if } A \cap B = \emptyset, \\ d(n, m) =&2. \end{aligned}$$

Thus (M, d) is a countable (in particular, separable) bounded, uniformly discrete metric space.

THEOREM 3.1: *Let X be a Banach space and suppose that there exist $D \in [1, 2)$ and $f : M \to X$ such that*

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

Then X contains a copy of ℓ_1.

Proof. We plan to use the Rosenthal theorem [2]. Thus, we have to find a sequence $(x_k) \subset X$ such that none of its subsequences is weakly Cauchy. We claim that if we put $x_k := f(k)$, $k \in \mathbb{N}$, then (x_k) will have this property. First, for any $a, b \in \mathbb{N} \subset M$ we consider the set

$$X_{a,b} = \{x^* \in B_{X^*} : \langle x^*, f(a) - f(b)\rangle \geq 4 - 2D\},$$

and we will show that for every disjoint $A, B \in F$,

$$X_{A,B} := \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset.$$

Indeed, let $x^* \in B_{X^*}$ be such that

$$\langle x^*, f(A) - f(B)\rangle = \|f(A) - f(B)\|.$$

Then $x^* \in X_{A,B}$ by the triangle inequality, the definition of M and the fact that $f : M \hookrightarrow_D X$.

Now, let $(k_i) \subset \mathbb{N}$ be given. We define

$$A_n = \{k_{2i} : i \leq n\} \quad \text{and} \quad B_n = \{k_{2i-1} : i \leq N\}$$

for every $n \in \mathbb{N}$ and observe that $(X_{A_n,B_n})_n$ is a decreasing sequence of non-empty w^*-compacts. Thus there exists $x^* \in \bigcap_{n=1}^{\infty} X_{A_n,B_n}$. It is clear that $\langle x^*, x_{k_{2n}} - x_{k_{2n+1}}\rangle \geq 4 - 2D$ for all $n \in \mathbb{N}$ and so $(x_{k_i})_{i=1}^{\infty}$ is not weakly Cauchy. ∎

Corollary 3.2: *Let X be a $C(K)$ space. Then the following assertions are equivalent:*

(i) *Every separable Banach space embeds linearly isometrically into X.*
(ii) $c_X(M) < 2$
(iii) *X is not Asplund, i.e., K is not scattered.*

This together with the aforementioned theorem of Kalton and Lancien [17] and the fact that c_0 is isometric to a closed subspace of any infinite dimensional $C(K)$ space answers completely Question 1 in [4], i.e., for an infinite (not necessarily metrizable) Hausdorff compact K we have

$$\sup\{c_{C(K)}(M) : M \text{ separable metric space}\} = \begin{cases} 1 & \text{if } K \text{ is not scattered,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The implication from (i) to (ii) is trivial as M embeds isometrically into a separable subspace of ℓ_∞. The implication from (ii) to (iii) follows from Theorem 3.1. Finally, the implication from (iii) to (i) is well known; let us sketch it for the convenience of the reader. If $C(K)$ is not Asplund, K is not scattered. By a result of Pełczyński and Semadeni [20] there is a continuous surjection of K onto $[0,1]$. Thus $C(K)$ contains isometrically $C([0,1])$ as a closed subspace. The proof is thus finished by the application of the Banach–Mazur theorem [2]. ∎

Corollary 3.2 inspires the following question: Let X be a Banach space. Assume that there is some constant $D < 2$ such that

$$c_X(M) \leq D$$

for every separable metric space M. Does then every separable Banach space linearly embed into X? Does at least c_0 linearly embed into X?

Also, we do not know whether the condition (ii) above could be replaced by the more natural condition

$$c_X(\ell_1) < 2.$$

(We refer to [5] for some lower bounds for $c_{C(K)}(\ell_1)$ when K is a countable compact of height less than ω.) We know that our methods cannot be employed in a direct way to achieve this because M does not embed well into ℓ_1 either. We will see this using Enflo's generalized roundness. Let us recall the definition as presented in [19].

Definition 3.1: A metric space (X,d) is said to have **generalized roundness** q, written $q \in gr(X,d)$, if for every $n \geq 2$ and all points $a_1,\dots,a_n,b_1,\dots,b_n \in X$ we have

$$\sum_{1\leq i<j\leq n}((d(a_i,a_j)^q + d(b_i,b_j)^q) \leq \sum_{1\leq i,j\leq n} d(a_i,b_j)^q.$$

PROPOSITION 3.3: *If $f:(M,d)\hookrightarrow_D(X,\delta)$ and there is $0<q\in gr(X,\delta)$, then $D\geq 2$.*

Proof. Let $a_1,\dots,a_n,b_1,\dots,b_n \in M$. We have

$$\sum_{1\leq i<j\leq n}((\delta(f(a_i),f(a_j))^q + \delta(f(b_i),f(b_j))^q) \leq \sum_{1\leq i,j\leq n} \delta(f(a_i),f(b_j))^q$$

and so

$$\sum_{1\leq i<j\leq n}((d(a_i,a_j)^q + d(b_i,b_j)^q) \leq D^q \sum_{1\leq i,j\leq n} d(a_i,b_j)^q.$$

If $a_1,\dots,a_n \in \mathbb{N}\subset M$ are arbitrary and $b_i=\{a_1,\dots,a_n\}\setminus\{a_i\}\in F$, the above inequality evaluates to

$$n(n-1)2^q \leq D^q n((n-1)+3^q),$$

which is possible for all n only if $D\geq 2$. ∎

COROLLARY 3.4: *The metric space M does not embed with distortion strictly less than 2 into any $L_1(\mu)$.*

Proof. This follows immediately from Proposition 3.3 and from the fact that $1\in gr(L_1(\mu))$ which is proved in [19, Corollary 2.6]. ∎

We do not know if $c_{\ell_1}(M)=2$.[2] By considering any mapping of M onto the unit vector basis of ℓ_1, it is clear that $c_{\ell_1}(M)\leq 4$. The next proposition does a little bit better; it shows that M lives isometrically in a space that is isomorphic to ℓ_1.

PROPOSITION 3.5: *There is an equivalent norm $|\cdot|$ on ℓ_1 such that M embeds isometrically into $(\ell_1,|\cdot|)$. More generally, if a Banach space contains a copy of ℓ_1 then X admits an equivalent norm $|\cdot|$ such that $M\hookrightarrow_1(X,|\cdot|)$.*

[2] *Added in proof:* $c_{\ell_1}(M)=3$. See [21].

Proof. The more general statement follows from the less general one by a standard argument: any equivalent norm on a subspace can be extended to an equivalent norm on the whole space.

Let us prove the less general statement. For the definition and basic properties of Lipschitz-free spaces see [13] and the references therein. The space M embeds isometrically into $\mathcal{F}(M)$ where $\mathcal{F}(M)$ is the Lipschitz-free space over M. The result thus follows from the following fact.

Fact: If (U, d) is a countable uniformly discrete bounded metric space, then $\mathcal{F}(U)$ is isomorphic to ℓ_1.

First, it is easy to observe that if we equip $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ with the distance $\rho(0, n) = 1$ for every $n \in \mathbb{N}$, and $\rho(n, m) = 2$ for every $n \neq m \in \mathbb{N}$, then $\mathcal{F}(\mathbb{N}_0)$ is isometrically isomorphic to ℓ_1 where the isomorphism $\overline{\varphi} : \mathcal{F}(\mathbb{N}_0) \to \ell_1$ is the unique linear extension of $\varphi : \mathbb{N}_0 \to \ell_1$ given by $\varphi(0) = 0$ and $\varphi(n) = e_n$. Second, it is clear that (U, d) is Lipschitz homeomorphic to $(\mathbb{N}_0, \rho)$. The free spaces $\mathcal{F}(U)$ and $\mathcal{F}(\mathbb{N}_0)$ are thus linearly isomorphic. This finishes the proof of the fact and of the proposition. ∎

Again, we do not know whether the Banach–Mazur distance between $\mathcal{F}(M)$ and ℓ_1 is 2. The above proof only shows that it is at most 4 while Corollary 3.4 shows that it is at least 2.[3]

4. Low distortion embeddings between $C(K)$ spaces

THEOREM 4.1: *For every ordinal $\mu < \omega_1$ there exists a countable uniformly discrete metric space $M_\mu \subset C([0, \omega^\mu])$ such that M_μ does not embed with distortion strictly less than 2 into $C(K)$ if $K^{(\mu)} = \emptyset$.*

We start by defining finite metric graphs (with 3 levels in the spirit of the infinite graph M from Section 3) that do not embed well into ℓ_∞^n if n is small. We then "glue" them together infinitely many times, via a relatively natural procedure that we call sup-amalgamation. The sup-amalgamation is done in a precise order which will be encoded by certain trees on $\mathbb{N}$.

[3] *Added in proof:* $d_{BM}(\mathcal{F}(M), \ell_1) = 3$, see [21].

4.1. Construction of 3-level metric graphs.

Definition 4.1: Let $C_1, \dots, C_h$ be pairwise disjoint finite sets not containing $\mathbf{0}$. We put

$$M(C_1, \dots, C_h) = \{\mathbf{0}\} \cup \bigcup_{i=1}^{h} C_i \cup F(C_1, \dots, C_h)$$

where $F(C_1, \dots, C_h) = \{\{c_1, \dots, c_h\} : c_i \in C_i\}$. The set $M(C_1, \dots, C_h)$ is finite; we turn it into a graph by putting an edge between $x, y \in M(C_1, \dots, C_h)$ iff $x = \mathbf{0}$ and $y \in \bigcup C_i$ or $x \in \bigcup C_i$, $y \in F(C_1, \dots, C_h)$ and $x \in y$. We consider the shortest path distance d on $M(C_1, \dots, C_h)$. For $n \neq m \in \bigcup C_i$ and $A \neq B \in F$, it has the following values:

$$\begin{aligned} d(\mathbf{0}, n) &= 1, & d(n, A) &= 1 \text{ if } n \in A, & d(n, A) &= 3 \text{ if } n \notin A, \\ d(\mathbf{0}, A) &= 2, & d(A, B) &= 2 \text{ if } A \cap B \neq \emptyset, & d(A, B) &= 4 \text{ if } A \cap B = \emptyset, \\ d(n, m) &= 2. \end{aligned}$$

Lemma 4.2: *Let $C_1, \dots, C_h$ be pairwise disjoint sets and let us assume that $C_1 = \{1, 2\}$. We denote $F := F(C_1, \dots, C_h)$. Then there is an isometric embedding $f : M(C_1, \dots, C_h) \to \ell_\infty(F)$ which satisfies*

- $f(\mathbf{0}) = 0$,
- $f(x)(A) \in \{\pm 1\}$ *for all* $x \in \bigcup_{i=1}^{h} C_i$ *and all* $A \in F$,
- $f(1)(A) = 1$ *and* $f(2)(A) = -1$ *for all* $A \in F$.

Proof. We define first a mapping $g : M(C_1, \dots, C_h) \to \ell_\infty(F)$ as

$$g(x)(A) = d(x, A) - d(\mathbf{0}, A).$$

It is clearly 1-Lipschitz on $M := M(C_1, \dots, C_h)$ and an isometry on F. Given $x, y \in M$, a case by case check shows that there are $A, B \in F$ so that x and y lie on a geodesic curve between A and B. As g is 1-Lipschitz on M and an isometry on F, the triangle inequality implies that it preserves the length of any sub-curve, hence, in particular, $d(g(x), g(y)) = d(x, y)$. Observe that $g(\mathbf{0}) = 0$ and also g satisfies the second additional property. For every $A \in F$ we have $g(1)(A) = +1$ iff $1 \notin A$ iff $2 \in A$ iff $g(2)(A) = -1$. We thus define

$$f(x)(A) := \begin{cases} g(x)(A) & \text{if } g(1)(A) = 1 \\ -g(x)(A) & \text{if } g(1)(A) = -1 \end{cases}$$

for all $x \in M(C_1, \dots, C_h)$. ∎

4.2. Sup-amalgam of metric spaces.

Definition 4.2: Let (M_n, d_n) be a sequence of uniformly bounded metric spaces, all containing a fixed metric space A as a subset. Denote the common metric on A by d, fix a distinguished point $\mathbf{0} \in A$ and assume for all n that

$$\sup\{d(x,\mathbf{0}) : x \in A\} \le \inf\{d_n(x,y) : n \in \mathbb{N}, x \in A \text{ and } y \in M_n \setminus A\}. \tag{1}$$

The **sup-amalgam of** (M_n) **with respect to** A is the space

$$M_A := A \cup \bigsqcup (M_n \setminus A)$$

with the original metric d_n on each M_n and $d(x,y) = \max\{d_n(x,\mathbf{0}), d_m(y,\mathbf{0})\}$ whenever $x \in M_n \setminus A$ and $y \in M_m \setminus A$ for $m \neq n$. By (1) and the triangle inequality this is indeed a metric. We shall say that the M_n's are "glued along A" and denote $M_A = (M_n)_{n=1}^{\infty}/A$.

We shall only apply the construction when $d(x,\mathbf{0}) = 1 \le d_n(x,y)$ for $\mathbf{0} \neq x \in A$ and $y \in M_n \setminus A$, so (1) will certainly hold.

Let M_n and A be as above and consider isometries $f_n : M_n \to X_n$ into Banach spaces X_n. Their amalgamation is the map $g : M_A \to (\bigoplus X_n)_\infty$ defined by

$$g(x) = \begin{cases} (0,\ldots,0,f_n(x),0,\ldots) & \text{for } x \in M_n \setminus A, \\ (f_1(x), f_2(x), \ldots) & \text{for } x \in A. \end{cases}$$

Note that (1) yields that g is an isometry.

We shall apply the amalgamation only when $X_n = C([0,\alpha_n])$ for some countable ordinals $\alpha_n > 0$. We then identify $(\bigoplus C([0,\alpha_n]))_\infty$ with $C_B([0,\sum \alpha_n))$, the space of bounded continuous functions on the half open interval $[0,\sum \alpha_n)$.

Lemma 4.3: *Let M_n, A, and α_n be as above and assume that A is finite and contains at least 3 points which we denote $\{\mathbf{0},1,2\}$. Assume that*

$$f_n : M_n \to C([0,\alpha_n])$$

are isometries satisfying $f_n(\mathbf{0}) \equiv 0$, $f_n(1) \equiv 1$, $f_n(2) \equiv -1$ and $f_n(a)(\beta) = \pm 1$ for all $a \in A^* = A \setminus \{\mathbf{0}\}$ and $\beta \le \alpha_n$.

Then there are $N \le 2^{|A^*|-2}$ and an isometry $f : M_A \to C([0,(\sum \alpha_n)\cdot N])$ satisfying $f(\mathbf{0}) \equiv 0$, $f(1) \equiv 1$, $f(2) \equiv -1$ and $f(a)(\beta) = \pm 1$ for all $a \in A^* = A \setminus \{\mathbf{0}\}$ and $\beta \le (\sum \alpha_n)\cdot N$.

In particular, $N = 1$ when $|A| = 3$, i.e., when $A = \{\mathbf{0},1,2\}$.

Proof. Let g be the amalgamation of the f_n's. Note that for $a \in M_A \setminus A^*$, the function $g(a)$ is eventually zero. Hence $\lim_{\beta\to\gamma} g(a)(\beta) = 0$ where $\gamma = \sum \alpha_n$. On the other hand, for $a \in A^*$ the function $g(a)$ attains only the values ± 1 on $[0,\gamma)$ and $\lim_{\beta\to\gamma} g(a)(\beta)$ does not have to exist. For each choice of signs $\varepsilon \in \{\pm 1\}^{A_*}$ we consider the set $T_\varepsilon = \bigcap_{a\in A_*}\{\beta \in [0,\gamma) : g(a)(\beta) = \varepsilon(a)\}$. Denoting $I = \{\varepsilon \in \{\pm 1\}^{A_*} : T_\varepsilon \neq \emptyset\}$, the sets $(T_\varepsilon)_{\varepsilon\in I}$ form a finite partition of $[0,\gamma)$ into clopen disjoint sets. Note that since $\varepsilon(1) = 1$ and $\varepsilon(2) = -1$ for every $\varepsilon \in I$ we have $|I| \le 2^{|A^*|-2}$.

If $x \in A_*$, we define $f(x)$ as the continuous function on $[0,\gamma] \times I$ such that $f(x)(\beta,\varepsilon) = \varepsilon(x)$ for every $\beta \le \gamma$ and every $\varepsilon \in I$, i.e., $f(x)$ is constant on each $[0,\gamma] \times \{\varepsilon\}$. If $x \in M_A \setminus A_*$ we define $f(x)$

$$f(x)(\beta,\varepsilon) = \begin{cases} g(x)(\beta) & \text{when } \beta \in T_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

for each $(\beta,\varepsilon) \in [0,\gamma] \times I$. Since T_ε are clopen in $[0,\gamma)$ and since

$$\lim_{\beta\to\gamma} g(x)(\beta) = 0,$$

the function $f(x)$ is continuous on $[0,\gamma] \times I$. Notice that we have $f(\mathbf{0}) \equiv 0$ and $f(x)(\beta,\varepsilon) = \pm 1$ for $x \in A_*$ and $(\beta,\varepsilon) \in [0,\gamma] \times I$. Let us check that f is an isometry. Using that g is an isometry and the definition of f and T_ε, it is obvious that

$$\begin{aligned} d(x,y) &= \|g(x) - g(y)\| \\ &= \sup\{|f(x)(\beta,\varepsilon) - f(y)(\beta,\varepsilon)| : (\beta,\varepsilon) \in [0,\gamma] \times I, \beta \in T_\varepsilon\}. \end{aligned}$$

On the other hand, checking the four possibilities ($x \in A_*$ or $x \notin A_*$) and ($y \in A_*$ or $y \notin A_*$) we see that

$$|f(x)(\beta,\varepsilon) - f(y)(\beta,\varepsilon)| \le d(x,y)$$

if $(\beta,\varepsilon) \in [0,\gamma] \times I$ are such that $\beta \notin T_\varepsilon$. Thus, f is an isometry from M_A into $C([0,\gamma] \times I)$. It is clear that $[0,\gamma] \times I$ is homeomorphic to $[0,\gamma \cdot N]$ where $N := |I|$. Hence f maps isometrically M_A into $C([0,\gamma \cdot N])$. ∎

4.3. The trees T_μ. We use trees here in a very basic fashion as index sets. For any unexplained notion and for further information, check [15]. For two finite sequences $\overline{m} = (m_1, \ldots, m_h)$ and $\overline{n} = (n_1, \ldots, n_l)$, we write

$$\overline{m}^\frown\overline{n} = (m_1, \ldots, m_h, n_1, \ldots, n_l)$$

for their concatenation. The empty sequence $\emptyset$ is the two-sided neutral element for the concatenation. We omit the parentheses for the sequences of length one, thus (n) is written as n. Let us construct for every ordinal $\mu < \omega_1$ the tree T_μ on $\mathbb{N}$ as follows:

- $T_0 = \{\emptyset\}$ (the tree containing only the empty sequence),
- if T_μ has been defined, we put $T_{\mu+1} = \{\emptyset\} \cup \bigcup_{n=1}^\infty n^\frown T_\mu$,
- if μ is limit and T_α has been defined for every $\alpha < \mu$, we choose some $\alpha_n \nearrow \mu$ and put

$$T_\mu = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} n^\frown T_{\alpha_n}$$

where $n^\frown T_\mu = \{n^\frown \overline{m} : \overline{m} \in T_\mu\}$. Clearly, for each μ, the tree T_μ is well founded

The following derivation on trees is standard:

$$T' = \{\overline{n} \in T : \overline{n}^\frown k \in T \text{ for some } k \in \mathbb{N}\} = T \setminus \max T.$$

We further put $T^{(0)} = T$, $T^{(\alpha+1)} = (T^{(\alpha)})'$ and $T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)}$ whenever α is a limit ordinal. We put $o(T) = \inf\{\alpha : T^{(\alpha)} = T_0\}$ if the set is nonempty, otherwise $o(T) = \infty$. We will compute the index of the trees T_μ above.

Lemma 4.4: *For each $\mu < \omega_1$ we have $o(T_\mu) = \mu$.*

Proof. We have clearly $o(T_0) = 0$ and $o(T_1) = 1$. Assume the claim to be true for all $\alpha < \mu$; we try to prove it for μ. If $\mu = \alpha + 1$ is non-limit, we have $T_\mu = \{\emptyset\} \cup \bigcup_{n=1}^\infty n^\frown T_\alpha$. It is thus clear that $T_\mu^{(\alpha)} = T_1$ and so $o(T_\mu) = \alpha+1 = \mu$. Finally assume that μ is limit, we have $T_\mu = \{\emptyset\} \cup \bigcup_{n=1}^\infty n^\frown T_{\alpha_n}$ for some $\alpha_n \nearrow \mu$. It is thus clear that $T_\mu^{(\alpha_m)} = \{\emptyset\} \cup \bigcup_{n=m}^\infty n^\frown T_{\alpha_n}^{(\alpha_m)}$ for all $m \in \mathbb{N}$. Hence $T_\mu^{(\mu)} = \bigcap_{\beta<\mu} T_\mu^{(\beta)} = T_0$, and so $o(T_\mu) = \mu$. ∎

4.4. Iterative construction of M_μ. Let us fix $\mu < \omega_1$ from now on. We will describe how the tree $T := T_\mu$ encodes a construction of a metric space. Observe first that for every $\overline{n} \in T$ the ordinal $r(\overline{n}) = \inf\{\alpha : \overline{n} \in \max T^{(\alpha)}\}$ is the unique ordinal α such that $\overline{n} \in \max T^{(\alpha)}$. Thus the sets $r^{-1}(\alpha) = \max T^{(\alpha)}$, $0 \le \alpha \le \mu$, form a partition of T.

Definition 4.3: Let $(C_i^k)_{k=1}^\infty$, $i \in \mathbb{N}$, be pairwise disjoint increasing sequences of finite sets with $|C_i^k| = k+1$ for all $i, k \in \mathbb{N}$, such that the set $\{\mathbf{0}, 1, 2\}$ is disjoint from their union. We denote $C_0 = \{1, 2\}$.

We will first define

$$M_{\overline{n}} = M(C_0, C_1^{n_1}, \dots, C_h^{n_h})$$

for each $\overline{n} = (n_1, \dots, n_h) \in \max T^{(0)}$. Then, once $M_{\overline{n}}$ has been defined for every $\overline{n} \in \bigcup_{\beta<\alpha} \max T^{(\beta)}$, we put

$$M_{\overline{n}} = (M_{\overline{n}^\frown k})_{k=1}^{\infty} / (\{\mathbf{0}\} \cup C_0 \cup C_1^{n_1} \cup \dots \cup C_h^{n_h})$$

for every $\overline{n} = (n_1, \dots, n_h) \in \max T^{(\alpha)}$. Since $r(\overline{n}^\frown k) < r(\overline{n}) = \alpha$, the definition makes sense.

Observe that $\max T^{(\mu)} = \{\emptyset\}$. We will show that $M_\emptyset$ satisfies the requirements of Theorem 4.1.

Proposition 4.5: *For every $0 \le \alpha \le \mu$ for every $\overline{n} = (n_1, \dots, n_h) \in \max T^{(\alpha)}$ (or rather $\overline{n} = \emptyset$ in the case $\alpha = \mu$) there are $N_{\overline{n}} < \infty$ and an isometric embedding $f : M_{\overline{n}} \to C([0, \omega^\alpha \cdot N_{\overline{n}}])$ satisfying*

$$f(\mathbf{0}) = 0, \quad f(1) \equiv 1, \quad f(2) \equiv -1 \quad \textit{and} \quad f(x)(\gamma) = \pm 1 \tag{2}$$

for $x \in C_0 \cup C_1^{n_1} \cup \dots \cup C_h^{n_h}$ and $\gamma \le \omega^\alpha \cdot N_{\overline{n}}$. When $\alpha = \mu$, we have $N_\emptyset = 1$, i.e., $M_\emptyset \hookrightarrow_1 C([0, \omega^\mu])$.

Proof. By the definition of the spaces $\{M_{\overline{n}} : \overline{n} \in \max T^{(0)}\}$, we get the claim for $\alpha = 0$ using Lemma 4.2. Here the ordinals $\omega^0 \cdot N_{\overline{n}} = N_{\overline{n}}$ are finite so the spaces $C([0, N_{\overline{n}}])$ and $\ell_\infty([0, N_{\overline{n}}])$ are the same.[4] For $\alpha > 1$, the proof is a standard transfinite induction argument exploiting Lemma 4.3. Let $0 < \alpha \le \mu$ be an ordinal. Let $\overline{n} = (n_1, \dots, n_h) \in \max T^{(\alpha)}$. By the inductive hypothesis, for each $k \in \mathbb{N}$, there are $\alpha_k < \alpha$, $N_k < \infty$ and an isometric embedding $f_k : M_{\overline{n}^\frown k} \hookrightarrow_1 C([0, \omega^{\alpha_k} \cdot N_k])$ satisfying (2) for all $x \in C_0 \cup C_1^{n_1} \cup \dots \cup C_h^{n_h} \cup C_{h+1}^k$ and all $\gamma \le \omega^{\alpha_k} \cdot N_k$. (If $\alpha = \beta + 1$, then $\alpha_k = \beta$ for all k. If α is limit, $\alpha_k \nearrow \alpha$.) Recalling the definition of $M_{\overline{n}}$ and applying Lemma 4.3 we get that there is some $N_{\overline{n}} < \infty$[5] and an isometric embedding

$$f : M_{\overline{n}} \underset{1}{\hookrightarrow} C\left(\left[0, \left(\sum_{k=1}^{\infty} \omega^{\alpha_k} \cdot N_k\right) \cdot N_{\overline{n}}\right]\right) = C([0, \omega^\alpha \cdot N_{\overline{n}}])$$

satisfying (2) for all $x \in C_0 \cup C_1^{n_1} \cup \dots \cup C_h^{n_h}$ and all $\gamma \le \omega^\alpha \cdot N_{\overline{n}}$.

[4] $N_{\overline{n}} = |F(C_0, C_1^{n_1}, \dots, C_h^{n_h})| = 2 \prod (n_i + 1)$.

[5] $N_{\overline{n}} \le 2^{(n_1+1)+\dots+(n_h+1)}$ by Lemma 4.3

Finally, when $\alpha = \mu$, the space $M_\emptyset$ is obtained by glueing along the three point set $\{\mathbf{0}, 1, 2\}$, thus $N_\emptyset = 1$. ∎

Given a metric space M, a mapping $f : M \to C(K)$, points $a, b \in M$ and a constant $1 \leq D < 2$, we denote

$$X^f_{a,b} := \{x^* \in K : |\langle x^*, f(a) - f(b)\rangle| \geq 4 - 2D\}.$$

The duality above means the evaluation at the point $x^* \in K$. We do not indicate the dependence on D since it will always be clear from the context which D we have in mind.

PROPOSITION 4.6: (i) *For each $1 \leq D < 2$ there exists a constant*

$$C_D = \left(\log\left(\left\lfloor \frac{D}{2-D} \right\rfloor + 1\right)\right)^{-1} > 0$$

such that for every $\alpha < \mu$ and every $\overline{n} = (n_1, \dots, n_h) \in \max T^{(\alpha)}$ and every $f : M_{\overline{n}} \hookrightarrow_D C(K)$ we have

$$\left| X^f_{1,2} \cap \bigcap_{i=1}^{h-1} X^f_{a_i,b_i} \cap K^{(r(\overline{n}))} \right| \geq C_D \log(n_h)$$

for all $a_i \neq b_i \in C_i^{n_i}$, $1 \leq i \leq h-1$ (with the obvious meaning when $h = 1$).

(ii) *For each $1 \leq D < 2$ and every $f : M_\emptyset \hookrightarrow_D C(K)$ we have*

$$X^f_{1,2} \cap K^{(\mu)} \neq \emptyset.$$

Proof. We will proceed by a transfinite induction on α. Let first $\alpha = 0$, $\overline{n} \in \max T$ and let $f : M_{\overline{n}} \hookrightarrow_D C(K)$. We may assume that $f(\mathbf{0}) = 0$. For given $a_i \neq b_i \in C_i^{n_i}$, $1 \leq i < h$ and $a \neq b \in C_h^{n_h}$ we put

$$A = \{1, a\} \cup \{a_i : 1 \leq i < h\} \quad \text{and} \quad B = \{2, b\} \cup \{b_i : 1 \leq i < h\}.$$

Let $x^*_{a,b} \in K$ be such that $|\langle x^*_{a,b}, f(A) - f(B)\rangle| = \|f(A) - f(B)\|$. Then, using the triangle inequality,

$$x^*_{a,b} \in X^f_{1,2} \cap \bigcap_{i=1}^{h-1} X^f_{a_i,b_i} \cap X^f_{a,b}.$$

Denoting $\Gamma = \{x^*_{a,b} : a, b \in C_h^{n_h}, a \neq b\}$ one can check that

$$\{(\langle f(a), \gamma\rangle)_{\gamma \in \Gamma} : a \in C_h^{n_h}\} \subset B_{\ell_\infty(\Gamma)}(0, D)$$

is a $(4-2D)$-separated set of cardinality n_h. We thus get that $|\Gamma| \geq C_D \log(n_h)$.

Let us now assume the result to be true for every $\beta < \alpha$ with the goal of proving it for $\alpha \le \mu$. Let $\overline{n} = (n_1, \ldots, n_h) \in \max T^{(\alpha)}$. Let $f : M_{\overline{n}} \hookrightarrow_D C(K)$. We have $M_{\overline{n}} = (M_{\overline{n}^\frown k})_{k=1}^{\infty} / (\{\mathbf{0}\} \cup C_0 \cup C_1^{n_1} \cup \cdots \cup C_h^{n_h})$. Since $M_{\overline{n}^\frown k} \subset M_{\overline{n}}$ canonically isometrically, the inductive hypothesis yields that

$$\left| X_{1,2}^f \cap \bigcap_{i=1}^{h} X_{a_i,b_i}^f \cap K^{(r(\overline{n}^\frown k))} \right| \ge C_D \log(k)$$

for every $k \in \mathbb{N}$ and every $a_i \ne b_i \in A_i^{n_i}$ $(1 \le i \le h)$. If α is a limit ordinal we have $r(\overline{n}^\frown k) \nearrow \alpha$, and if $\alpha = \beta + 1$ we have $r(\overline{n}^\frown k) = \beta$ for all k. In both cases

$$X_{1,2}^f \cap \bigcap_{i=1}^{h} X_{a_i,b_i}^f \cap K^{(\alpha)} \ne \emptyset$$

by compactness. In the case when $\alpha = \mu$ we can stop as we have proved point (ii). In the remaining case we have $h \ge 1$ and so we get by the same volume argument as above that

$$\left| X_{1,2}^f \cap \bigcap_{i=1}^{h-1} X_{a_i,b_i}^f \cap K^{(r(\overline{n}))} \right| \ge C_D \log(n_h). \qquad \blacksquare$$

Proof of Theorem 4.1. Let $\mu < \omega_1$ be given. We put $M_\mu := M_\emptyset$. This space embeds isometrically into $C([0, \omega^\mu])$ by Proposition 4.5. By Proposition 4.6 (ii) we see that if $M_\emptyset \hookrightarrow_D C(K)$, $D < 2$, then $K^{(\mu)} \ne \emptyset$. $\blacksquare$

5. Further consequences of Theorem 4.1

Remark 5.1: Let $\mu < \omega_1$ and let K be a Hausdorff compact space such that $K^{(\mu)} \ne \emptyset$. We denote by $C_0(K)$ the closed subspace of $C(K)$ of the functions whose restrictions on $K^{(\alpha)}$ are identically zero. Assume that M_μ embeds into $C_0(K)$ with distortion strictly less than 2. Then $f(1)$ and $f(2)$ are null on $K^{(\mu)}$, thus $X_{1,2}^f \cap K^{(\mu)} = \emptyset$ contradicting Proposition 4.6 (ii). In particular, $C([0, \omega^\mu])$ does not embed with distortion strictly less than 2 into $C_0([0, \omega^\mu])$. For $\mu = 1$ the last statement means that c does not embed with distortion strictly less than 2 into c_0. This also follows from [17, Proposition 3.1] as an easy but entertaining exercise.

Remark 5.2: Let us recall the following result of Bessaga and Pełczyński [8, 15]: Let $\omega \le \alpha \le \beta < \omega_1$. Then $C([0, \alpha])$ is linearly isomorphic to $C([0, \beta])$ iff $\beta < \alpha^\omega$. It is a longstanding open problem whether $C([0, \beta])$ can be Lipschitz homeomorphic to a subspace of $C([0, \alpha])$ if $\beta > \alpha^\omega$. Theorem 4.1 implies that the distortion of any such Lipschitz homeomorphism onto a subspace, if it exists, must be at least 2.

5.1. Relation to the non-linear Amir–Cambern theorem. The reader can consult Section 2 for the definition of the distances between Banach spaces that we are about to use. The theorem of Amir [3] and Cambern [9] is the following generalization of Banach–Stone theorem: *Let K and L be two compact spaces. If $d_{BM}(C(K), C(L)) < 2$, then K and L are homeomorphic.* A result of Cohen [10] shows that the constant 2 above is optimal but at the present it is not clear whether one could draw the same conclusion under the weaker hypothesis of $d_L(C(K), C(L)) < 2$ or even $d_N(C(K), C(L)) < 2$. The results of Jarosz [16], resp. Dutrieux and Kalton [12], resp. Górak [14], show that K and L are homeomorphic if $d_L(C(K), C(L)) < 1+\varepsilon$ (with $\varepsilon > 0$ universal but small), resp. $d_N(C(K), C(L)) < 17/16$, resp. $d_N(C(K), C(L)) < 6/5$. Theorem 4.1 also implies that if K and L are two countable compacts, then assuming $d_L(C(K), C(L)) < 2$ it implies that K and L have the same height (see also the stronger Corollary 5.2). Observe that we do not require surjectivity in order to get this result.

We do not know whether one has

$$c_{C([0,\omega^\mu \cdot m])}(C([0, \omega^\mu \cdot n])) = 2$$

for $1 \le m < n < \omega$, but we get as a byproduct of the proof of our main result the following proposition.

Proposition 5.1: *Let $1 \le D < 2$ be given. Then for every $1 \le m < \omega$ there is $1 \le n < \omega$ such that for all $\mu < \omega_1$ the space $C([0, \omega^\mu \cdot n])$ does not embed into the space $C([0, \omega^\mu \cdot m])$ with distortion D. More precisely, there is a uniformly discrete and bounded countable metric space $M \subset C([0, \omega^\mu \cdot n])$ which does not embed into the space $C([0, \omega^\mu \cdot m])$ with distortion D.*

Proof. We find $k \in \mathbb{N}$ such that $C_D \log(k) > m$. Let $n = 2^{k+1}$. Let $T = T_{\mu+1}$ and consider $\overline{k} = (k) \in T$. Then $r(\overline{k}) = \mu$. Proposition 4.6 (i) yields that

$$|X^f_{1,2} \cap K^{(\mu)}| \ge C_D \log(k) > m$$

for every $f : M_{\overline{k}} \hookrightarrow_D C(K)$. Hence no such embedding f can exist as $K^{(\mu)} = \{\omega^\mu \cdot 1, \ldots, \omega^\mu \cdot m\}$. On the other hand $M_{\overline{k}} \hookrightarrow_1 C([0, \omega^\mu \cdot n])$ by Proposition 4.5, the footnotes in its proof and the choice of n. Consequently $C([0, \omega^\mu \cdot n])$ does not embed into $C(K)$ with distortion D. ∎

The next corollary answers partially Problem 4.2 in [14].

Corollary 5.2: *Let $\gamma \neq \alpha < \omega_1$ and $n, m \in \mathbb{N}$. Then*

$$d_N(C([0, \omega^\gamma \cdot n]), C([0, \omega^\alpha \cdot m])) \geq 2.$$

Proof. In fact, we are going to prove the stronger claim that for $\beta < \omega^\alpha$ and for every net $\mathcal{N}$ in $C([0, \omega^\alpha])$ there is no Lipschitz embedding $f : \mathcal{N} \to C([0, \beta])$ such that $\operatorname{dist}(f) < 2$.

Let us suppose that such f and $\mathcal{N}$ exist. Assume that $\mathcal{N}$ is an (a, b)-net and consider a mapping $\pi : C([0, \omega^\alpha]) \to \mathcal{N}$ such that $\|x - \pi(x)\| \leq b$. Let us consider M_α as a subset of $C([0, \omega^\alpha])$ which we can by Theorem 4.1. Since $d(x, y) \geq 1$ for all $x \neq y \in M_\alpha$, we have, for $\lambda > 2b$, that

$$\operatorname{dist}(\pi \restriction_{\lambda M_\alpha}) \leq \Big(1 + \frac{2b}{\lambda}\Big)\Big(1 + \frac{2b}{\lambda - 2b}\Big).$$

Thus it is clear that for λ large enough we have $\operatorname{dist}(g) < 2$ for the embedding $g : M_\alpha \to C([0, \beta])$ defined as

$$g(x) = \frac{1}{\lambda} f(\pi(\lambda x))$$

for $x \in M_\alpha$. According to Theorem 4.1, such embedding cannot exist. Contradiction. ∎

Corollary 5.3: *Let $1 \leq D < 2$ be given. Then for every $1 \leq m < \omega$ there is $1 \leq n < \omega$ such that*

$$d_N(C([0, \omega^\mu \cdot n]), C([0, \omega^\mu \cdot m])) \geq D$$

for all $\mu < \omega_1$.

The proof follows the lines of the proof of Corollary 5.2 with Theorem 4.1 replaced by Proposition 5.1. The details are left to the reader.

5.2. Lower bounds on the Szlenk index. Finally, we will give a lower bound on the Szlenk index of a Banach space X that admits a certain M_α with distortion strictly less than 2 (for the definition and properties of the Szlenk index, the reader can consult [15, 18]).

Theorem 5.4: *Let X be an Asplund space and assume that M_{ω^α} embeds into X with distortion strictly less than 2 for an ordinal $\alpha < \omega_1$. Then*

$$\operatorname{Sz}(X) \geq \omega^{\alpha+1}.$$

For the proof we will need the following version of Zippin's lemma as presented in [7, page 27]; see also [23, Lemma 5.11].

Zippin's lemma: *Let X be a separable Banach space with separable dual and let $\frac{1}{2} > \varepsilon > 0$. Then there exist a compact K, an ordinal $\beta < \omega^{Sz(X,\frac{\varepsilon}{8})+1}$, a subspace Y of $C(K)$, isometric to $C([0,\beta])$ and an embedding $i : X \to C(K)$ with $\|i\|\|i^{-1}\| < 1+\varepsilon$ such that for any $x \in X$ we have*

$$\operatorname{dist}(i(x), Y) \leq 2\varepsilon\|i(x)\|.$$

Proof of Theorem 5.4. Let us assume that $M_{\omega^\alpha} \hookrightarrow_D X$ with $D < 2$. Let $\varepsilon > 0$ be small enough so that $D' = D(1+\varepsilon) < 2$ and also that $\eta := 2\varepsilon D' < \frac{1}{2}$ and $\frac{1+4\varepsilon}{1-2\eta} D' < 2$. Let K and $\beta < \omega^{\mathrm{Sz}(X,\frac{\varepsilon}{8})+1}$ be as in Zippin's lemma. Then M_{ω^α} embeds into $C(K)$ with distortion $D' < 2$ via some embedding g such that $d(x,y) \leq \|g(x) - g(y)\| \leq D'd(x,y)$ and, without loss of generality, that $g(\mathbf{0}) = 0$. Thus for every $x \in M_{\omega^\alpha}$ we have $\|g(x)\| \leq 2D'$. We know that for each $x \in M_{\omega^\alpha}$ there is $f(x) \in C([0,\beta])$ such that

$$\|g(x) - f(x)\| \leq \eta.$$

This implies that $\|g(x)-g(y)\|-2\eta \leq \|f(x)-f(y)\| \leq \|g(x)-g(y)\|+2\eta$. Now since $1 \leq d(x,y)$ we have

$$d(x,y)(1-2\eta) \leq \|f(x) - f(y)\| \leq d(x,y)D'(1+4\varepsilon).$$

This proves that f is a Lipschitz embedding of M_{ω^α} into $C([0,\beta])$ with distortion strictly less than 2 and so, according to Theorem 4.1, we have $\beta \geq \omega^{\omega^\alpha}$. This implies that $\mathrm{Sz}(X) > \omega^\alpha$ and so $\mathrm{Sz}(X) \geq \omega^{\alpha+1}$ by [15, Theorem 2.43]. ∎

An interesting immediate consequence of the above theorem is the fact that, for every $\gamma < \alpha < \omega_1$ and for every equivalent norm $|\cdot|$ on $C([0,\omega^{\omega^\gamma}])$, the space M_{ω^α} does not embed with distortion strictly less than 2 into $(C([0,\omega^{\omega^\gamma}]), |\cdot|)$.

We do not know if every Banach space X such that $\mathrm{Sz}(X) \geq \omega^{\alpha+1}$ admits an equivalent norm $|\cdot|$ such that $M_{\omega^\alpha} \hookrightarrow X$ isometrically or at least with distortion strictly less than 2.

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(Tony Procházka, Besançon, March 16, 2017)

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