ON THE LIPSCHITZ CONTINUITY OF CERTAIN QUASIREGULAR MAPPINGS BETWEEN SMOOTH JORDAN DOMAINS

BY

Jiaolong Chen

Department of Mathematics, Shantou University
Shantou, Guangdong 515063, People's Republic of China
e-mail: jiaolongchen@sina.com

AND

Peijin Li

Department of Mathematics, Hunan Normal University
Changsha, Hunan 410081, People's Republic of China
e-mail: wokeyi99@163.com

AND

Swadesh Kumar Sahoo

Discipline of Mathematics, Indian Institute of Technology
Indore, Simrol, Indore 453 552, India
e-mail: swadesh@iiti.ac.in

AND

Xiantao Wang*

Department of Mathematics, Shantou University
Shantou, Guangdong 515063, People's Republic of China
e-mail: xtwang@stu.edu.cn

---

* Corresponding author.

ABSTRACT

We first investigate the Lipschitz continuity of (K, K')-quasiregular C^2 mappings between two Jordan domains with smooth boundaries, satisfying certain partial differential inequalities concerning Laplacian. Then two applications of the obtained result are given: As a direct consequence, we get the Lipschitz continuity of ρ-harmonic (K, K')-quasiregular mappings, and as the other application, we study the Lipschitz continuity of (K, K')-quasiconformal self-mappings of the unit disk, which are the solutions of the Poisson equation $\Delta w = g$. These results generalize and extend several recently obtained results by Kalaj, Mateljević and Pavlović.

1. Introduction and statement of the main results

1.1. PRELIMINARIES.

1.1.1. (K, K')-*quasiregular mappings.* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

We will consider the matrix norm

$$|A| = \max\{|Az| : \ z \in \mathbb{C}, |z| = 1\}$$

and the matrix function

$$l(A) = \min\{|Az| : \ z \in \mathbb{C}, |z| = 1\}.$$

Let D and Ω be subdomains of the complex plane $\mathbb{C}$, and let $w = u + iv : D \to \Omega$ be a function that has both partial derivatives at $z = x + iy$ in D. The notation Dw always denotes the Jacobian matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}.$$

Obviously,

$$|Dw| = |w_z| + |w_{\overline{z}}| \quad \text{and} \quad l(Dw) = ||w_z| - |w_{\overline{z}}||.$$

We say that a function $w : D \to \mathbb{C}$ is **absolutely continuous on lines**, **ACL** in brief, in the region D if for every closed rectangle $R \subset D$ with sides parallel to the axes x and y, w is absolutely continuous on almost every horizontal line and almost every vertical line in R. Such a function has, of course, partial

derivatives w_x and w_y a.e. in D. Further, we say $w \in ACL^2$ if $w \in ACL$ and its partial derivatives are locally L^2 integrable in D.

We say that a sense-preserving continuous mapping $w : D \to \Omega$ is (K, K')-**quasiregular** if

(1) w is ACL^2 in D and $J_w \neq 0$ a.e. in D;

(2) there are constants $K \geq 1$ and $K' \geq 0$ such that

$$|Dw|^2 \leq K J_w + K',$$

where J_w denotes the Jacobian of w, which is given by

$$J_w = |w_z|^2 - |w_{\overline{z}}|^2 = |Dw|l(Dw).$$

In particular, w is called (K, K')-**quasiconformal** if w is a (K, K')-quasiregular homeomorphism; and if w is a K-quasiregular homeomorphism, then w is K-quasiconformal.

If $K' = 0$, then "(K, K')-quasiregular" (resp. "(K, K')-quasiconformal") mappings reduce to "K-quasiregular" (resp. "K-quasiconformal") ones.

We remark that there are (K, K')-quasiregular mappings which are not K_1-quasiregular for any $K_1 \geq 1$, and also there are (K, K')-quasiconformal mappings whose inverses are not (K_1, K_1')-quasiconformal for any $K_1 \geq 1$ and $K_1' \geq 0$. See the examples in Sections 2 and 4 for the details.

The following result easily follows from the proof of Lemma 2.1 in [1].

LEMMA 1.1: *Suppose w is a (K, K')-quasiregular mapping. Then*

$$|Dw| \leq K l(Dw) + \sqrt{K'}.$$

A mapping $f : D \to \Omega$ is **proper** if the preimage of every compact set in Ω is compact in D. It is known that if $D = \Omega = \mathbb{D} = \{z : |z| < 1\}$, then a mapping f is proper if and only if $|f(z)| \to 1$ as $|z| \to 1$ (cf. [16]).

1.1.2. *Lipschitz continuity.* We say that a mapping $f : D \to \Omega$ is in Lip_α if there exists a constant L_1 and an exponent $\alpha \in (0, 1]$ such that for all $z, w \in D$,

$$|f(z) - f(w)| \leq L_1 |z - w|^\alpha.$$

Such mappings are also called α-**Hölder continuous**.

In particular, if $\alpha = 1$, then we say that f is **Lipschitz continuous**.

A mapping f is said to be **coLipschitz continuous** if there exists a constant L_2 such that for all $z, w \in D$,

$$|f(z) - f(w)| \geq L_2|z - w|.$$

1.1.3. *Jordan domains.* A **Jordan curve** is a set in the complex plane $\mathbb{C}$ which is homeomorphic to a circle. For a **Jordan domain**, we mean a domain whose boundary is a Jordan curve. In this paper, unless specially stated, all Jordan domains are assumed to be bounded.

Denote by $\ell(\gamma)$ the length of γ, and let $\Gamma : [0, \ell(\gamma)] \to \gamma$ be the arc length parameterization of γ, i.e., the parameterization satisfying the condition

$$|\Gamma'(s)| = 1 \quad \text{a.e. in } [0, \ell(\gamma)].$$

We say that γ is of **class** $C^{n,\alpha}$ for some $n \in \mathbb{N}$ and $\alpha \in (0, 1]$ if Γ is of class C^n and

$$\sup_{t \neq s \in [0, \ell(\gamma)]} \frac{|\Gamma^{(n)}(t) - \Gamma^{(n)}(s)|}{|t - s|^\alpha} < \infty,$$

where $\Gamma^n(t)$ denotes the nth derivative of $\Gamma(t)$ with respect to t. The Jordan domain D is called a $C^{n,\alpha}$ **domain** if its boundary ∂D is a $C^{n,\alpha}$ Jordan curve.

Let $\gamma \in C^{1,\alpha}$ be a closed Jordan curve, and $d_\gamma(\Gamma(s), \Gamma(t))$ the distance between $\Gamma(s)$ and $\Gamma(t)$ along the curve γ, i.e.,

$$d_\gamma(\Gamma(s), \Gamma(t)) = \min\{|s - t|, \ell(\gamma) - |s - t|\}.$$

A closed rectifiable Jordan curve γ is said to enjoy a **b-chord-arc condition** if there exists $b > 1$ such that for all $z_1, z_2 \in \gamma$,

$$d_\gamma(z_1, z_2) \leq b|z_1 - z_2|.$$

We remark that the unit circle $\mathbb{S}$ enjoys a $\frac{\pi}{2}$-chord-arc condition.

1.1.4. *Normalized mappings.* For a closed curve β in $\mathbb{C}$, three points a_0, a_1 and a_2 in β are said to be **well-distributed** if for $i \in \{0, 1\}$,

$$\ell(\beta[a_i, a_{i+1}]) = \ell(\beta[a_{i+1}, a_{i+2}]),$$

where $a_3 = a_0$ and $\beta[a_i, a_{i+1}]$ denotes the part of β with endpoints a_i and a_{i+1}.

Let D be a Jordan domain with rectifiable boundary. We will say that a mapping $f : \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S} \to \overline{\Omega}$ is **normalized** if there are three well-distributed points t_0, t_1, t_2 in $\mathbb{S}$; their images $f(t_0)$, $f(t_1)$ and $f(t_2)$ under f are also well-distributed in $\partial\Omega = f(\mathbb{S})$ (cf. [14]).

1.2. LIPSCHITZ CONTINUITY FOR CERTAIN (K, K')-QUASIREGULAR MAPPINGS.
If a (K, K')-quasiregular (resp. (K, K')-quasiconformal) mapping is harmonic,
then it is said to be **harmonic (K, K')-quasiregular** (resp. **harmonic (K, K')-quasiconformal**). Martio [19] was the first who considered harmonic quasiconformal mappings in $\mathbb{C}$. Recent papers [1, 8, 12, 14, 15, 18, 22] and references
therein together bring much light on the topic of harmonic quasiconformal mappings in $\mathbb{C}$. See [9, 11] for the discussions in this line in the space. In [17, 20],
the Lipschitz characteristic of harmonic quasiconformal mappings has been discussed. See [24, 25, 26, 27] for similar discussions in this line. In [3], Finn
and Serrin discussed the Hölder continuity of a class of elliptic mappings which
satisfy the following partial differential inequality

$$|w_z|^2 + |w_{\overline{z}}|^2 \le KJ_w + K',$$

where $K \ge 1$ and $K' \ge 0$ are constants. See also [21].

Recently, in [14], Kalaj and Mateljević discussed the Lipschitz continuity
of (K, K')-quasiconformal harmonic mappings. In [12], they considered the
Lipschitz continuity of a quasiconformal C^2 diffeomorphism $w : D \to \Omega$ which
satisfies the partial differential inequality

(1.1) $|\Delta w| \le M|w_z \cdot w_{\overline{z}}|,$

where $M \ge 0$ is a constant and D (resp. Ω) denotes a $C^{1,\alpha}$ (resp. $C^{2,\alpha}$) Jordan
domain, and in [10], as a generalization of the discussions in [12], Kalaj studied
the Lipschitz continuity of a K-quasiregular C^2 mapping $w : D \to \Omega$ which
satisfies the partial differential inequality

(1.2) $|\Delta w| \le M|Dw|^2 + N$

for some constants $M \ge 0$ and $N \ge 0$.

Obviously, if a mapping satisfies the partial differential inequality (1.1), then
it also satisfies (1.2). Observe that if $M = N = 0$ in (1.2), then w is harmonic.
The reader is referred to [2] for the properties of this class of mappings.

As the first aim of this paper, we consider the Lipschitz continuity of (K, K')-quasiregular C^2 solutions of (1.2). Our result is as follows.

THEOREM 1.1: *Suppose w is a proper (K, K')-quasiregular C^2 mapping of a
Jordan domain D with $C^{1,\alpha}$ boundary onto a Jordan domain Ω with $C^{2,\alpha}$
boundary. If w satisfies the partial differential inequality (1.2) for constants
$M \ge 0$ and $N \ge 0$, then w has bounded partial derivatives in D. In particular,
w is Lipschitz continuous.*

We remark that Theorem 1.1 is a substantial generalization of [10, Theorem 1.2] and [12, Theorem 1.3] (The example in Section 2 demonstrates this.) This theorem will be proved in Section 3.

1.3. LIPSCHITZ CONTINUITY FOR CERTAIN ρ-HARMONIC (K,K')-QUASIREGULAR MAPPINGS. A two times continuously differentiable complex-valued function f between two domains D and Ω in $\mathbb{C}$ is said to be ρ-**harmonic** if it satisfies the Euler–Lagrange equation

$$(1.3) \qquad\qquad f_{z\bar{z}} + ((\log \rho)_w \circ w) f_z f_{\bar{z}} = 0,$$

where $w = f(z)$, and $\rho(w)|dw|$ is an arbitrary conformal C^1-metric defined in Ω.

If φ is a holomorphic mapping different from 0 and if $\rho = |\varphi|$ in Ω, we call w a φ-**harmonic mapping**.

Since $\rho^2 = \varphi\overline{\varphi}$, an elementary computation yields $2(\log\rho)_w = (\log\varphi)'$. It follows from (1.3) that if f is φ-harmonic, then

$$f_{z\bar{z}} + \Big(\frac{\varphi'}{2\varphi} \circ w\Big) f_z f_{\bar{z}} = 0.$$

In [12], Kalaj and Mateljević considered the Lipschitz continuity of ρ-harmonic quasiconformal mappings. As a direct consequence of Theorem 1.1, we obtain the Lipschitz continuity of ρ-harmonic (K, K')-quasiregular mappings. Our first result concerns the Lipschitz continuity of φ-harmonic (K, K')-quasiregular mappings, which is as follows.

COROLLARY 1.1: *Suppose w is a φ-harmonic mapping of $\mathbb{D}$ onto a $C^{2,\alpha}$ Jordan domain Ω. If $|(\log\varphi)'|_\infty = \sup\{|(\log\varphi(z))'| : z \in \mathbb{D}\} < \infty$ and w is a proper (K, K')-quasiregular mapping, then w has bounded partial derivatives. In particular, w is Lipschitz continuous.*

Our next result, concerning approximately analytic metrics, generalizes Corollary 1.1, where a C^1 function h is said to be **approximately analytic** if there is a constant $C \geq 0$ such that $|h_{\bar{z}}| \leq C|h|$.

COROLLARY 1.2: *Suppose w is a ρ-harmonic mapping of $\mathbb{D}$ onto a $C^{2,\alpha}$ Jordan domain Ω. Further, if ρ is approximately analytic in Ω and w is a proper (K, K')-quasiregular mapping, then w has bounded partial derivatives. In particular, w is Lipschitz continuous.*

We remark that Corollary 1.1 (resp. Corollary 1.2) is a generalization of [12, Theorem 3.1] (resp. [12, Theorem 3.3]).

1.4. Lipschitz continuity for certain (K, K')-quasiconformal self-mappings of the unit disk.

This subsection deals with the discussion on the Lipschitz continuity of (K, K')-quasiconformal solutions of the Poisson equation (1.4) (see below) in $\mathbb{D}$. We start with some necessary concepts. Let P be the Poisson kernel, i.e., the function

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},$$

and let G denote the Green function of $\mathbb{D}$, i.e.

$$G(z, \omega) = \frac{1}{2\pi} \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right|,$$

where $z \in \mathbb{D} \setminus \{\omega\}$. Obviously, P is harmonic in $\mathbb{D}$ (cf. [2]), and G is harmonic in $\mathbb{D} \setminus \{\omega\}$.

Let $f : \mathbb{S} \to \mathbb{C}$ be a bounded integrable function in $\mathbb{S}$, and let $g : \mathbb{D} \to \mathbb{C}$ be continuous. It is known that the solutions of the Poisson equation

$$(1.4) \qquad\qquad\qquad \Delta w = g$$

in $\mathbb{D}$ satisfying the boundary condition $w|_{\mathbb{S}} = f \in L^1(\mathbb{S})$ has the following expression:

$$w = P[f] - G[g],$$

where

$$P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\varphi}) f(e^{i\varphi}) d\varphi,$$

$$G[g](z) = \int_{\mathbb{D}} G(z, \omega) g(\omega) \mathrm{dm}(\omega)$$

and $\mathrm{dm}(\omega)$ denotes the Lebesgue measure in $\mathbb{C}$. Also, it is known that if f and g are continuous in $\mathbb{S}$ and $\overline{\mathbb{D}}$, respectively, then w has a continuous extension $\tilde{w}$ to $\mathbb{S}$ and $\tilde{w}|_{\mathbb{S}} = f$ (cf. [7]).

For convenience, in the following, we always set

$$\mathcal{P} = P[f] \quad \text{and} \quad \mathcal{G} = G[g].$$

Let $\mathcal{D}_{\mathbb{D}}(g)$ denote the family of all solutions w of the Poisson equation (1.4) from $\mathbb{D}$ onto $\mathbb{D}$, which satisfy that (1) each element w is a C^2 diffeomorphism;

(2) each restriction $w|_{\mathbb{S}} = f$ is normalized; (3) every function $f(e^{it}) = e^{i\psi(t)}$ is homeomorphic and absolutely continuous; and (4) $\psi(2\pi) = \psi(0) + 2\pi$.

Observe that any element in $\mathcal{D}_{\mathbb{D}}(g)$ is proper and satisfies the partial differential inequality (1.2) with $M = 0$ and $N = |g|_\infty$.

In [16], Kalaj and Pavlović discussed the Lipschitz continuity of quasiconformal self-mappings of $\mathbb{D}$ satisfying the Poisson equation (1.4). As the main application of Theorem 1.1, we study the Lipschitz continuity of (K, K')-quasiconformal solutions of (1.4). The aim is to generalize the arguments in [16] to the case of (K, K')-quasiconformal solutions of (1.4). The following is our result.

THEOREM 1.2: *Suppose that $w \in \mathcal{D}_{\mathbb{D}}(g)$ is a (K, K')-quasiconformal mapping and that $g \in C(\overline{\mathbb{D}})$.*

(1) *Then for all z_1 and $z_2 \in \mathbb{D}$,*

$$|w(z_1) - w(z_2)| \le M|z_1 - z_2|,$$

where $M = M(K, K', |g|_\infty)$, which means that the constant M depends only on K, K' and $|g|_\infty$.

(2) *If w^{-1} is also (K, K')-quasiconformal, then for all z_1 and $z_2 \in \mathbb{D}$,*

$$|w(z_1) - w(z_2)| \ge N|z_1 - z_2|,$$

where $N = N(K, K', |g|_\infty)$. Here N is positive if K' and $\|g\|_\infty$ are small enough.

By comparing with [16, Theorem 1.2], a natural question is whether the assumption "w^{-1} being (K, K')-quasiconformal" in the second statement in Theorem 1.2 is necessary or not. We will construct an example (Example 4.1 below) to show that there is a solution of the Poisson equation such that it is (K, K')-quasiconformal, its inverse is not (K, K')-quasiconformal for any $K \ge 1$ and $K' \ge 0$, and it is not coLipschitz continuous. This fact shows that the mentioned assumption in Theorem 1.2 is necessary. Section 4 is devoted to the proof of Theorem 1.2 together with the statement and the proof of Example 4.1.

In Section 2, we will construct an example to show that Theorem 1.1 is a substantial generalization of the corresponding results in [10] and [12], respectively.

2. An example

In this section, we will construct an example to show the existence of the (K, K')-quasiregular solutions of the partial differential inequality (1.2), which satisfy the requirements in Theorem 1.1, but fail to satisfy the assumptions in the corresponding results in [10] and [12], respectively.

Example 2.1: Let $w(z) = 2|z|^4 z^2 - |z|^{10} z^2$ in $\mathbb{D}$. Then

   (1) w is a $(1, 144)$-quasiregular mapping of $\mathbb{D}$ onto $\mathbb{D}$;
   (2) w is proper;
   (3) w is not K-quasiregular for any $K \geq 1$;
   (4) w satisfies $|\Delta w| \leq |Dw|^2 + 76$;
   (5) w doesn't satisfy $|\Delta w| \leq M_1 |Dw|^2$ for any $M_1 \geq 0$;
   (6) w doesn't satisfy $|\Delta w| \leq M_2 |w_z w_{\overline{z}}|$ for any $M_2 \geq 0$;
   (7) w is Lipschitz continuous.

Proof. Obviously, $w(e^{i\theta}) = e^{2i\theta}$ for $\theta \in [0, 2\pi]$,

$$w_z = 8|z|^4 z - 7|z|^{10} z \quad \text{and} \quad w_{\overline{z}} = 4|z|^2 z^3 - 5|z|^8 z^3.$$

It follows that

$$|w_z(z)| - |w_{\overline{z}}(z)| = \begin{cases} 2|z|^5(2 - |z|^6) > 0, & \text{if } 0 < |z|^6 < 4/5, \\ 12|z|^5(1 - |z|^6) > 0, & \text{if } 4/5 \leq |z|^6 < 1, \end{cases}$$

and

$$|Dw|^2 - J_w \leq |Dw|^2 = \begin{cases} 144|z|^{10}(1 - |z|^6)^2 < 144, & \text{if } 0 < |z|^6 < 4/5, \\ 4|z|^{10}(2 - |z|^6)^2 < 16, & \text{if } 4/5 \leq |z|^6 < 1. \end{cases}$$

Let $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+\pi)}$ with $0 < r < 1$. Then $z_1 \neq z_2 \in \mathbb{D}$ and $w(z_1) = w(z_2)$. Hence we have proved that w satisfies the first two assertions in the example.

The limits

$$\lim_{|z| \to 1^-} \frac{|w_{\overline{z}}|}{|w_z|} = \lim_{|z| \to 1^-} \frac{5|z|^{11} - 4|z|^5}{8|z|^5 - 7|z|^{11}} = 1$$

and

$$\lim_{|z| \to 1^-} (|w_z(z)| + |w_{\overline{z}}(z)|) = 2$$

tell us that w is not K-quasiregular for any $K \geq 1$, which implies that the third assertion is satisfied.

Obviously,

$$|\Delta w| = \left| 64|z|^4 - 140|z|^{10} \right| \leq 76 \leq |Dw|^2 + 76$$

and

$$\lim_{|z| \to 0} \frac{|\Delta w|}{|Dw|^2} = \lim_{|z| \to 0} \frac{16 - 35|z|^6}{36|z|^6(1 - |z|^6)^2} = +\infty.$$

Hence the assertions (4)–(6) in the example hold.

It remains to show the Lipschitz continuity of w. This easily follows from the estimate $|Dw| < 12$ in $\mathbb{D}$, and so the proof of the example is complete. ∎

3. Lipschitz continuity for certain (K, K')-quasiregular mappings between Jordan domains

The main purpose of this section is to prove Theorem 1.1. Before the proof, some preparation is needed.

3.1. SOME AUXILIARY RESULTS. We start this subsection with a lemma.

LEMMA 3.1: *Suppose that w is a (K, K')-quasiregular mapping from D to Ω, where both D and Ω are Jordan domains in $\mathbb{C}$, and that w has the decomposition $w = \rho S$, where $\rho = |w|$. Then we have*

$$(3.1) \qquad |Dw| \leq K|\nabla \rho| + \sqrt{K'}$$

and

$$(3.2) \qquad \frac{|\nabla \rho| - \sqrt{K'}}{K} \leq \rho|DS| \leq K|\nabla \rho| + \sqrt{K'}$$

a.e. in D.

Proof. First, we prove the inequality (3.1). Since $\rho = |w|$, a similar approach as in the proof of [10, Lemma 2.2] implies that

$$\nabla \rho = \frac{1}{|w|} w^T Dw,$$

where T denotes the transpose of matrices. Here and hereafter, we regard $w = u + iv$ as not only a number in $\mathbb{C}$, but also a 2×1 column vector, where both u and v are real. Then

$$(3.3) \qquad \nabla \rho h = \frac{1}{|w|} \langle Dwh, w \rangle,$$

and so

$$(3.4) \qquad |\nabla\rho| = \max_{|h|=1} |\nabla\rho h| = \max_{|h|=1} \frac{|\langle Dwh, w\rangle|}{|w|} \le |Dw|.$$

It follows from Lemma 1.1 that to prove (3.1), it suffices to show

$$l(Dw) \le |\nabla\rho|.$$

Since $J_w = \det(Dw) \ne 0$ a.e. in D, we know that there exists h_1 such that

$$(3.5) \qquad Dwh_1 = \frac{w}{|w|} \quad \text{a.e. in } D.$$

Then we infer from (3.3) that

$$\nabla\rho h_1 = \frac{\langle \frac{w}{|w|}, w\rangle}{|w|} = 1,$$

and so for $h = \frac{h_1}{|h_1|}$, it easily follows that $\nabla\rho h = \frac{1}{|h_1|}$. Also, by (3.5), we get $h_1 = (Dw)^{-1}\frac{w}{|w|}$ a.e. in D, and thus

$$|\nabla\rho| \ge |\nabla\rho h| = \frac{1}{|h_1|} = \left|(Dw)^{-1}\frac{w}{|w|}\right|^{-1}.$$

Since

$$\left|(Dw)^{-1}\frac{w}{|w|}\right| \le |(Dw)^{-1}| \quad \text{and} \quad |(Dw)^{-1}| = \frac{1}{\min\{|Dwh| : |h| = 1\}},$$

we obtain that

$$(3.6) \qquad |\nabla\rho| \ge |(Dw)^{-1}|^{-1} = \min\{|Dwh| : |h| = 1\} = l(Dw),$$

as required. Hence the inequality (3.1) holds.

Now, we check the truth of the right side of the inequality (3.2). We infer from a similar argument as in the proof of [10, Lemma 2.2] that

$$DS = \frac{Dw}{|w|} - ((Dw)^T w) \otimes \frac{w}{|w|^3},$$

and so for all $h \in \mathbb{C}$,

$$DSh = \frac{Dwh}{|w|} - \frac{w\langle Dwh, w\rangle}{|w|^3},$$

where $\otimes$ is the tensor product between column vectors, i.e., for two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, $\overrightarrow{a} \otimes \overrightarrow{b} = \overrightarrow{b} \cdot (\overrightarrow{a})^T$. Then

$$(3.7) \qquad \rho^2|DSh|^2 = |Dwh|^2 - \left\langle Dwh, \frac{w}{|w|}\right\rangle^2 \le |Dw|^2|h|^2,$$

whence

$$\rho|DS| = \sup_{|h| \neq 0} \left\{ \rho \left| DS \frac{h}{|h|} \right| \right\} \leq |Dw|.$$

Hence we obtain from Lemma 1.1 and (3.6) that

$$\rho|DS| \leq |Dw| \leq Kl(Dw) + \sqrt{K'} \leq K|\nabla\rho| + \sqrt{K'},$$

as needed.

Next, we check the truth of the left side of (3.2). Since, obviously, there is an h_1 such that $\langle Dwh_1, \frac{w}{|w|} \rangle = 0$ with $|h_1| = 1$, we see from (3.7) that

$$\rho^2|DSh_1|^2 = |Dwh_1|^2,$$

which, together with Lemma 1.1, guarantees that

$$\rho|DS| \geq \rho \left| DS \frac{h_1}{|h_1|} \right| = \left| Dw \frac{h_1}{|h_1|} \right| \geq l(Dw) \geq \frac{|Dw| - \sqrt{K'}}{K},$$

and then we get from (3.4) that

$$\rho|DS| \geq \frac{|Dw| - \sqrt{K'}}{K} \geq \frac{|\nabla\rho| - \sqrt{K'}}{K},$$

which is what we want. Therefore, the proof of the lemma is complete. ∎

Remark 3.1: Lemma 3.1 is a generalization of [10, Lemma 2.2].

The proof of Theorem 1.1 also needs the following known results.

LEMMA A ([10, Proposition 2.5] or [4, 23, 28, 29]): *If D and Ω are Jordan domains having $C^{n,\alpha}$ $(n \geq 1)$ boundary and if ω is a conformal mapping of D onto Ω, then*

(a) $|\omega'(z)| \geq \inf\{|\omega'(\zeta)| : \zeta \in D\} > 0$ *for $z \in D$;*
(b) $\omega^{(n)} \in C^\alpha(\overline{D})$. *In particular $|\omega^{(n)}|_\infty = \sup\{|\omega^n(z)| : z \in D\} < \infty$.*

LEMMA B ([5, Theorem 4′] or [13, Proposition 3.2]): *Suppose u is a continuous function from $\overline{\mathbb{D}}$ into the real axis $\mathbb{R}$ and satisfies the following conditions:*

(1) *u is C^2 in $\mathbb{D}$;*
(2) *$u_b(\theta) = u(e^{i\theta})$ is C^2; and*
(3) *$|\Delta u| \leq M_0|\nabla u|^2 + N_0$ in $\mathbb{D}$ for some constants M_0 and N_0.*

Then $|\nabla u|$ is bounded in $\mathbb{D}$.

3.2. THE PROOF OF THEOREM 1.1. We are ready to prove the Lipschitz continuity of w. We divide the proof into two steps. In the first step, we construct a (K_1, K_1')-quasiregular self-mapping τ in $\mathbb{D}$ satisfying the partial differential inequality (1.2) for some constants M_1 and N_1. In the second step, by applying the mapping τ, we show that $|Dw|$ is bounded in D, which completes the proof.

STEP 3.1: *The construction of τ.*

Let φ be a conformal mapping of $\mathbb{D}$ onto D, g a conformal mapping of Ω onto $\mathbb{D}$, and

$$\tau = g \circ w \circ \varphi.$$

In the following, we apply Lemma A to show that this τ satisfies our requirements as mentioned in the first paragraph in this subsection.

First, it follows from

$$(3.8) \qquad |D\tau| = |\tau_z| + |\tau_{\overline{z}}| \quad \text{and} \quad |D\tau| = |g'\varphi'||Dw|$$

that

$$J_\tau = |g'\varphi'|^2 J_w.$$

Meanwhile, by Lemma A, we know that the function $|g'|$ (resp. $|\varphi'|$) is bounded from above and below. Hence we easily know that τ is a $(K, K'|g'|_\infty^2|\varphi'|_\infty^2)$-quasiregular self-mapping in $\mathbb{D}$.

Second, since

$$|4\tau_z\tau_{\overline{z}}| \leq (|\tau_z| + |\tau_{\overline{z}}|)^2 = |D\tau|^2,$$

$$|\Delta w| \leq M|Dw|^2 + N = M\frac{|D\tau|^2}{|g'|^2|\varphi'|^2} + N,$$

and since elementary computations lead to

$$\Delta\tau = (4g''w_zw_{\overline{z}} + g'\Delta w)|\varphi'|^2 = \Big(\frac{4g''\tau_z\tau_{\overline{z}}}{g'^2} + g'|\varphi'|^2\Delta w\Big),$$

we have

$$(3.9) \qquad |\Delta\tau| \leq \Big(\frac{|g''|_\infty}{|g'|_1^2} + \frac{M}{|g'|_1}\Big)|D\tau|^2 + |g'|_\infty|\varphi'|_\infty^2 N,$$

where

$$|g'|_1 = \inf\{|g'(z)| : z \in \Omega\}.$$

The boundedness of $|g''|_\infty$ follows from Lemma A. Hence τ satisfies (1.2), and so this τ is what we need.

STEP 3.2: $|Dw|$ is bounded in D.

It follows from (3.8) that

$$|Dw| = |D\tau||g'\varphi'|^{-1},$$

and then we know from Lemma A that to prove the boundedness of $|Dw|$ in D, it suffices to show the boundedness of $|D\tau|$ in $\mathbb{D}$. Now, we are going to prove the boundedness of $|D\tau|$. Obviously, it follows from the fact $w \in C^2(\mathbb{D})$ that $\tau \in C^2(\mathbb{D})$, and then $|D\tau|$ is bounded in $\overline{\mathbb{D}}_r$ for any $r \in (0,1)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \le r\}$. To prove the boundedness of $|D\tau|$ in $\mathbb{D}$, it is enough to show that $|D\tau|$ is bounded in $\mathbb{D} \setminus \overline{\mathbb{D}}_r$ for some $r \in (0,1)$. For this, we let

$$\tau = \rho S,$$

where $\rho = |\tau|$. Then the inequality (3.1) in Lemma 3.1 makes sure that if $|\nabla \rho|$ is bounded, then so is $|D\tau|$. To prove that $|D\tau|$ is bounded in $\mathbb{D}$, it is sufficient to find an $r \in (0,1)$ such that $|\nabla \rho|$ is bounded in $\mathbb{D} \setminus \overline{\mathbb{D}}_r$.

In the following, we apply Lemma B to show the existence of the needed r. To reach this aim, we need the following existence of a function related to ρ.

CLAIM 3.1: There is a function ρ_2 in $\mathbb{D}$ such that

   (1) ρ_2 satisfies all assumptions in Lemma B; and
   (2) ρ_2 and ρ coincide with each other in $\mathbb{D} \setminus \overline{\mathbb{D}}_r$ for some $r \in (0,1)$.

We will apply Whitney's Theorem [30, Theorem 1] to construct such a function ρ_2. Since w is a proper (K, K')-quasiregular mapping, it follows that τ is a proper self-mapping of $\mathbb{D}$. Thus

$$\lim_{|z| \to 1^-} \rho(z) = 1.$$

Therefore, there exists an $r_1 > 0$ such that $r_1 \le |z| \le 1$ implies

$$\rho(z) \ge 1/2.$$

Let $0 < r_1 < r_2 < 1$ and $r_2 > 1/2$. Since $\rho \in C^2(A)$, where

$$A = \{z : r_1 \le |z| \le r_2\},$$

according to Whitney's Theorem, there exists an extension ρ_1 of the restriction $\rho|_A$ such that $\rho_1 \in C^2(\mathbb{C})$. Let

$$\rho_2 = \begin{cases} \rho, & z \in \mathbb{D} \setminus \overline{\mathbb{D}}_{\frac{r_1+r_2}{2}}, \\ \rho_1, & z \in \overline{\mathbb{D}}_{\frac{r_1+r_2}{2}}. \end{cases}$$

Since

$$\rho_2(e^{i\theta}) = \lim_{r \to 1^-} \rho_2(re^{i\theta}) = \lim_{r \to 1^-} \rho(re^{i\theta}) = 1,$$

ρ_2 obviously satisfies the assumptions (1) and (2) in Lemma B. In order to show that this ρ_2 is what we need, it remains to check that ρ_2 satisfies (1.2), i.e., the third assumption in Lemma B. We will apply Lemma 3.1 to reach this goal.

It follows from [10, Lemma 2.4] that

$$|\Delta\rho| = \left|\frac{1}{2}\langle\Delta\tau, S\rangle + 2\rho|DS|^2\right| \le \frac{1}{2}|\Delta\tau| + 2\rho|DS|^2.$$

Then the inequality (3.2) in Lemma 3.1 along with (3.9) implies

$$\begin{aligned}|\Delta\rho| \le &\frac{1}{2}|\Delta\tau| + \frac{2}{\rho}(K|\nabla\rho| + \sqrt{K'})^2\\ \le &\frac{1}{2}\Big(\frac{|g''|_\infty}{|g'|_1^2} + \frac{M}{|g'|_1}\Big)|D\tau|^2 + \frac{1}{2}|g'|_\infty|\varphi'|_\infty^2 N + \frac{4}{\rho}(K^2|\nabla\rho|^2 + K'),\end{aligned}$$

and further, we get from the inequality (3.1) in Lemma 3.1 that

$$|\Delta\rho| \le N_1(\rho)|\nabla\rho|^2 + M_1(\rho),$$

where

$$N_1(\rho) = \Big(\frac{4}{\rho} + \frac{|g''|_\infty}{|g'|_1^2} + \frac{M}{|g'|_1}\Big)K^2$$

and

$$M_1(\rho) = \Big(\frac{4}{\rho} + \frac{|g''|_\infty}{|g'|_1^2} + \frac{M}{|g'|_1}\Big)K' + \frac{1}{2}|g'|_\infty|\varphi'|_\infty^2 N.$$

Since for all $z \in \mathbb{D} \setminus \overline{\mathbb{D}}_{r_1}$, $\rho(z) \ge \frac{1}{2}$, we see that

$$|\Delta\rho_2| \le N_1\Big(\frac{1}{2}\Big)|\nabla\rho_2|^2 + M_1\Big(\frac{1}{2}\Big)$$

in $\mathbb{D} \setminus \overline{\mathbb{D}}_{r_1}$.

Let

$$M_0 = \max\{|\Delta\rho_2(z)| : z \in \overline{\mathbb{D}}_{\frac{r_1+r_2}{2}}\}.$$

Since $\rho_1 \in C^2(\mathbb{C})$, by the definition of ρ_2, we see that $M_0 < +\infty$, and so for all $z \in \mathbb{D}$,

$$|\Delta\rho_2| \le N_1\Big(\frac{1}{2}\Big)|\nabla\rho_2|^2 + M_1\Big(\frac{1}{2}\Big) + M_0,$$

which shows that ρ_2 satisfies the third assumption in Lemma B.

Since the definition of ρ_2 implies that $\rho_2 = \rho$ in $\mathbb{D} \setminus \overline{\mathbb{D}}_{\frac{r_1+r_2}{2}}$, we see that this ρ_2 justifies our need and the proof of our claim is complete.

It follows from Claim 3.1 and Lemma B that $|\nabla\rho_2|$ is bounded in $\mathbb{D}$. Hence $|\nabla\rho|$ is bounded in $\mathbb{D} \setminus \overline{\mathbb{D}}_{\frac{r_1+r_2}{2}}$, and so this radius $\frac{r_1+r_2}{2}$ is what we want.

We see from the existence of the radius $r = \frac{r_1+r_2}{2}$ that $|\nabla\rho|$ is bounded in $\mathbb{D}$, and so the proof of the theorem is complete.   ∎

4. Lipschitz continuity for certain (K, K')-quasiconformal self-mappings of the unit disk

This section is devoted to the proof of Theorem 1.2 together with the statement and the proof of Example 4.1. We start with a lemma.

4.1. A LEMMA. By Theorem 1.1, the following assertions easily follow from Lemmas 2.7 and 2.8 in [16].

LEMMA C: *Under the assumptions of Theorem 1.2, we have*

   (i) $\lim_{r \to 1^-} D\mathcal{G}(re^{i\theta}) = D\mathcal{G}(e^{i\theta})$;
   (ii) $\lim_{r \to 1^-} D\mathcal{P}(re^{i\theta}) = D\mathcal{P}(e^{i\theta})$ *a.e. in* $[0, 2\pi]$;
   (iii) $\lim_{r \to 1^-} Dw(re^{i\theta}) = Dw(e^{i\theta})$ *a.e. in* $[0, 2\pi]$;
   (iv) $\max_{z \in \mathbb{D}}\{|\mathcal{G}_z(re^{i\theta})|, \ |\mathcal{G}_{\overline{z}}(re^{i\theta})|\} \leq \frac{1}{3}|g|_\infty$;
   (v) $\max_{0 \leq \theta \leq 2\pi}\{|\mathcal{G}_z(e^{i\theta})|, \ |\mathcal{G}_{\overline{z}}(e^{i\theta})|\} \leq \frac{1}{4}|g|_\infty$; *and*
   (vi) $|J_w(e^{i\theta}) - \frac{\psi'(\theta)}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi| \leq \frac{1}{2}\psi'(\theta)|g|_\infty$,

where $w|_{\mathbb{S}} = f$ *and* $f(e^{i\theta}) = e^{i\psi(\theta)}$.

4.2. THE PROOF OF THEOREM 1.2. First, we prove the first statement of the theorem, i.e., the Lipschitz continuity of w in $\mathbb{D}$. Let

$$M_1 = \sup_{z \in \mathbb{D}}|Dw(z)|.$$

Obviously, we only need to show that M_1 has an upper bound. Since $w + \mathcal{G}$ is harmonic, we see from [11, Lemma 2.2] and Lemma C that for all $z \in \mathbb{D}$,

$$|Dw(z)| \leq |Dw(z) + D\mathcal{G}(z)| + |D\mathcal{G}(z)|$$

$$\leq \mathrm{esssup}_{0 \leq \theta \leq 2\pi}|Dw(e^{i\theta}) + D\mathcal{G}(e^{i\theta})| + |D\mathcal{G}(z)|$$

$$\leq \mathrm{esssup}_{0 \leq \theta \leq 2\pi}|Dw(e^{i\theta})| + \frac{7}{6}|g|_\infty,$$

which implies that for every $\varepsilon > 0$, there exists a θ_ε such that

(4.1) $$M_1 \leq (1 + \varepsilon)|Dw(e^{i\theta_\varepsilon})| + \frac{7}{6}|g|_\infty.$$

Obviously, to estimate M_1, it is sufficient to estimate the quantity $|Dw(e^{i\theta_\varepsilon})|$.

Now, we are going to estimate $|Dw(e^{i\theta_\varepsilon})|$. It follows from Lemma C and the assumption w being a (K, K')-quasiconformal mapping that

(4.2)
$$|Dw(e^{i\theta})|^2 = \lim_{r \to 1^-} |Dw(re^{i\theta})|^2 \le K \lim_{r \to 1^-} J_w(re^{i\theta}) + K'$$
$$\le K\psi'(\theta)\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2}d\varphi + \frac{1}{2}|g|_\infty\right) + K'$$

a.e. in $[0, 2\pi]$.

Now, we need a relationship between $\psi'(\theta)$ and $|Dw(e^{i\theta})|$. Since Theorem 1.1 guarantees that $|Dw(z)|$ is bounded by a constant in $\mathbb{D}$, we deduce from the Lebesgue Dominated Convergence Theorem that

$$f(e^{i\theta}) = \lim_{r \to 1^-} w(re^{i\theta}) = \lim_{r \to 1^-} \int_{\theta_0}^{\theta} \frac{\partial}{\partial\varphi} w(re^{i\varphi})d\varphi + f(e^{i\theta_0})$$
$$= \int_{\theta_0}^{\theta} \lim_{r \to 1^-} \left(\frac{\partial}{\partial\varphi} w(re^{i\varphi})\right)d\varphi + f(e^{i\theta_0})$$
$$= \int_{\theta_0}^{\theta} \lim_{r \to 1^-} (rDw(re^{i\varphi})ie^{i\varphi})d\varphi + f(e^{i\theta_0}).$$

Since $f(e^{i\theta}) = e^{i\psi(\theta)}$ is absolutely continuous, by differentiating in θ, we obtain

(4.3)
$$\frac{d}{d\theta}f(e^{i\theta}) = \lim_{r \to 1^-} \frac{\partial}{\partial\theta}w(re^{i\theta}) = \lim_{r \to 1^-}(rDw(re^{i\theta}))ie^{i\theta}$$

and

(4.4)
$$\frac{d}{d\theta}f(e^{i\theta}) = i\psi'(\theta)e^{i\psi(\theta)}$$

a.e. in $[0, 2\pi]$, whence combining Lemma C, we have

(4.5)
$$\psi'(\theta) = \lim_{r \to 1^-} |Dw(re^{i\theta})| = |Dw(e^{i\theta})|,$$

which is our desired relationship between $\psi'(\theta)$ and $|Dw(e^{i\theta})|$.

Using (4.5), the relation (4.2) is changed into the following form:

$$|Dw(e^{i\theta})|^2 \le K|Dw(e^{i\theta})|\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2}d\varphi + \frac{1}{2}|g|_\infty\right) + K',$$

which, necessarily, implies that

$$|Dw(e^{i\theta})| \le \frac{K}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2}d\varphi + \frac{K}{2}|g|_\infty + \sqrt{K'},$$

and thus we easily know from (4.1) that

$$
\begin{aligned}
M_1 \leq (1+\varepsilon) &\Big[\frac{K}{2\pi} \int_0^{2\pi} M_1^{1-\mu} |e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2 + \mu - 2} \frac{|f(e^{i\theta_\varepsilon}) - f(e^{i\varphi})|^{1+\mu}}{|e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2 + \mu}} \\
&\times \frac{|f(e^{i\theta_\varepsilon}) - f(e^{i\varphi})|^{1-\mu}}{M_1^{1-\mu}} d\varphi \Big] \\
&+ (1+\varepsilon) \Big[\frac{K}{2} |g|_\infty + \sqrt{K'} \Big] + \frac{7}{6} |g|_\infty,
\end{aligned}
\tag{4.6}
$$

where

$$
\mu = \frac{1}{K(1+\pi)^2}.
\tag{4.7}
$$

Now, we need an auxiliary result which is (4.8) below: Since for any $z_1 = re^{i\theta}$ and $z_2 = re^{i\eta}$ in $\mathbb{D}$,

$$
|w(z_1) - w(z_2)| \leq \int_{[z_1, z_2]} |Dw(z)||dz| \leq M_1 |z_1 - z_2|,
$$

by letting $r \to 1^-$, we obtain

$$
|f(e^{i\theta}) - f(e^{i\eta})| \leq M_1 |e^{i\theta} - e^{i\eta}|.
\tag{4.8}
$$

Let us continue the proof. Since $\mathbb{S}$ enjoys the $\frac{\pi}{2}$-chord-arc condition, we get from (4.6) along with (4.8) and [14, Lemma 2.4] that

$$
\begin{aligned}
M_1 \leq (1+\varepsilon) &\Big[\frac{1}{2\pi} K P_{\mathbb{S}}^{1+\mu} \int_0^{2\pi} M_1^{1-\mu} |e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2 - 1} d\varphi \Big] \\
&+ (1+\varepsilon) \Big[\frac{1}{2} K |g|_\infty + \sqrt{K'} \Big] + \frac{7}{6} |g|_\infty,
\end{aligned}
$$

where

$$
P_{\mathbb{S}} = 4(1+\pi) 2^\mu \sqrt{\max\Big\{ \frac{2\pi^2 K}{\log 2}, \frac{2\pi K'}{K(1+\pi)^2 + 4} \Big\}}.
\tag{4.9}
$$

Hence

$$
M_1 \leq (1+\varepsilon)[M_2 M_1^{1-\mu}] + (1+\varepsilon) \Big[\frac{1}{2} K |g|_\infty + \sqrt{K'} \Big] + \frac{7}{6} |g|_\infty,
$$

where

$$
M_2 = \frac{1}{2\pi} K P_{\mathbb{S}}^{1+\mu} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2 - 1} d\varphi.
$$

For the convergence of the integral $\int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2-1} d\varphi$, the reader is referred to [9, Lemma 1.6]. Also, we easily know that M_2 does not depend on θ_ϵ. By letting $\varepsilon \to 0$, we get

$$(4.10) \qquad M_1 \leq M_2 M_1^{1-\mu} + \Big(\frac{1}{2}K + \frac{7}{6}\Big)|g|_\infty + \sqrt{K'}.$$

To get an estimate on M_1, we need a lower bound on M_1 which is (4.11) below. Since

$$\int_0^{2\pi} \psi'(\theta)d\theta = \psi(2\pi) - \psi(0) = 2\pi,$$

we know that

$$\mathrm{esssup}_{0 \leq \theta \leq 2\pi} \psi'(\theta) \geq 1.$$

Since

$$\psi'(\theta) = |\psi'(\theta)| = \Big|\frac{\partial f(e^{i\theta})}{\partial e^{i\theta}} \frac{\partial e^{i\theta}}{\partial \theta}\Big| = \Big|\frac{\partial f(e^{i\theta})}{\partial e^{i\theta}}\Big|$$

and

$$\mathrm{esssup}_{0 \leq \theta \leq 2\pi} \lim_{\eta \to \theta} \Big|\frac{f(e^{i\eta}) - f(e^{i\theta})}{e^{i\eta} - e^{i\theta}}\Big| \leq \mathrm{esssup}_{0 \leq \theta \neq \eta < 2\pi} \Big|\frac{f(e^{i\eta}) - f(e^{i\theta})}{e^{i\eta} - e^{i\theta}}\Big|,$$

it follows from the inequality (4.8) that

$$(4.11) \qquad 1 \leq \mathrm{esssup}_{0 \leq \theta \leq 2\pi} \psi'(\theta) \leq \mathrm{esssup}_{0 \leq \theta \neq \eta < 2\pi} \Big|\frac{f(e^{i\eta}) - f(e^{i\theta})}{e^{i\eta} - e^{i\theta}}\Big| \leq M_1.$$

Now, we are able to get an upper bound for M_1. Using (4.11), the relation (4.10) implies

$$M_1 \leq \Big(M_2 + \Big(\frac{1}{2}K + \frac{7}{6}\Big)|g|_\infty + \sqrt{K'}\Big)M_1^{1-\mu},$$

and so

$$M_1 \leq \Big(M_2 + \frac{1}{2}K|g|_\infty + \frac{7}{6}|g|_\infty + \sqrt{K'}\Big)^{K(1+\pi)^2} = C_0.$$

Moreover, by [16, Lemma 2.9] and (4.10), we see that if

$$(1-\mu)M_2 = \frac{1}{2\pi}\Big(1 - \frac{1}{K(1+\pi)^2}\Big)KP_\mathbb{S}^{1+\mu} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2-1} d\varphi < 1,$$

then

$$M_1 \leq \frac{M_2 + \frac{1}{2}K|g|_\infty + \frac{7}{6}|g|_\infty + \sqrt{K'} - (1-\mu)M_2}{1 - (1-\mu)M_2} = C_1.$$

Let

$$M = \begin{cases} C_0, & \text{if } (1-\mu)M_2 \geq 1, \\ \min\{C_0, C_1\}, & \text{if } (1-\mu)M_2 < 1. \end{cases}$$

Then we see that

$$M_1 \leq M,$$

and so the proof of the first statement of the theorem is complete.

Next, we are going to prove the second statement of the theorem, i.e., the coLipschitz continuity of w under the assumption that w^{-1} is also (K, K')-quasiconformal. It follows from Lemma C that

$$(4.12) \qquad l(Dw) \geq l(D\mathcal{P}) - |D\mathcal{G}| \geq l(D\mathcal{P}) - \frac{2}{3}|g|_\infty$$

a.e. in $\mathbb{D}$.

Obviously, to prove the coLipschitz continuity of w, it is sufficient to find the lower bound of $l(Dw)$ in $\mathbb{D}$, and (4.12) implies that it is enough to find the lower bound of $l(D\mathcal{P})$. For this purpose, we need the following claim.

CLAIM 4.1: *For a.e. $\theta \in [0, 2\pi]$, we have*

$$K\psi'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{1}{2}|g|_\infty - \sqrt{K'} \geq N_1,$$

where

$$N_1 = \frac{1}{2\pi} P_{\mathbb{S}}^{-\frac{2}{\mu}} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{\frac{2}{\mu}-2} d\varphi - \frac{1}{2}|g|_\infty - \sqrt{K'}.$$

First, we prove the inequality

$$(4.13) \qquad K\psi'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{1}{2}|g|_\infty - \sqrt{K'}$$

a.e. in $[0, 2\pi]$.

It follows from (4.3) and (4.4) that

$$\psi'(\theta) = \left| \frac{df(e^{i\theta})}{d\theta} \right| = \left| \lim_{r \to 1^-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right|.$$

Since Lemma C guarantees that

$$\psi'(\theta) = \left| \lim_{r \to 1^-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right| \geq |w_z(e^{i\theta})| - |w_{\overline{z}}(e^{i\theta})| = l(Dw(e^{i\theta}))$$

and

$$\frac{J_w(e^{i\theta})}{\psi'(\theta)} \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{1}{2}|g|_\infty$$

a.e. in $[0, 2\pi]$, we have

$$
\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{1}{2}|g|_\infty \leq \frac{J_w(e^{i\theta})}{\psi'(\theta)} \leq |Dw(e^{i\theta})|
$$
$$
\leq Kl(Dw(e^{i\theta})) + \sqrt{K'}
$$
$$
\leq K\psi'(\theta) + \sqrt{K'}
$$

a.e. in $[0, 2\pi]$, as required.

Next, we get an estimate on the integral in Claim 4.1, which is as follows:

$$
(4.14) \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi \geq \frac{1}{2\pi} P_{\mathbb{S}}^{-\frac{2}{\mu}} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{\frac{2}{\mu}-2} d\varphi.
$$

By [14, Lemma 2.4] and the assumptions that "w^{-1} is (K, K')-quasiconformal and w is normalized", we have that for all z_1 and $z_2 \in \mathbb{S}$,

$$
|z_1 - z_2| \leq P_{\mathbb{S}} |w(z_1) - w(z_2)|^\mu,
$$

i.e.,

$$
|w(z_1) - w(z_2)| \geq P_{\mathbb{S}}^{-\frac{1}{\mu}} |z_1 - z_2|^{\frac{1}{\mu}},
$$

where $P_{\mathbb{S}}$ and μ are the same as in (4.7) and (4.9). Then we have

$$
\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi \geq N_2,
$$

where

$$
N_2 = \frac{1}{2\pi} P_{\mathbb{S}}^{-\frac{2}{\mu}} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{\frac{2}{\mu}-2} d\varphi \leq P_{\mathbb{S}}^{-\frac{2}{\mu}} 2^{\frac{2}{\mu}-2}.
$$

Here we remark that by using the substitution in the integral, we easily see that N_2 is independent of θ, i.e., $N_2 = N_2(K, K')$.

Obviously, the proof of Claim 4.1 follows from (4.13) and (4.14).

Now, we are ready to finish the proof of our theorem by applying Claim 4.1. It follows from Claim 4.1, together with the inequalities (4.5), Lemmas 1.1 and C, that

$$
N_1 \leq K\psi'(\theta) = K|Dw(e^{i\theta})| \leq K^2 l(Dw(e^{i\theta})) + K\sqrt{K'}
$$

a.e. in $[0, 2\pi]$, whence again Lemma C implies

$$
\begin{aligned}
l(D\mathcal{P}(e^{i\theta})) = \lim_{r \to 1^-} l(D\mathcal{P}(re^{i\theta})) &= \lim_{r \to 1^-} (|\mathcal{P}_z(re^{i\theta})| - |\mathcal{P}_{\overline{z}}(re^{i\theta})|) \\
&\geq \lim_{r \to 1^-} (|w_z(re^{i\theta}) + \mathcal{G}_z(re^{i\theta})| - |w_{\overline{z}}(re^{i\theta}) + \mathcal{G}_{\overline{z}}(re^{i\theta})|) \\
&\geq \lim_{r \to 1^-} (l(Dw(re^{i\theta})) - |D\mathcal{G}(re^{i\theta})|) \\
&\geq \frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty.
\end{aligned}
$$

(4.15)

Let

$$
N = \frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{7}{6}|g|_\infty.
$$

Then $N = N(K, K', |g|_\infty)$ since $N_1 = N_1(K, K', |g|_\infty)$, and next, we are going to show the following.

CLAIM 4.2: *The inequality $l(Dw) \geq N$ holds in $\mathbb{D}$.*

Without loss of generality, we assume that $N > 0$. Obviously,

$$
\frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty > 0.
$$

Under this assumption, we need to get a lower bound for $l(D\mathcal{P})$ in $\mathbb{D}$ (see (4.16) below). We will employ the famous Heinz Theorem [6] to reach this aim.

Since $f(e^{i\theta}) = e^{i\psi(\theta)}$ is an increasing homeomorphism on $\mathbb{S}$, we see from the Choquet–Radó–Kneser Theorem (cf. [2, p. 29]) that $\mathcal{P}$ is a sense-preserving harmonic diffeomorphism. Then the Heinz Theorem implies

$$
2|\mathcal{P}_z|^2 \geq |\mathcal{P}_z|^2 + |\mathcal{P}_{\overline{z}}|^2 \geq \frac{1}{\pi^2}.
$$

Let

$$
\varphi(z) = \frac{\overline{\mathcal{P}_{\overline{z}}(z)}}{\mathcal{P}_z(z)} \quad \text{and} \quad \phi(z) = \frac{1}{\mathcal{P}_z(z)} \Big(\frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty \Big).
$$

Then both φ and ϕ are holomorphic, $|\varphi(z)| < 1$, and further

$$
|\phi| \leq \sqrt{2}\pi \Big(\frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty \Big)
$$

in $\mathbb{D}$. Since (4.15) leads to

$$
|\varphi(e^{i\theta})| + |\phi(e^{i\theta})| = \frac{|\mathcal{P}_{\overline{z}}(e^{i\theta})| + (\frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty)}{|\mathcal{P}_z(e^{i\theta})|} \leq 1,
$$

we see that

$$
|\varphi(z)| + |\phi(z)| \leq P[|\varphi|_{\mathbb{S}}](z) + P[|\phi|_{\mathbb{S}}](z) \leq 1,
$$

which implies

$$|\mathcal{P}_{\overline{z}}(z)| + \frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty \leq |\mathcal{P}_z(z)|$$

in $\mathbb{D}$, i.e.,

(4.16)
$$l(D\mathcal{P}) \geq \frac{N_1}{K^2} - \frac{\sqrt{K'}}{K} - \frac{1}{2}|g|_\infty$$

in $\mathbb{D}$, as required.

Now, it follows from (4.12) and (4.16) that

$$l(Dw) \geq N,$$

and hence the claim is proved.

Since the second statement in Theorem 1.2 (2) easily follows from Claim 4.2, we see that the proof of the theorem is complete. ∎

4.3. AN EXAMPLE. The following example shows that the assumption "w^{-1} being (K, K')-quasiconformal" in the second assertion in Theorem 1.2 is necessary.

Example 4.1: Let $w(z) = \frac{1}{2n}((2n+1)z - z|z|^{2n})$ in $\mathbb{D}$, where $n \geq 1$. Then

    (1) w is a (K, K')-quasiconformal mapping of $\mathbb{D}$ onto $\mathbb{D}$;
    (2) w is not K-quasiconformal for any $K \geq 1$;
    (3) w^{-1} is not a (K, K')-quasiconformal mapping for any $K \geq 1$ and $K' \geq 0$;
    (4) w is Lipschitz continuous but not coLipschitz.

Proof. Elementary calculations show that

$$w_z(z) = \frac{1}{2n}((2n+1) - (n+1)|z|^{2n}) \quad \text{and} \quad w_{\overline{z}}(z) = -\frac{1}{2}|z|^{2n-2}z^2.$$

It follows from

$$J_w(z) = \frac{1}{4n^2}(|w_z(z)|^2 - |w_{\overline{z}}(z)|^2) = \frac{1}{4n^2}(2n+1)(1-|z|^{2n})(2n+1-|z|^{2n}) > 0,$$

together with the fact $w(e^{i\theta}) = e^{i\theta}$ and the degree principle, that w is a sense-preserving homeomorphism from $\mathbb{D}$ onto $\mathbb{D}$. Furthermore,

$$l(Dw)(z) = \frac{1}{2n}((2n+1) - (2n+1)|z|^{2n}) \quad \text{and} \quad |Dw(z)| = \frac{1}{2n}((2n+1) - |z|^{2n}).$$

Then w is a $(1, (1+\frac{1}{2n})^2)$-quasiconformal mapping since

$$|Dw|^2 \leq J_w + |Dw|^2 \leq J_w + \left(1 + \frac{1}{2n}\right)^2.$$

The limit

$$\lim_{|z|\to 1^-} \frac{|Dw|^2}{J_w} = +\infty$$

tells us that w is not a K-quasiconformal mapping for any $K \geq 1$. The Lipschitz continuity of w easily follows from the estimate $|Dw| \leq 1 + \frac{1}{2n}$.

It is well known that for the nonsingular matrix Dw, we have

$$|Dw^{-1}| = 1/l(Dw) \quad \text{and} \quad l(Dw^{-1}) = 1/|Dw| \quad (\text{cf. [11]}).$$

Hence

$$|Dw^{-1}(z)| = \frac{2n}{(2n+1) - (2n+1)|z|^{2n}} \quad \text{and} \quad l(Dw^{-1}(z)) = \frac{2n}{(2n+1) - |z|^{2n}}.$$

Then for any $K \geq 1$,

$$\lim_{|z|\to 1^-} (|Dw^{-1}(z)|^2 - KJ_{w^{-1}}(z)) = \lim_{|z|\to 1^-} |Dw^{-1}|(|Dw^{-1}| - Kl(\mathfrak{D}w^{-1})) = +\infty.$$

This shows that w^{-1} is not a (K, K')-quasiconformal mapping.

Let $\partial_\alpha w(z)$ denote the directional derivative of w. Note that if w is Lipschitz continuous with Lipschitz constant C, then

$$|\partial_\alpha w(z)| = \left| \lim_{r\to 0} \frac{w(z + re^{i\alpha}) - w(z)}{r} \right| = \lim_{r\to 0} \frac{|w(z + re^{i\alpha}) - w(z)|}{r} \leq C.$$

Hence it follows from the obvious fact $|Dw(z)| = \max_\alpha |\partial_\alpha w(z)|$ that

$$|Dw| \leq C,$$

namely, "w being Lipschitz continuous" is equivalent to "$|Dw|$ being bounded". Since

$$\lim_{|z|\to 1^-} |D(w^{-1})(z)| = +\infty,$$

we see that w^{-1} is not Lipschitz continuous and so w is not coLipschitz continuous. The proof is finished. ∎

References

[1] M. Chen and X. Chen, (K, K')-quasiconformal harmonic mappings of the upper half plane onto itself, Annales Academiæ Scientiarum Fennicæ. Mathematica **37** (2012), 265–276.

[2] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, Vol. 156, Cambridge University Press, Cambridge, 2004.

[3] R. Finn and J. Serrin, *On the Hölder continuity of quasiconformal and elliptic mappings*, Transactions of the American Mathematical Society **89** (1958), 1–15.

[4] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, RI, 1969

[5] E. Heinz, *On certain nonlinear elliptic differential equations and univalent mappings*, Journal d'Analyse Mathématique **5** (1956/57), 197–272.

[6] E. Heinz, *On one-to-one harmonic mappings*, Pacific Journal of Mathematics **9** (1959), 101–105.

[7] L. Hörmander, *Notions of Convexity*, Progress in Mathematics, Vol. 127, Birkhäuser Boston, Boston, MA, 1994.

[8] D. Kalaj, *Quasiconformal and harmonic mappings between Jordan domains*, Matheamtische Zeitschrift **260** (2008), 237–252.

[9] D. Kalaj, *On harmonic quasiconformal self-mappings of the unit ball*, Annales Academiæ Scietiarum Fennicæ. Mathematica **33** (2008), 261–271.

[10] D. Kalaj, *On quasiregular mappings between smooth Jordan domains*, Journal of Mathematical Analysis and Applications **362** (2010), 58–63.

[11] D. Kalaj, *On the quasiconformal self-mappings of the unit ball satisfying the Poisson differential equation*, Annales Academiæ Scientiarum Fennicæ. Mathematica **36** (2011), 177–194.

[12] D. Kalaj and M. Mateljević, *Inner estimate and quasiconformal harmonic maps between smooth domains*, Journal d'Analyse Mathématique **100** (2006), 117–132.

[13] D. Kalaj and M. Mateljević, *On certain nonlinear elliptic PDE and quasiconfomal maps between Euclidean surfaces*, Potential Analysis **34** (2011), 13–22.

[14] D. Kalaj and M. Mateljević, (K, K')-quasiconformal harmonic mappings, Potential Analysis **36** (2012), 117–135.

[15] D. Kalaj and M. Pavlović, *Boundary correspondence under harmonic quasiconformal diffeomorphisms of a half-plane*, Annales Academiæ Scientiarum Fennicæ. Mathematica **30** (2005), 159–165.

[16] D. Kalaj and M. Pavlović, *On quasiconformal self-mappings of the unit disk satisfying Poisson equation*, Transactions of the American Mathematical Society **363** (2011), 4043–4061.

[17] M. Knežević and M. Mateljević, *On the quasi-isometries of harmonic quasiconformal mappings*, Journal of Mathematical Analysis and Applications **334** (2007), 404–413.

[18] P. Li, J. Chen and X. Wang, *Quasiconformal solutions of Poisson equations*, Bulletin of the Australian Mathematical Society **92** (2015), 420–428.

[19] O. Martio, *On harmonic quasiconformal mappings*, Annales Academiæ Scientiarum Fennicæ. Mathematica **425** (1968), 3–10.

[20] M. Mateljević and M. Vuorinen, *On harmonic quasiconformal quasi-isometries*, Journal of Inequalities and Applications (2010), Art. ID 178732, 19.

[21] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Communications on Pure and Applied Mathematics **6** (1953), 103–156.

[22] M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk*, Annales Academiæ Scientiarum Fennicæ. Mathematica **27** (2002), 365–372.

[23] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der Mathematischen Wissenschaften, Vol. 299, Springer-Verlag, Berlin, 1992.

[24] L. F. Tam and T. Y. H. Wan, *Harmonic diffeomorphisms into Cartan-Hadamard surfaces with prescribed Hopf differentials*, Communications in Analysis and Geometry **2** (1994), 593–625.

[25] L. F. Tam and T. Y. H. Wan, *Quasi-conformal harmonic diffeomorphism and the universal Teichmüller space*, Journal of Differential Geometry **42** (1995), 368–410.

[26] L. F. Tam and T. Y. H. Wan, *On harmonic quasiconformal maps*, Pacific Journal of Mathematics **182** (1998), 359–383.

[27] T. Y. H. Wan, *Constant mean curvature surface, harmonic maps and universal Teichmüller space*, Journal of Differential Geometry **35** (1992), 643–657.

[28] S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Transactions of the American Mathematical Society **38** (1935), 310–340.

[29] S. E. Warschawski, *On differentiability at the boundary in conformal mapping*, Proceedings of the American Mathematical Society **12** (1961), 614–620.

[30] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society **36** (1934), 63–89.