

RIGIDITY OF INFINITESIMAL MOMENTUM MAPS

BY

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ABSTRACT

In this paper we prove rigidity theorems for Poisson Lie group actions on Poisson manifolds. In particular, we prove that close infinitesimal momentum maps associated to Poisson Lie group actions are equivalent using a normal form theorem for SCI spaces. When the Poisson structure of the acted manifold is integrable, this yields rigidity also for lifted actions to the symplectic groupoid.

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1. Introduction

In 1961 Palais proved that close actions of compact Lie groups on compact manifolds can be conjugated by a diffeomorphism [29]. The interest of this rigidity theorem relies in the approximation of actions by nearby ones. As an application of this rigidity theorem of Palais we can recover normal form theorems such as Bochner’s linearization theorem [5]. Several generalizations of this result have been obtained in [15] and [25] for symplectic actions on symplectic manifolds and in [26] for Hamiltonian actions of semisimple Lie algebras on Poisson manifolds.

In this paper we generalize a rigidity result from [26] to the context of Poisson Lie groups and pre-Hamiltonian actions. This addresses a question considered by Ginzburg in [14].

Poisson Lie group actions on Poisson manifolds with non-trivial Poisson structures appear naturally in the study of R matrices. As explained in [21] this generalization of Hamiltonian actions is useful to take into account the properties of the dressing transformations under the hidden symmetry group in the case of R -matrices. For these, the notion of momentum mapping for Poisson manifolds coincides with the monodromy matrix of the associated linear system. Thus, rigidity for Poisson Lie group actions can be useful to understand the stability of the integrable systems associated to R -matrices. On the other hand, the generalization to Poisson Lie group actions represents a first step towards quantization of symmetries, as studied in [12]. The first result of this paper is the rigidity of Hamiltonian actions, i.e., Poisson Lie actions generated by a G -equivariant momentum map $J : M \rightarrow G^*$.

THEOREM 1.1: *Let G be a compact semisimple Poisson Lie group G acting on a compact Poisson manifold (M, π) in a Hamiltonian fashion with momentum map $J_0 : M \rightarrow G^*$.*

There exist a positive integer l and two positive real numbers α and β (with $\beta < 1 < \alpha$) such that, if J_1 is another momentum map on M with respect to the same Poisson structure and Poisson Lie group, satisfying

$$(1) \quad \|J_0 - J_1\|_{2l-1} \leq \alpha \quad \text{and} \quad \|J_0 - J_1\|_l \leq \beta,$$

then there exists a Poisson diffeomorphism ψ of class \mathcal{C}^k , for all $k \geq l$, on M such that

$$J_1 \circ \psi = J_0.$$

In other words, close Hamiltonian actions are equivalent. The proof uses a global linearization theorem due to Ginzburg and Weinstein [15] and the rigidity result for Hamiltonian actions on Poisson manifolds obtained in [26]. As a consequence of this rigidity theorem for momentum maps in the Poisson Lie group setting, we obtain rigidity for lifted actions to the symplectic groupoid when the Poisson structure is integrable.

As pointed out in [14], in many cases it is enough to consider the infinitesimal version of the momentum map. Poisson actions generated by an infinitesimal momentum map are called pre-Hamiltonian actions. The advantage of considering the infinitesimal momentum map relies on the fact that existence and uniqueness are much simpler to prove and we have a big class of examples given by semisimple Lie algebras [14]. Infinitesimal momentum maps are the local counterpart to momentum maps and topology on the acted manifold is an obstruction to its integration to global momentum maps. This is also the case when the Poisson structure on G is not trivial but there are additional obstructions as shown in [13].

The second result of this paper is the rigidity of the infinitesimal momentum map.

THEOREM 1.2: *Let us consider a pre-Hamiltonian action of a semisimple compact Poisson Lie group (G, π_G) on a compact Poisson manifold (M, π) with infinitesimal momentum map α .*

There exist a positive integer l and two positive real numbers a and b (with $b < 1 < a$) such that, if $\tilde{\alpha}$ is another infinitesimal momentum map on M with

respect to the same Poisson structure, satisfying

$$(2) \quad \|\alpha - \tilde{\alpha}\|_{2l-1} \leq a \quad \text{and} \quad \|\alpha - \tilde{\alpha}\|_l \leq b,$$

then there exists a map $\Phi : \Omega^1(M) \rightarrow \Omega^1(M)$ of class \mathcal{C}^k , for all $k \geq l$, preserving the Lie algebra structure of $\Omega^1(M)$ and the differential d , such that $\Phi(\alpha_X) = \tilde{\alpha}_X$.

The proof uses techniques native to geometrical analysis and an abstract normal form theorem from [26]. This abstract normal form encapsulates a Newton's iterative method used by Moser and Nash to prove the inverse function theorem in infinite dimensions (see for example [20]). Newton's method is used to prove normal form results by approximating the solution by means of an iterative process. The solution is then presented as a limit. The abstract normal form for SCI spaces (where SCI stands for **scaled of \mathcal{C}^∞ -type**) presented in [26] allows to prove normal forms results (and, in particular, linearization and rigidity theorems) without having to plunge into the details of the iterative method. In this paper we provide a new application of this normal form for SCI spaces. The abstract normal form theorem in [26] has had other applications in the theory of generalized complex manifolds (see [2] and [3]) and a variant of it to normal forms in a neighbourhood of a symplectic leaf of a Poisson manifold [28]. In this paper we provide a new application of this normal form for SCI spaces. As in [26] we first prove an infinitesimal rigidity result and then we apply the SCI normal form theorem to conclude rigidity. Our theorem can be seen as another reincarnation of Mather's principle "*infinitesimal stability implies stability*" (see [24] and its sequel).

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2. Preliminaries: Poisson Lie groups and pre-Hamiltonian actions

In this section we introduce a generalization of the notion of Hamiltonian actions in the setting of Poisson Lie groups acting on Poisson manifolds. Let us recall that a Poisson Lie group is a pair (G, π_G) , where G is a Lie group and π_G is a multiplicative Poisson structure. The Lie algebra \mathfrak{g} corresponding to the Lie

group G is equipped with the 1-cocycle,

$$(3) \quad \delta = d_e \pi_G : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g},$$

which defines a Lie algebra structure on the dual vector space \mathfrak{g}^* . For this reason, the pair (\mathfrak{g}, δ) is said to be a Lie bialgebra. If G is connected and simply connected there is a one-to-one correspondence between the Poisson Lie group (G, π_G) and the Lie bialgebra (\mathfrak{g}, δ) , as proven in [10] (for this reason we assume this hypothesis to hold throughout this paper). The dual Poisson Lie group G^* is defined to be the Lie group associated to the Lie algebra \mathfrak{g}^* . Given a Poisson Lie group (G, π_G) and a Poisson manifold (M, π) , we introduce the following

Definition 2.1: The action of (G, π_G) on (M, π) is called **Poisson action** if the map $\Phi : G \times M \rightarrow M$ is Poisson, that is

$$(4) \quad \{f \circ \Phi, g \circ \Phi\}_{G \times M} = \{f, g\}_M \circ \Phi \quad \forall f, g \in \mathcal{C}^\infty(M)$$

where the Poisson structure on $G \times M$ is given by $\pi_G \oplus \pi$.

Observe that if G carries the zero Poisson structure $\pi_G = 0$, the action is Poisson if and only if it preserves π . In general, when $\pi_G \neq 0$, the structure π is not invariant with respect to the action.

Let $\Phi : G \times M \rightarrow M$ be a Poisson action and \widehat{X} the fundamental vector field associated to any element $X \in \mathfrak{g}$. For each $X \in \mathfrak{g}$ we can also define the left invariant 1-form θ_X on G^* with value X at e .

Definition 2.2 (Lu, [22], [23]): A **momentum map** for the Poisson action $\Phi : G \times M \rightarrow M$ is a map $J : M \rightarrow G^*$ such that

$$(5) \quad \widehat{X} = \pi^\#(J^*(\theta_X)).$$

Let us denote by λ_g the (left) dressing action of G on its dual G^* .

Definition 2.3: A momentum map for the Poisson action Φ is said to be **G -equivariant** if for every $g \in G$ we have $J \circ \Phi_g = \lambda_g \circ J$.

We recall that, as proved in [22, Theorem 3.18], a momentum map is G -equivariant if and only if it is a Poisson map given that G is connected, i.e.,

$$(6) \quad J_* \pi = \pi_{G^*}.$$

Finally, we can say that a **Hamiltonian action** in this context is a Poisson action given by an equivariant momentum map. This definition generalizes Hamiltonian actions in the canonical setting of Lie groups acting on Poisson manifolds. Indeed, we notice that, if the Poisson structure on G is trivial, the dual G^* corresponds to the dual of the Lie algebra \mathfrak{g}^* , the one-form $J^*(\theta_X)$ is exact and the infinitesimal generator \widehat{X} is a Hamiltonian vector field.

The notion of Hamiltonian action can be further generalized by using a weaker definition of momentum map, first introduced by Ginzburg in [14] and further developed in [12]. The basic idea is to consider the infinitesimal version of an equivariant momentum map generating the fundamental vector fields of a Poisson action. For this purpose, it is useful to recall that given a Poisson structure π , the anchor map π^\sharp defined as $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ defines a skew-symmetric operation $[\cdot, \cdot]_\pi : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$. This operation is given by the general formula

$$(7) \quad [\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

Furthermore, it provides $\Omega^1(M)$ with a Lie algebra structure such that $\pi^\sharp : T^*M \rightarrow TM$ is a Lie algebra homomorphism (e.g., see Theorem 4.1 in [30]).

Definition 2.4: Let (M, π) be a Poisson manifold and (G, π_G) a Poisson Lie group.

- (i) An **infinitesimal momentum map** is a map $\alpha : \mathfrak{g} \rightarrow \Omega^1(M) : X \mapsto \alpha_X$ such that it generates the action by

$$(8) \quad \widehat{X} = \pi^\sharp(\alpha_X)$$

and satisfies the conditions to be a Lie algebra homomorphism

$$(9) \quad \alpha_{[X, Y]} = [\alpha_X, \alpha_Y]_\pi$$

and a cochain map

$$(10) \quad d\alpha_X = \alpha \wedge \alpha \circ \delta(X).$$

- (ii) A **pre-Hamiltonian action** is an action of (G, π_G) on (M, π) defining an infinitesimal momentum map $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$.

A similar definition can be found in [14], where the author defines the **cotangential action** as an infinitesimal action $\mathfrak{g} \rightarrow \Gamma(TM) : X \mapsto \widehat{X}$ that can be lifted to $\Omega^1(M)$. In other words, there exists a linear map $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$ such

that $\widehat{X} = \pi^\sharp(\alpha_X)$, called **cotangent lift** (or **pre-momentum map**). This definition differs from the definition above since the equivariance (9) and the condition (10) are not required. Moreover, the author in [14] does not require the action to be Poisson. From our definition it follows immediately that the infinitesimal momentum map defines a Poisson map, as proved in [8]. Thus, in the light of Theorem 3.18 in [22], condition (9) is the analogue to equivariance.

Clearly this notion is weaker than the Hamiltonian notion, as it does not reduce to the canonical one when the Poisson structure on G is trivial. In fact, if $\pi_G = 0$ we have $\delta = 0$ and Eq. (10) implies that α_X is a closed form, but in general this form is not exact. If M is not simply connected we can get examples in the symplectic realm like rotations on a torus or more sophisticated ones for general Poisson structures.

Example 2.1: Consider the torus \mathbb{T}^2 , with Poisson structure $\pi = \sin \theta_1 \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}$ where the coordinates on the torus are $\theta_1, \theta_2 \in [0, 2\pi]$. This Poisson structure is symplectic away from the set $Z = \{\theta_1 \in \{0, \pi\}\}$ and the Poisson structure satisfies a transversality condition at the vanishing set. This Poisson structure pertains to a class called b -Poisson structures (or b -symplectic structures) studied in [18]. The circle action of rotation on the θ_2 coordinate defines a pre-Hamiltonian action on \mathbb{T}^2 . Indeed it is possible to associate a b -symplectic form to this Poisson structure (see [18]) and work with b -symplectic actions, in this case $\frac{1}{\sin \theta_1} d\theta_1 \wedge d\theta_2$. The circle action of rotation on the θ_2 coordinate is pre-Hamiltonian and the associated one-form is $\frac{1}{\sin \theta_1} d\theta_1$ (see [19] for properties of these actions on b -Poisson manifolds).

Furthermore, it is clear that any Hamiltonian action is pre-Hamiltonian because the equivariant momentum map J induces the infinitesimal one α by $\alpha_X = J^*(\theta_X)$. But not every infinitesimal momentum map arises from a momentum map J . Here we recall an example of pre-Hamiltonian action which is not Hamiltonian (see Remark 3.3 in [14]).

The study of the conditions in which an infinitesimal momentum map determines a momentum map can be found in [13]. The authors here proved that if M and G are simply connected and G is compact, then $\mathcal{D} = \{\alpha_\xi - \theta_\xi, \xi \in \mathfrak{g}\}$ generates an involutive distribution on $M \times G^*$ and a leaf $\mu_{\mathcal{F}}$ of \mathcal{D} is a graph of a momentum map if

$$(11) \quad \pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta)|_{\mathcal{F}} = 0, \quad \xi, \eta \in \mathfrak{g}.$$

In other words, if M and G are simply connected and G is compact the authors in [13] found the obstructions for a pre-Hamiltonian action to be Hamiltonian. The advantage of working with infinitesimal momentum map is that the study of its existence and uniqueness is much simpler than for the G -equivariant momentum map of Definition 2.2. In particular, it has been proven in [14] that any action of a compact group with $H^2(\mathfrak{g}) = 0$ admits an infinitesimal momentum map. Since assuming \mathfrak{g} semisimple implies automatically $H^2(\mathfrak{g}) = 0$, we can conclude that whenever \mathfrak{g} is semisimple the action admits an infinitesimal momentum map.

3. Rigidity of Hamiltonian actions

As introduced in Section 1, by **rigidity** of the actions we mean that close actions are equivalent, i.e., they are conjugated by a diffeomorphism. In what follows, we recall the rigidity theorem in the context of Hamiltonian actions and, as a consequence, we prove the rigidity of Hamiltonian actions.

3.1. THE CASE OF ZERO POISSON STRUCTURE ON THE POISSON LIE GROUP G . Let us consider a Lie group G acting on a Poisson manifold M and assume that the action is Hamiltonian, i.e., there exists a G -equivariant momentum map $\mu : M \rightarrow \mathfrak{g}^*$. The rigidity of Hamiltonian actions was proved in [26] via a rigidity theorem for the momentum map. More precisely,

THEOREM 3.1 (Miranda, Monnier, Zung [26]): *Consider a compact Poisson manifold (M, π) and a Hamiltonian action on M given by the momentum map $\lambda : M \rightarrow \mathfrak{g}^*$ where \mathfrak{g} is a semisimple Lie algebra of compact type.*

There exist a positive integer l and two positive real numbers α and β (with $\beta < 1 < \alpha$) such that, if μ is another momentum map on M with respect to the same Poisson structure and Lie algebra, satisfying

$$(12) \quad \|\lambda - \mu\|_{2l-1} \leq \alpha \quad \text{and} \quad \|\lambda - \mu\|_l \leq \beta,$$

then there exists a diffeomorphism ψ of class \mathcal{C}^k , for all $k \geq l$, on M such that $\mu \circ \psi = \lambda$.

Remark 3.1: This theorem holds in the \mathcal{C}^p -category assuming λ is of class \mathcal{C}^{2p-1} and the statement also holds in the \mathcal{C}^∞ -category. From the construction in [26] the diffeomorphism ψ can be chosen to depend continuously on μ in the terms of [17](Appendix B) as it happens in Palais' construction [29].

The main idea to prove this theorem is to use Newton’s iterative method applied to momentum maps. As explained in [17], a first approach to prove the equivalence of Lie group actions on manifolds would be the following steps. In general, a Lie group action gives an element in $\mathcal{M} = \text{Hom}(G, \text{Diff}(M))$ and we can consider the additional action

$$\begin{aligned} \beta : \text{Diff}(M) \times \mathcal{M} &\longmapsto \mathcal{M} \\ (\phi, \alpha) &\longmapsto \phi \circ \alpha \circ \phi^{-1}. \end{aligned}$$

Two actions α_0 and α_1 are conjugate if they are on the same orbit under β so, in particular, if β has open orbits the action is rigid. Observe that:

- The tangent space to the orbit of β coincides with 1-coboundaries of the group cohomology with coefficients in $V = \text{Vect}(M)$ and the tangent space to \mathcal{M} are the 1-cocycles.
- The generalized Whitehead lemma implies that for compact G the cohomology group $H^1(G; \text{Vect}(M))$ vanishes. This phenomenon is known as infinitesimal rigidity. In this case the tangent space to the orbit equals the tangent space to \mathcal{M} .
- If \mathcal{M} is a manifold (or tame Fréchet) we can apply the inverse function theorem of Nash–Moser to go from the tangent space to the manifold. We can use this fact to prove that β has open manifolds and thus the action is rigid.

In general, it is hard to verify the “tame Fréchet” condition but we can apply the method used in the proof of Nash–Moser’s theorem (Newton’s iterative method). This method allows to prove several results of type *infinitesimal rigidity implies rigidity*. For Hamiltonian actions on Poisson manifold the authors [26] consider the Chevalley–Eilenberg complex associated to the representation given by the momentum map following the next steps:

- (1) Assume that the close momentum maps are $\mu : M \rightarrow \mathfrak{g}^*$ and $\lambda : M \rightarrow \mathfrak{g}^*$. The difference

$$\phi = \mu - \lambda$$

defines a 1-cochain of the complex which is a near 1-cocycle.

- (2) We define Φ as the time-1 map of the Hamiltonian vector field $X_{S_t(h(\phi))}$ with h the homotopy operator of the Chevalley–Eilenberg complex and S_t is a smoothing operator.

(3) The Newton iteration is given by

$$\phi_d = \phi_{X_{S_t(h(\eta_d))}}^1$$

with $\eta_d = (\mu - \lambda) \circ \phi_{d-1}$. This converges to a Poisson diffeomorphism that conjugates both actions.

Convergence is a hard part of the proof. In order to circumvent these difficulties, in [26] a strong use of geometric analysis tools is performed to check convergence using the paraphernalia of SCI spaces. In particular, the theorem needed to prove convergence is an abstract normal form presented in Section 4.2.

3.2. THE CASE OF NON-TRIVIAL POISSON STRUCTURE ON THE POISSON LIE GROUP G . In this section we prove rigidity of Hamiltonian actions, i.e., we show that two close Hamiltonian actions with momentum maps $J_0, J_1 : M \rightarrow G^*$ are equivalent. This result has been obtained by combining well-known results of Ginzburg and Weinstein concerning linearization of Poisson Lie groups with the rigidity theorem 3.1 for canonical momentum maps.

Observe that since a Poisson structure on a Poisson Lie group (with Poisson structure π_G) must vanish at $e \in G$, its linearization at e is well-defined (recall that $d_e\pi_G : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$). The following theorem says that if G is compact and semisimple, the Poisson structure π_G is linearizable, thus equivalent to $d_e\pi_G$ by diffeomorphisms.

THEOREM 3.2 (Ginzburg, Weinstein [15]): *Let G be a compact semisimple Poisson Lie group. Then the dual Poisson Lie group G^* is globally diffeomorphic to \mathfrak{g}^* with the linear Poisson structure defined as $\{f, g\}_\eta = \langle \eta, [df_\eta, dg_\eta] \rangle$.*

Combining Theorem 3.2 and Theorem 3.1 we obtain rigidity for the Hamiltonian action, as stated below.

THEOREM 3.3: *Let G be a compact semisimple Poisson Lie group G acting on a compact Poisson manifold (M, π) in a Hamiltonian fashion given by the momentum map $J_0 : M \rightarrow G^*$.*

There exist a positive integer l and two positive real numbers α and β (with $\beta < 1 < \alpha$) such that, if J_1 is another momentum map on M with respect to the same Poisson structure and Poisson Lie group, satisfying

$$(13) \quad \|J_0 - J_1\|_{2l-1} \leq \alpha \quad \text{and} \quad \|J_0 - J_1\|_l \leq \beta,$$

then there exists a Poisson diffeomorphism ψ of class \mathcal{C}^k , for all $k \geq l$, on M such that $J_1 \circ \psi = J_0$.

Proof. Denote by Φ the linearizing Poisson diffeomorphism¹ $\Phi : G^* \rightarrow \mathfrak{g}^*$ given by Theorem 3.2 and consider the compositions $\mu_0 = \Phi \circ J_0$ and $\mu_1 = \Phi \circ J_1$. The mappings $\mu_0 : M \rightarrow \mathfrak{g}^*$ and $\mu_1 : M \rightarrow \mathfrak{g}^*$ are canonical momentum maps and we may consider the infinitesimal Hamiltonian actions of \mathfrak{g} (β_0 and β_1). These actions integrate to infinitesimal standard Hamiltonian actions of the Lie group G which preserve the Poisson structure on M . We may now apply Theorem 3.1 to obtain a diffeomorphism ψ such that $\mu_1 \circ \psi = \mu_0$ and therefore $J_1 \circ \psi = J_0$. ■

This result is just telling us that rigidity of the standard momentum map implies rigidity of Lu’s momentum map. In general, this will work whenever we have a linearization theorem for the Poisson Lie group G .

METATHEOREM 3.1: *Whenever the Poisson Lie group structure in G is linearizable, the rigidity of the momentum map $\mu : M \rightarrow \mathfrak{g}^*$ implies the rigidity of the momentum map $J : M \rightarrow G^*$ for linearizable Poisson Lie groups G .*

The linearization of Poisson Lie groups has been studied by Enriquez, Etingof and Marshall [11] in the context of quasitriangular Poisson Lie groups and further generalized to coboundary Poisson Lie groups by Alekseev and Meinrenken [1]. In particular, for coboundary Poisson Lie groups the authors define a modified exponential $E : \mathfrak{g}^* \rightarrow G^*$ and prove that it is a Poisson diffeomorphism. If a rigidity result would work for coboundary Poisson Lie groups, then the metatheorem above would imply rigidity for this class too because $J_i = E \circ \mu_i$. To the authors’ knowledge, such a rigidity result for Hamiltonian actions is not known to hold in general for non-semisimple Lie groups.

Remark 3.2: It would be possible to relax the SCI hypotheses in order to prove rigidity for Poisson Lie group actions on compact manifolds. The SCI apparatus is indeed thought for the semilocal case (neighbourhood of a compact invariant submanifold). However, thanks to the SCI scheme the rigidity statement for compact manifolds is automatically valid in the semilocal setting (due to the need to control the convergence of the radii of shrank neighbourhoods in the iterative process).

¹ The differentiability class can be assumed to be k by the construction in the proof of Theorem 3.2.

3.2.1. *An application to groupoids.* Theorem 3.1 has a direct application to the study of momentum maps lifted to symplectic groupoids. Let us consider an integrable Poisson manifold M and its symplectic groupoid $\Sigma(M)$ over M . We recall that, as proved in [31], if one has a Hamiltonian action of (G, π_G) on (M, π) with momentum map $J : M \rightarrow G^*$, then $J^{\Sigma(M)} : \Sigma(M) \rightarrow G^*$ is exact:

$$(14) \quad J^{\Sigma(M)}(x) = J(t(x))J(s(x))^{-1},$$

where s, t are the source and target maps. Thus, using the fact that Poisson morphisms can be integrated (see [7]) we get the following

COROLLARY 3.4: *Given two close momentum maps $J_i : M \rightarrow G^*$, $i = 1, 2$ on an integrable Poisson manifold M , then there exists a symplectic groupoid morphism ϕ on $\Sigma(M)$ such that the corresponding lifted moment maps $J_i^{\Sigma(M)}$ satisfy $J_1^{\Sigma(M)} = J_2^{\Sigma(M)} \circ \phi$.*

In other words, rigidity of the momentum maps implies rigidity of the corresponding lifted momentum maps. The general case of momentum maps on symplectic groupoids is still open and this corollary gives a motivating example to investigate the rigidity of $J^{\Sigma(M)}$ when J does not exist.

4. Rigidity of pre-Hamiltonian actions

In this section we prove rigidity of pre-Hamiltonian actions. More precisely, we consider two Poisson actions generated by the infinitesimal momentum maps α and $\tilde{\alpha}$ and we prove a rigidity property: *close implies equivalent*. The proof follows the same lines discussed in Section 3.1.

First, we have to set up the concepts of close and equivalent for infinitesimal momentum maps. We can define the topology by using the associated infinitesimal momentum maps, i.e., we can also use the \mathcal{C}^k -norm of the infinitesimal momentum map

$$\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$$

and work with α_X , for $X \in \mathfrak{g}$ as mappings

$$\alpha_X : M \rightarrow T^*M.$$

On the other hand, two infinitesimal momentum maps are said to be **equivalent** if there exists a morphism of Lie algebras conjugating them.

As in the Hamiltonian setting, we aim to prove that infinitesimal rigidity implies rigidity, thus the first step is to consider the Chevalley–Eilenberg complex associated to the infinitesimal momentum map. The first cohomology group of this complex can be interpreted as infinitesimal deformations and when it vanishes we obtain infinitesimal rigidity. Then, using the techniques of SCI spaces we can prove the equivalence of infinitesimal momentum maps via Lie algebra morphisms. More explicitly, let α and $\tilde{\alpha}$ be two close infinitesimal momentum maps. The idea is to construct a sequence α_n which are equivalent, with $\alpha_0 = \alpha$ and such that α_n tends to $\tilde{\alpha}$ when n tends to $+\infty$.

- (1) We consider the difference $\beta = \alpha - \tilde{\alpha}$, which defines a 1-cochain of the associated complex which is a near 1-cocycle.
- (2) We define Φ as the time-1 map of the vector field

$$X_{h(\beta)} = \pi^\sharp(h(\beta))$$

with h the homotopy operator.

- (3) The Newton iteration is given by,

$$\Phi_n = \phi_{X_{h(\beta_n)}}^1$$

with $\beta_n = \Phi_{n-1} \circ (\alpha - \tilde{\alpha})$. This converges to a Lie algebra morphism that conjugates both momentum maps.

Instead of checking convergence of this sequence, we are going to use a normal form theorem for SCI spaces.

4.1. A CHEVALLEY–EILENBERG COMPLEX AND INFINITESIMAL RIGIDITY.

First we define the Chevalley–Eilenberg complex associated to an infinitesimal momentum map $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$ and discuss the properties that will be used to prove the rigidity theorem. The infinitesimal momentum map defines a representation of the Lie algebra \mathfrak{g} on the space of 1-forms on M as we prove in the following

LEMMA 4.1: *Let $\alpha : \mathfrak{g} \rightarrow \Omega^1(M) : X \mapsto \alpha_X$ be the infinitesimal momentum map. It defines a representation ρ of \mathfrak{g} on $\Omega^1(M)$ by*

$$(15) \quad \rho_X(\beta) := [\alpha_X, \beta]_\pi$$

for any $X \in \mathfrak{g}$.

Proof. This is a direct consequence of properties of the Lie bracket $[\cdot, \cdot]_\pi$ and of α since we have

$$\begin{aligned} \rho_X \rho_Y(\beta) - \rho_Y \rho_X(\beta) &= [\alpha_X, [\alpha_Y, \beta]_\pi]_\pi - [\alpha_Y, [\alpha_X, \beta]_\pi]_\pi \\ &= [[\alpha_X, \alpha_Y]_\pi, \beta]_\pi \\ &= [\alpha_{[X, Y]}, \beta]_\pi \\ &= \rho_{[X, Y]}(\beta). \end{aligned}$$

This concludes the proof. ■

Thus, for $q \in \mathbb{N}$, $C^q(\mathfrak{g}, \Omega^1(M)) = \text{Hom}(\wedge^q \mathfrak{g}, \Omega^1(M))$ is the space of alternating q -linear maps from \mathfrak{g} to $\Omega^1(M)$, with the convention

$$C^0(\mathfrak{g}, \Omega^1(M)) = \Omega^1(M).$$

The associated differential is denoted by ∂_i . Explicitly, we have

$$\Omega^1(M) \xrightarrow{\partial_0} C^1(\mathfrak{g}, \Omega^1(M)) \xrightarrow{\partial_1} C^2(\mathfrak{g}, \Omega^1(M))$$

where

$$(16) \quad \begin{aligned} \partial_0(\beta)(X) &= \rho_X(\beta) = [\alpha_X, \beta]_\pi, \\ \partial_1(\gamma)(X \wedge Y) &= \rho_X(\gamma(Y)) - \rho_Y(\gamma(X)) - \gamma([X, Y]), \end{aligned}$$

for any $\beta \in \Omega^1(M)$, $\gamma \in C^1(\mathfrak{g}, \Omega^1(M))$ and $X, Y \in \mathfrak{g}$. These differentials satisfy $\partial_i \circ \partial_{i-1} = 0$ and we can define the quotients

$$H^i(\mathfrak{g}, \Omega^1(M)) = \ker(\partial_i) / \text{im}(\partial_{i-1}) \quad \forall i \in \mathbb{N}.$$

The first cohomology group can be interpreted as infinitesimal deformations of the infinitesimal momentum maps modulo trivial deformations. In the compact semisimple case, it is known that the first and second cohomology groups vanish, so we have the infinitesimal rigidity. To prove that infinitesimal rigidity implies rigidity we need to prove that our spaces comply with the SCI spaces requirement (for more details about SCI spaces see [26]). In particular, certain inequalities have to be checked for the homotopy operators, necessary to control the loss of differentiability in the iterative process. In the Hamiltonian case, the trick used in [26] and [6] in order to prove the desired inequalities is to first use Sobolev metrics and then Sobolev inequalities and then take the real part. For the Chevalley–Eilenberg complex defined above we need those inequalities applied to mappings $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$ and work with α_X , for $X \in \mathfrak{g}$, as mappings

$\alpha_X : M \rightarrow T^*M$. Since M is compact, Sobolev inequalities holds too. A different way to do this is to consider Sobolev norms in the space of one-forms² and \mathcal{C}^k -topology for the space of one-forms (see for instance [9] or [16]) and adapt the same steps.

LEMMA 4.2: *In the Chevalley–Eilenberg complex associated to ρ :*

$$\Omega^1(M) \xrightarrow{\partial_0} C^1(\mathfrak{g}, \Omega^1(M)) \xrightarrow{\partial_1} C^2(\mathfrak{g}, \Omega^1(M))$$

there exists a chain of homotopy operators

$$\Omega^1(M) \begin{matrix} \xrightarrow{\partial_0} \\ \xleftarrow{h_0} \end{matrix} C^1(\mathfrak{g}, \Omega^1(M)) \begin{matrix} \xrightarrow{\partial_1} \\ \xleftarrow{h_1} \end{matrix} C^2(\mathfrak{g}, \Omega^1(M))$$

such that

$$\begin{aligned} \partial_0 \circ h_0 + h_1 \circ \partial_1 &= \text{id}_{C^1(\mathfrak{g}, \Omega^1(M))}, \\ \partial_1 \circ h_1 + h_2 \circ \partial_2 &= \text{id}_{C^1(\mathfrak{g}, \Omega^1(M))}. \end{aligned}$$

Moreover, for each k , there exists a real constant $C_k > 0$ such that

$$(17) \quad \|h_j(S)\|_{k,r} \leq C_k \|S\|_{k+s,r}, \quad j = 0, 1, 2$$

for all $S \in C^{j+1}(\mathfrak{g}, \Omega^1(M))$.

Proof. We apply the same strategy of [26] replacing the Sobolev inequalities for smooth functions by the analogous expressions for differential forms. A key point is that those Sobolev norms are invariant by the action of the Lie group which is linear. The linearity of the action is needed to decompose the Hilbert space into spaces which are invariant.

In our case we can assume that this action is also linear using an appropriate G -equivariant embedding by virtue of the Mostow–Palais theorem ([27], [29]).³ As was done in [26], we can check the regularity properties of the homotopy operators with respect to these Sobolev norms and then deduce, as a consequence, regularity properties of the initial norms by looking at the real part. The proof

² For one-forms on oriented manifolds, we may consider the higher degree versions of the following norm: $\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$ where $*\beta$ stands for the Hodge dual of β .

³ Using an orthonormal basis in the vector space E for this action, we can define the corresponding Sobolev norms in the ambient spaces provided by the Mostow–Palais embedding theorem. This norm is invariant by the action of G (we can even assume G is a subgroup of the orthogonal group).

holds step by step by replacing the standard Sobolev inequalities by the ones for differential one-forms. ■

Remark 4.1: If we restrict to exact forms, it follows immediately that

$$(18) \quad \partial_0(\beta)(X) = [\alpha_X, \beta]_\pi = [dH_X, df]_\pi = d\{H_X, f\}.$$

Thus, the Chevalley–Eilenberg complex $C^q(\mathfrak{g}, \Omega^1(M))$ recovers $C^q(\mathfrak{g}, \mathcal{C}^\infty(M))$.

4.2. AN ABSTRACT NORMAL FORM FOR SCI SPACES. As announced, we need to recall the normal form theorem proved in [26] for SCI spaces. SCI spaces (where SCI stands for **scaled \mathcal{C}^∞ -type**) are a generalization of scaled spaces and tame Fréchet spaces. This analytical apparatus is needed to prove normal form theorems in the most possible general setting which includes the neighbourhood of a point, a compact invariant submanifold or a compact manifold. We refer to [26] for the basic definitions of SCI spaces, SCI groups and SCI actions. It is good to keep in mind the following archetypical example: an example of SCI spaces is the set of Poisson structures, an example of an SCI group is the group of diffeomorphisms (which can be germified, semilocal or global), and in this case an example of SCI action is the push-forward of a Poisson structure via a diffeomorphism.

The scheme of proof of a normal form theorem in this abstract setting is the following:

- (1) \mathcal{G} (for instance, diffeomorphisms) which acts on a set \mathcal{S} (of structures).
- (2) We consider the subset of structures in normal forms \mathcal{N} inside \mathcal{S} .
- (3) The equivalence of an element in \mathcal{S} to a normal form is understood in the following way: for each element $f \in \mathcal{S}$ there is an element $\phi \in \mathcal{G}$ such that $\phi \cdot f \in \mathcal{N}$.

For practical purposes it is convenient to assume that a \mathcal{S} (in the example above, the set of Poisson structures) is a subset of a linear space \mathcal{T} (in the example above \mathcal{T} would be the set of bivector fields). The SCI group \mathcal{G} acts on \mathcal{T} and the set of normal forms $\mathcal{N} = \mathcal{F} \cap \mathcal{S}$, where \mathcal{F} is a linear subspace of \mathcal{T} .

The following theorem is an abstract normal form theorem for SCI spaces. In order to apply it to particular situations, we need to identify the sets \mathcal{S} , \mathcal{F} , \mathcal{T} and the SCI group \mathcal{G} in each case. We also need to identify \mathcal{G}_0 , a closed subgroup of \mathcal{G} which is not necessarily an SCI subgroup. As a consequence the equivalence to the normal form is given by the existence of $\psi \in \mathcal{G}$ (or in a closed subgroup, \mathcal{G}_0) for each $f \in \mathcal{S}$ such that $\psi \cdot f \in \mathcal{N}$.

THEOREM 4.3 (Miranda, Monnier, Zung [26]): *Let \mathcal{T} be a SCI space, \mathcal{F} a SCI subspace of \mathcal{T} , and \mathcal{S} a subset of \mathcal{T} . Denote $\mathcal{N} = \mathcal{F} \cap \mathcal{S}$. Assume that there is a projection $\pi : \mathcal{T} \rightarrow \mathcal{F}$ (compatible with restriction and inclusion maps) such that for every f in $\mathcal{T}_{k,\rho}$, the element $\zeta(f) = f - \pi(f)$ satisfies*

$$(19) \quad \|\zeta(f)\|_{k,\rho} \leq \|f\|_{k,\rho} \text{Poly}(\|f\|_{[(k+1)/2],\rho})$$

for all $k \in \mathbb{N}$ (or at least for all k sufficiently large), where $[\cdot]$ is the integer part.

Let \mathcal{G} be an SCI group acting on \mathcal{T} by a linear left SCI action and let \mathcal{G}^0 be a closed subgroup of \mathcal{G} formed by elements preserving \mathcal{S} . Let \mathcal{H} be a SCI space and assume that there exist maps $\mathbf{H} : \mathcal{S} \rightarrow \mathcal{H}$ and $\Phi : \mathcal{H} \rightarrow \mathcal{G}^0$ and an integer $s \in \mathbb{N}$ such that for every $0 < \rho \leq 1$, every f in \mathcal{S} and g in \mathcal{H} , and for all k in \mathbb{N} (or at least for all k sufficiently large), we have the three properties:

$$(20) \quad \begin{aligned} \|\mathbf{H}(f)\|_{k,\rho} &\leq \|\zeta(f)\|_{k+s,\rho} \text{Poly}(\|f\|_{[(k+1)/2]+s,\rho}) \\ &+ \|f\|_{k+s,\rho} \|\zeta(f)\|_{[(k+1)/2]+s,\rho} \text{Poly}(\|f\|_{[(k+1)/2]+s,\rho}), \end{aligned}$$

$$(21) \quad \|\Phi(g) - \text{id}\|_{k,\rho'} \leq \|g\|_{k+s,\rho} \text{Poly}(\|g\|_{[(k+1)/2]+s,\rho})$$

and

$$(22) \quad \begin{aligned} \|\Phi(g_1) \cdot f - \Phi(g_2) \cdot f\|_{k,\rho'} \\ \leq \|g_1 - g_2\|_{k+s,\rho} \|f\|_{k+s,\rho} \text{Poly}(\|g_1\|_{k+s,\rho}, \|g_2\|_{k+s,\rho}) \\ + \|f\|_{k+s,\rho} \text{Poly}_{(2)}(\|g_1\|_{k+s,\rho}, \|g_2\|_{k+s,\rho}) \end{aligned}$$

if $\rho' \leq \rho(1 - c\|g\|_{2,\rho})$ in (21) and $\rho' \leq \rho(1 - c\|g_1\|_{2,\rho})$ and $\rho' \leq \rho(1 - c\|g_2\|_{2,\rho})$ in (22).

Finally, for every f in \mathcal{S} denote

$$\phi_f = \text{id} + \chi_f = \Phi(\mathbf{H}(f)) \in \mathcal{G}^0$$

and assume that there is a positive real number δ such that we have the inequality

$$(23) \quad \|\zeta(\phi_f \cdot f)\|_{k,\rho'} \leq \|\zeta(f)\|_{k+s,\rho}^{1+\delta} Q(\|f\|_{k+s,\rho}, \|\chi_f\|_{k+s,\rho}, \|\zeta(f)\|_{k+s,\rho}, \|f\|_{k,\rho})$$

(if $\rho' \leq \rho(1 - c\|\chi_f\|_{1,\rho})$), where Q is a polynomial of four variables and whose degree in the first variable does not depend on k , and with positive coefficients.

Then there exist $l \in \mathbb{N}$ and two positive constants α and β with the following property: for all $p \in \mathbb{N} \cup \{\infty\}$, $p \geq l$, and for all $f \in \mathcal{S}_{2p-1,R}$ with $\|f\|_{2l-1,R} < \alpha$ and $\|\zeta(f)\|_{l,R} < \beta$, there exists $\psi \in \mathcal{G}_{p,R/2}^0$ such that $\psi \cdot f \in \mathcal{N}_{p,R/2}$.

Here we use the following notation:

- $\text{Poly}(\|f\|_{k,r})$ stands for a polynomial term in $\|f\|_{k,r}$ where the polynomial has positive coefficients and does not depend on f (though it may depend on k and on r continuously).
- The notation $\text{Poly}_{(p)}(\|f\|_{k,r})$, where p is a strictly positive integer, denotes a polynomial term in $\|f\|_{k,r}$, where the polynomial has positive coefficients and does not depend on f (though it may depend on k and on r continuously) and which contains terms of degree greater than or equal to p .

4.3. RIGIDITY OF INFINITESIMAL MOMENTUM MAPS. Finally, we can state the main theorem of this paper in which we prove the rigidity of pre-Hamiltonian actions of Poisson Lie groups on Poisson manifolds with infinitesimal momentum map $\alpha : \mathfrak{g} \rightarrow \Omega^1(M)$. In Section 4.1 we introduced the associated Chevalley–Eilenberg complex and the infinitesimal rigidity. In what follows we use Theorem 4.3 to prove the equivalence of two close momentum maps. In order to prove that our spaces satisfy the hypotheses of Theorem 4.3 we need some technical lemmas (they are generalizations of the Lemmas in [26, Appendix 2]).

LEMMA 4.4: *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. Consider a one-form ω on a ball $B_{r(1+\eta)} \in \mathbb{R}^n$ and a smooth map $\chi : B_r \rightarrow \mathbb{R}^n$ such that $\chi(0) = 0$ and $\|\chi\|_{1,r} < \eta$. Then the composition $(id + \chi^*) \circ \omega$ is a one-form on a ball B_r which satisfies the following inequality:*

$$(24) \quad \|(id + \chi^*) \circ \omega\|_{k,r} \leq \|\omega\|_{k,r(1+\eta)}(1 + P_k(\|\chi\|_{k,r})),$$

where P_k is a polynomial of degree k with vanishing constant term (and which is independent of ω and χ).

Proof. The proof follows the same lines of [26, Lemma B.1]. ■

Before stating the second technical lemma we need to recall some basic results of Poisson calculus, following [4]. In particular, we introduce a Lie derivative \mathcal{L}_α in the direction of a 1-form α . We will see that \mathcal{L}_α integrates to a flow Φ_t^* on $\Omega^1(M)$ which preserves the Lie algebra structure. Recall from Section 2 that the space of one-forms on M is endowed with a Lie bracket $[\cdot, \cdot]_\pi$ defined by Eq. (7). We set

$$(25) \quad \mathcal{L}_\alpha \beta = [\alpha, \beta]_\pi.$$

It is clear that this operation gives $\Omega^1(M)$ the structure of an $\Omega^1(M)$ -module, since it is a Lie algebra. Now, let us consider a vector field $\pi^\sharp(\alpha)$ and assume that its flow ϕ_t is defined for all $t \in \mathbb{R}$.

THEOREM 4.5 (Ginzburg, [14]): *There exist families of fiber-wise linear automorphisms Φ_t^* of the vector bundle $p : T^*M \rightarrow M$ covering ϕ_{-t} by $p \circ \Phi_t^* = \phi_{-t}$ such that*

(i) Φ_t^* is a flow:

$$\Phi_{t_1+t_2}^* = \Phi_{t_1}^* \Phi_{t_2}^*$$

for any $t_1, t_2 \in \mathbb{R}$.

(ii) For any $\beta \in \Omega^1(M)$ the time-dependent form $\beta(t) = \Phi_t^* \beta$ is a unique solution of the differential equation

$$(26) \quad \frac{d\beta(t)}{dt} = \mathcal{L}_\alpha \beta(t), \quad \beta(0) = \beta.$$

From now on we say that the flow Φ_t^* is a **Poisson flow** and it is known that it has many interesting features; here we just recall the one that will be necessary for our purposes.

PROPOSITION 4.6 (Ginzburg, [14]): *The Poisson flow Φ_t^* has the following properties:*

(i) It preserves the algebra structure:

$$(27) \quad \Phi_t^*(\beta_1 \wedge \beta_2) = \Phi_t^* \beta_1 \wedge \Phi_t^* \beta_2.$$

(ii) $\Phi_t^* : \Omega^1(M) \rightarrow \Omega^1(M)$ is a Lie algebra morphism:

$$(28) \quad \Phi_t^*[\beta_1, \beta_2]_\pi = [\Phi_t^* \beta_1, \Phi_t^* \beta_2]_\pi.$$

The definition and the properties of Poisson flows allow us to prove the following

LEMMA 4.7: *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. With the notation above, we have the following two properties:*

(i) For any positive integer k we have

$$(29) \quad \|\partial(\alpha - \tilde{\alpha})\|_{k,r} \leq c \|\alpha - \tilde{\alpha}\|_{k+1,r}^2,$$

where ∂ denotes the Chevalley–Eilenberg differential introduced in Section 4.1, while c is a positive constant independent of α and $\tilde{\alpha}$.

(ii) *There exists a constant $a > 0$ such that, if $\|\alpha - \tilde{\alpha}\|_{s+2,r(1+\eta)} < a\eta$, then we have, for any positive integer k ,*

$$(30) \quad \|\Phi^* \circ \alpha - \tilde{\alpha}\|_{k,r} \leq \|\alpha - \tilde{\alpha}\|_{k+s+2,r(1+\eta)}^2 P(\|\alpha - \tilde{\alpha}\|_{k+s+1,r(1+\eta)}),$$

where P is a polynomial with positive coefficients, independent of α and $\tilde{\alpha}$.

Proof. (i) Let us consider a basis $\{X_1, \dots, X_n\}$ of the Lie algebra \mathfrak{g} and the structure constants of the Lie algebra c_{ij}^k defined by

$$[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k.$$

Let us denote by α_i the one-form α_{X_i} associated to the element $X_i \in \mathfrak{g}$ and by β the difference $\alpha - \tilde{\alpha}$ of two infinitesimal momentum maps (with respect to the same Poisson structure). Using the definition of the Chevalley–Eilenberg differential ∂ introduced in Section 4.1, we have

$$(31) \quad \partial\beta(X_i \wedge X_j) = [\alpha_i, \beta_j]_\pi - [\alpha_j, \beta_i]_\pi - \beta([X_i, X_j]),$$

where $[\cdot, \cdot]_\pi$ is the Lie bracket induced on $\Omega^1(M)$ by the Poisson structure π on M . Thus we can write the following equality:

$$(32) \quad [\beta_i, \beta_j]_\pi = [\alpha_i, \alpha_j]_\pi - [\alpha_i, \tilde{\alpha}_j]_\pi - [\tilde{\alpha}_i, \alpha_j]_\pi + [\tilde{\alpha}_i, \tilde{\alpha}_j]_\pi.$$

Since α and $\tilde{\alpha}$ are infinitesimal momentum maps, we can use Eq. (9) to obtain

$$(33) \quad [\alpha_i, \alpha_j]_\pi = \sum_{k=1}^n c_{ij}^k \alpha_k \quad \text{and} \quad [\tilde{\alpha}_i, \tilde{\alpha}_j]_\pi = \sum_{k=1}^n c_{ij}^k \tilde{\alpha}_k.$$

This yields

$$(34) \quad \partial\beta(X_i \wedge X_j) = [\beta_i, \beta_j]_\pi.$$

Finally, we just write the following estimates:

$$(35) \quad \|\partial\beta\|_{k,r} \leq n(n-1)\|\pi\|_{k,r}\|\beta\|_{k+1,r}^2.$$

(ii) The difference $\beta = \alpha - \tilde{\alpha}$ can be seen as a 1-cochain in the Chevalley–Eilenberg complex $C^\bullet(\mathfrak{g}, \Omega^1(M))$. Thus $\mathfrak{h}(\beta)$ is an element of $\Omega^1(M)$ that we can contract with the Poisson structure π to get a vector field

$\pi^\sharp(\mathfrak{h}(\beta))$. Let Φ_t^* be the associated Poisson flow introduced in Theorem 4.5 and consider

$$(36) \quad \Phi^* \alpha_i - \tilde{\alpha}_i = \Phi^* \alpha_i - \Phi^* \tilde{\alpha}_i + \Phi^* \tilde{\alpha}_i - \tilde{\alpha}_i.$$

Using the definition of Poisson flow given by Eq. (26) we have, for each $i \in \{1, \dots, n\}$,

$$(37) \quad \begin{aligned} \Phi^* \tilde{\alpha}_i - \tilde{\alpha}_i &\stackrel{(26)}{=} \int_0^1 \Phi_t^* \mathcal{L}_{\mathfrak{h}(\beta)} \tilde{\alpha}_i \, dt \\ &\stackrel{(25)}{=} \int_0^1 \Phi_t^* [\mathfrak{h}(\beta), \tilde{\alpha}_i] \, dt \\ &\stackrel{(16)}{=} - \int_0^1 \Phi_t^* \partial(\mathfrak{h}(\beta))_i \, dt. \end{aligned}$$

From Lemma 4.2, we have the equality

$$(38) \quad \beta = \partial \mathfrak{h}(\beta) + \mathfrak{h} \partial(\beta).$$

Then, substituting Eq. (38) in Eq. (37), we obtain

$$(39) \quad \Phi_t^*(\tilde{\alpha}_i) - \tilde{\alpha}_i = - \int_0^1 \Phi_t^* \beta_i \, dt + \int_0^1 \Phi_t^* \mathfrak{h} \partial(\beta)_i \, dt.$$

Thus

$$(40) \quad \begin{aligned} \Phi^*(\alpha_i) - \tilde{\alpha}_i &\stackrel{(36)}{=} \Phi^*(\alpha_i) - \Phi^*(\tilde{\alpha}_i) + \Phi^*(\tilde{\alpha}_i) - \tilde{\alpha}_i \\ &\stackrel{(39)}{=} \Phi^* \beta_i - \int_0^1 \Phi_t^* \beta_i \, dt + \int_0^1 \Phi_t^* \mathfrak{h} \partial(\beta)_i \, dt \\ &\stackrel{(37)}{=} \int_0^1 \int_t^1 \Phi_\tau^* [\mathfrak{h}(\beta), \beta_i] \, d\tau \, dt + \int_0^1 \Phi_t^* \mathfrak{h} \partial(\beta)_i \, dt. \end{aligned}$$

The first integral can be estimated just using estimate (24) for the Chevalley–Eilenberg differential of the difference of two infinitesimal momentum maps. To estimate the second integral we first need to apply the estimate for the homotopy operator (17) and then again (24). Finally, combining these estimates we obtain the following:

$$(41) \quad \|\Phi^* \circ \alpha - \tilde{\alpha}\|_{k,r} \leq \|\alpha - \tilde{\alpha}\|_{k+s+2,r(1+\eta)}^2 P(\|\alpha - \tilde{\alpha}\|_{k+s+1,r(1+\eta)}),$$

which concludes our proof. ■

These estimates finally let us prove

THEOREM 4.8: *Consider a pre-Hamiltonian action of a semisimple compact Poisson Lie group (G, π_G) on a compact Poisson manifold (M, π) , generated by an infinitesimal momentum map α .*

There exist a positive integer l and two positive real numbers a and b (with $b < 1 < a$) such that, if $\tilde{\alpha}$ is another infinitesimal momentum map on M with respect to the same Poisson structure, satisfying

$$(42) \quad \|\alpha - \tilde{\alpha}\|_{2l-1} \leq a \quad \text{and} \quad \|\alpha - \tilde{\alpha}\|_l \leq b,$$

then there exists a Lie algebra morphism $\Phi : \Omega^1(M) \rightarrow \Omega^1(M)$ preserving the chain map property (10) of class \mathcal{C}^k , for all $k \geq l$, on M such that $\Phi(\alpha_X) = \tilde{\alpha}_X$.

Remark 4.2: This theorem holds in the \mathcal{C}^∞ -category (there is also a \mathcal{C}^p statement assuming α is of class \mathcal{C}^{2p-1}). The resulting morphism Φ can be chosen to depend continuously on $\tilde{\alpha}$.

Proof. Here we just need to check that the hypotheses of Theorem 4.3 are satisfied. First, we make the following identifications:

- The SCI space \mathcal{T} is defined to be the space \mathcal{T}_k of \mathcal{C}^k -differentiable maps from \mathfrak{g} to $\Omega(M)$.
- The subset \mathcal{S} is given by the infinitesimal momentum maps.
- The origin of the affine space is given by α and $\mathcal{F} = \mathcal{N} = 0$, so that the estimate (19) is obvious.
- The SCI group \mathcal{G} consists of the \mathcal{C}^k -differentiable maps from $\Omega^1(M)$ to itself, where the action is $\psi \cdot \alpha = \psi \circ \alpha$, with $\psi \in \mathcal{G}$ and $\alpha \in \mathcal{T}$.
- The closed subgroup \mathcal{G}_0 of \mathcal{G} is given by the Lie algebra morphisms. The elements of \mathcal{G}_0 preserve \mathcal{S} .
- The SCI space \mathcal{H} is by the space of one-forms on M .

Let us consider the difference of two infinitesimal momentum maps $\beta = \alpha - \tilde{\alpha}$ as a 1-cochain in the Chevalley–Eilenberg complex $C^q(\mathfrak{g}, \Omega^1(M))$, i.e., an element in \mathcal{S} . Thus the image of β by the map $\mathbf{H} : \mathcal{S} \rightarrow \mathcal{H}$ is simply $\mathbf{h}_0(\beta)$. Using the estimate of the homotopy operator Eq. (17), the relation (20) is obvious. As stated in Proposition 4.6 the flow ϕ_t of the vector field $\widehat{X} = \pi^\sharp(\mathbf{h}(\beta))$ can be recovered by a fiber-wise linear automorphism Φ_t^* on $\Omega^1(M)$, which is a Lie algebra morphism (see Eq. (28)). By construction, Φ_t^* commutes with the

differential so it preserves Eq. (10):

$$\begin{aligned}
 d\tilde{\alpha}_X &= d\Phi_t^*\alpha_X \\
 &= \Phi_t^*d\alpha_X \\
 &\stackrel{(10)}{=} \Phi_t^*(\alpha \wedge \alpha \circ \delta(X)) \\
 &= \Phi_t^*(\alpha_{X_i} \wedge \alpha_{X_j}) \\
 &\stackrel{(27)}{=} \Phi_t^*\alpha_{X_i} \wedge \Phi_t^*\alpha_{X_j} \\
 &= \tilde{\alpha}_{X_i} \wedge \tilde{\alpha}_{X_j} \\
 &= \tilde{\alpha} \wedge \tilde{\alpha} \circ \delta(X).
 \end{aligned}$$

Thus it provides the map $\Phi : \mathcal{H} \rightarrow \mathcal{G}_0$. The estimates (21)–(22)–(23) are direct consequences of Lemmas 4.4 and 4.7. ■

Finally, we observe that the equivalence of two infinitesimal momentum maps implies the equivalence of the corresponding actions under an additional assumption.

COROLLARY 4.9: *Let α and $\tilde{\alpha}$ be two infinitesimal momentum maps generating the fundamental vector fields $\widehat{X} = \pi^\sharp(\alpha_X)$ and $\widehat{X}' = \pi^\sharp(\tilde{\alpha})$ of two different actions, respectively. Under the assumptions of Theorem 4.8, if α_X vanishes on the symplectic leaves, then $\widehat{X}' = \phi^*\widehat{X}$.*

Proof. From Theorem 4.8 we know that the two momentum maps are equivalent, i.e., $\tilde{\alpha}_X = \Phi_t^*\alpha$. It is clear that if α_X vanishes on the symplectic leaves, we have $\Phi_t^*\alpha_X = \phi_t^*\alpha_X$, where ϕ_t is the flow on M underlying Φ_t^* (see Theorem 4.5). Thus

$$\begin{aligned}
 \widehat{X}' &= \pi^\sharp(\tilde{\alpha}) \\
 &= \pi^\sharp(\Phi_t^*\alpha_X) \\
 &= \pi^\sharp(\phi_t^*\alpha_X) \\
 &= \phi_*^t\pi^\sharp(\alpha_X) \\
 &= \phi_*^t\widehat{X}. \quad \blacksquare
 \end{aligned}$$

Since we have used the machinery of SCI spaces, the analogue of Theorem 4.8 also holds in the local and semilocal case (the neighbourhood of an invariant compact submanifold). Thus, in the same spirit of [26] we also obtain rigidity for

pre-Hamiltonian actions for actions in a neighbourhood of an invariant compact submanifold (which can be reduced to a single point in the case of fixed points for the action).

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