ISRAEL JOURNAL OF MATHEMATICS **219** (2017), 1–70
DOI: 10.1007/s11856-017-1473-3

IWASAWA MODULES AND p-MODULAR REPRESENTATIONS OF GL2

BY

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ABSTRACT

Let F be a finite extension of $\mathbf{Q}_p$. We associate, to certain smooth p-modular representations π of $\mathbf{GL}_2(F)$, a module $\mathfrak{S}(\pi)$ on the mod-p Iwasawa algebra of the standard Iwahori subgroup I of $\mathbf{GL}_2(F)$. When F is unramified, we obtain a module on a suitable formally smooth $\mathbf{F}_q$-algebra, endowed with an action of $\mathscr{O}_F^\times$ (the units in the ring of integers of F) and an $\mathscr{O}_F^\times$ equivariant, Frobenius semilinear endomorphism which turns out to be p-étale. We study the torsion properties of such a module, as well as its Iwahori-radical filtration.

CONTENTS

Received August 18, 2013 and in revised form July 21, 2014

1. Introduction

The p-modular Langlands program. Let F be a p-adic field, $\mathscr{O}_F$ its ring of integers and k_F its residue field. The p-adic Langlands program has the ambition to establish a dictionary between n-dimensional p-adic Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and certain p-adic Banach space representations of $\mathrm{GL}_n(F)$. Such correspondence is expected to be compatible with mod-p reduction of coefficients, to be realized in appropriate cohomologies of Shimura curves and to be compatible with deformation-theoretic techniques.

This correspondence is well understood in the special case of $\mathbf{GL}_2(\mathbf{Q}_p)$. The first breakthrough was Breuil's classification of p-modular supercuspidal representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ (cf. [Bre03a]), yielding a natural parametrization of their isomorphism classes by means of irreducible 2-dimensional Galois representations. The second breakthrough was the realization, by Colmez, of a functor from smooth, finite length admissible p-modular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ to Fontaine's (φ, Γ)-modules ([Col], §IV).

The last years have experienced extensive research for a p-modular correspondence for GL_2 over finite extensions of $\mathbf{Q}_p$. The most striking phenomenon is the proliferation of supercuspidal representations (as showed by the work of Breuil and Paskunas [BP] and Hu [Hu]), which does not seem to find any justification on the Galois side (the mod-p, absolutely irreducible Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ are finitely many up to isomorphism).

Although many problems in the category of smooth p-modular representations of $\mathbf{GL}_2(F)$ are extremely delicate, investigations in the last years showed that their approach by Iwasawa theoretical methods can be fruitful (cf. [HMS], [Sch]).

The aim of this paper is to develop this approach, describing a way to associate to a universal p-modular representation of GL_2 a module over a power series ring of characteristic p (the Iwasawa algebra of the integral points of a unipotent radical of $\mathbf{GL}_2$) endowed with commuting semilinear actions of $\mathscr{O}_F^\times$ and a Frobenius morphism $\mathscr{F}$, and study some of its properties when F is unramified.

It turns out that such a module is torsion free, the Frobenius action is p-étale and its quotients by certain non-zero submodules have dense torsion.

1.1. Description of the main results. All representations are smooth, over k-linear spaces, where k is a (sufficiently large) finite extension of k_F. By classical results of Barthel and Livné [BL94] a **supersingular** representation π of $\mathbf{GL}_2(F)$ is (up to twist) an admissible quotient of an explicit universal representation $\pi(\sigma,0)$. The latter is defined to be the cokernel of a certain $\mathbf{GL}_2(F)$-equivariant endomorphism (or **Hecke operator**) on the compact induction $\mathrm{ind}_{\mathbf{GL}_2(\mathscr{O}_F)F^\times}^{\mathbf{GL}_2(F)}\sigma$ where σ is an irreducible smooth representation of $\mathbf{GL}_2(\mathscr{O}_F)F^\times$ with trivial action of the uniformizer $\varpi\in F^\times$ (i.e. σ is a **Serre weight**).

More precisely (cf. [Mo1], Theorems 1.1 and 1.2) we have a $\mathbf{GL}_2(\mathscr{O}_F)$-equivariant decomposition $\pi(\sigma,0)|_{\mathbf{GL}_2(\mathscr{O}_F)}\cong R_{\infty,0}\oplus R_{\infty,1}$ and the smooth representations $R_{\infty,0}$ $R_{\infty,1}$ fit into an exact sequence:

$$0\to V_\bullet\to \mathrm{ind}_I^{\mathbf{GL}_2(\mathscr{O}_F)}\big(R^-_{\infty,\bullet}\big)\to R_{\infty,\bullet}\to 0$$

where $V_\bullet$, is an explicit subquotient of the smooth induction $\mathrm{ind}_I^{\mathbf{GL}_2(\mathscr{O}_F)}\chi_\bullet$, the smooth character $\chi_\bullet$ depending in a simple way on the highest weight of σ (cf. §2 for more details).

Therefore a first step to understand the irreducible quotients of $\pi(\sigma,0)$ consists in a precise control of the representations $R^-_{\infty,i}$.

The universal Iwasawa module and its torsion properties. Let $\mathfrak{S}^0_\infty$, $\mathfrak{S}^1_\infty$ be the Pontryagin duals of $R^-_{\infty,0}$, $R^-_{\infty,1}$ respectively. They are profinite modules over the Iwasawa algebra $k[[I]]$ of I. By restriction, they can equivalently be seen as modules for the Iwasawa algebra A associated to the p-adic analytic group $\mathrm{U}^-(\varpi)\overset{\mathrm{def}}{=}\left[\begin{smallmatrix}1&0\\ \varpi\mathscr{O}_F&1\end{smallmatrix}\right]$, endowed with continuous actions of the groups

$$\Gamma\overset{\mathrm{def}}{=}\begin{bmatrix}1&0\\0&1+\varpi\mathscr{O}_F\end{bmatrix},\quad \mathrm{U}^+\overset{\mathrm{def}}{=}\begin{bmatrix}1&\mathscr{O}_F\\0&1\end{bmatrix},$$

and

$$\mathbf{T}(k_F)\overset{\mathrm{def}}{=}\left\{\begin{bmatrix}a&0\\0&d\end{bmatrix},\ a,d\in k_F^\times\right\}$$

(where we consider $\mathbf{T}(k_F)$ as a subgroup of $\mathbf{GL}_2(\mathscr{O}_F)$ via the Teichmüller section of the natural reduction morphism $\mathbf{T}(\mathscr{O}_F)\to\mathbf{T}(k_F)$).

The actions of Γ, $\mathbf{T}(k_F)$ on $\mathfrak{S}^0_\infty$, $\mathfrak{S}^1_\infty$ are **semilinear**, as the former groups normalize $\mathrm{U}^-(\varpi)$. On the other hand, the group U^+ only acts by continuous k-linear endomorphisms and its action is extremely subtle. Its (partial) control is one of the technical heart of the paper (cf. Corollary 4.9, Proposition 6.1).

The first result of this paper is a precise description of $\mathfrak{S}^\bullet_\infty$ (for $\bullet \in \{0,1\}$), as a projective limit of finite-dimensional A-modules endowed with continuous actions of Γ, $\mathbf{T}(k_F)$, U^+ (i.e. finite dimensional $k[[I]]$-modules). From now on, we assume F to be unramified over $\mathbf{Q}_p$ and we write $\underline{r} \in \{0, \ldots, p-1\}^f$ for the f-tuple which parametrizes the isomorphism class of $\sigma|_{\mathbf{SL}_2(k_F)}$ (cf. (4) for such parametrization).

Moreover, we assume the Serre weight σ to be **regular**, i.e. that $\underline{r} \in \{1, \ldots, p-2\}^f$. We remark that some of the results of this paper can be modified to obtain similar statements in non-regular cases, but the technicality of the arguments in their proofs have convinced the author not to include them in this paper.

THEOREM 1.1 ((Proposition 3.7)): *Let $\bullet \in \{0,1\}$. The $k[[I]]$-module $\mathfrak{S}^\bullet_\infty$ is obtained as the limit of a projective system of finite length $k[[I]]$-modules $\left\{\mathfrak{S}^\bullet_{n+1}\right\}_{n\in 2\mathbf{N}+1+\bullet}$ where, for all $n \in 2N+1+\bullet$, $n \geq 2$, the transition morphisms $\mathfrak{S}^\bullet_{n+1} \twoheadrightarrow \mathfrak{S}^\bullet_{n-1}$ fit into the following commutative diagram:*

$$\begin{tikzcd}
\mathfrak{S}^\bullet_{n-1} \arrow[r, hook] & A/\langle X_i^{p^{n-2}(r_{i+n-2}+1)}, i\rangle \arrow[d, hook] & 1 \arrow[d, mapsto] \\
 & A/\langle X_i^{p^{n-1}(r_{i+n-1}+1)}, i\rangle & \prod_{i=0}^{f-1} X_i^{p^{n-2}(p(r_{i+n-1}+1)-(r_{i+n-2}+1))} \\
\mathfrak{S}^\bullet_{n+1} \arrow[uu, two heads] \arrow[r, hook] & A/\langle X_i^{p^{n}(r_{i+n}+1)}, i\rangle \arrow[u, two heads, "proj_{n+1}"'] & \\
\ker(proj_{n+1}) \arrow[u, hook] \arrow[r, equal] & \ker(proj_{n+1}) \arrow[u, hook] &
\end{tikzcd} \tag{1}$$

where the left vertical complex is exact and $proj_{n+1}$ denotes the natural projection.

We make precise the content of Theorem 1.1. The Iwasawa algebra A can be seen, by the Iwahori decomposition, as a $k[[I]]$-module. We recall that A is a complete local regular k algebra and we determine (Lemma 3.2) a regular system of parameters $X_0, \ldots, X_{f-1} \in A$, which give rise to a system of eigenvectors for the action of $\mathbf{T}(k_F)$ on the tangent space of A. All the morphisms in the diagram (1) are $k[[I]]$-equivariant and it is shown (§3) that the ideals

$\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\rangle_A$ are stable under the actions of Γ, $\mathbf{T}(k_F)$, U^+ (where the indices $i+n$ appearing in r_{i+n} are understood to be elements in $\mathbf{Z}/f\mathbf{Z}$).

Moreover, we can describe precisely the monomorphisms

$$\mathfrak{S}^{\bullet}_{n+1} \hookrightarrow A/\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\rangle,$$

deducing an explicit family of A-generators $\mathscr{G}^{\bullet}_{n+1}$ for $\mathfrak{S}^{\bullet}_{n+1}$.

The families $\mathscr{G}^{\bullet}_{n+1}$ are compatible with the transition maps, yielding a set $\mathscr{G}^{\bullet}_{\infty}$ of topological A-generators for $\mathfrak{S}^{\bullet}_{\infty}$ which is finite if and only if $F=\mathbf{Q}_p$ (in which case it is a one-point set). In other words, we have an A-linear (and $\mathbf{T}(k_F)$-equivariant) continuous morphism with dense image

$$\prod_{e\in\mathscr{G}^{\bullet}_{\infty}} A\cdot e \twoheadrightarrow \mathfrak{S}^{\bullet}_{\infty} \tag{2}$$

and the next step is to investigate the torsion properties of $\mathfrak{S}^{\bullet}_{\infty}$:

Theorem 1.2 ((Propositions 7.5, 7.6, 7.7)): *For $\bullet\in\{0,1\}$ the module $\mathfrak{S}^{\bullet}_{\infty}$ is torsion free over A and contains a dense A-submodule of rank one over Frac(A).*

Finally, if $x\in\mathfrak{S}^{\bullet}_{\infty}\setminus\{0\}$ is in the image of $\bigoplus_{e\in\mathscr{G}^{\bullet}_{\infty}} A\cdot e\to\mathfrak{S}^{\bullet}_{\infty}$ the torsion submodule of $\mathfrak{S}^{\bullet}_{\infty}/\langle x\rangle_A$ is dense in $\mathfrak{S}^{\bullet}_{\infty}/\langle x\rangle_A$.

Even if $\mathfrak{S}^{\bullet}_{\infty}$ is not of finite type over A (unless $F=\mathbf{Q}_p$) it is possible to determine a $k[[I]]$-submodule of finite co-length, which is finitely generated over an appropriate skew power series ring. More precisely, A is endowed with a Frobenius endomorphism $\mathscr{F}:A\to A$, which is k-linear, Γ, $\mathbf{T}(k_F)$-equivariant and characterized by the condition $\mathscr{F}(X_i)=X_{i-1}^p$.

We set $\mathfrak{S}^{\geq 1}_{\infty}\overset{\text{def}}{=}\ker\left(\mathfrak{S}^0_{\infty}\twoheadrightarrow\mathfrak{S}^0_0\right)$ and, similarly, $\mathfrak{S}^{\geq 2}_{\infty}\overset{\text{def}}{=}\ker\left(\mathfrak{S}^1_{\infty}\twoheadrightarrow\mathfrak{S}^1_1\right)$; the result is then the following:

Theorem 1.3 ((Proposition 5.11, 5.12)): *The module $\mathfrak{S}^{\geq 1}_{\infty}\oplus\mathfrak{S}^{\geq 2}_{\infty}$ is a submodule of $\mathfrak{S}^0_{\infty}\oplus\mathfrak{S}^1_{\infty}$ of finite co-length endowed with an $\mathscr{F}$-semilinear, Γ, $\mathbf{T}(k_F)$-equivariant endomorphism $\mathscr{F}_{\infty}$.*

The topological A-linearization of $\mathscr{F}_{\infty}$

$$A\otimes_{\mathscr{F},A}\left(\mathfrak{S}^{\geq 1}_{\infty}\oplus\mathfrak{S}^{\geq 2}_{\infty}\right)\xrightarrow{id\otimes\mathscr{F}_{\infty}}\mathfrak{S}^{\geq 1}_{\infty}\oplus\mathfrak{S}^{\geq 2}_{\infty}$$

has an image of finite co-length.

Finally, $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ *(resp.* $\mathfrak{S}_{\infty}^{\geq \bullet+1}$*) admits a finite family of generators as a module over the skew power series ring* $A[[\mathscr{F}]]$ *(resp.* $A[[\mathscr{F}^2]]$*), consisting of* $[F : \mathbf{Q}_p]$ *distinct eigencharacters for the* $\mathbf{T}(k_F)$*-action.*

We refer the reader to the paper [Ven], §2 for the definitions and basic properties of the skew power series ring $A[[\mathscr{F}]]$.

The case $F = \mathbf{Q}_p$. If $F = \mathbf{Q}_p$ we have a precise Galois theoretic description of the Iwasawa module $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ in terms of Wach modules.

Fix an embedding $k \hookrightarrow \overline{\mathbf{F}}_p$ and let $\omega_2 : G_{\mathbf{Q}_{p^2}} \to k$ be a choice for the Serre fundamental character of niveau 2, where $G_{\mathbf{Q}_{p^2}}$ is the absolute Galois group of the quadratic unramified extension $\mathbf{Q}_{p^2}$ of $\mathbf{Q}_p$. For $0 \leq r \leq p-1$ we write $\mathrm{ind}(\omega_2^{r+1})$ for the unique (absolutely) irreducible $G_{\mathbf{Q}_p}$-representation whose restriction to the inertia $I_{\mathbf{Q}_p}$ is described by $\omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ and whose determinant is ω^{r+1} (where ω is the mod-p cyclotomic character). Under the p-modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$ ([Bre03a], Definition 4.2.4), the Galois representation $\mathrm{ind}(\omega_2^{r+1})$ is associated to the supersingular representation $\pi(\sigma_r, 0)$.

In section 7.3 we verify that the $\mathscr{F}_{\infty}$-module $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ associated to $\pi(\sigma_r, 0)$ is compatible with the p-modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$. Indeed, the explicit description of the elements in $\mathscr{G}_{\infty}^{\bullet}$ lets us control the $\mathscr{F}_{\infty}$-action on $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ and one can compare the $\mathscr{F}_{\infty}$-action with the Frobenius action on the Wach modules associated to crystalline Galois representations.

Proposition 1.4 ((Proposition 7.9)): *Let* $0 \leq r \leq p-1$*, and write* $\chi_{(0,1)}$ *for the crystalline character of* $G_{\mathbf{Q}_{p^2}}$ *such that* $\chi_{(0,1)}(p) = 1$ *and with labelled Hodge–Tate weights* $-(0,1)$ *(for a choice of an embedding* $\mathbf{Q}_{p^2} \hookrightarrow \overline{\mathbf{Q}}_p$*). Define the crystalline representation* $V_{r+1} \overset{def}{=} \mathrm{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \chi_{(0,1)}^{r+1}$.

Then we have an isomorphism of φ*-modules*

$$\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2} \xrightarrow{\sim} \mathbf{N}\big(\overline{V}_{r+1}\big)$$

where $\mathbf{N}(\overline{V}_{r+1})$ *is the mod-*p *reduction of the Wach module associated to* V_{r+1} *and* $\mathfrak{S}_{\infty}^{\geq 1} \oplus \mathfrak{S}_{\infty}^{\geq 2}$ *is the Iwasawa module of Theorem 1.3.*

We recall that the mod-p reduction of V_{r+1} is the dual of $\mathrm{ind}(\omega_2^{r+1})$, in particular the statement of Proposition 1.4 is consistent with the p-modular Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$.

Radical filtration on the universal Iwasawa module. We focus on the radical filtration for the $k[[I]]$-module $\mathfrak{S}_\infty^\bullet$. Indeed, provided the surjection (2), we point out that none of the composite morphisms $A \hookrightarrow \mathfrak{S}_\infty^\bullet$ are equivariant for the extra actions of Γ, U^+.

If $[F : \mathbf{Q}_p](p-1) \leq p(p-2)$ our results give a new, much simplified proof of the main theorems of [Mo2] describing the socle filtration for the representations $R^-_{\infty,\bullet}$, avoiding almost completely manipulations on Witt vectors.

The first result in this direction is the proof that the A-radical filtration on A is stable by the actions of Γ, U^+.

Proposition 1.5 ((Corollary 4.8)): *For any $k \in \mathbf{N}$ the k-th power $\mathfrak{m}^k$ of the maximal ideal $\mathfrak{m}$ of A is endowed with a continuous action of* Γ, U^+, *which is trivial on the quotient*

$$\mathfrak{m}^k/\mathfrak{m}^{k+(p-2)}.$$

In particular, the $k[[I]]$-radical filtration on A coincides with the $\mathfrak{m}$-adic filtration.

Since $\mathfrak{S}_\infty^\bullet|_A$ is free of rank one if $F = \mathbf{Q}_p$, Proposition 1.5 (together with [Mo1], Theorem 1.2) gives another proof of [Bre03a], Théorème 3.2.4 :

Theorem 1.6 (([Bre03a], Théorème 3.2.4, Corollaire 4.1.4)): *Assume $F = \mathbf{Q}_p$ and write $I(1)$ for the pro-p Sylow subgroup of the Iwahori I. For any $r \in \{0, \ldots, p-1\}$ we have*

$$\dim\left(\pi(r,0)^{I(1)}\right) = 2.$$

The action of Γ being by k-algebra endomorphisms, the main difficulty to deduce Proposition 1.5 consists in the control of the U^+-action; this is done by a delicate induction argument (Proposition 4.7). The statement of Proposition 1.5 is expected to be false as soon as F ramifies over $\mathbf{Q}_p$ (the integral torus $\mathbf{T}(\mathscr{O}_F)$ does not act semisimply on the tangent space of A).

Similarly as we did in the paper [Mo2], the next step is to control the action of $k[[I]]$ on the graded pieces $\mathcal{K}er_{n+1} \overset{\text{def}}{=} \ker\left(\mathfrak{S}_{n+1}^\bullet \twoheadrightarrow \mathfrak{S}_{n-1}^\bullet\right)$. The result is the following:

Proposition 1.7 ((Proposition 6.1)): *Let $n \geq 2$. For any $k \geq 0$ the A-submodule $\mathfrak{m}^k\mathcal{K}er_{n+1}$ is endowed with a discrete action of* Γ, U^+, *which is* *trivial on the quotient*

$$\mathfrak{m}^k\mathcal{K}er_{n+1}/\mathfrak{m}^{k+(p-2)}\mathcal{K}er_{n+1}.$$

In particular, the $k[[I]]$-radical filtration on $\mathcal{K}er_{n+1}$ coincides with its A-radical filtration.

As for Proposition 1.5, the main difficulty in Proposition 1.7 is the control of the action of U^+ and we use in a crucial way some of the properties of the Frobenius $\mathscr{F}$ on A.

The last step in order to recover the $k[[I]]$-radical filtration on $\mathfrak{S}_\infty^\bullet$ consists in an appropriate "gluing" of the filtrations obtained by Proposition 1.7 on the subquotients $\{\mathcal{K}er_{n+1}\}_{n\in 2\mathbf{N}+1+\bullet}$. The argument is now mainly formal (as happened for the representation theoretic approach in [Mo2]).

Theorem 1.8 ((Proposition 7.1)): *Let $\bullet\in\{0,1\}$ and, for $k\in\mathbf{N}$, write $\mathscr{I}_k$ for the closure of $\mathfrak{m}^k\mathfrak{S}_\infty^\bullet$ in $\mathfrak{S}_\infty^\bullet$.*

Then the A-linear filtration $\{\mathscr{I}_k\}_{k\in\mathbf{N}}$ coincides with the $k[[I]]$-radical filtration on $\mathfrak{S}_\infty^\bullet$.

We remark that we can find an explicit submodule $\mathfrak{S}_\infty^{\geq 3}\overset{\mathrm{def}}{=}\ker\big(\mathfrak{S}_\infty^0\twoheadrightarrow\mathfrak{S}_2^0\big)$ of finite colength on which the Γ and U^+ actions are trivial on the quotients $\mathscr{I}_k/\mathscr{I}_{k+(p-2)}$.

We finally describe the isotypical components of $\mathrm{cosoc}_{k[[I]]}\big(\mathfrak{S}_\infty^\bullet\big)$, the $k[[I]]$-cosocle of $\mathfrak{S}_\infty^\bullet$. If χ is an irreducible $k[[I]]$-module, we write $V(\chi)$ to denote the χ-isotypical component of the $k[[I]]$-cosocle of $\mathfrak{S}_\infty^\bullet$ and the result is the following:

Corollary 1.9 ((Corollary 7.3)): *Assume that σ is a regular Serre weight. Then*

$$cosoc_{k[[I]]}\big(\mathfrak{S}_\infty^0\big)=V(\chi_{-\underline{r}})\oplus\bigoplus_{i=0}^{f-1}V(\chi_{\underline{r}}\mathrm{det}^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}),$$

$$cosoc_{k[[I]]}\big(\mathfrak{S}_\infty^1\big)=V(\chi_{\underline{r}}\mathrm{det}^{-\underline{r}})\oplus\bigoplus_{i=0}^{f-1}V(\chi_{\underline{r}}\mathrm{det}^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}),$$

where

$$\dim(V(\chi_{-\underline{r}})) = \dim(V(\chi_{\underline{r}}\det^{-\underline{r}})) = 1,$$

$$\dim(V(\chi_{\underline{r}}\det^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)})) = \begin{cases} \infty & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } F \neq \mathbf{Q}_p, \\ 0 & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } F = \mathbf{Q}_p. \end{cases}$$

Here, $\chi_{\underline{r}}$, $\mathfrak{a}$ are the smooth characters of I characterized by

$$\chi_{\underline{r}}\left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}\right) = a^{\sum_{i=0}^{f-1} p^i r_i}, \qquad \mathfrak{a}\left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}\right) = ad^{-1}.$$

Organization of the paper. In section 2 we recall the structure theorems for the universal p-modular representations of $\mathbf{GL}_2$ (Theorem 2.1), describing the construction of the representations $R^-_{\infty,0}$, $R^-_{\infty,1}$ as it appears in [Mo1], §3.

We subsequently dualize these constructions in §3. After recalling the main formal properties of Pontryagin duality for compact p-adic analytic groups, we determine the dual of a Serre weight (§3.2), thanks to an appropriate choice of a regular system of parameters for the Iwasawa algebra A. The description of the universal modules $\mathfrak{S}^\bullet_\infty$ follows finally from the construction of the Hecke operator T (3.3). The main result is Proposition 3.7, where we give a precise account of $\mathfrak{S}^\bullet_\infty$ as a $k[[I]]$-module.

Section 4 is devoted to the investigation of the A-radical filtration on A with respect to the extra action of the groups Γ, U^+ and the main result is Corollary 4.8.

In §5 we study the Frobenius $\mathscr{F}$ on A and its relations with the universal modules $\mathfrak{S}^\bullet_\infty$. After its formal definition and its first properties, we recall the constructions of [Ven] on the skew power series ring $A[[\mathscr{F}]]$ (§5.1.1). We subsequently deduce, in §5.2, the behavior of $\mathscr{F}$ with respect to certain modules (associated to the projective system defining $\mathfrak{S}^\bullet_\infty$) and we conclude (section 5.3) with the construction of a Frobenius, with a p-étale action, on an appropriate submodule of $\mathfrak{S}^\bullet_\infty$ of finite co-length. Moreover, we show that such submodule is of finite type over the skew power series ring $A[[\mathscr{F}]]$.

Section 6 is concerned with the $k[[I]]$-radical filtration for certain subquotients $\mathcal{K}er_{n+1}$ of $\mathfrak{S}^\bullet_\infty$. The techniques are similar to those of §4 and the new ingredient (in order to control the action of U^+) is the crucial use of the properties of the Frobenius. We remark that the behavior of $\mathcal{K}er_{n+1}$ is different for $n = 1$ and $n \geq 2$. The main result is Proposition 6.1.

Finally, the results of §6 and §4 are used in section 7.1 in order to recover the $k[[I]]$-radical filtration on $\mathfrak{S}^\bullet_\infty$ (7.1). In section 7.2 we conclude describing the torsion properties of the universal module $\mathfrak{S}^\bullet_\infty$.

The paper ends (§8) with a brief comment on the parallel constructions for the principal and special series representations for $\mathbf{GL}_2(F)$, where all the results are much simplified.

1.2. Notation. Let p be an odd prime. We consider a p-adic field F, with ring of integers $\mathscr{O}_F$, uniformizer ϖ and residue field k_F. We assume that $[k_F : \mathbf{F}_p] = f$ is finite. We write $val : F \to \mathbf{Z}$ for the valuation on F, normalized by $val(\varpi) = 1$, $x \mapsto \overline{x}$ for the reduction morphism $\mathscr{O}_F \to k_F$ and $\overline{x} \mapsto [\overline{x}]$ for the Teichmüller lift $k_F^\times \to \mathscr{O}_F^\times$ (we set $[0] \overset{\text{def}}{=} 0$).

Consider the general linear group $\mathbf{GL}_2$. We fix the maximal torus $\mathbf{T}$ of diagonal matrices and the unipotent radical $\mathbf{U}$ of upper unipotent matrices, so that $\mathbf{B} \overset{\text{def}}{=} \mathbf{T} \ltimes \mathbf{U}$ is the Borel subgroup of upper triangular matrices. We similarly write $\overline{\mathbf{U}}$ for the opposite unipotent radical and $\overline{\mathbf{B}} \overset{\text{def}}{=} \mathbf{T} \ltimes \overline{\mathbf{U}}$ for the opposite Borel.

Let $\mathscr{T}$ denote the Bruhat–Tits tree associated to $\mathbf{GL}_2(F)$ (cf. [Ser77]) and consider the hyperspecial maximal compact subgroup $K \overset{\text{def}}{=} \mathbf{GL}_2(\mathscr{O}_F)$. The following subgroups of K will play an important role in this article:

$$\mathrm{U}^-(\varpi^j) \overset{\text{def}}{=} \overline{\mathbf{U}}(\varpi^j \mathscr{O}_F) \text{ (where } j \in \mathbf{N}),\ \Gamma \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1+\varpi\mathscr{O}_F \end{bmatrix},\ \mathrm{U}^+ \overset{\text{def}}{=} \mathbf{U}(\mathscr{O}_F).$$

The natural reduction map $\mathbf{T}(\mathscr{O}_F) \twoheadrightarrow \mathbf{T}(k_F)$ has a section (induced by the Teichmüller lift) and we identify $\mathbf{T}(k_F)$ as a subgroup of $\mathbf{T}(\mathscr{O}_F)$. Concretely, $\mathbf{T}(k_F) \cong \left\{ \left[\begin{smallmatrix} [a] & 0 \\ 0 & [d] \end{smallmatrix}\right] \in K,\ a, d, \in k_F^\times \right\}$.

For notational convenience, we introduce the following objects:

$$s \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{GL}_2(F),\ \alpha \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} \in \mathbf{GL}_2(F),\ K_0(\varpi) \overset{\text{def}}{=} red^{-1}\big(\mathbf{B}(k_F)\big)$$

(where $red : K \to \mathbf{GL}_2(k_F)$ is the reduction morphism).

Let E be a p-adic field, with ring of integers $\mathscr{O}$ and finite residue field k (the "coefficient field"). Up to enlarging E, we can assume that $\mathrm{Card}\big(\mathrm{Hom}_{\mathbf{F}_p}(k_F, k)\big) = [k_F : \mathbf{F}_p]$.

A representation σ of a subgroup H_1 of $\mathbf{GL}_2(\mathbf{Q}_p)$ is always understood to be smooth with coefficients in k. If $h \in H_1$, we sometimes write $\sigma(h)$ to denote the k-linear automorphism induced by the action of h on the underlying vector space of σ. We denote by $(\sigma)^{H_1}$ the space of H_1 invariant vectors of σ and by $(\sigma)_{H_1}$ the space of H_1 co-invariant vectors.

Let $H_2 \leq H_1$ be compact open subgroups of K. For a smooth representation σ of H_2 we write $\mathrm{ind}_{H_2}^{H_1}\sigma$ to denote the (compact) induction of σ from H_2 to H_1. If $v \in \sigma$ and $h \in H_1$ we write $\big[h, v\big]$ for the unique element of $\mathrm{ind}_{H_2}^{H_1}\sigma$ supported in H_2h^{-1} and sending h to v. We deduce in particular the following equalities:

$$h' \cdot \big[h, v\big] = \big[h'h, v\big], \qquad \big[hk, v\big] = \big[h, \sigma(k)v\big] \tag{3}$$

for any $h' \in H_1$, $k \in H_2$.

If $Z \cong F^\times$ is the center of $\mathbf{GL}_2(F)$ and σ is a representation of KZ, we similarly write $\mathrm{ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ for the subspace of the full induction $\mathrm{Ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ consisting of functions which are compactly supported modulo the center Z (cf. [Bre03a], §2.3). For $g \in \mathbf{GL}_2(F)$, $v \in \sigma$ we use the same notation $\big[g, v\big]$ for the element of $\mathrm{ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ having support in KZg^{-1} and sending g to v; the element $\big[g, v\big]$ verifies similar compatibility relations as in (3).

A **Serre weight** is an absolutely irreducible representation of K. Up to isomorphism they are of the form

$$\bigotimes_{\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)} \big(\det{}^{t_\tau} \otimes_{k_F} \mathrm{Sym}^{r_\tau} k_F^2\big) \otimes_{k_F, \tau} k$$

where $r_\tau, t_\tau \in \{0, \ldots, p-1\}$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ and $t_\tau < p-1$ for at least one τ. This gives a bijective parametrization of the isomorphism classes of Serre weights by $2f$-tuples of integers $r_\tau, t_\tau \in \{0, \ldots, p-1\}$ such that $t_\tau < p-1$ for at least one τ. The Serre weight characterized by $t_\tau = 0$, $r_\tau = p-1$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ will be referred as the **Steinberg** weight and denoted by $\overline{St}$.

Recall that the K representations $\mathrm{Sym}^{r_\tau} k_F^2$ can be identified with the homogeneous component of degree r_τ of the polynomial algebra $k_F[X, Y]$. In this case, the action of K is described by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X^{r_\tau - i} Y^i \stackrel{\mathrm{def}}{=} (\overline{a}X + \overline{c}Y)^{r_\tau - i} (\overline{b}X + \overline{d}Y)^i$$

for any $0 \leq i \leq r_\tau$.

We fix once and for all a field homomorphism $k_F \hookrightarrow k$. The results of this paper do not depend on this choice. Up to twist, a Serre weight has now the more concrete expression

$$\sigma_{\underline{r}} \cong \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{r_i} k^2\right)^{\mathrm{Frob}^i} \tag{4}$$

where $\underline{r} = (r_0, \ldots, r_{f-1}) \in \{0, \ldots, p-1\}^f$ and $\left(\mathrm{Sym}^{r_i} k^2\right)^{\mathrm{Frob}^i}$ is the K-representation obtained from $\mathrm{Sym}^{r_i} k^2$ via the homomorphism $\mathbf{GL}_2(k_F) \to \mathbf{GL}_2(k_F)$ induced by the i-th Frobenius $x \mapsto x^{p^i}$ on k_F. We usually extend the action of K to the group KZ, by imposing the scalar matrix $\varpi \in Z$ to act trivially.

Let G be a compact p-adic analytic group (cf. [DDSMS], §8.4). It is a profinite topological group, with an open pro-p subgroup of finite rank.

The Iwasawa algebra $k[[G]]$ associated to G is the limit of the group algebras associated to the finite quotients of G:

$$\Omega_G \overset{\text{def}}{=} \varprojlim_{U} k[G/U]$$

where the limit is taken over the open normal subgroups U of G (cf. [AB] for the main properties of Iwasawa algebras). If G is pro-p, the associated Iwasawa algebra is a local noetherian regular domain, whose maximal ideal $\mathfrak{m}$ is the augmentation ideal:

$$\mathfrak{m} = \ker(\Omega_G \twoheadrightarrow k) = \langle x - 1,\ x \in G \rangle_{\Omega_G}$$

(note that the abstract ideal on the RHS is automatically closed since Ω_G is noetherian and compact). In this case the Krull dimension of the associated graded ring $\mathrm{gr}\big(\Omega_G\big)$ equals the dimension of the group G. If moreover G is a finitely generated free abelian pro-p-group, then $\dim(G)$ is the Krull dimension of Ω_G.

A module M over the Iwasawa algebra Ω_G is always understood to be a profinite left Ω_G-module (i.e. an inverse limit of finite left Ω_G-modules). If M, N are profinite right and left Ω_G modules respectively, their **completed tensor product** is the profinite k-module defined by

$$M \widehat{\otimes}_{\Omega_G} N \overset{\text{def}}{=} \varprojlim_{M', N'} M/M' \otimes_{\Omega_G} N/N'$$

where the projective limit is taken over the open Ω_G-submodules M', N' of M, N respectively. We refer the reader to [RZ], §5.5 or [Wil], §7.7 for the basic properties of the completed tensor product of profinite modules.

A k-valued character χ of the torus $\mathbf{T}(k_F)$ will be considered, by inflation, as a smooth character of any subgroup of $K_0(\varpi)$. We write χ^s to denote the conjugate character of χ, defined by

$$\chi^s(t) \stackrel{\text{def}}{=} \chi(sts^{-1})$$

for $t \in \mathbf{T}(k_F)$. Similarly, if τ is any representation of $K_0(\varpi)$, we write τ^s to denote the conjugate representation, defined by

$$\tau^s(h) = \tau(\alpha h \alpha)$$

for any $h \in K_0(\varpi)$.

If $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$ is an f-tuple we define the characters of $\mathbf{T}(k_F)$:

$$\chi_{\underline{r}}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \stackrel{\text{def}}{=} a^{\sum_{i=0}^{f-1} p^i r_i}, \qquad \mathfrak{a}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \stackrel{\text{def}}{=} ad^{-1}.$$

If τ is a semisimple representation of $K_0(\varpi)$ we write $V_\tau(\chi)$ (or simply $V(\chi)$ if the representation τ is clear from the context) for the χ-isotypical component of τ; thus

$$\tau = \bigoplus_{\chi \in X^*(\mathbf{T}(k_F))} V_\tau(\chi).$$

Let $\mathscr{C}$ be an abelian category and write $\mathscr{C}^{ss}$ for the full subcategory consisting of semisimple objects; if $X \in \mathscr{C}$ we can consider the functor

$$\begin{array}{ccc} \mathscr{C}^{ss} & \longrightarrow & \mathscr{S}ets \\ Y & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{array}$$

If the functor is representable, by a couple $X \to Q$, we define the **radical** $\mathrm{Rad}(X)$ of X to be the kernel

$$\mathrm{Rad}(X) \stackrel{\text{def}}{=} \ker(X \to Q).$$

If R is a ring which is semisimple modulo its Jacobson ideal J and $\mathscr{C}$ is a full subcategory of the category of left-R modules, then the radical of an object in $\mathscr{C}$ always exists and we have

$$\mathrm{Rad}(M) = J \cdot M$$

for any $M \in \mathscr{C}$. In particular, for any object $M \in \mathscr{C}$ we can define, by induction, the **radical filtration** $\{\mathrm{Rad}^n(M)\}_{n \in \mathbf{N}}$ by $\mathrm{Rad}^0(M) \stackrel{\text{def}}{=} M$ and $\mathrm{Rad}^n(M) \stackrel{\text{def}}{=} J \cdot \mathrm{Rad}^{n-1}(M)$ for $n \geq 1$.

The dual notion of the radical filtration is the **socle filtration**.

We recall some conventions on the multi-index notations. We write $\underline{\alpha} \overset{\text{def}}{=} (\alpha_0, \dots, \alpha_{f-1})$ to denote an f-tuple $\underline{\alpha} \in \mathbf{N}^f$ and if $\underline{\alpha}, \underline{\beta}$ are f-tuples we define

i) $\underline{\alpha} \geq \underline{\beta}$ if and only if $\alpha_s \geq \beta_s$ for all $s \in \{0, \dots, f-1\}$;

ii) $\underline{\alpha} \pm \underline{\beta} \overset{\text{def}}{=} (\alpha_0 \pm \beta_0, \dots, \alpha_{f-1} \pm \beta_{f-1})$ (where the difference $\underline{\alpha} - \underline{\beta}$ is defined only if $\underline{\alpha} \geq \underline{\beta}$).

The **length** of an f-tuple $\underline{\alpha}$ is defined as $|\underline{\alpha}| \overset{\text{def}}{=} \sum_{s=0}^{f-1} \alpha_s$ and, for $s \in \{0, \dots, f-1\}$, we define the element $e_s \overset{\text{def}}{=} (0, \dots, 0, 1, 0, \dots, 0)$ where the only non-zero coordinate appears in position s.

If $A = k[[X_0, \dots, X_{f-1}]]$, $\lambda \in k_F$ and $\underline{\alpha} \in \mathbf{N}^f$ is an f-tuple we write

$$\underline{X}^{\underline{\alpha}} \overset{\text{def}}{=} \prod_{s=0}^{f-1} X_s^{\alpha_s}, \qquad \lambda^{\underline{\alpha}} \overset{\text{def}}{=} \lambda^{\sum_{s=0}^{f-1} p^s \alpha_s}$$

with the usual convention $0^0 \overset{\text{def}}{=} 1$.

Finally, we recall that if S is any set, and $s_1, s_2 \in S$, the Kronecker delta $\delta_{(s_1,s_2)}$ is defined by

$$\delta_{(s_1,s_2)} \overset{\text{def}}{=} \begin{cases} 0 & \text{if} s_1 \neq s_2, \\ 1 & \text{if} s_1 = s_2. \end{cases}$$

2. Reminders on the universal representation for $\mathbf{GL}_2$

We recall here the definition of the universal representation for $\mathbf{GL}_2$, and we specialize its construction by means of certain amalgamated sums of finite inductions. The main upshot is Theorem 2.1, which shows that in order to control the universal representation it is sufficient to consider a suitable subrepresentation of the Iwahori subgroup of K. The reader is invited to refer to [Mo1], §2.1 and §3.1 for the omitted details.

We fix an f-tuple $\underline{r} \in \{0, \dots, p-1\}^f$ and write $\sigma = \sigma_{\underline{r}}$ for the associated Serre weight described in (4). In particular, the highest weight space of σ affords the character $\chi_{\underline{r}}$. We recall ([BL94], [Her1]) that the Hecke algebra $\mathscr{H}_{KZ}(\sigma)$ is commutative and isomorphic to a polynomial algebra:

$$\mathscr{H}_{KZ}(\sigma) \overset{\sim}{\to} k[T].$$

The Hecke operator T is supported on the double coset $K\alpha KZ$ and is completely determined as a suitable linear projection on σ (cf. [Her1], Theorem 1.2); it admits an explicit description in terms of the Bruhat–Tits tree of $\mathbf{GL}_2(F)$ (cf. [Bre03a], §2.5).

The universal representation $\pi(\sigma,0)$ for $\mathbf{GL}_2$ is then defined[1] by the exact sequence

$$0\to \mathrm{ind}_{KZ}^G\sigma \xrightarrow{T} \mathrm{ind}_{KZ}^G\sigma\to \pi(\sigma,0)\to 0.$$

Using the Mackey decomposition for the KZ-restriction, we are able to describe $\pi(\sigma,0)|_{KZ}$ as a compact induction from an explicit $K_0(\varpi)$-representation, as we outline in the following lines.

Let $n\in\mathbf{N}$. We consider the anti-dominant co-weight $\lambda_n\in X(\mathbf{T})_*$ characterized by

$$\lambda_n(\varpi)=\begin{bmatrix}1 & 0\\ 0 & \varpi^n\end{bmatrix}$$

and we introduce the subgroups

$$K_0(\varpi^n)\overset{\mathrm{def}}{=}\left(\lambda_n(\varpi)K\lambda_n(\varpi)^{-1}\right)\cap K=\left\{\begin{bmatrix}a & b\\ \varpi^n c & d\end{bmatrix}\in K\right\}.$$

The element $\lambda_n(\varpi)s=\left[\begin{smallmatrix}0 & 1\\ \varpi^n & 0\end{smallmatrix}\right]$ normalizes $K_0(\varpi^n)$ and we define the $K_0(\varpi^n)$-representation $\sigma^{(n)}$ as the $K_0(\varpi^n)$-restriction of σ endowed with the $K_0(\varpi^n)$-action twisted by the element $\left[\begin{smallmatrix}0 & 1\\ \varpi^n & 0\end{smallmatrix}\right]$. Explicitly,

$$\sigma^{(n)}\left(\begin{bmatrix}a & b\\ \varpi^n c & d\end{bmatrix}\right)\cdot \underline{X}^{\underline{r-j}}\underline{Y}^{\underline{j}}\overset{\mathrm{def}}{=}\sigma\left(\begin{bmatrix}d & c\\ \varpi^n b & a\end{bmatrix}\right)\underline{X}^{\underline{r-j}}\underline{Y}^{\underline{j}}.$$

Finally, for $n\geq 1$ we write

$$R_n^-(\sigma)\overset{\mathrm{def}}{=}\mathrm{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}\left(\sigma^{(n)}\right),\qquad R_0^-\overset{\mathrm{def}}{=}\mathrm{cosoc}_{K_0(\varpi)}(\sigma^{(1)})\cong \mathrm{soc}_{K_0(\varpi)}(\sigma^{(0)}).$$

For notational convenience, we write $\underline{Y}^{\underline{r}}$ for a basis of R_0^-. If the Serre weight σ is clear from the context, we write R_n^- instead of $R_n^-(\sigma)$.

The interest of the representations R_n^- is that they realize the Mackey decomposition for $\mathrm{ind}_{KZ}^G\sigma$:

$$\left(\mathrm{ind}_{KZ}^G\sigma\right)|_{KZ}\quad\xrightarrow{\sim}\quad \sigma^{(0)}\oplus\bigoplus_{n\geq 1}\mathrm{ind}_{K_0(\varpi)}^K\left(R_n^-\right).$$

The interpretation in terms of the tree of $\mathbf{GL}_2$ is clear: the $k[K_0(\varpi)]$-module R_n^- maps isomorphically onto the space of elements of $\mathrm{ind}_{KZ}^G\sigma$ having support

[1] In the current literature the universal representation is written $\pi(\sigma,0,1)$. We decided to write $\pi(\sigma,0)$ in order to lighten the notations.

on the double coset $K_0(\varpi)\lambda_n(\varpi)KZ$. In particular, if σ is the trivial weight, a basis for R_n^- is parametrized by the vertices of $\mathscr{T}$, belonging to the negative part of the tree and lying at distance n from the central vertex.

The Hecke morphism T induces, by transport of structure, a family of $K_0(\varpi)$-equivariant morphisms $\{(T_n)^{\mathrm{neg}}\}_{n\geq 1}$ defined on the $k[K_0(\varpi)]$-modules R_n^-: for $n\geq 2$ we have $(T_n)^{\mathrm{neg}} \overset{\mathrm{def}}{=} T|_{R_n^-}$ and, for $n=1$, we define

$$(T_1)^{\mathrm{neg}}: R_1^- \xrightarrow{T|_{R_1^-}} R_2^- \oplus R_0^-;$$

this is possible since the image of $T|_{R_1^-}$ lies in $R_2^- \oplus \mathrm{soc}_{K_0(\varpi)}(\sigma^{(0)})$, cf. [Mo2], Lemma 2.7.

More expressively, one shows (cf. [Mo2], §2.1) that for any $n\geq 1$ the Hecke operator $(T_n)^{\mathrm{neg}}$ admits a decomposition $(T_n)^{\mathrm{neg}} = T_n^+ \oplus T_n^-$ where[2] the morphisms $T_n^{\pm}: R_n^- \to R_{n\pm 1}^-$ are obtained by compact induction (from $K_0(\varpi^n)$ to $K_0(\varpi)$) from the following morphisms:

$$\begin{aligned} t_n^+ : \sigma^{(n)} &\hookrightarrow \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)}\sigma^{(n+1)} \\ \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}} &\mapsto \sum_{\lambda_n\in k_F}(-\lambda_n)^{\underline{j}}\begin{bmatrix}1 & 0\\ \varpi^n[\lambda_n] & 1\end{bmatrix}[1,\underline{X}^{\underline{r}}]; \end{aligned} \tag{5}$$

and

$$\begin{aligned} t_{n+1}^- : \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)}\sigma^{(n+1)} &\twoheadrightarrow \sigma^{(n)} \\ [1,\underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}}] &\mapsto \delta_{\underline{j},\underline{r}}\underline{Y}^{\underline{r}} \end{aligned} \tag{6}$$

and, for $n=0$, we have the natural epimorphism

$$\begin{aligned} T_1^- : R_1^- &\twoheadrightarrow R_0^- \\ \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}} &\mapsto \delta_{\underline{j},\underline{r}}\underline{Y}^{\underline{r}} \end{aligned}$$

(this shows that T_n^+ are monomorphisms and T_n^- epimorphisms for all $n\geq 1$).

[2] According to [Mo1], the morphisms $T_n^{\pm}$ should be written as $(T_n^{\pm})^{\mathrm{neg}}$. We decided to use here the lighter notation $T_n^{\pm}$.

The Hecke operators $T_n^{\pm}$ can be used to construct a family of amalgamated sums, in the following way. We define $R_0^- \oplus_{R_1^-} R_2^-$ as the push out:

$$\begin{tikzcd} R_1^- \arrow[r, hook, "T_1^+"] \arrow[d, "-T_1^-"'] & R_2^- \arrow[d, dotted, two heads, "pr_2"] \\ R_0^- \arrow[r, hook, dotted] & R_0^- \oplus_{R_1^-} R_2^- \end{tikzcd}$$

and, assuming we have inductively constructed $pr_{n-1} : R_{n-1}^- \twoheadrightarrow R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-$ (where $n \geq 3$ is odd), we define the amalgamated sum $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ by the following co-cartesian diagram:

$$\begin{tikzcd} R_n^- \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"'] & R_{n+1}^- \arrow[d, dotted, two heads, "pr_{n+1}"] \\ R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^- \arrow[r, hook, dotted] & R_0^- \oplus_{R_1^-} R_2^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^-. \end{tikzcd}$$

The amalgamated sums $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ (where n is odd) form, in an evident manner, an inductive system and we define

$$R_{\infty,0}^- \stackrel{\text{def}}{=} \varinjlim_{n\in 2\mathbf{N}+1} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

We can repeat the previous construction for n even, defining an inductive system of $K_0(\varpi)$-representations $R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-$ and we write

$$R_{\infty,1}^- \stackrel{\text{def}}{=} \varinjlim_{n\in 2\mathbf{N}+2} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

The relation between the representations $R_{\infty,\bullet}^-$ and the universal representation $\pi(\sigma, 0)$ is described by the following

Theorem 2.1 (([Mo1], Theorem 1.1)): *Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight. The KZ-restriction of the universal representation $\pi(\sigma,0)$ decomposes as $\pi(\sigma,0)|_{KZ} = R_{\infty,0} \oplus R_{\infty,1}$ and we have short exact sequences of K-representations*

$$0 \to Rad(\chi_{\underline{r}}) \to \mathrm{ind}_{K_0(\varpi)}^K\big(R_{\infty,0}^-\big) \to R_{\infty,0} \to 0$$
$$0 \to Soc(\chi_{\underline{r}}^s) \to \mathrm{ind}_{K_0(\varpi)}^K\big(R_{\infty,1}^-\big) \to R_{\infty,1} \to 0$$

where $Rad(\chi_{\underline{r}})$, $Soc(\chi_{\underline{r}}^s)$ *are defined by*

$$\begin{array}{lll} Rad(\chi_{\underline{r}}) \overset{\text{def}}{=} \overline{St}, & Soc(\chi_{\underline{r}}^s) \overset{\text{def}}{=} 1 & \text{if } \underline{r} = \underline{0} \\ Rad(\chi_{\underline{r}}) \overset{\text{def}}{=} 1, & Soc(\chi_{\underline{r}}^s) \overset{\text{def}}{=} \overline{St} & \text{if } \underline{r} = \underline{p-1} \\ Rad(\chi_{\underline{r}}) \overset{\text{def}}{=} Rad\big(\mathrm{ind}_{K_0(\varpi)}^K \chi_{\underline{r}}\big), & Soc(\chi_{\underline{r}}^s) \overset{\text{def}}{=} Soc\big(\mathrm{ind}_{K_0(\varpi)}^K \chi_{\underline{r}}^s\big) & \text{otherwise.} \end{array}$$

Proof. This is Corollary 3.4 in [Mo1]. ∎

We shall remark that the representations $R_{\infty,0}^-$ $R_{\infty,1}^-$ can also be used to control the action of the normalizer of the Iwahori subgroup, cf. [Mo1], Proposition 3.8.

3. Dual translation

The first step in order to control the representations $R_{\infty,0}^-$, $R_{\infty,1}^-$ consists in a precise knowledge of their Pontryagin duals $\mathfrak{S}_\infty^0$, $\mathfrak{S}_\infty^1$. We start by recalling some well-known results about the duality between smooth representations of compact p-adic analytic groups and profinite modules (§3.1) and we specialize the construction to the group $\mathrm{U}^-(\varpi)$. In particular, we determine a family of $\mathbf{T}(k_F)$-eigenvectors for the tangent space of $k[[\mathrm{U}^-(\varpi)]]$, which lets us easily deduce the dual of a Serre weight (3.2). The description of $\mathfrak{S}_\infty^0$, $\mathfrak{S}_\infty^1$ follows then by a formal construction, which is detailed in section 3.3.

We fix throughout this section a Serre weight $\sigma = \sigma_{\underline{r}}$. In particular, the highest weight space of σ affords the $K_0(\varpi)$-character $\chi_{\underline{r}}$.

3.1. Review of Pontryagin duality. The aim of this section is to give a precise survey of the main formal properties of Pontryagin duality for compact p-adic analytic groups. The subject is classical and we invite the reader to refer to the work of Emerton [Eme], §2.2 or Ribes-Zalesskii [RZ], §5.1 for more details.

Let A be a complete, Noetherian local $\mathscr{O}_F$-algebra with finite residue field and let G be a compact p-adic analytic group (cf. [DDSMS], §8.4).

The category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ of smooth, A-linear G-representations is defined as the category of locally Artinian A-modules endowed with the discrete topology and a continuous action of G. On the other hand, we have the category $\mathrm{Mod}_G^{\mathrm{pro}}(A)$ of profinite $A[[G]]$-modules.

We recall the following result

THEOREM 3.1 ((Pontryagin Duality)): *For any compact-open subgroup K of G we have an involutive anti-equivalence of categories*

$$\begin{array}{ccc} Mod_K^{sm}(A) & \stackrel{\sim}{\longleftrightarrow} & Mod_K^{pro}(A) \\ V & \longrightarrow & V^{\vee}. \end{array}$$

Moreover, the equivalence is compatible with restriction and induction: if $K_1 \leq K_2$ are two compact open subgroups of G and $V \in Mod_{K_1}^{sm}(A)$ then

$$\left(\mathrm{ind}_{K_1}^{K_2} V\right)^{\vee} = A[[K_2]] \otimes_{A[[K_1]]} (V)^{\vee}$$

and the functor $\mathrm{ind}_{K_1}^{K_2}$ is right adjoint to the restriction to K_1 (Frobenius reciprocity).

We content ourselves to recall that the dual of $V \in \mathrm{Mod}_K^{\mathrm{sm}}(A)$ is defined as $V^{\vee} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathscr{O}_F}(V, F/\mathscr{O}_F)$, the latter endowed with the compact-open topology (hence the topology of the simple convergence as the $\mathscr{O}_F$-modules $V, F/\mathscr{O}_F$ are endowed with the discrete topology) and the action of K given by $(g \cdot f)(v) \stackrel{\mathrm{def}}{=} f(g^{-1}v)$ for any $g \in K$, $v \in V$, $f \in V^{\vee}$.

Conversely, if $M \in \mathrm{Mod}_K^{\mathrm{pro}}(A)$ one considers the topological dual $M^{\vee} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathscr{O}_F}^{\mathrm{cont}}(M, F/\mathscr{O}_F)$, endowed with the discrete topology and the (continuous) contragradient action of G.

Let $H \trianglelefteq G$ be compact open, and $V \in \mathrm{Mod}_{G/H}^{\mathrm{sm}}(A)$ (which will be considered as an element of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ by inflation). Let $\{K_n\}_{n \in \mathbf{N}}$ be a family of compact open subgroups of G such that $K_\infty = \bigcap_{n \in \mathbf{N}} K_n$ is closed and such that, for any compact open subgroup $U \leq G$ one has $H \cdot U \supseteq K_{n(U)}$ for some $n(U) \in \mathbf{N}$. Then any continuous function $f : G \to V$ which is left K_∞-equivariant is automatically left K_n-equivariant for some n (depending on f), and we have $\mathrm{ind}_{K_\infty}^G V|_{K_\infty} = \varinjlim_n \mathrm{ind}_{K_n}^G V|_{K_n}$. Hence

$$\begin{aligned} \left(\mathrm{ind}_{K_\infty}^G V|_{K_\infty}\right)^{\vee} &= \varprojlim_n \left(\mathrm{ind}_{K_n}^G V|_{K_n}\right)^{\vee} = \varprojlim_n A[[G]] \otimes_{A[[K_n]]} (V|_{K_n})^{\vee} \\ &= A[[G]] \otimes_{A[[K_\infty]]} (V|_{K_\infty})^{\vee} \end{aligned}$$

(the last equality clearly holds if G is discrete, and one passes to the inverse limit over the open compact normal subgroups of G, cf. [RZ], Theorem 6.10.8).

We deduce, using the continuity of the restriction functor and the Mackey decomposition, that for a closed subgroup U of G we have an isomorphism of profinite $A[[U]]$-modules

$$(7)\quad \left(A[[G]]\otimes_{A[[K_\infty]]}\big(V|_{K_\infty}\big)^\vee\right)|_{A[[U]]} \cong \prod_{e\in U\backslash G/K_\infty} A[[U]]\otimes_{A[[eK_\infty e^{-1}\cap U]]}\big(e(V|_{K_\infty})^\vee\big)|_{A[[eK_\infty e^{-1}\cap U]]}$$

We can now specialize the previous construction to our situation. For $n\geq m\geq 1$ let us define:

$$A_{m,n}\overset{\text{def}}{=}k[\mathrm{U}^-(\varpi^m)/\mathrm{U}^-(\varpi^{n+1})],\qquad\qquad A_m\overset{\text{def}}{=}k[[\mathrm{U}^-(\varpi^m)]].$$

They are regular noetherian local k-algebras and we write $A\overset{\text{def}}{=}A_1$ to ease notations.

By the Iwasawa decomposition, the Verma modules

$$k[[K_0(\varpi^m)]]\otimes_{k[[K_0(\varpi^{n+1})]]}1,\qquad\qquad k[[K_0(\varpi^m)]]\otimes_{k[[K_0(\varpi^\infty)]]}1$$

are free of rank one as $A_{m,n}$, A_m-modules respectively (where $K_0(\varpi^\infty)\overset{\text{def}}{=}\bigcap_{n\in\mathbf{N}}K_0(\varpi^n)$).

Hence $A_{m,n}$, A_m are naturally endowed with a continuous action of Γ, U^+, $\mathbf{T}(k_F)$, and we would like to describe such actions in terms of regular parameters for $A_{m,n}$, A_m.

Note that the objects

$$S^m_{n+1}(\sigma)\overset{\text{def}}{=}k[[K_0(\varpi^m)]]\otimes_{k[[K_0(\varpi^{n+1})]]}\left(\sigma^{(n+1)}\right)^\vee$$

are pseudo-compact modules of finite length over $k[[\mathrm{U}^-(\varpi)]]$ admitting an explicit description in terms of $k[[K_0(\varpi)]]$-stable ideals of A (cf. Proposition 3.3). To ease notations, we omit the Serre weight σ if this is clear from the context.

The universal module $\mathfrak{S}_\infty\overset{\text{def}}{=}(I^-(\pi(\sigma,0))^\vee$ (where $I^-(\pi(\sigma,0))$ is the I-subrepresentation of the universal module $\pi(\sigma,0)$ defined in the work of Hu [Hu1]) is then obtained as an appropriate gluing of the "Verma" modules $S_{n+1}(\sigma)$ along the Hecke operators $(T_n^\pm)^\vee$ (cf. Proposition 3.7). Therefore, in order to obtain any pertinent information of the I-quotients of $\pi(\sigma,0)$ (or, rather, of $I^-(\pi(\sigma,0))$) it is important to understand the I-action on the Iwasawa modules $S_{n+1}(\sigma)$.

Note that the natural action of $\Gamma,\mathrm{U}^+,\mathbf{T}(k_F)$ on A_m is induced by conjugation on the elements of $\mathrm{U}^-(\varpi^m)\subset A^\times$ (and similarly happens for $A_{m,n}$, S_{n+1}, S^m_{n+1}). Therefore, as $\mathbf{T}$ normalizes $\overline{\mathbf{U}}$ it is easy to see that $\Gamma,\mathbf{T}(k_F)$ act by local k-algebra automorphisms on $A_{m,n}$, A_m (and semilinearly on S_{n+1}, S^m_{n+1}).

The finite torus $\mathbf{T}(k_F)$ acts semi-simply on the tangent space of A and we are able to determine, in the unramified case, a regular system of parameters for the maximal ideal $\mathfrak{m}$ of A, formed by $\mathbf{T}(k_F)$-eigenvectors.

Lemma 3.2: *Assume that $F/\mathbf{Q}_p$ is unramified. Define, for $i \in \{0, \dots, f-1\}$, the following elements of A:*

$$X_i \stackrel{\mathrm{def}}{=} \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \in A.$$

The family $\{X_0, \dots, X_{f-1}\}$ is a regular system of parameters for the maximal ideal $\mathfrak{m}$ in A and $\mathbf{T}(k_F)$ acts on X_i by the character $\mathfrak{a}^{-p^i}$

Proof. We have to show that the elements X_i form a basis for the tangent space of A. This is equivalent to asking that the discrete A module

$$k[\mathrm{U}^-(p)/\mathrm{U}^-(p^2)] \cong k[\mathrm{U}^-(p)] \otimes_{k[\mathrm{U}^-(p^2)]} 1$$

admits the images of the elements X_i as a basis for the first graded piece in its radical filtration.

We can now apply [Mo2], Proposition 4.4 (with $m = n = 1$), noticing that X_i is nothing but the element $F^{(1)}_{\underline{p-1}-e_i}$ in the notation of loc. cit.

The statement about the action of $\mathbf{T}(k_F)$ is an easy check. ∎

For $n \geq 1$ consider the natural injection $K_0(p^{n+1}) \hookrightarrow K_0(p)$. It induces a monomorphism of Iwasawa algebras, hence a morphism of Verma modules

$$k[[K_0(p^{n+1})]] \otimes_{k[[K_0(p^\infty)]]} 1 \to k[[K_0(p)]] \otimes_{k[[K_0(p^\infty)]]} 1. \tag{8}$$

This provides, by restriction to $k[[\mathrm{U}^-(p)]]$, a monomorphism of Iwasawa algebras

$$A_{n+1} \hookrightarrow A$$

and it is immediate to see that the elements

$$X_{i-n}^{p^n} = \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p^{n+1}[\lambda] & 1 \end{bmatrix} \in k[[\mathrm{U}^-(p^{n+1})]]$$

form a regular system of parameters for the maximal ideal of A_{n+1}, and $X_{i-n}^{p^n}$ is an $\mathbf{T}(k_F)$-eigenvector, of associated eigencharacter $\mathfrak{a}^{-p^i}$.

3.2. The dual of a Serre weight. We describe here the dual of the Serre weight $\sigma = \sigma_{\underline{r}}$ as an explicit $k[[K_0(p)]]$-quotient of A. We use in a crucial way the $\mathbf{T}(k_F)$-eigenvectors decomposition of the tangent space of A given in Lemma 3.2. Thus, from now until the end of the paper, **we assume that F is unramified**.

Proposition 3.3: *Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight and fix isomorohisms $\iota : \chi_{\underline{r}} \overset{\sim}{\to} (\sigma)^{K_0(p)}$, $\iota^s : (\sigma)_{K_0(p)} \overset{\sim}{\to} \chi_{\underline{r}}^s$. Let $n \geq 1$.*

If $\dim_k(\sigma) \neq q$ then the following Hom spaces are 1-dimensional,

$$Hom_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s) = \langle \phi_n \rangle_k;$$
$$Hom_{K_0(p^n)}(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}, \sigma^{(n)}) = \langle \psi_n \rangle_k$$

and ϕ_n (resp. ψ_n) is a monomorphism (resp. epimorphism) which, as a k-linear map, depends only on ι (resp. ι^s). If $\dim_k(\sigma) = q$ the following Hom spaces are 2-dimensional:

$$Hom_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s) = \langle \phi_n, \widetilde{\phi}_n \rangle_k;$$
$$Hom_{K_0(p^n)}(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}, \sigma^{(n)}) = \langle \psi_n, \widetilde{\psi}_n \rangle_k$$

and ϕ_n, ψ_n are isomorphisms (depending only on ι, ι^s as k-linear maps) while $\widetilde{\phi}_n$, $\widetilde{\psi}_n$ have one dimensional image.

Finally, we have exact sequences of $k[[K_0(p^n)]]$-modules:

(9)
$$0 \to \left\langle X_{i-n+1}^{p^{n-1}(r_i+1)},\ i = 0, \dots, f-1 \right\rangle \to k[[K_0(p^n)]] \otimes_{k[[K_0(p^{n+1})]]} (\chi_{\underline{r}} \det{}^{-\underline{r}}) \to \left(\sigma^{(n)}\right)^\vee \to 0$$

(10)
$$\begin{aligned} 0 &\to \left(\sigma^{(n)}\right)^\vee \to k[[K_0(p^n)]] \otimes_{k[[K_0(p^{n+1})]]} (\chi_{\underline{r}}^s \det{}^{-\underline{r}}) \\ &\to \left(k[[K_0(p^n)]] \otimes_{k[[K_0(p^{n+1})]]} (\chi_{\underline{r}}^s \det{}^{-\underline{r}})\right) / \left\langle \prod_{i=0}^{f-1} X_{i-n+1}^{p^{n-1}(p-1-r_i)} \right\rangle \to 0. \end{aligned}$$

Note that, in the hypotheses of Proposition 3.3, one has

$$k[[K_0(p^n)]] \otimes_{k[[K_0(p^{n+1})]]} (\chi_{\underline{r}} \det{}^{-\underline{r}}) = k[\mathrm{U}^-(p^n)/\mathrm{U}^-(p^{n+1})] \otimes_k (\chi_{\underline{r}} \det{}^{-\underline{r}})$$

as $A_{n,n}$, $\mathbf{T}(k_F)$-modules.

Proof. We start from the exact sequence (9).

We see that the $K_0(p^{n+1})$-restriction of $\sigma^{(n)}$ is described by

$$\sigma^{(n)}|_{K_0(p^{n+1})} = \bigoplus_{0\leq \underline{j}\leq \underline{r}} \chi_{\underline{r}}^s \mathfrak{a}^{\underline{j}}.$$

Note now that the isomorphism ι induces, by conjugation by the element $\lambda_n(p)s$, an isomorphism $\iota^{(n)} : \left(\sigma^{(n)}\right)^{K_0(p^n)} \xrightarrow{\sim} \chi_{\underline{r}}^s$. In particular, we have $\left(\sigma^{(n)}\right)^{K_0(p^n)} = (\sigma)^{K_0(p)}$ as k-linear spaces, and therefore if we define ϕ_n to be the image of $\iota^{(n)}$ by the Frobenius reciprocity isomorphism

$$\mathrm{Hom}_{K_0(p^{n+1})}(\sigma^{(n)}, \chi_{\underline{r}}^s) \cong \mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s) \tag{11}$$

we see that ϕ_n does not depend on n as a k-linear morphism.

If $\dim(\sigma) \neq q$, the $\mathbf{T}(k_F)$-characters of σ are all distinct, so that, by (11), the Hom space $\mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s)$ is one-dimensional.

Moreover, ϕ_n is an **injective** morphism: by construction, $\left(\phi_n(v_{\underline{r}})\right)(1)$ is a linear generator of $\chi_{\underline{r}}^s$ if $v_{\underline{r}} \in (\sigma)^{K_1(p)}$ is non-zero. Thus $\left(\ker(\phi_n)\right)^{\mathrm{U}^-(p^n)} = 0$, and the claim follows as $\mathrm{U}^-(p^n)$ is a pro-p group.

If $\dim(\sigma) = q$ then the lowest weight vector $v^{(\underline{r})}$ and the highest weight vector $v_{\underline{r}}$ in σ are the only $\mathbf{T}(k_F)$-eigenvectors of $\sigma^{(n)}$ affording the character $\chi_{\underline{r}}^s$. We deduce two linearly independent morphisms

$$\phi_n, \widetilde{\phi}_n \in \mathrm{Hom}_{K_0(p^n)}(\sigma^{(n)}, \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s)$$

characterized by $\left(\phi_n(v_{\underline{r}})\right)(1) = e$ and $\left(\widetilde{\phi}_n(v^{(\underline{r})})\right)(1) = e$ for a linear generator e of $\chi_{\underline{r}}^s$. As above, we see that ϕ_n is a monomorphism (independent from n as a k-linear map) and hence an isomorphism for dimension reasons; and moreover that $\mathrm{soc}(\sigma^{(n)})$ is a subspace of $\ker(\widetilde{\phi}_n)$.

As the characters of the socle filtration for $\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}}^s$ are all distinct except for those appearing in the socle and the cosocle we deduce that $\widetilde{\phi}_n$ has to factor via $\mathrm{cosoc}(\sigma^{(n)})$ into a non-zero morphism.

Passing to duals we deduce an epimorphism of $k[[K_0(p)]]$-modules:

$$\phi_n^\vee : k[\mathrm{U}^-(p^n)/\mathrm{U}^-(p^{n+1})] \otimes_k (\chi_{\underline{r}}\det{}^{-\underline{r}}) \twoheadrightarrow \left(\sigma^{(n)}\right)^\vee$$

which is an isomorphism if $\dim(\sigma) = q$.

Assume $\dim(\sigma) \neq q$. By counting dimensions, the equality

$$\ker(\phi_n^\vee) = \left\langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i = 0, \ldots, f-1 \right\rangle$$

is established once we show that $X_{i-n+1}^{p^{n-1}(r_i+1)} \in \ker(\phi^\vee)$ for any $i = 0, \ldots, f-1$.

This is immediate, since the $\mathbf{T}(k_F)$ eigencharacters of $\big(\sigma^{(n)}\big)^\vee$ (which are all distinct) are described by

$$\big(\sigma^{(n)}\big)^\vee|_{\mathbf{T}(k_F)} = \bigoplus_{0\leq \underline{j}\leq \underline{r}} (\chi_{\underline{r}}\det{}^{-\underline{r}})\mathfrak{a}^{-\underline{j}}$$

while $\mathbf{T}(k_F)$ acts on $X_{i-n+1}^{p^{n-1}(r_i+1)}$ by $\big(\chi_{\underline{r}}\det{}^{-\underline{r}}\big)\mathfrak{a}^{-(r_i+1)e_i}$.

The proof of the existence of the natural exact sequence (10) is similar and left to the reader, noticing that $\operatorname{cosoc}\big((\sigma^{(n)})^\vee\big) \cong \big(\operatorname{soc}(\sigma^{(n)})\big)^\vee = (\chi_{\underline{r}}^s)^\vee$ and that the $\mathbf{T}(k_F)$-eigencharacter of $\prod_{i=0}^{f-1} X_{i-n+1}^{p^{n-1}(p-1-r_i)}$ is $(\chi_{\underline{r}}^s)^\vee$. ∎

3.3. The dual of the universal module. In this section we complete the dictionary between the representations $R^-_{\infty,\bullet}$ and the corresponding Pontryagin dual $\mathfrak{S}^\bullet_\infty$.

We first describe the dual of the Hecke morphisms $T_n^\pm$ in terms of $k[[K_0(p)]]$-modules. For each finite level of the dual operators $(T_n^\pm)^\vee$ let us glue the modules S_{n+1}, obtaining certain $k[[K_0(p)]]$-subquotients $\mathfrak{S}^\bullet_{n+1}$ of A, i.e. $K_0(p)$-stable ideals generated by an explicit family of monomials in A. The universal modules $\mathfrak{S}^\bullet_\infty$ are obtained as a limit of the modules $\mathfrak{S}^\bullet_{n+1}$ via appropriate transition maps; they are not of finite type over A (except if $F = \mathbf{Q}_p$).

Let $n \geq 2$. Recall (§2) that the Hecke morphism T^+_{n-1} is obtained as the induction, from $K_0(p^{n-1})$ to $K_0(p)$, of the $K_0(p^{n-1})$-equivariant morphism

$$t^+_{n-1} : \sigma^{(n-1)} \to \operatorname{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \sigma^{(n)}.$$

Let $K_1(p^n)$ be the maximal pro-p subgroup of $K_0(p^n)$ and fix an isomorphism $\iota : (\sigma)^{K_0(p)} \xrightarrow{\sim} \chi_{\underline{r}}$ as in the statement of Proposition 3.3.

Since $K_1(p^n)$ is normal in $K_0(p^{n-1})$ and it acts trivially on $\sigma^{(n-1)}$ we deduce a factorization

(12)
$$\begin{tikzcd} \sigma^{(n-1)} \arrow[r] \arrow[dr, hook] & \operatorname{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \sigma^{(n)} \\ & \operatorname{ind}_{K_0(p^n)}^{K_0(p^{n-1})} \chi_{\underline{r}}^s \arrow[u, hook] \end{tikzcd}$$

where the vertical arrow is induced from the $\lambda_n(p^n)s$-twist of ι.

Note that t_n^+ does not depend on n as a k-linear map and, thanks to its explicit definition given in (5), we may and do assume that the isomorphism ι is such that the diagonal arrow in the diagram (12) is precisely ϕ_{n-1}, independently from n.

Dualizing (12) and using Lemma 3.3 we obtain

$$\begin{tikzcd}
\left(\sigma^{(n-1)}\right)^{\vee} & \left(\chi_{\underline{r}}^{s}\right)^{\vee} \otimes_k k[[\mathrm{U}^{-}(p^{n-1})]] \otimes k[[\mathrm{U}^{-}(p^{n})]]/\left\langle X_{i-n+1}^{p^{n-1}(r_i+1)},\, i\right\rangle \arrow[l, two heads] \arrow[d] \\
 & k[[\mathrm{U}^{-}(p^{n-1})]] \otimes \left(\chi_{\underline{r}}^{s}\right)^{\vee} \arrow[ul, two heads]
\end{tikzcd}$$

where the vertical arrow is induced by base change from the projection $\left(\sigma^{(n)}\right)^{\vee} \twoheadrightarrow (\chi_{\underline{r}}^{s})^{\vee}$ and the tensor product is over $k[[\mathrm{U}^{-}(p^{n})]]$ (otherwise stated). We conclude that $(T_{n-1}^{+})^{\vee}$ is the natural projection for any $n \geq 2$.

We turn our attention to $(T_n^{-})^{\vee}$. As above, we fix an isomorphism $\iota^s : (\sigma)_{K_0(p)} \overset{\sim}{\to} \chi_{\underline{r}}^{s}$ and recall that the Hecke morphism T_n^{-} is obtained as the induction, from $K_0(p^n)$ to $K_0(p)$, of the $K_0(p^n)$-equivariant morphism

$$t_n^{-} : \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} \to \sigma^{(n)}.$$

Since $K_1(p^{n+1})$ (the maximal pro-p subgroup of $K_0(p^{n+1})$) acts trivially on $\sigma^{(n)}$ and is normal in $K_0(p^n)$, we deduce a factorization

(13)
$$\begin{tikzcd}
\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \sigma^{(n+1)} \arrow[r, two heads] \arrow[d, two heads] & \sigma^{(n)} \\
\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_{\underline{r}} \arrow[ur, two heads] &
\end{tikzcd}$$

where the vertical arrow is induced from the $\lambda_{n+1}(p)s$-twist of ι^s.

As before we may and do fix the isomorphism ι^s in such a way that the diagonal arrow in the diagram (13) is precisely ψ_n for any $n \geq 1$.

Dualizing (13) and using Lemma 3.3 we obtain

$$\begin{tikzcd}
(\chi_{\underline{r}})^{\vee} \otimes_k k[[\mathrm{U}^{-}(p^{n})]] \otimes k[[\mathrm{U}^{-}(p^{n+1})]]/\langle X_{i-n}^{p^{n}(r_i+1)}, i\rangle & (\chi_{\underline{r}})^{\vee} \otimes_k k[[\mathrm{U}^{-}(p^{n})]]/\langle X_{i-n+1}^{p^{n-1}(r_i+1)}, i\rangle \arrow[l, hook'] \arrow[dl, hook'] \\
k[[\mathrm{U}^{-}(p^{n})]] \otimes (\chi_{\underline{r}})^{\vee} \arrow[u, hook] &
\end{tikzcd}$$

where the vertical arrow is induced by base change from the injection

$$\begin{aligned}\big(\chi_{\underline{r}}\big)^\vee &\hookrightarrow (\chi_{\underline{r}})^\vee \otimes_k k[[\mathrm{U}^-(p^{n+1})]]/\big\langle X_{i-n}^{p^n(r_i+1)},\, i\big\rangle\\ 1 &\mapsto \prod_{l=0}^{f-1} X_{i-n}^{p^n r_i},\end{aligned}$$

the diagonal arrow is deduced from Lemma 3.3 and the tensor product in the RHS is over $k[[\mathrm{U}^-(p^{n+1})]]$ (otherwise stated).

We obtain

Proposition 3.4: *Let $n \geq 1$.*

The dual of the partial Hecke operator $T_n^+ : R_n^- \otimes \chi_{-\underline{r}}^s \to R_{n+1}^- \otimes \chi_{-\underline{r}}^s$ is the natural surjection:

$$A/\big\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\big\rangle \twoheadrightarrow A/\big\langle X_i^{p^{n-1}(r_{i+n-1}+1)},\, i=0,\dots,f-1\big\rangle.$$

The dual of the partial Hecke operator $T_{n+1}^- : R_{n+1}^- \otimes \chi_{-\underline{r}}^s \to R_n^- \otimes \chi_{-\underline{r}}^s$ is the monomorphism:

$$\begin{aligned}A/\big\langle X_i^{p^{n-1}(r_{i+n-1}+1)},\, i=0,\dots,f-1\big\rangle &\hookrightarrow A/\big\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\big\rangle\\ 1 &\mapsto \prod_{i=0}^{f-1} X_i^{p^{n-1}(p(r_{i+n}+1)-(r_{i+n-1}+1))}.\end{aligned}$$

Finally $T_1^- : R_1 \to R_0$ is dualized to:

$$\begin{aligned}\big(\chi_{\underline{r}}\big)^\vee &\hookrightarrow A \otimes_k \big(\chi_{\underline{r}}^s\big)^\vee/\big\langle X_i^{(r_i+1)},\, i=0,\dots,f-1\big\rangle\\ 1 &\mapsto \prod_{i=0}^{f-1} X_i^{r_i}.\end{aligned}$$

Proof. This is deduced from the previous discussion for $n \geq 2$. The case $n = 1$ is immediate. ∎

We can now describe the Pontryagin dual $\mathfrak{S}_\infty^\bullet$ of $R_{\infty,\bullet}^-$ as a projective limit of certain $k[[K_0(p)]]$-modules of finite length, which are explicit $K_0(p)$-stable ideals of A.

Indeed, we can introduce a system of $k[[K_0(p)]]$-modules obtained by a recursive fibered product along the dual Hecke morphisms: if we assume the injection $S_\bullet \times_{S_{\bullet+1}} \cdots {}_{S_{n-2}} S_{n-1} \hookrightarrow S_{n-1}$ being constructed, we define $S_\bullet \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}$

through the following cartesian diagram

$$\begin{tikzcd}
S_\bullet \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1} \arrow[r, dotted, two heads] \arrow[dd, dotted, hook] & S_\bullet \times_{S_{\bullet+1}} \cdots \times_{S_{n-1}} S_{n-1} \arrow[d, hook] \\
 & S_{n-1} \arrow[d, hook, "(T_n^-)^\vee"] \\
S_{n+1} \arrow[r, two heads, "(T_n^+)^\vee"] & S_n
\end{tikzcd}$$

where the upper (resp. left) dotted arrow is an epimorphism (resp. monomorphism) by base change.

Definition 3.5: For $\bullet \in \{0,1\}$ let $m, n \in 2\mathbf{N}+1+\bullet$ be such that $n \geq m$. We define the $k[[K_0(p)]]$-modules

$$\mathfrak{S}^\bullet_{n+1} \overset{\text{def}}{=} S_\bullet \times_{S_{\bullet+1}} \cdots \times_{S_n} S_{n+1}, \qquad \mathfrak{S}^{\geq m}_{n+1} \overset{\text{def}}{=} \ker\left(\mathfrak{S}^\bullet_{n+1} \twoheadrightarrow \mathfrak{S}^\bullet_{m-1}\right).$$

For $n \geq 1$ define the integer

$$m_{k,l,n}(\sigma) \overset{\text{def}}{=} \sum_{s=k}^{n-1} p^s (r_{l+s}+1)(-1)^{s+k+1}$$

and we write $m_{k,l}$ if n, σ are clear from the context. Note that for all $j \geq 1$ one has

$$\mathcal{K}er_{j+1} \overset{\text{def}}{=} \ker\bigl(\mathfrak{S}^\bullet_{j+1} \to \mathfrak{S}_{j-1}\bigr) = \bigl\langle X_i^{p^{j-1}(r_{i+j-1}+1)},\, i = 0, \ldots, f-1 \bigr\rangle$$

and therefore by the description of the Hecke operators given in Proposition 3.4, one obtains:

Proposition 3.6: *For* $\bullet \in \{0,1\}$ *let* $m, n \in 2\mathbf{N}+1+\bullet$ *be such that* $n \geq m$. *The elements*

$$e_{2(j+1)+\bullet,\, i} \overset{\text{def}}{=} X_i^{p^{2j+\bullet}(r_{i+2j+\bullet}+1)} \prod_{l=0}^{f-1} X_l^{m_{2j+1+\bullet,l}}$$

for $i = 0, \ldots, f-1$, $j = 0, \ldots, \frac{n-1-\bullet}{2}$ *and*

a) *the element* $e_0 \overset{\text{def}}{=} \prod_{l=0}^{f-1} X_l^{-1-m_{0,l}(\sigma)}$ *if* $\bullet = 0$,
b) *the element* $e_1 \overset{\text{def}}{=} \prod_{l=0}^{f-1} X_l^{m_{0,l}(\sigma)}$ *if* $\bullet = 1$,

form a family $\mathscr{G}^{\bullet}_{n+1}$ of A-generators for the $k[[K_0(p)]]$-stable ideal $\mathfrak{S}_{n+1} \lhd A_{1,n+1}$.

Similarly, the set

$$\mathscr{G}^{\geq m}_{n+1} \stackrel{\text{def}}{=} \Big\{ e_{2(j+1)+\bullet,i}, \text{ for } \frac{m-1-\bullet}{2} \leq j \leq \frac{n-1-\bullet}{2}, \, i \in \{0, \ldots, f-1\} \Big\}$$

is a family of A-generators for the $k[[K_0(p)]]$-stable ideal $\mathfrak{S}^{\geq m}_{n+1} \lhd A_{1,n+1}$.

Note that, a priori, it is not at all obvious that the submodule of S_{n+1} generated by the elements listed in $\mathscr{G}^{\geq m}_{n+1}$ is stable under the action of Γ, U^+.

We can summarize the preceding discussion in the following:

Proposition 3.7: *Let $\sigma = \sigma_{\underline{r}}$ be a Serre weight. For $\bullet \in \{0, 1\}$ there is an inductive system of $k[[K_0(p)]]$-modules*

$$\cdots \twoheadrightarrow \mathfrak{S}_{n+1} \twoheadrightarrow \mathfrak{S}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow S_{\bullet}$$

such that, for all $n \geq 1$, the transition morphisms fit into a commutative diagram with exact rows:

(14)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}er_{n+1} \arrow[r] \arrow[dd, equal] & \mathfrak{S}_{n+1} \arrow[r] \arrow[dd, hook] & \mathfrak{S}_{n-1} \arrow[r] \arrow[d, hook] & 0 \\
 & & & S_{n-1} \arrow[d, hook, "(T_n^-)^\vee"] & \\
0 \arrow[r] & \mathcal{K}er_{n+1} \arrow[r] & S_{n+1} \arrow[r, "(T_n^+)^\vee"] & S_n \arrow[r] & 0
\end{tikzcd}$$

and where $\mathcal{K}er_{n+1} = \big\langle X^{p^{n-1}(r_i+1)}_{i-n+1}, \, i = 0, \ldots, f-1 \big\rangle \leq S_{n+1}$.

By letting

$$\mathfrak{S}^{\bullet}_{\infty} \stackrel{\text{def}}{=} \varprojlim_{n \in 2\mathbf{N}+1+\bullet} \mathfrak{S}^{\bullet}_{n+1}, \quad \big(\mathit{resp}\ \mathfrak{S}^{\geq m}_{\infty} \stackrel{\text{def}}{=} \ker\big(\mathfrak{S}^{\bullet}_{\infty} \twoheadrightarrow \mathfrak{S}^{\bullet}_{m-1}\big) \text{ if } m \in 2\mathbf{N}+1+\bullet\big)$$

we have a family $\mathscr{G}^{\bullet}_{\infty}$ (resp. $\mathscr{G}^{\geq m}_{\infty}$) of topological A-generators for $\mathfrak{S}^{\bullet}_{\infty}$ (resp. $\mathscr{G}^{\geq m}_{\infty}$)

$$\mathscr{G}^{\bullet}_{\infty} = \big\{ e_{2(j+1)+\bullet,i}, \, e_{\bullet}, \text{ for } j \in \mathbf{N}, \, i \in \{0, \ldots, f-1\} \big\}$$
$$\Big(\text{resp. } \mathscr{G}^{\geq m}_{\infty} \stackrel{\text{def}}{=} \Big\{ e_{2(j+1)+\bullet,i}, \text{ for } j \geq \frac{m-1-\bullet}{2}, \, i \in \{0, \ldots, f-1\} \Big\}\Big)$$

which is compatible at each finite level with the families $\mathscr{G}^{\bullet}_{n+1}$, $\mathscr{G}^{\geq m}_{n+1}$, i.e. for any $n \in 2\mathbf{N}+1+\bullet$, $i \in \{0, \ldots, f-1\}$, $0 \leq j \leq \frac{n-1-\bullet}{2}$ we have $e_{2(j+1)+\bullet,i} \mapsto e_{2(j+1)+\bullet,i}$, $e_{\bullet} \mapsto e_{\bullet}$ via $\mathfrak{S}^{\bullet}_{\infty} \twoheadrightarrow \mathfrak{S}^{\bullet}_{n+1}$.

Proof. Everything is clear from the previous discussion, the only non-trivial assertion being the compatibility of the elements in $\mathscr{G}^{\bullet}_{n+1}$, $\mathscr{G}^{\bullet}_{n-1}$ via the transition morphisms $\mathfrak{S}^{\bullet}_{n+1} \twoheadrightarrow \mathfrak{S}^{\bullet}_{n-1}$.

However, this is an elementary check using the explicit definition of the elements $e_{2(j+1)+\bullet,i}$ and of the morphism $(T_n^-)^\vee$.

For instance, in the particular case where $\bullet = 0$ (i.e. n odd) and $j = 0$ one has

$$X_i^{r_i+1} \prod_l X_l^{m_{1,l,n}} = \Big(X_i^{r_i+1} \prod_l X_l^{m_{1,l,n-2}}\Big) \cdot \prod_l X_l^{p^{n-2}(p(r_{l+n-1}+1)-(r_{l+n-2}+1))}$$

in other words $(T_n^+)^\vee(e_{2,i}) = (T_n^-)^\vee(e_{2,i})$ (where $e_{2,i} \in \mathscr{G}^{\bullet}_{n+1}$ in the LHS and $e_{2,i} \in \mathscr{G}^{\bullet}_{n-1}$ in the RHS), which is precisely the required compatibility. ∎

Remark 3.8: The bare definition of the elements in $\mathscr{G}^{\bullet}_{n+1}$ may look complicated, but it becomes very natural if one visualizes the transition morphisms $(T_n^{\pm})^\vee$ in terms of monomials in A: see the example in Figure 1.

Remark 3.9: Note that if in the statement of Proposition 3.6 we moreover assume that $r_i < p-1$ for all i, then the elements in $\mathscr{G}^{\bullet}_{n+1}$ are all non-zero in $\mathfrak{S}^{\bullet}_{n+1}$.

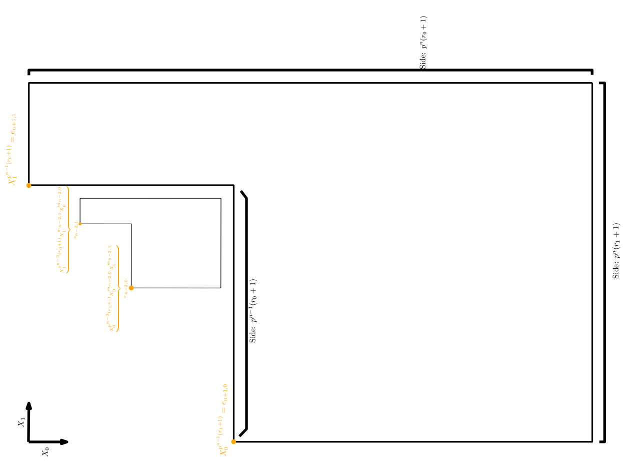

Figure 1. The figure represents the fibered product $\mathcal{K}er_{n+1} \times_{S_n} \mathcal{K}er_{n-1}$ when $f = 2$, $r_0 > r_1$, $n \in 2\mathbf{N}+1$.

4. A filtration on monogenic Iwasawa modules

The aim of this section is to give a first partial control of the Γ-and U^+-action on the $\mathfrak{m}$-adic filtration on A. Even though the Γ-action is comparatively simple to control (as the action is through k-algebra homomorphisms), the U^+ is extremely subtle and its partial control (Corollary 4.9) is one of the technical hearts of this paper.

Recall that A has a structure of a $k[[K_0(p)]]$-module via the isomorphism

$$\big(k[[K_0(p)]]\otimes_{k[[K_0(p^\infty)]]}1\big)|_A \xrightarrow{\sim} A$$

and it is endowed with the $\mathfrak{m}$-adic valuation:

$$\begin{array}{rcl} A & \overset{\mathrm{ord}}{\longrightarrow} & \mathbf{N}\cup\{\infty\} \\ \sum\limits_{\underline{j}}\kappa_{\underline{j}}\underline{X}^{\underline{j}} & \longmapsto & \min\big\{|\underline{j}|,\ \ \underline{j}\text{ s.t. }\kappa_{\underline{j}}\neq 0\big\}. \end{array}$$

We are going to show that ord is compatible with the action of Γ, U^+ and, even more precisely,

Proposition 4.1: *Let $g\in\Gamma$, U^+ and $P(\underline{X})\in A$. Then*

$$ord\big((g-1)\cdot P(\underline{X})\big)\geq ord\big(P(\underline{X})\big)+(p-2).$$

The proof occupies the rest of this section.

Recall that A is endowed with a Frobenius homomorphism ϕ defined by $\phi(X_i)=X_i^p$. For $k\in\mathbf{N}$ we write $\big(\phi^k(\mathfrak{m})\big)$ for the ideal of A generated by the image of $\mathfrak{m}$ via the k-th composite of the Frobenius ϕ^k. Note that in particular any element $P(\underline{X})$ in $\big(\phi^k(\mathfrak{m})\big)$ verifies

$$\mathrm{ord}\big(P(\underline{X})\big)\geq p^k.$$

The following observation will be used constantly:

Lemma 4.2: *Let $n\in\mathbf{N}$ and $z\in\mathscr{O}_F$. In the Iwasawa algebra A we have*

$$\begin{bmatrix} 1 & 0\\ pz & 1\end{bmatrix}\in 1+\phi^{val(z)}(\mathfrak{m}).$$

Proof. Writing $z = p^{val(z)} z_0$ we have

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 \\ pz_0 & 1 \end{bmatrix} \right)^{p^{val(z)}} \\ &= \left(\begin{bmatrix} 1 & 0 \\ pz_0 & 1 \end{bmatrix} - 1 \right)^{p^{val(z)}} + 1\end{aligned}$$

and the result follows since the maximal ideal of A is the augmentation ideal. ∎

We start with the action of Γ:

Lemma 4.3: *Let $\gamma \in 1 + p\mathscr{O}_F \cong \Gamma$. Then, for any $i = 0, \dots, f-1$ we have*

$$\gamma \cdot X_i \in X_i + \left(\phi^{val(x)+1}(\mathfrak{m}) \right).$$

In particular, if $\underline{j} \in \mathbf{N}^f$ we have

$$\gamma \cdot \underline{X}^{\underline{j}} \in \underline{X}^{\underline{j}} + \sum_{\underline{i} \gneqq 0} \underline{X}^{\underline{j}-\underline{i}} \left(\phi^{val(x)+1}(\mathfrak{m}) \right)^{|\underline{i}|}.$$

Proof. Let us write $\gamma = 1 + px$ and $z \stackrel{\text{def}}{=} p[\lambda]x$ for an element $x \in \mathscr{O}_F$. We have

$$\begin{aligned}\gamma \cdot X_i &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] + p^2[\lambda]x & 1 \end{bmatrix} \\ &= \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} \\ &\in \sum_{\lambda \in k_F^\times} \lambda^{-p^i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \left(1 + \phi^{val(z)}(\mathfrak{m}) \right),\end{aligned}$$

where the last equality follows from Lemma 4.2.

The second statement is then clear, as the elements of Γ act by k-algebra endomorphisms. ∎

4.1. Digression on some regular elements and Fourier sums. The action of U^+ on A is more subtle: it is only k-linear and difficult to explicitly describe in terms of monomials in A. Nevertheless, we are able to approximate the monomials in A by means of certain discrete Fourier series in $\mathscr{C}^\infty\left(\mathrm{U}^-(p), k\right)$ and this is enough to get a first estimate on the U^+-action.

Let $n \geq m \geq 1$. For $i \in \{m, \dots, n\}$ let $\underline{l}_i \in \{0, \dots, p-1\}^{\mathrm{Hom}(k_F, k)}$ be an f-tuple, say $\underline{l}_i = \{l_{i,\tau}\}_{\tau \in \mathrm{Hom}(k_F,k)}$. We introduce the following elements of $A_{m,n}$:

$$F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \stackrel{\mathrm{def}}{=} \sum_{\lambda_m \in \mathbf{F}_q} \prod_{\tau} \tau(\lambda_m)^{l_{m,\tau}} \begin{bmatrix} 1 & 0 \\ p^m[\varphi^{-m+1}(\lambda_m)] & 1 \end{bmatrix}$$
$$\dots \sum_{\lambda_n \in \mathbf{F}_q} \prod_{\tau} \tau(\lambda_n)^{l_{n,\tau}} \begin{bmatrix} 1 & 0 \\ p^n[\varphi^{-n+1}(\lambda_n)] & 1 \end{bmatrix},$$
$$X_{m,\tau} \stackrel{\mathrm{def}}{=} \sum_{\lambda_m \in \mathbf{F}_q} \tau(\lambda_m)^{-1} \begin{bmatrix} 1 & 0 \\ p^m[\varphi^{-m+1}(\lambda_m)] & 1 \end{bmatrix},$$

where φ is the absolute Frobenius on k_F and the products appearing in the definition of $F^{(m,n)}$ are taken over all the embeddings $\tau : k_F \hookrightarrow k$ (with the usual convention that $0^0 \stackrel{\mathrm{def}}{=} 1$).

We fix an embedding $\tau_0 : k_F \hookrightarrow k$ and write more expressively $l_{i,j} \stackrel{\mathrm{def}}{=} l_{i,\tau_0 \circ \varphi^j}$ so that

$$\prod_{\tau} \tau(\lambda_i)^{l_{i,\tau}} = \lambda_i^{\sum_{j=0}^{f-1} p^j l_{i,j}} \stackrel{\mathrm{def}}{=}: \lambda_i^{\underline{l}_i}.$$

If the level m is clear, we just write X_i instead of $X_{m,\tau_0 \circ \varphi^i}$ to ease the notations.

Recall from [Mo1], §4.1.1 that $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ (and $\underline{X}^{\underline{l}}$ as well) can be identified by an element in $\mathbf{N}^f$:

$$F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \longleftrightarrow \left(\sum_{i=m}^{n} p^{i-m} l_{i,\tau} \right)_{\tau}.$$

Define the quantity

$$\kappa_{\underline{l}}^{-1} \stackrel{\mathrm{def}}{=} (\kappa_{\underline{l}_m,\dots,\underline{l}_n})^{-1} \stackrel{\mathrm{def}}{=} \prod_{i=m}^{n} \left((-1)^{f-1} \prod_{j=0}^{f-1} (p-1-l_{i,j})! \right) \in \mathbf{F}_p^{\times}.$$

The following proposition provides us with a dictionary between the Fourier sums defined above and the monomial elements in A:

Proposition 4.4: *Let $n \geq m \geq 0$ and let $\underline{l} = (\underline{l}_m, \dots, \underline{l}_n) \in \left\{\{0, \dots, p-1\}^f\right\}^{(n-m)}$ be an $(n-m)$-tuple of f-tuples.*

Then one has the following equality in $A_{m,n}$:

$$\underline{X}^{\underline{l}} \equiv \kappa_{\underline{l}} F^{(m,n)}_{\underline{p-1}-\underline{l}_m,\dots,\underline{p-1}-\underline{l}_n} \bmod \mathfrak{m}^{|\underline{l}|+(p-1)}$$

where

$$\underline{X}^{\underline{l}} = \prod_{j=0}^{f-1} X_j^{\sum_{i=m}^{n} p^{i-m} l_{i,j}}$$

and

$$\underline{p-1} - \underline{l_i} \overset{\text{def}}{=} (p-1-l_{i,j})_{j=0}^{f-1}$$

for all $i = m, \dots, n$.

We invite the reader to compare the statement of Proposition 4.4 with [Bre], Théorème 7.1, where we have a similar statement for the image of the elements X_i in $k[[X]]$ via the morphism $\mathscr{O}_F \to \mathbf{Z}_p$ induced by the trace (and $\mathfrak{m}^{|\underline{j}|+(p-1)}$ is replaced by $\mathfrak{m}^{|\underline{j}|+1}$ in loc. cit.).

Proof. The proof is divided into two steps: the residual case ($n-m=1$) and a dévissage. Note that for $n-m=1$ the statement is clear up to the explicit multiplicative constant, by looking at the action of the finite torus.

Lemma 4.5: *Keep the setting of Proposition 4.4 and assume that* $n-m=1$.

For any f-tuple $\underline{l} \in \{0,\dots,p-1\}^f$ *we have the following equality in* $A_{m,m+1}$*:*

$$\underline{X}^{\underline{l}} = \begin{cases} \kappa_{\underline{l}} F_{\underline{p-1}-\underline{l}} & \text{if} |\underline{l}| > 0, \\ \kappa_0 F_{\underline{p-1}} + (-1)^{f-1} \underline{X}^{\underline{p-1}} & \text{else.} \end{cases}$$

Proof. Note first that

$$\kappa_{\underline{l}+e_i} = (p-1-l_i)\kappa_{\underline{l}} \tag{15}$$

and that $\kappa_{e_i} = 1$ for all $i \in \{0,\dots,f-1\}$. The statement is therefore an immediate induction using Lemma 4.6 below. ∎

Lemma 4.6: *Keep the hypotheses of Lemma 4.5. Assume moreover that* $\underline{l}+e_i \leq \underline{p-1}$. *Then:*

$$F_{\underline{p-1}-e_i} F_{\underline{p-1}-\underline{l}} = (p-1-l_i) F_{\underline{p-1}-(\underline{l}+e_i)}.$$

Proof. By the very definition of the elements $F_{\underline{p-1}-e_i}$, $F_{\underline{p-1}-\underline{l}}$ we have

$$\begin{aligned} F_{\underline{p-1}-e_i} F_{\underline{p-1}-\underline{l}} &= \sum_{\lambda,\mu \in k_F} \lambda^{\underline{p-1}-e_i} (\mu-\lambda)^{\underline{p-1}-\underline{l}} \begin{bmatrix} 1 & 0 \\ p^m[\varphi^{-m+1}(\lambda)] & 1 \end{bmatrix} \\ &= \sum_{\underline{j} \leq \underline{p-1}-\underline{l}} \binom{\underline{p-1}-\underline{l}}{\underline{j}} (-1)^{\underline{j}} \sum_{\lambda \in k_F} \lambda^{\underline{p-1}-e_i+\underline{j}} F_{\underline{p-1}-\underline{l}-\underline{j}} \end{aligned}$$

and the result follows since

$$\sum_{\lambda\in k_F}\lambda^{\underline{p-1}-e_i+\underline{j}}=-\delta_{\underline{j},e_i}. \qquad \blacksquare$$

We consider now the dévissage. Recall that the inclusion $p^{m+1}\mathscr{O}_F/p^n\mathscr{O}_F\hookrightarrow p^m\mathscr{O}_F/p^n\mathscr{O}_F$ induces an injective k-algebra homomorphism:

$$\begin{aligned}\iota: A_{m+1,n} &\hookrightarrow A_{m,n}\\ X_{m+1,i} &\mapsto X_{m,i}^p.\end{aligned}$$

In order to emphasize the inductive argument, we write $\mathfrak{m}$, $\mathfrak{m}_1$ to denote the maximal ideal of A, A_1 respectively (so that, in particular, $\iota(\mathfrak{m}_1)=\mathfrak{m}^p$).

Given a monomial $\underline{X}^{\underline{l}}\in A_{m,n}$, we can write

$$\underline{X}^{\underline{l}}=\underline{X}^{\underline{l}^{(1)}}\iota\big(\underline{X}^{\underline{l}^{(2)}}\big)$$

for $\underline{l}^{(1)}\in\{0,\dots,p-1\}^f$, $\underline{l}^{(2)}\in\mathbf{N}^f$ verifying $\underline{l}=\underline{l}^{(1)}+p\underline{l}^{(2)}$.

By the inductive hypothesis on $A_{m+1,n}$ we have

$$\iota\big(\underline{X}^{\underline{l}^{(2)}}\big)\in\kappa_{\underline{l}^{(2)}}F^{(m+1,n)}_{\underline{p-1}-\underline{l}^{(2)}}+\iota\big(\mathfrak{m}_1^{|\underline{l}^{(2)}|+(p-1)}\big)=\kappa_{\underline{l}^{(2)}}F^{(m+1,n)}_{\underline{p-1}-\underline{l}^{(2)}}+\mathfrak{m}^{p|\underline{l}^{(2)}|+p(p-1)} \tag{16}$$

and we claim that

Claim: *In the situation above, we have*

$$\underline{X}^{\underline{l}^{(1)}}\in\kappa_{\underline{l}^{(1)}}F^{(m)}_{\underline{p-1}-\underline{l}^{(1)}}\mod\mathfrak{m}^{|\underline{l}^{(1)}|+(p-1)}. \tag{17}$$

This will imply the statement of Proposition 4.4, since from (16) and (17) we easily get

$$\underline{X}^{\underline{l}}\equiv\kappa_{\underline{l}}F^{(m,n)}_{\underline{p-1}-\underline{l}_m,\dots,\underline{p-1}-\underline{l}_n}+\mathfrak{m}^{|\underline{l}|+(p-1)}.$$

Proof of the Claim By Lemma 4.5 we have, in $A_{m,n}$,

$$\underline{X}^{\underline{l}^{(1)}}\in\kappa_{\underline{l}^{(1)}}F^{(m)}_{\underline{p-1}-\underline{l}^{(1)}}+\sum_{i=0}^{f-1}X_i^p\cdot A_{m,n}. \tag{18}$$

Let us consider a monomial $X_i^p\underline{X}^{\underline{t}}$ appearing with a non-zero coefficient in the sum $\sum_{i=0}^{f-1}X_i^pA_{m,n}$ in the RHS of (18). As the finite torus $\mathbf{T}(k_F)$ acts semisimply on $A_{m,n}$ and $\underline{X}^{\underline{l}^{(1)}}$, X_i^p are eigenvectors, we deduce that the f-tuple $\underline{t} \in \mathbf{N}$ verifies

$$\sum_{j=0}^{f-1} p^j r_j \equiv \sum_{j=0}^{f-1} p^j l_j^{(1)} - p^{i+1} \bmod q-1.$$

This implies $|\underline{t}| \equiv |\underline{l}^{(1)}| - 1 \bmod p-1$, hence the Claim. ∎

We are now ready to analyze the U^+-action on the monomials in A. Note that for $x \in \mathscr{O}_F$ the action of $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$ on the Iwasawa algebra $k[[\mathrm{U}^-(p)]]$ is obtained, by linearity, from the following continuous maps

$$\begin{aligned} \mathscr{O}_F &\xrightarrow{\delta_{x,j}} \mathscr{O}_F \\ z &\longmapsto \frac{z}{1+pzx} \end{aligned}$$

via

$$\begin{aligned} (19)\quad &\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^j z & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ p^j \delta_{x,j}(z) & 1 \end{bmatrix} \begin{bmatrix} 1+p^j xz & 0 \\ 0 & 1-p^j x \delta_{x,j}(z) \end{bmatrix} \begin{bmatrix} 1 & x(1+p^j xz)^{-1} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Proposition 4.7: *For any $x \in \mathscr{O}_F$ and $\underline{X}^{\underline{j}} \in A$ we have*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^{\underline{j}} \in \underline{X}^{\underline{j}} + \mathfrak{m}^{|\underline{j}|+(p-2)}.$$

Proof. It is enough to prove the statement in $k[K_0(p)] \otimes_{k[K_0(p^n)]} 1$ for any finite level $n \in \mathbf{N}$.

In the latter case, the statement is exactly the statement of [Mo2], Proposition 4.7 (with $m=1$), provided by the dictionary given by Proposition 4.4, which lets us identify the Fourier series used in [Mo2] with monomials in A.

If f verifies $f(p-1) \leq p(p-2)$ we can nevertheless avoid the reference to [Mo2] and use instead an inductive argument via the embeddings

$$A_{n-1,n} \overset{\iota}{\hookrightarrow} \cdots \overset{\iota}{\hookrightarrow} A_{2,n} \overset{\iota}{\hookrightarrow} A.$$

Let us write

$$\underline{X}^{\underline{j}} = \underline{X}^{\underline{j}^{(1)}} \iota(\underline{X}^{\underline{j}^{(2)}})$$

where $\underline{j} = \underline{j}^{(1)} + p\underline{j}^{(2)}$, $\underline{j}^{(1)} \in \{0, \dots, p-1\}$. Assume the statement holds for $A_{2,n}$.

As $|\underline{j}| + (p-1) \leq p|\underline{j}^{(2)}| + p(p-1)$, we deduce from Proposition 4.4 and (19) that:

$$
\begin{aligned}
&(\kappa_{\underline{j}}^{-1}\kappa_{\underline{j}^{(2)}}) \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^{\underline{j}} \\
&\equiv \sum_{\lambda \in k_F} \lambda^{\underline{j}^{(1)}} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^2[-\lambda^2\mu] & 1 \end{bmatrix} \cdot \\
&\qquad \cdot \iota\Big(\begin{bmatrix} 1 + p[\lambda\mu] + p^2 * & [\mu] \\ p^3 * & 1 - p[\lambda\mu] + p^2 * \end{bmatrix} \cdot \underline{X}^{\underline{j}^{(2)}} \Big)
\end{aligned}
$$

modulo $\mathfrak{m}^{|\underline{j}|+(p-1)}$. Using Lemmas 4.2, 4.3 and the inductive hypothesis on $A_{2,n}$, we have

$$
\begin{bmatrix} 1 + p[\lambda\mu] + p^2 * & [\mu] \\ p^3 * & 1 - p[\lambda\mu] + p^2 * \end{bmatrix} \cdot \underline{X}^{\underline{j}^{(2)}} \in \underline{X}^{\underline{j}^{(2)}} + \mathfrak{m}_2^{|\underline{j}^{(2)}|+(p-2)} \tag{20}
$$

(again, $\mathfrak{m}_2$ denotes the maximal ideal in $A_{2,n}$) and since we assume $f(p-1) \leq p(p-2)$ we obtain

$$
\iota\Big(\begin{bmatrix} 1 + p[\lambda\mu] & [\mu] \\ p^3 * & 1 - p[\lambda\mu] \end{bmatrix} \cdot \underline{X}^{\underline{j}^{(2)}} \Big) \equiv \iota\big(\underline{X}^{\underline{j}^{(2)}}\big) \text{ modulo } \mathfrak{m}^{|\underline{j}|+(p-2)}.
$$

Moreover,

$$
\iota\Big(\begin{bmatrix} 1 & 0 \\ p^2[-\lambda^2\mu] & 1 \end{bmatrix} \cdot \underline{X}^{\underline{j}^{(2)}} \Big) \equiv \iota\Big(\sum_{\underline{i} \geq 0} (\lambda^2\mu)^{p\underline{i}} \varepsilon_{\underline{i}} \underline{X}^{\underline{j}^{(2)}+\underline{i}} \Big) \text{ modulo } \iota\big(\mathfrak{m}_2^{|\underline{j}^{(2)}|+(p-1)}\big)
$$

where the scalars $\varepsilon_{\underline{i}} \in \mathbf{F}_p$ depend on $\underline{i}, \underline{j}^{(2)}$.

As in our situation we have $p(|\underline{j}^{(2)}| + (p-1)) \geq |\underline{j}| + (p-2)$, we finally obtain

$$
\begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix} \cdot \underline{X}^{\underline{j}} \equiv \sum_{\underline{i} \geq 0} \nu_{\underline{i}} \mu^{p\underline{i}} X^{p\underline{j}^{(2)}+p\underline{i}} \sum_{\lambda \in k_F} \lambda^{\underline{j}^{(1)}+2p\underline{i}} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \text{ modulo } \mathfrak{m}^{|\underline{j}|+(p-2)}
$$

for some scalars $\nu_{\underline{i}} \in \mathbf{F}_p$ and we conclude using Proposition 4.4. ∎

As a corollary, we get

Corollary 4.8: *For any $k \geq 0$ the ideal $\mathfrak{m}^k \lhd A$ is a $k[[K_0(p)]]$-submodule of A.*

Moreover, the action of $\mathbf{T}(1+p\mathscr{O}_F) \ltimes \mathrm{U}^+$ is trivial on the quotients

$$\mathfrak{m}^k/\mathfrak{m}^{k+(p-2)}.$$

Proof. It follows from Lemma 4.3 and Proposition 4.7. ∎

Since for all $n \geq 1$ we have an epimorphism of $k[[K_0(p)]]$-modules $A \twoheadrightarrow S_{n+1}(\sigma)$, we immediately deduce

Corollary 4.9: *Let $n, k \geq 1$. Let σ be a Serre weight and write $\overline{\mathfrak{m}}^k$ for the image of $\mathfrak{m}^k$ via the projection $A \twoheadrightarrow S_{n+1}(\sigma)$. The filtration $\{\overline{\mathfrak{m}}^k\}_k$ is stable by the action of Γ, U^+ and $\mathbf{T}(k_F)$. Moreover, the action of Γ, U^+ is trivial on the quotient*

$$\overline{\mathfrak{m}}^k/\overline{\mathfrak{m}}^{k+(p-2)}.$$

5. The twisted Frobenius

In this section we construct a "twisted" Frobenius morphism between the graded pieces $\mathcal{K}er_{n+1}$ of the natural filtration on $\mathfrak{S}_\infty^\bullet$. This morphism is Γ, $\mathbf{T}(k_F)$-equivariant and it is obtained from the twisted Frobenius $\mathscr{F}$ on A (the latter induced from conjugation by the element αs).

The main properties of the twisted Frobenius are listed in Propositions 5.4 and 5.7: roughly speaking, this morphism lets us translate information from $S_2(\sigma)$, where computations are still accessible, to higher-dimensional quotients $S_{n+1}(\sigma)$, where things get considerably more complicated.

In section 5.3 we determine an explicit $k[[K_0(p)]]$-submodule of $\mathfrak{S}_\infty^0 \oplus \mathfrak{S}_\infty^1$ of finite colength, endowed with an $\mathscr{F}$-semilinear, Γ, $\mathbf{T}(k_F)$-equivariant endomorphism, which turns out to be p-étale. The main result is summarized in Proposition 5.11. As a corollary, we deduce that such a submodule is of finite type over the skew polynomial ring $A[[\mathscr{F}]]$ (Corollary 5.12).

We remark that some of the statements of §5.1, 5.2, which refer to the $k[[K_0(p^{n+1})]]$-modules $\left(\sigma^{(n+1)}\right)^\vee$, hold in greater generality for any $k[[K_0(p^{n+1})]]$-module. Nevertheless, we believe that the specialized statements of Lemmas 5.1 and 5.6 are more expressive for the subsequent applications to the universal module $\mathfrak{S}_\infty^\bullet$.

5.1. Analysis for the trivial weight. Let $l \geq j \geq 1$ and consider the $k[[K_0(p^j)]]$-module

$$\mathbf{S}_l^{(j)} \overset{\text{def}}{=} \left(\operatorname{ind}_{K_0(p^l)}^{K_0(p^j)} 1\right)^{\vee};$$

recall that, as a $k[[\mathrm{U}^-(p^j)]]$-algebra, $\mathbf{S}_l^{(j)}$ is nothing but $A_{j,l-1}$.

Define $\mathrm{B}^-(p^j) \overset{\text{def}}{=} \overline{\mathbf{B}}(\mathscr{O}_F) \cap K_0(p^j)$ so that restriction to $k[[\mathrm{B}^-(p^j)]]$ provides, by the Iwahori decomposition, the following isomorphism of $k[[\mathrm{B}^-(p^j)]]$-modules:

$$\mathbf{S}_l^{(j)}|_{k[[\mathrm{B}^-(p^j)]]} \cong k[[[\mathrm{B}^-(p^j)]] \otimes_{k[[\mathrm{B}^-(p^l)]]} 1;$$

we can thus define a Frobenius morphism

$$\mathscr{F}_{j,l} : \mathbf{S}_l^{(j)} \hookrightarrow \mathbf{S}_{l+1}^{(j)}$$

induced by conjugation by $\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ (we omit the indexes j, l in $\mathscr{F}_{j,l}$ to ease notations). The following lemma is then immediate:

Lemma 5.1: *The morphism $\mathscr{F}$ respects the natural k-algebra structures on $\mathbf{S}_l^{(j)}$, $\mathbf{S}_{l+1}^{(j)}$. It is injective, with image $\mathbf{S}_{l+1}^{(j+1)}$ and it is described explicitly by*

$$\begin{array}{rcl} \mathscr{F} : \mathbf{S}_l^{(j)} & \longrightarrow & \mathbf{S}_{l+1}^{(j)} \\ X_i^{p^{j-1}} & \longmapsto & X_{i-1}^{p^j}. \end{array}$$

Moreover it is $\mathbf{T}(k_F)$ and Γ-equivariant.

Proof. This is clear. ∎

We can give a very rough estimate on the compatibility between the Frobenius and the U^+ action on $\mathbf{S}_l^{(j)}$. Indeed, we have

Lemma 5.2: *The action of U^+ on $\mathbf{S}_l^{(j)}$ is trivial if and only if $l \leq 2j$.*

Proof. Recall that for $z \in \mathscr{O}_F$ the elements

$$\begin{bmatrix} 1 & 0 \\ p^j z & 1 \end{bmatrix} \otimes 1$$

give a family of generators for the module $k[[K_0(p^j)]] \otimes_{k[[K_0(p^l)]]} 1$. If $x \in \mathscr{O}_F$ we have

$$p^j \delta_{x,j}(z) = p^j z + p^{2j} z'$$

for a suitable $z' \in \mathscr{O}_F$ with $val(z') = val(xz^2)$ and the statement is then clear from the equality (19). ∎

We therefore deduce

Corollary 5.3: *Let $m \geq l \geq j \geq 1$. The $(m-l)$-composite of $\mathscr{F}$,*

$$\mathbf{S}_l^{(j)} \xrightarrow{\mathscr{F}^{m-l}} \mathbf{S}_m^{(j)},$$

factors through the U^+ *invariants of* $\mathbf{S}_m^{(j)}$ *if and only if* $0 \leq 2(j-l)+m$.

Proof. It suffices to remark that $\mathscr{F}^{m-l}$ induces an isomorphism of $\mathbf{S}_l^{(j)}$ onto the subalgebra $\mathbf{S}_m^{(j+m-l)}$ of $\mathbf{S}_m^{(j)}$ and use Lemma 5.2. ∎

5.1.1. *The skew power series ring* $A[[\mathscr{F}]]$. The evident, similar constructions of the previous section, with $K_0(p^l)$ replaced by $K_0(p^\infty)$, give us a Frobenius endomorphism on A:

Proposition 5.4: *We have a* Γ, $\mathbf{T}(k_F)$*-equivariant monomorphism of local* k*-algebras* $\mathscr{F} : A \to A$ *described by*

$$\begin{aligned} \mathscr{F} : A &\longrightarrow A \\ X_i &\longmapsto X_{i-1}^p. \end{aligned}$$

In particular, we have a decomposition

$$A \cong \bigoplus_{0 \leq \underline{i} \leq \underline{p-1}} \mathscr{F}(A)\underline{X}^{\underline{i}},$$

and $\mathscr{F}$ *is a flat endomorphism of* A.

Proof. The relation $\mathscr{F}(X_i) = X_{i-1}^p$ comes from a direct computation on the definition of X_i and $\mathscr{F}$. Moreover, one has

$$A \cong \bigoplus_{g \in \mathrm{U}^-(p)/\mathrm{U}^-(p^2)} k[[\mathrm{U}^-(p^2)]] \cdot g$$

which shows that $\mathscr{F}$ is an injective and flat endomorphism on A. ∎

We recall (cf. [Ven], §2) the skew power series ring $A[[\mathscr{F}]]$, whose elements are formal power series $\sum_{i=0}^\infty a_i \mathscr{F}^i$ with $a_i \in A$ and multiplication law induced by

$$\mathscr{F} \cdot a \stackrel{\mathrm{def}}{=} \mathscr{F}(a)\mathscr{F}$$

for any $a \in A$. It is a local ring, endowed with a structure of a complete, separated topological ring, a basis of open neighborhoods of 0 being described by

$$\prod_{i=0}^{k-1} \mathfrak{m}^k \mathscr{F}^i \times \prod_{i=k}^{\infty} A \mathscr{F}^i$$

for $k \in \mathbf{N}$. In particular, the skew polynomial ring $A[\mathscr{F}]$ is a dense subring of $A[[\mathscr{F}]]$.

We introduce the following notion (cf. [Fon], §B 1.3):

Definition 5.5: An $\mathscr{F}$-semilinear morphism φ of profinite A-modules $D_1 \stackrel{\varphi}{\to} D_2$ is p**-étale** if the image of the natural map

$$A \otimes_{A,\mathscr{F}} D_1 \stackrel{id \otimes \varphi}{\longrightarrow} D_2$$

has finite colength.

5.2. Analysis for a general Serre weight. The aim of this section is to endow the modules $S_n(\sigma)$ with a Frobenius morphism $\mathscr{F}_n$ and collect some basic properties which are useful to obtain a Frobenius on subobjects of the universal module $\mathfrak{S}_\infty$ (the universal module itself does not have a Frobenius action).

As usual, we let σ be a Serre weight whose highest weight space affords the trivial character of $K_0(p)$. Recall that we have defined, for $l \geq j \geq 1$, the modules

$$S_l^{(j)}(\sigma) \stackrel{\text{def}}{=} k[[K_0(p^j)]] \otimes_{k[[K_0(p^l)]]} (\sigma)^\vee.$$

We have

Lemma 5.6: *Let $l \geq j \geq 1$. There exists a unique morphism $\mathscr{F}_\sigma : S_l^{(j)}(\sigma) \to S_{l+1}^{(j)}(\sigma)$ of k-algebras making the following diagram commute*

(21)
$$\begin{tikzcd} \mathbf{S}_{l+1}^{(j)} \arrow[r, hook, "\mathscr{F}"] \arrow[d] & \mathbf{S}_{l+2}^{(j)} \arrow[d] \\ S_l^{(j)}(\sigma) \arrow[r, hook, dotted, "\mathscr{F}_\sigma"] & S_{l+1}^{(j)}(\sigma) \end{tikzcd}$$

where the vertical arrows are induced by the morphisms $(\phi_l)^\vee$ and $(\phi_{l+1})^\vee$ of Proposition 3.3.

Proof. It suffices to use the explicit definition of $\mathscr{F}$ and to recall that the kernel of the vertical arrow on the RHS (resp. on the LHS) is the ideal generated by the elements $X_{i-l}^{p^l(r_i+1)}$ (resp. $X_{i-l+1}^{p^{l-1}(r_i+1)}$) for $i = 0, \ldots, f-1$. ∎

As $\mathbf{S}^{(j)}_{l+1} \twoheadrightarrow S^{(j)}_l(\sigma)$ is a morphism of $k[[K_0(p^j)]]$-modules we get

Proposition 5.7: *Let $l \geq j \geq 1$. We have a monomorphism of k-algebras*

$$\begin{array}{rcl} \mathscr{F}_\sigma : S^{(j)}_l(\sigma) & \longrightarrow & S^{(j)}_{l+1}(\sigma) \\ X_i^{p^{j-1}} & \longmapsto & X_{i-1}^{p^j} \end{array}$$

verifying the following properties:

i) *The morphism $\mathscr{F}_\sigma$ is Γ and $\mathbf{T}(k_F)$-equivariant.*
ii) *The morphism $\mathscr{F}_\sigma$ admits the factorization*

$$\begin{tikzcd} S^{(j)}_l(\sigma) \arrow[r, hook, "\mathscr{F}_\sigma"] \arrow[dr, "\cong"'] & S^{(j)}_{l+1}(\sigma) \\ & S^{(j+1)}_{l+1}(\sigma) \arrow[u, hook] \end{tikzcd}$$

with the vertical arrow being the natural inclusion.

iii) *Let $m \geq l$. If $0 \leq 2(j-l)+m-1$, then the composite morphism*

$$S^{(j)}_l(\sigma) \xrightarrow{\mathscr{F}^{m-l}} S^{(j)}_m(\sigma)$$

factors through the U^+*-invariants of $S^{(j)}_m(\sigma)$.*

Proof. Parts i) and ii) follow from the properties of the morphism $\mathscr{F}$ (Lemma 5.1) and from Lemma 5.6, recalling that the vertical arrows in the diagram (21) are morphisms of $k[[K_0(p^{j+1})]]$-modules.

Property iii) follows from Corollary 5.3 using the epimorphism $\mathbf{S}^j_{m+1} \twoheadrightarrow S^{(j)}_m(\sigma)$ (which is U^+-equivariant). ∎

5.3. The twisted Frobenius on the universal Iwasawa module. The aim of this section is to construct, from the twisted Frobenii $\mathscr{F}_\sigma$ of Proposition 5.7, a Frobenius morphism $\mathscr{F}$ on a suitable $k[[K_0(p)]]$-submodule $\mathfrak{S}^{\geq 1}_\infty \oplus \mathfrak{S}^{\geq 2}_\infty \subseteq \mathfrak{S}^0_\infty \oplus \mathfrak{S}^1_\infty$. Such a submodule is of finite co-length and the action of $\mathscr{F}$ is p-étale. Moreover, $\mathfrak{S}^{\geq 1}_\infty \oplus \mathfrak{S}^{\geq 2}_\infty$ is of finite rank on the skew power series ring $A[[\mathscr{F}]]$. Throughout this section $\sigma = \sigma_{\underline{r}}$ is a fixed Serre weight.

We start from the following

Lemma 5.8: *For $n \geq 3$ we have commutative diagrams of k-linear spaces:*

$$\begin{tikzcd} S_{n-1} \arrow[r, hook, "\mathscr{F}_\sigma"] & S_n \\ S_{n-2} \arrow[u, hook] \arrow[r, hook, "\mathscr{F}_\sigma"] & S_{n-1} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} S_{n-1} \arrow[r, hook, "\mathscr{F}_\sigma"] \arrow[d, two heads] & S_n \arrow[d, two heads] \\ S_{n-2} \arrow[r, hook, "\mathscr{F}_\sigma"] & S_{n-1} \end{tikzcd}$$

where the horizontal arrows are the monomorphisms of k-algebras of Proposition 5.7 (with $j = 1$) and the vertical arrows are the morphisms of $k[[K_0(p)]]$-modules defined in Proposition 3.4.

Proof. The commutativity of the diagrams can be checked directly, using the definition of the morphisms in terms of the regular parameters X_i (noticing that $\mathscr{F}_\sigma$ is a morphism of k-algebras). The details are left to the reader. ∎

Since the $k[[K_0(p)]]$-modules $\mathfrak{S}^\bullet_{n+1}$ admit an explicit family of A-generators, we easily see that $\mathscr{F}_\sigma : S_{n+1} \to S_{n+2}$ induces a morphism between appropriate submodules of $\mathfrak{S}_{n+1}$ and $\mathfrak{S}_{n+2}$.

Proposition 5.9: *Let $n \in 2\mathbf{N} + 1$ and $e_{2(j+1),i} \in \mathscr{G}^{\geq 1}_{n+1}$ (cf. Definition 3.5). The morphism $\mathscr{F}_\sigma : S_{n+1} \hookrightarrow S_{n+2}$ verifies*

$$\mathscr{F}_\sigma(e_{2(j+1),i}) = e_{2(j+1)+1,i-1} \in \mathscr{G}^{\geq 2}_{n+2}.$$

Similarly, for $m \in 2\mathbf{N} + 2$ one has

$$\mathscr{F}_\sigma(e_{2(j+1)+1,i}) = e_{2(j+2),i-1} \in \mathscr{G}^{\geq 1}_{m+2}.$$

In particular, we have the following commutative diagrams:

$$\begin{tikzcd} S_{n+1} \arrow[r, hook, "\mathscr{F}_\sigma"] & S_{n+2} \\ \mathfrak{S}^{\geq 1}_{n+1} \arrow[u, hook] \arrow[r, hook, "\mathscr{F}"] & \mathfrak{S}^{\geq 2}_{n+2} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} S_{m+1} \arrow[r, hook, "\mathscr{F}_\sigma"] & S_{m+2} \\ \mathfrak{S}^{\geq 2}_{m+1} \arrow[u, hook] \arrow[r, hook, "\mathscr{F}"] & \mathfrak{S}^{\geq 1}_{m+2} \arrow[u, hook] \end{tikzcd}$$

and the morphisms

$$\mathfrak{S}^{\geq 1}_{n+1} \overset{\mathscr{F}}{\hookrightarrow} \mathfrak{S}^{\geq 2}_{n+2}, \qquad \mathfrak{S}^{\geq 2}_{m+1} \overset{\mathscr{F}}{\hookrightarrow} \mathfrak{S}^{\geq 1}_{m+2}$$

are Γ, $\mathbf{T}(k_F)$ equivariant, $\mathscr{F}$-semilinear and p-étale over A.

Proof. The first part of the statement follows from an elementary computation on the elements $e_{2(j+1),i} \in \mathscr{G}^{\geq 1}_{n+1}$ (resp. $e_{2(j+1)+1,i} \in \mathscr{G}^{\geq 2}_{m+1}$).

We deduce the factorization of the morphism $S_{n+1} \overset{\mathscr{F}_\sigma}{\hookrightarrow} S_{n+2}$ (resp. $S_{m+1} \hookrightarrow S_{m+2}$), as the module $\mathfrak{S}^{\geq 2}_{n+2}$ (resp. $\mathfrak{S}^{\geq 1}_{m+2}$) is generated, over A, by the elements $e_{2(j+1)+1,i} \in \mathscr{G}^{\geq 2}_{n+2}$ (resp. by the elements $e_{2(j+1),i} \in \mathscr{G}^{\geq 1}_{m+2}$).

The induced morphisms on the $k[[K_0(p)]]$-modules are clearly $\mathscr{F}$-semilinear and Γ, $\mathbf{T}(k_F)$ equivariant. Their p-étale nature follows again by noticing that the A-generators of $\mathfrak{S}^{\geq 2}_{n+2}$ (resp. $\mathfrak{S}^{\geq 3}_{m+2}$) are the elements $e_{2(j+1)+1,i} \in \mathscr{G}^{\geq 2}_{n+2}$ (resp. the elements $e_{2(j+1),i} \in \mathscr{G}^{\geq 3}_{m+2}$) and the cokernel of $\mathfrak{S}^{\geq 3}_{m+2} \hookrightarrow \mathfrak{S}^{\geq 1}_{m+2}$ is the finite A-module $\mathcal{K}er_2$. ∎

We are now left to prove that the morphisms of Proposition 5.9 are compatible with the transition maps of the projective system defining the universal modules $\mathfrak{S}^{\geq 1}_{\infty}$, $\mathfrak{S}^{\geq 2}_{\infty}$.

Proposition 5.10: *Let $n \in 2\mathbf{N} + 3$. We have a commutative diagram*

$$\begin{tikzcd}
\mathfrak{S}^{\geq 1}_{n+1} \arrow[rr, hook] \arrow[dr, hook] \arrow[ddd] & & \mathfrak{S}^{\geq 2}_{n+2} \arrow[dr, hook] \arrow[ddd] & \\
& S_{n+1} \arrow[rr, hook] \arrow[d] & & S_{n+2} \arrow[d] \\
& S_n \arrow[rr, hook] & & S_{n+1} \\
\mathfrak{S}^{\geq 1}_{n-1} \arrow[rr, hook] \arrow[dr, hook] & & \mathfrak{S}^{\geq 2}_{n} \arrow[dr, hook] & \\
& S_{n-1} \arrow[rr, hook] \arrow[uu, hook] & & S_n \arrow[uu, hook]
\end{tikzcd}$$

where the horizontal arrows are the previously defined Frobenii morphisms.

We have a similar result for $m \in 2\mathbf{N} + 4$ and the diagram

$$\begin{tikzcd}
\mathfrak{S}^{\geq 2}_{m+1} \arrow[rr, hook] \arrow[dr, hook] \arrow[ddd, two heads] & & \mathfrak{S}^{\geq 1}_{m+2} \arrow[dr, hook] \arrow[ddd, two heads] & \\
& S_{m+1} \arrow[rr, hook] \arrow[d, two heads] & & S_{m+2} \arrow[d, two heads] \\
& S_m \arrow[rr, hook] & & S_{m+1} \\
\mathfrak{S}^{\geq 2}_{m-1} \arrow[rr, hook] \arrow[dr, hook] & & \mathfrak{S}^{\geq 1}_{m} \arrow[dr, hook] & \\
& S_{m-1} \arrow[rr, hook] \arrow[uu, hook] & & S_m \arrow[uu, hook]
\end{tikzcd}$$

Proof. The top and bottom squares of the diagram are commutative, by Proposition 5.9; the squares on the left and right sides are commutative by the construction of the fibered products $\mathfrak{S}^{\geq 1}_{n+1}$, $\mathfrak{S}^{\geq 2}_{m+1}$. Finally, the front square is commutative by Lemma 5.8.

The commutativity of the back square follows by an easy diagram chase, noticing that the composite morphism $\mathfrak{S}^{\geq 2}_{n} \to S_{n+1}$ (resp. $\mathfrak{S}^{\geq 1}_{m} \to S_{m+1}$) is a monomorphism. ∎

We therefore deduce

Proposition 5.11: *We have a Γ, $\mathbf{T}(k_F)$-equivariant, $\mathscr{F}$-semilinear morphism*

$$\mathscr{F} : \mathfrak{S}^{\geq 1}_{\infty} \quad \hookrightarrow \quad \mathfrak{S}^{\geq 2}_{\infty},$$

which is p-étale and verifies

$$\mathscr{F}(e_{2(j+1),i}) = e_{2(j+1)+1,i-1} \in \mathscr{G}^{\geq 2}_{\infty}$$

for all $e_{2(j+1),i} \in \mathscr{G}^{\geq 1}_{\infty}$

Similarly, we have a Γ, $\mathbf{T}(k_F)$-equivariant, $\mathscr{F}$-semilinear morphism

$$\mathscr{F} : \mathfrak{S}^{\geq 2}_{\infty} \quad \hookrightarrow \quad \mathfrak{S}^{\geq 1}_{\infty},$$

which is p-étale and verifies

$$\mathscr{F}(e_{2(j+1)+1,i}) = e_{2(j+2)+1,i-1} \in \mathscr{G}^{\geq 1}_{\infty}$$

for all $e_{2(j+1),i} \in \mathscr{G}_{\infty}^{\geq 2}$.

Proof. The assertions follow from Proposition 5.9 and the compatibility with the transition morphisms given by Proposition 5.10. For the p-étale property of the second morphism we just remark that, from the proof of Proposition 5.9, we have an exact sequence

$$A\otimes_{\mathscr{F},A}\mathfrak{S}_{m+1}^{\geq 2} \to \mathfrak{S}_{m+2}^{\geq 1} \to \mathcal{K}er_2 \to 0$$

for all $m \in 2\mathbf{N}+2$, and by passing to the limit we get a complex

$$A\otimes_{\mathscr{F},A}\mathfrak{S}_{\infty}^{\geq 2} \to \mathfrak{S}_{\infty}^{\geq 1} \to \mathcal{K}er_2 \to 0$$

which is again exact (the transition morphisms in the projective system are all epi). ∎

In particular, we deduce a finiteness property for the modules $\mathfrak{S}_{\infty}^{\geq 1}$, $\mathfrak{S}_{\infty}^{\geq 2}$ on the twisted polynomial algebra $A[[\mathscr{F}^2]]$:

Corollary 5.12: *For* $\bullet \in \{0,1\}$ *we have a* $A[[\mathscr{F}^2]]$*-equivariant surjection*

$$\begin{array}{rcl} \bigoplus_{i=0}^{f-1} A[[\mathscr{F}^2]]e_{2+\bullet,i} & \longrightarrow & \mathfrak{S}_{\infty}^{\geq \bullet+1} \\ e_{2+\bullet,i} & \mapsto & e_{2+\bullet,i}. \end{array}$$

Proof. To ease notation, we consider the case where $\bullet = 0$. It is clear by Proposition 5.11 that for all $l \in \mathbf{N}$ we have a semilinear morphism

$$\mathscr{F}^{2l} : \mathfrak{S}_{\infty}^{\geq 1} \quad \hookrightarrow \quad \mathfrak{S}_{\infty}^{\geq 1}$$

which verifies

$$\mathscr{F}^{2l}(e_{2(j+1)+1,i}) = e_{2(j+l+1),i-2l}$$

for all $j \in \mathbf{N}$, $i \in \{0,\ldots,f-1\}$. We deduce that the natural morphism

$$\begin{array}{rcl} \bigoplus_{i=0}^{f-1} A[\mathscr{F}^2]e_{2+\bullet,i} & \longrightarrow & \mathfrak{S}_{\infty}^{\geq 1} \\ e_{2+\bullet,i} & \mapsto & e_{2+\bullet,i} \end{array}$$

is $A[\mathscr{F}^2]$-linear, continuous and with dense image. Since the completion $A[[\mathscr{F}^2]]$ is compact and $\mathfrak{S}_{\infty}^{\geq \bullet+1}$ is separated, the statement follows. ∎

A speculation. One can verify that the universal module $\mathfrak{S}^0_\infty \oplus \mathfrak{S}^1_\infty$ contains many A-submodules of finite rank which are $\mathscr{F}$-stables. The theory of Wach modules suggests that such $(A,\mathscr{F})$-submodules may be related to finite dimensional G_∞-representation, where G_∞ is the absolute Galois group of the Kummer extension $F_\infty \stackrel{\text{def}}{=} \bigcup_{n\in\mathbf{N}} F(\pi_i)$ for a compatible system $(\pi_i)_{i\in\mathbf{N}}$ of p-roots of the uniformizer $p\in F$ (i.e. $\pi_i^p=\pi_{i-1}$ if $i\geq 1$, $\pi_0 \stackrel{\text{def}}{=} p$). Indeed, there are many non-canonical morphisms of k-algebras $A\to k\otimes_{\mathbf{F}_p} k_F[[X]]$ and one could try to (arbitrarily) construct some $(\varphi, k\otimes_{\mathbf{F}_p} k_F[[X]])$-modules of finite rank from $(A,\mathscr{F})$-submodules in $\mathfrak{S}^0_\infty\oplus\mathfrak{S}^1_\infty$.

For instance, let us define

$$f_1 \stackrel{\text{def}}{=} e_{2,0}\prod_{i\neq 0} X_i^{(r_i+1)} = e_{2,j}\prod_{i\neq j} X_i^{(r_i+1)} \in \mathfrak{S}^0_\infty$$

(the equalities can be verified at any finite level $\mathfrak{S}^0_{n+1}$).

We have $f_1\in\mathfrak{S}^{\geq 1}_\infty$ and an easy computation (which can be performed at any finite level) gives

$$\mathscr{F}^2(f_1) = f_1\cdot\prod_{l=0}^{f-1} X_l^{p(r_{l+1}+1)-(r_l+1)}.$$

Hence the module $Af_1\oplus A\mathscr{F}(f_1)$ is an $\mathscr{F}$-stable, rank 2 A-submodule of $\mathfrak{S}^0_\infty\oplus\mathfrak{S}^1_\infty$, with an explicit action of $\mathscr{F}$ on the A-generators $(f_1,\mathscr{F}(f_1))$.

One should expect to find a huge number of other finite rank, $\mathscr{F}$-stable submodules in $\mathfrak{S}^0_\infty\oplus\mathfrak{S}^1_\infty$; the meaning of this phenomenon in Galois theoretical terms remains, at present, mysterious.

6. A filtration on the ideals $\mathcal{K}er_{n+1}$

We recall that, for $\bullet\in\{0,1\}$, the universal module $\mathfrak{S}^\bullet_\infty$ is a pseudo-compact module over A, with a separated filtration consisting of open neighborhoods of 0 whose graded pieces are isomorphic to $\mathcal{K}er_{n+1}$ for $n\in 2\mathbf{N}+1+\bullet$. For $n\geq 1$ the description of the graded pieces $\mathcal{K}er_{n+1}$ in terms of the regular parameters X_i is deduced by Proposition 3.3:

$$\mathcal{K}er_{n+1} = \langle e_{n+1,\,i},\ i\in\{0,\dots,f-1\}\rangle_A$$

where $e_{n+1,\,i}=\mathscr{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1})$ and $X_{i+n-1}^{r_{i+n-1}+1}=e_{2,i+n-1}\in S_2(\sigma)$.

As we did for the monogenic modules $S_{n+1}(\sigma)$, we endow the module $\mathcal{K}er_{n+1}$ with a natural A-linear filtration $\{I_{k,n+1}\}_{k\in\mathbf{N}}$, which turns out to be $k[[K_0(p)]]$-stable.

The $\mathbf{T}(k_F)$, Γ-stability of $I_{k,n+1}$ follows easily, as these groups act by algebra homomorphisms on $S_{n+1}(\sigma)$. The action of U^+ is, again, more delicate and we need to use in a crucial way that the A-generators $e_{n+1,i}$ of $\mathcal{K}er_{n+1}$ lie in the image of the twisted Frobenius $\mathscr{F}^{n-1}$, in particular that they are fixed under the U^+-action if $n \geq 2$ (cf. Proposition 5.7, iii)).

When $n = 1$ the situation is slightly more complicated and we need some extra arguments (cf. Lemma 6.5).

We fix a Serre weight $\sigma = \sigma_{\underline{r}}$; for the rest of this section we further assume that σ is **weakly regular**, i.e. $0 \leq r_i \leq p-2$ for all $i = 0, \dots, f-1$ (cf. [Gee] Definition 2.1.5). Up to twist we may and do further assume that the highest weight space of σ affords the trivial character.

For any $n \geq 1$ we have an epimorphism of A-modules

$$\begin{array}{rcl} \bigoplus_{i=0}^{f-1} A \cdot e_{n+1,\,i} & \longrightarrow & \mathcal{K}er_{n+1} \\ e_{n+1,\,i} & \longmapsto & \mathscr{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1}) \end{array} \tag{22}$$

which is $\mathbf{T}(k_F)$-equivariant if we make $\mathbf{T}(k_F)$ act by the character $\mathfrak{a}^{-p^i(r_i+1)}$ on $e_{n+1,\,i-n+1}$.

The $k[[K_0(p)]]$-module $\bigoplus_{i=0}^{f-1} A\cdot e_{n+1,\,i}$ is endowed with the valuation ord_{n+1} of the infimum

$$ord_{n+1}\Big(\sum_{i=0}^{f-1} P_i(\underline{X})e_{n+1,\,i}\Big) \stackrel{\mathrm{def}}{=} \min\{ord(P_i(\underline{X})),\, i = 0, \dots, f-1\}$$

hence with an A-linear filtration $\{I^0_{k,n+1}\}_k$. Let $\{I_{k,n+1}\}_k$ be the filtration on $\mathcal{K}er_{n+1}$ induced from $\{I^0_{k,n+1}\}_k$ via the morphism (22). Concretely, one has $I_{k,n+1} = \sum_{i=0}^{f-1} \mathfrak{m}^k e_{n+1,i}$. As the morphism (22) is not Γ, U^+-equivariant, there is no reason for which $\{I_{k,n+1}\}_k$ should be a filtration of $k[[K_0(p)]]$-modules on $\mathcal{K}er_{n+1}$.

We define

$$h \stackrel{\mathrm{def}}{=} h(\sigma) \quad \stackrel{\mathrm{def}}{=} \quad \max\big\{|r_{i_1} - r_{i_2}|,\ \text{for } i_1,\, i_2 \in \{0, \dots, f-1\}\big\}.$$

The result is the following:

Proposition 6.1: *Assume that σ is a weakly regular Serre weight. Let $n \geq 1$ and consider the induced A-linear filtration $\left\{I_{k,n+1}\right\}_{k\in\mathbf{N}}$ on $\mathcal{K}er_{n+1}$.*

For each $k \in \mathbf{N}$ the A-submodule $I_{k,n+1}$ is stable under the $K_0(p)$-action and, moreover, the Γ, U^+-actions are trivial on the quotients

$$I_{k,2}/I_{k+(p-2-h),2}, \qquad I_{k,n+1}/I_{k+(p-2),n+1} \text{ if } n \geq 2.$$

In particular, for all $n \geq 2$ the filtration $\left\{I_{k,n+1}\right\}_{k\in\mathbf{N}}$ defines the $k[[K_0(p)]]$-radical filtration on $\mathcal{K}er_{n+1}$ and the same result holds true for $n = 1$ if $h \neq p-2$.

As the morphism (22) is A-linear and $\mathbf{T}(k_F)$-equivariant, it is clear that the filtration $\left\{I_{k,n+1}\right\}_{k\in\mathbf{N}}$ is $\mathbf{T}(k_F)$-stable and defines the A-radical filtration on $\mathcal{K}er_{n+1}$.

With some additional work, using that Γ acts semilinearly and commutes with the $\mathbf{T}(k_F)$-action, one could indeed state a more precise result concerning the Γ-action on $S_2(\sigma)$. Moreover, the statement of Proposition 6.1 can be proved to hold true even in some non-regular situations. As the proofs of such results are very technical and do not add any substantial improvements to the main results of this paper, we decided to omit them.

The rest of this section is devoted to the proof of the Γ and U^+-stability of $\left\{I_{k,n+1}\right\}_{k\in\mathbf{N}}$; the techniques are similar to those introduced in section 4, using now in a crucial way the properties of the twisted Frobenius $\mathscr{F}$. Indeed, as the A-generators of $\mathcal{K}er_{n+1}$ lie in the image of the twisted Frobenius, it suffices to investigate the Γ, U_0^+ action on the Iwasawa module $S_2(\sigma)$ (where the computations are still accessible) to get the control of the filtration $\left\{I_{k,n+1}\right\}_{k\in\mathbf{N}}$ for a general n.

6.1. On the Γ-action. We start our analysis with a careful study of the Γ-action on the filtration $\{I_{k,n+1}\}_{k\in\mathbf{N}}$, when $n = 1$. By the properties of the twisted Frobenius $\mathscr{F}$ (cf. Proposition 5.7), this lets us detect the behavior of $\{I_{k,n+1}\}_{k\in\mathbf{N}}$ for arbitrary $n \in \mathbf{N}$.

We start with the following:

Lemma 6.2: *Let $i \in \{0, \dots, f-1\}$ and $\mu \in k_F$.*

We have the following equality in $A_{1,3} \cong A/\langle X_i^{p^2}, i = 0, \dots, f-1\rangle$:

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{bmatrix} - 1\right) \cdot X_i = \mu X_{i-1}^p + \sum_{\underline{0}<\underline{s}\leq\underline{p-1}} P_{\underline{s}}(\mu)\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}$$

where

i) $\underline{\ell}(\underline{s}) \in \{0,\dots,p-1\}^f$ *is the unique non-zero f-tuple such that* $\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}$ *affords the eigencharacter* $\mathfrak{a}^{-e_i}$*;*

ii) $P_{\underline{s}}(\mu)$ *is a "polynomial" in μ of total degree* $|\underline{s}|$*:*

$$P_{\underline{s}}(\mu) = \sum_{\underline{0}\leq\underline{\alpha}\leq\underline{p-1}} \nu_{\underline{s},\underline{\alpha}}\mu^{\underline{\alpha}}$$

for some $\nu_{\underline{s},\underline{\alpha}} \in k$ *such that* $\nu_{\underline{s},\underline{\alpha}} = 0$ *as soon as* $|\underline{\alpha}| > |\underline{s}|$.

Proof. We compute

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{bmatrix} \cdot X_i &= \begin{bmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{bmatrix} \cdot \sum_{\lambda\in k_F} \lambda^{\underline{p-1}-e_i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix}\\ &= \sum_{\lambda\in k_F} \lambda^{\underline{p-1}-e_i} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^2[\lambda\mu] & 1 \end{bmatrix}.\end{aligned}$$

We note that, for any $x \in k_F^{\times}$, we have the following equality in $A_{1,2}$:

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ p[x] & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & [x] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} &= 1 + \begin{bmatrix} 1 & 0 \\ 0 & [x] \end{bmatrix} \sum_{\underline{0}<\underline{s}\leq\underline{p-1}} \nu_{\underline{s}}\underline{X}^{\underline{s}}\\ &= 1 + \sum_{\underline{0}<\underline{s}\leq\underline{p-1}} x^{\underline{s}}\nu_{\underline{s}}\underline{X}^{\underline{s}}\end{aligned}$$

for some $\nu_{\underline{s}} \in k$. Therefore

$$\begin{bmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{bmatrix} \cdot X_i = X_i + \sum_{\underline{0}<\underline{s}\leq\underline{p-1}} \mu^{\underline{s}[1]}\nu_{\underline{s}}\underline{X}^{p\underline{s}} \sum_{\lambda\in k_F} \lambda^{\underline{p-1}-e_i+\underline{s}[1]} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix}$$

where $\underline{s}[1]$ denotes the shifted f-tuple associated to $\underline{s}$, defined by $(\underline{s}[1])_i = (\underline{s})_{i-1}$.

Define $[-\underline{s}[1]+e_i] \in \{0,\dots,p-1\}^f$ as the unique non-zero f-tuple such that $\lambda^{[-\underline{s}[1]+e_i]} = \lambda^{-\underline{s}[1]+e_i}$ for all $\lambda \in k_F$. Since $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{smallmatrix}\right] \cdot X_i$ is an $\mathfrak{a}^{-e_i}$-eigenvector for $\mathbf{T}(k_F)$, we can use Proposition 4.4 to deduce

$$\sum_{\lambda\in k_F} \lambda^{\underline{p-1}-e_i+\underline{s}[1]} \begin{bmatrix} 1 & 0 \\ p[\lambda] & 1 \end{bmatrix} \equiv \underline{X}^{[-\underline{s}[1]+e_i]}\kappa^{-1}_{[-\underline{s}[1]+e_i]}$$

modulo $\left(\mathfrak{m}^{(p-1)+|\underline{s}|-p^i}\right)_{\mathfrak{a}^{-e_i+\underline{s}[1]}}$ (the $\mathfrak{a}^{-e_i+\underline{s}[1]}$-isotypical component of $\mathfrak{m}^{(p-1)+|\underline{s}|-p^i}$).

The conclusion follows, as for any $\underline{s} \neq e_{i-1}$, there is precisely one non-zero f-tuple $\underline{\ell}(\underline{s}) \in \{0, \ldots, p-1\}^f$ such that $\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}$ affords the $\mathbf{T}(k_F)$-character $\mathfrak{a}^{-e_i}$. ∎

As a corollary, we obtain a precise description of the Γ-action on the A-generators for $\mathcal{K}er_2$:

Corollary 6.3: *Let $i \in \{0, \ldots, f-1\}$, $\mu \in k_F$. We have the following equality in $S_2(\sigma)$:*

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1+p[\mu] \end{bmatrix} - 1\right) \cdot X_i^{r_i+1} = \mu^{r_i+1} X_{i-1}^{p(r_i+1)} + \sum_{\underline{0}<\underline{s}\leq \underline{p-1}} P_{\underline{s}}(\mu) \underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}$$

where

i) *$\underline{\ell}(\underline{s}) \in \{0, \ldots, p-1\}^f$ is the unique non-zero f-tuple such that $\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}$ affords the eigencharacter $\mathfrak{a}^{-e_i(r_i+1)}$;*

ii) *$P_{\underline{s}}(\mu)$ is a "polynomial" in μ of degree $|\underline{s}|$:*

$$P_{\underline{s}}(\mu) = \sum_{\underline{0}\leq\underline{\alpha}\leq\underline{p-1}} \nu_{\underline{s},\underline{\alpha}} \mu^{\underline{\alpha}}$$

for some $\nu_{\underline{s},\underline{\alpha}} \in k$ such that $\nu_{\underline{s},\underline{\alpha}} = 0$ as soon as $|\underline{\alpha}| > |\underline{s}|$.

Proof. We recall that Γ acts by k-algebra endomorphism on A; the result follows from a direct computation on binomial developments via Lemma 6.2. ∎

We can now use the properties of the twisted Frobenius to deduce, from Corollary 6.3, the behavior of the filtration $\{I_{k,n+1}\}_k$ with respect to the Γ-action.

Lemma 6.4: *Let $\underline{X}^{\underline{j}} \in \mathfrak{m}^k$ and let $\gamma \in \Gamma$.*

For any $i \in \{0, \ldots, f-1\}$ we have the following relation in $S_{n+1}(\sigma)$:

$$\gamma \cdot (\underline{X}^{\underline{j}} e_{n+1,\,i}) \in \big(\gamma \cdot \underline{X}^{\underline{j}}\big)\Big(e_{n+1,\,i} + \sum_s e_{n+1,s}\, \mathscr{F}^{n-1}(\mathfrak{m}^{p-1-h})\Big). \tag{23}$$

In particular, the Γ-action is trivial on the quotients

$$I_{k,2}/I_{k+(p-1-h),2}; \qquad I_{k,n+1}/I_{k+(p-1),n+1} \quad \text{for } n \geq 2.$$

Proof. Recall that $e_{n+1,\,i} = \mathscr{F}^{n-1}(X_{i+n-1}^{r_{i+n-1}+1})$ with $X_{i+n-1}^{r_{i+n-1}+1} \in S_2(\sigma)$.

As Γ acts by conjugation on A (recall the isomorphism $k[[K_0(p)]]\otimes_{k[[K_0(p^\infty)]]} 1 \cong A$) and $\mathscr{F}$ is Γ-equivariant, we deduce

$$\gamma \cdot (\underline{X}^{\underline{j}} e_{n+1,i}) \in \big(\gamma \cdot \underline{X}^{\underline{j}}\big)\big(\mathscr{F}(\gamma \cdot e_{2,i+n-1})\big).$$

We claim that

Claim: *Let $\gamma \in \Gamma$. We have the following relation in $S_2(\sigma)$:*

$$(\gamma - 1) \cdot X_i^{r_i+1} \in \sum_s X_s^{r_s+1} \mathfrak{m}^{(p-1)-h}.$$

Provided the *Claim*, the statement of Lemma 6.4 follows.

Proof of the Claim. By Lemma 4.3 we have

$$(\gamma - 1) \cdot X_i^{r_i+1} \in \qquad \mathfrak{m}^{p-1+r_i+1}. \tag{24}$$

We moreover recall that $\mathcal{K}er_2$ is a $k[[K_0(p)]]$-submodule in $S_2(\sigma)$, with A generators given by $X_s^{r_s+1}$ for $s = 0, \dots, f-1$.

As $X_i^{r_i+1} \in \mathcal{K}er_2$ we deduce that the left-hand side in (24) is indeed in $\mathfrak{m}^{p-1+r_i+1} \cap \mathcal{K}er_2$ and the result follows by the definition of h. ∎

6.2. On the U^+-action. We turn our attention to the action of the upper unipotent radical U^+. Once again, the statements are much more delicate to prove and we now need the precise description of the Γ-action provided by Corollary 6.3. It is at this point that we require σ to be weakly regular. It is actually possible to treat the non-regular case, but the proofs become much more technical and we decided not to include them here.

The following Lemma is analogous to the Claim in the proof of Lemma 6.4, using Corollary 4.9 instead of Lemma 4.3:

Lemma 6.5: *Let $x \in \mathscr{O}_F$, $i \in \{0, \dots, f-1\}$ and let $\underline{j} \in \mathbf{N}^f$ be an f-uplet. We have the following relation in $S_2(\sigma)$:*

$$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} - 1\right) \cdot \underline{X}^{\underline{j}} X_i^{r_i+1} \in \sum_s X_s^{r_s+1} \mathfrak{m}^{|\underline{j}|+(p-2)-h}.$$

Proof. By Corollary 4.9 we have:

$$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} - 1\right) \cdot \underline{X}^{\underline{j}} X_i^{r_i+1} \quad \in \quad \mathfrak{m}^{p-2+r_i+1+|\underline{j}|}. \tag{25}$$

We moreover recall that $\mathcal{K}er_2$ is a $k[[K_0(p)]]$-submodule in $S_2(\sigma)$, with A generators given by $X_s^{r_s+1}$ for $s = 0, \dots, f-1$.

As $\underline{X}^{\underline{j}} X_i^{r_i+1} \in \mathcal{K}er_2$, we deduce that the left-hand side in (25) is indeed in $\mathfrak{m}^{p-2+r_i+1+|\underline{j}|} \cap \mathcal{K}er_2$ and the result follows by the definition of h. ∎

Lemma 6.6: *Let $n \geq 2$ and $x \in \mathcal{O}_F$. For any $i \in \{0, \ldots, f-1\}$ we have the following relation in $S_{n+1}(\sigma)$:*

$$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} - 1 \right) \cdot (\underline{X}^{\underline{j}} e_{n+1,i-n+1}) \in \sum_s e_{n+1,s} \mathfrak{m}^{p-2+|\underline{j}|} = I_{p-2+|\underline{j}|,n+1}. \tag{26}$$

In particular, for any $k \in \mathbf{N}$, the U^+*-action of* U^+ *is trivial on the quotients*

$$I_{k,n+1}/I_{k+(p-2),n+1} \text{ if } n \geq 2; \qquad\qquad I_{k,2}/I_{k+p-2-h,2}.$$

Proof. We only need to prove the statement when $n \geq 2$. In this case we deduce by Proposition 5.7 iii) that $e_{n+1,i-n+1} = \mathscr{F}(X_i^{r_i+1}) = \mathscr{F}(e_{2,i})$ is fixed by U^+.

According to Proposition 4.4 we have

$$\kappa_{\underline{j}}^{-1} \underline{X}^{\underline{j}} \equiv \sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ pz(\underline{\lambda}) & 1 \end{bmatrix} \qquad \text{modulo } \mathfrak{m}^{|\underline{j}|+(p-1)}$$

where we used the evident compact notations $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (k_F)^n$, $pz(\underline{\lambda}) \overset{\text{def}}{=} \sum_{i=1}^{n-1} p^i [\varphi^{-i+1}(\lambda_i)]$ and, if $\underline{j} = \sum_{i=1}^{n} p^{i-1} \underline{j}^{(i)}$ with $\underline{j}^{(i)} \in \{0, \ldots, p-1\}^f$, we define

$$\underline{\lambda}^{\underline{p-1-j}} \overset{\text{def}}{=} \lambda_1^{\underline{p-1-j}^{(1)}} \lambda_2^{\underline{p-1-j}^{(2)}} \cdot \lambda_n^{\underline{p-1-j}^{(n)}}.$$

As U^+ stabilizes $\mathfrak{m}^{|\underline{j}|+(p-1)}$ and fixes $e_{n+1,i}$ we can therefore write

$$\begin{aligned} &\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\kappa_{\underline{j}}^{-1} \underline{X}^{\underline{j}} e_{n+1,i-n+1}) \\ &\quad = \sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \begin{bmatrix} 1+pxz & 0 \\ 0 & 1-px\delta_x(z) \end{bmatrix} \cdot \mathscr{F}^{n-1}(e_{2,i}). \end{aligned}$$

modulo $I_{p-2+|\underline{j}|,n+1}$.

As $\mathscr{F}$ is Γ-equivariant and Γ^p acts trivially on $S_2(\sigma)$, we are left to understand the quantity

$$\begin{bmatrix} 1+p[\overline{x}\lambda_1] & 0 \\ 0 & 1-p[\overline{x}\lambda_1] \end{bmatrix} \cdot X_i^{r_i+1} \in S_2(\sigma).$$

We now use the notations and the statement of Corollary 6.3, which provides us with the equality

$$\begin{aligned}(27)\quad &\begin{bmatrix} 1+pxz & 0 \\ 0 & 1-px\delta_x(z) \end{bmatrix} \cdot e_{n+1,i-n+1} \\ &= \mathscr{F}^{n-1}\Big(X_i^{r_i+1} + \sum_{\underline{0}<\underline{s}\leq \underline{p-1}} P_{\underline{s}}(\lambda_1\overline{x})\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})} + (-2\lambda_1\overline{x})^{r_i+1}X_{i-1}^{p(r_i+1)}\Big).\end{aligned}$$

Let us fix an f-tuple $\underline{s}$ appearing in the RHS in (27) and write

$$P_{\underline{s}}(\lambda_1\overline{x}) = \sum_{\underline{0}\leq\underline{\alpha}\leq\underline{p-1}} \nu_{\underline{\alpha}}(\lambda_1\overline{x})^{\underline{\alpha}}$$

where $\nu_\alpha = \nu_{\underline{s},\underline{\alpha}}$ verify $\nu_{\underline{\alpha}} = 0$ as soon as $|\underline{\alpha}| > |\underline{s}|$.

As $|\underline{s}| > 0$, there exists an $m \in \{0,\dots,f-1\}$ (depending on $\underline{s}$) such that the element

$$\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})-(r_m+1)}e_{2,m}$$

is well defined and belongs to $\mathcal{K}er_2$.

Therefore we can write

$$\begin{aligned}(28)\quad &\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \mathscr{F}^{n-1}\big(P_{\underline{s}}(\lambda_1\overline{x})\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}\big) \\ &= \underline{X}^{p^n\underline{s}+p^{n-1}\underline{\ell}(\underline{s})-p^{n-1}(r_m+1)}e_{n+1,m+n-1}\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} P_{\underline{s}}(\lambda_1\overline{x})\end{aligned}$$

Let us further develop the RHS in (28). In the development of $P_{\underline{s}}(\lambda_1\overline{x})$ we fix an $\underline{\alpha}$ such that $|\underline{\alpha}| \leq |\underline{s}|$ and we obtain

$$\underline{x}^{\underline{\alpha}}\nu_{\underline{\alpha}}\underline{X}^{p^n\underline{s}+p^{n-1}\underline{\ell}(\underline{s})-p^{n-1}(r_m+1)}e_{n+1,m+n-1}\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}}\lambda_1^{\underline{\alpha}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix}$$

and, by Proposition 4.4 and the definition of the U^+-action on A, we have

$$\begin{aligned}(29)\quad &\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}}\lambda_1^{\underline{\alpha}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \\ &\qquad\in \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\Big(\underline{X}^{\underline{j}^{\geq 2}+[\underline{j}^{(1)}-\underline{\alpha}]} + \mathfrak{m}^{|\underline{j}^{\geq 2}|+|[\underline{j}^{(1)}-\underline{\alpha}]|+(p-1)}\Big)\end{aligned}$$

where $\underline{j}^{\geq 2} \overset{\mathrm{def}}{=} p\underline{j}^{(2)} + \cdots + p^{n-1}\underline{j}^{(n)}$ and $[\underline{j}^{(1)} - \underline{\alpha}] \in \{0, \dots, p-1\}^f$ is the unique non-zero f-tuple such that $\lambda_1^{\underline{j}^{(1)}-\underline{\alpha}} = \lambda_1^{[\underline{j}^{(1)}-\underline{\alpha}]}$ for all $\lambda_1 \in k_F$. In particular, (29) is in $\mathfrak{m}^{|\underline{j}|-|\underline{\alpha}|+n_\alpha(p-1)}$ where $n_\alpha \in \mathbf{N}$ is such that $|\underline{j}^{(1)}|-|n_\alpha|+n_\alpha(p-1) \geq 0$ (this follows from the definition of $|[\underline{j}^{(1)} - \underline{\alpha}]|$).

If we show that

$$p^n|\underline{s}| + p^{n-1}|\underline{\ell}(\underline{s})| - p^{n-1}(r_m + 1) - |\underline{\alpha}| \geq p - 2, \tag{30}$$

we finally obtain by Corollary 4.9 (as $\underline{s}$ and $\underline{\alpha}$ were arbitrary)

$$\mathscr{F}^{n-1}\Bigg(\sum_{\underline{0}<\underline{s}\leq \underline{p-1}} P_{\underline{s}}(\lambda_1\overline{x})\underline{X}^{p\underline{s}+\underline{\ell}(\underline{s})}\Bigg) \equiv 0 \tag{31}$$

modulo $I_{|\underline{j}|+(p-2),n+1}$.

But since $|\underline{\alpha}| \leq |\underline{s}|$ and $r_m \leq p-2$ (as σ is weakly regular), the inequality (30) is obvious.

In the very same manner one shows that

$$\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \mathscr{F}^{n-1}\big(\lambda_1^{r_i+1}X_{i-1}^{r_i+1}\big) \equiv 0 \tag{32}$$

modulo $I_{|\underline{j}|+(p-2),n+1}$.

Therefore, by (31) and (32) we obtain

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot (\kappa_{\underline{j}}^{-1}\underline{X}^{\underline{j}} e_{n+1,i-n+1}) \equiv \sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \cdot \mathscr{F}^{n-1}(e_{2,i}),$$

modulo $I_{p-2+|\underline{j}|,n+1}$.

Again by Proposition 4.4 and the definition of the U^+-action on A we have

$$\sum_{\underline{\lambda}} \underline{\lambda}^{\underline{p-1-j}} \begin{bmatrix} 1 & 0 \\ p\delta_x(z(\underline{\lambda})) & 1 \end{bmatrix} \in \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Big(\kappa_{\underline{j}}^{-1}\underline{X}^{\underline{j}} + \mathfrak{m}^{|\underline{j}|+(p-1)}\Big)$$

and the result finally follows from Corollary 4.9. ∎

7. The universal Iwasawa module

We can finally analyze some properties of the universal Iwasawa module $\mathfrak{S}_\infty^\bullet$. We first focus on the Iwahori radical filtration (§7.1). The main result is Proposition 7.1, where we show that, for a regular Serre weight σ, the A-radical filtration on $\mathfrak{S}_\infty^\bullet$ coincides with the $k[[K_0(p)]]$-radical filtration. In Corollary

7.3 we deduce the isotypical components of the cosocle of $\mathfrak{S}^{\bullet}_{\infty}$: we have a 2-dimensional isotypical space, together with some other infinite-dimensional spaces as soon as $F \neq \mathbf{Q}_p$.

In section 7.2 we study some torsion properties for the universal module $\mathfrak{S}^{\bullet}_{\infty}$ proving that it is torsion free over A and it contains a dense submodule of rank one over $\mathrm{Frac}(A)$ (Proposition 7.7).

We fix a Serre weight $\sigma = \sigma_{\underline{r}}$ as in section 6; in particular, σ is weakly regular: $0 \leq r_i \leq p-2$ for all $i \in \{0, \dots, f-1\}$. We say that σ is **regular** if we further have $r_i \geq 1$ for all $i \in \{0, \dots, f-1\}$; in particular, for a regular Serre weight we have $h(\sigma) \neq 0$ (cf. section 6 for the definition of $h(\sigma)$). We remark that our definition of regular Serre weight differs slightly from [Gee] (cf. loc. cit., Definition 2.1.5). Once again, some of the results of this section hold true in greater generality, but the proofs in non-regular cases become more technical (and we decided not to include them here).

Recall that, by Proposition 3.7, we have an A-linear morphism with dense image

$$\mathcal{M}^{\bullet}_{\infty} \overset{\mathrm{def}}{=} \Bigg(\bigoplus_{e_{2(j+1)+\bullet,i} \in \mathscr{G}^{\bullet}_{\infty}} A \cdot e_{2(j+1)+\bullet,i} \Bigg) \oplus A \cdot e_{\bullet} \overset{\Psi_{\infty}}{\twoheadrightarrow} \mathfrak{S}^{\bullet}_{\infty}$$

as well as a family of compatible commutative diagrams

$$\begin{tikzcd} \mathcal{M}^{\bullet}_{\infty} \arrow[r, "\Psi_{\infty}"] \arrow[d] & \mathfrak{S}^{\bullet}_{\infty} \arrow[d, "pr_{n+1}"] \\ \mathcal{M}^{\bullet}_{n+1} \arrow[r, two heads, "\Psi_{n+1}"] & \mathfrak{S}^{\bullet}_{n+1} \end{tikzcd} \tag{33}$$

where $\mathcal{M}^{\bullet}_{n+1} \overset{\mathrm{def}}{=} \Bigg(\bigoplus_{e_{2(j+1)+\bullet,i} \in \mathscr{G}^{\bullet}_{+}} A \cdot e_{2(j+1)+\bullet,i} \Bigg) \oplus A \cdot e_{\bullet}$.

We make $\mathbf{T}(k_F)$ act by $\mathfrak{a}^{-p^{i+2j+\bullet}(r_{i+2j+\bullet}+1)}(\chi^s_{\underline{r}})^{\vee}$ on $e_{2(j+1)+\bullet,i}$ and by $(\chi_{\underline{r}})^{\vee}$ on e_0 (resp. by $(\chi^s_{\underline{r}})^{\vee}$ on e_1); in this way the morphisms Ψ_{∞}, Ψ_{n+1} become $\mathbf{T}(k_F)$-equivariant.

7.1. Filtration on the fibered products. We endow the $k[[K_0(p)]]$-module $\mathcal{M}^{\bullet}_{\infty}$ with the valuation of the infimum, i.e. with the A-linear filtration

$\{\mathscr{J}_k^\bullet\}_k$ defined by

$$\mathscr{J}_k^\bullet \stackrel{\text{def}}{=} \Big\{ \sum_{i,j} P_{i,j}(\underline{X}) e_{2(j+1)+\bullet,i} + P_\bullet(\underline{X}) e_\bullet \in \mathcal{M}_\infty^\bullet,\ \min\{ord(P_{i,j}(\underline{X})), ord(P_\bullet(\underline{X}))\} \geq k \Big\}.$$

By Proposition 4.8 the filtration $\{\mathscr{J}_k^\bullet\}_k$ is $k[[K_0(p)]]$-stable and realizes the $k[[K_0(p)]]$-radical filtration on $\mathcal{M}_\infty^\bullet$. We define in the analogous, evident way the filtration $\{\mathscr{J}_{k,n+1}^\bullet\}_k$ on the modules $\mathcal{M}_{n+1}^\bullet$ for $n \geq 1$.

We define in the obvious way the A-linear filtration $\{\mathscr{I}_k^\bullet\}_k$ on $\mathfrak{S}_\infty^\bullet$:

$$\mathscr{I}_k^\bullet \stackrel{\text{def}}{=} \overline{\Psi_\infty(\mathscr{J}_k^\bullet)}.$$

By letting $\mathscr{I}_{k,n+1}^\bullet \stackrel{\text{def}}{=} \Psi_{n+1}(\mathscr{J}_{k,n+1}^\bullet)$, the commutative diagram (33) lets us write more expressively

$$\mathscr{I}_k^\bullet = \bigcap_{n \in 2\mathbf{N}+1+\bullet} pr_{n+1}^{-1}(\mathscr{I}_{k,n+1}^\bullet). \tag{34}$$

As the morphisms Ψ_∞, Ψ_{n+1} are not Γ, U^+-equivariant, there is no reason, a priori, for the filtration on $\mathfrak{S}_\infty^\bullet$, $\mathfrak{S}_{n+1}^\bullet$ to be $k[[K_0(p)]]$-stable; as we did in sections 4 and 6 the aim of this section is to prove that this is indeed the case, i.e. that $\mathscr{I}_k^\bullet$ is a $k[[K_0(p)]]$-submodule in $\mathfrak{S}_\infty^\bullet$ for any $k \geq 0$.

This follows (almost) directly from Corollary 4.9 and Proposition 6.1, using a formal argument on the valuation ord_{n+1} on the modules $S_{n+1}(\sigma)$.

A remark for the case $\bullet = 0$. In order to get a better result for the behavior of the filtrations $\mathscr{I}_k^0$, $\mathscr{I}_{k,n+1}^0$ we need to slightly refine their construction. This is because the $k[[K_0(p)]]$-module $\mathfrak{S}_2^0$ behaves in a slightly different way than the modules $\mathfrak{S}_{2t}^{\geq 2m+1}$ $(m, t \geq 1)$, cf. Lemma 6.5.

Thus, assume that $n \geq 3$ is odd.

Write $\mathfrak{S}_*^{\geq 3} \stackrel{\text{def}}{=} \ker\left(\mathfrak{S}_*^0 \twoheadrightarrow \mathfrak{S}_2^0\right)$ for $* \in \{n+1, \infty\}$ and set $\mathcal{M}_*^{\geq 3}$ for the stadard complement of $\mathcal{M}_2^0$ in $\mathcal{M}_*^\bullet$. The morphism Ψ_∞ restricts to an A-linear, $\mathbf{T}(k_F)$-equivariant morphism with dense image $\Psi_\infty^{\geq 3} : \mathcal{M}_\infty^{\geq 3} \to \mathfrak{S}_\infty^{\geq 3}$ (resp. an A-linear, $\mathbf{T}(k_F)$-equivariant epimorphism $\Psi_{n+1}^{\geq 3} : \mathcal{M}_{n+1}^{\geq 3} \to \mathfrak{S}_{n+1}^{\geq 3}$) and we have the evident compatibility between $\Psi_\infty^{\geq 3}$ and $\Psi_{n+1}^{\geq 3}$ as in (33).

We define in an analogous way the filtrations $\{\mathscr{I}_k^{\geq 3}\}_k$, $\{\mathscr{I}_{k,n+1}^{\geq 3}\}_k$ on $\mathfrak{S}_\infty^{\geq 3}$ and $\mathfrak{S}_{n+1}^{\geq 3}$, having

$$\mathscr{I}_k^{\geq 3} = \bigcap_{n\in 2\mathbf{N}+3} pr_{n+1}^{-1}(\mathscr{I}_{k,n+1}^{\geq 3}). \tag{35}$$

Let $\mathscr{I}_k^{\leq 2}$ be the image of $\mathscr{J}_{k,2}^0$ in $\mathfrak{S}_\infty^0$ via $\Psi_\infty|_{\mathcal{M}_2^0}$. It is a closed A-submodule of $\mathfrak{S}_\infty^0$, as $\mathcal{M}_2^0$ is finitely generated. Since $\mathfrak{S}_\infty^0 = \mathfrak{S}_\infty^{\geq 3} \times_{S_3} \mathfrak{S}_2^0$ and $\mathscr{I}_k^{\geq 3}$, $\mathscr{I}_k^{\leq 2}$ are closed in $\mathfrak{S}_\infty^0$, we have

$$\mathscr{I}_k^0 = \mathscr{I}_k^{\geq 3} + \mathscr{I}_k^{\leq 2},$$

and similarly, $\mathscr{I}_{k,n+1}^0 = \mathscr{I}_{k,n+1}^{\geq 3} + \mathscr{I}_{k,n+1}^{\leq 2}$ (with the obvious definition of $\mathscr{I}_{k,n+1}^{\leq 2}$).

We are now ready to describe the behavior of $\mathscr{I}_k^\bullet$ with respect to the Γ, U^+ actions:

Proposition 7.1: *Assume σ is a weakly regular Serre weight.*

a) *The A-linear filtration $\{\mathscr{I}_k^1\}_k$ on $\mathfrak{S}_\infty^1$ (resp. $\{\mathscr{I}_k^{\geq 3}\}_k$ on $\mathfrak{S}_\infty^{\geq 3} \subseteq \mathfrak{S}_\infty^0$) is $k[[K_0(p)]]$-stable.*

b) *For all $k \in \mathbf{N}$, the Γ, U^+ actions are trivial on the subquotients*

$$\mathscr{I}_k^1/\mathscr{I}_{k+(p-2)}^1, \qquad \mathscr{I}_k^{\geq 3}/\mathscr{I}_{k+(p-2)}^{\geq 3}$$

of $\mathfrak{S}_\infty^1$, $\mathfrak{S}_\infty^{\geq 3}$ respectively.

c) *Assume further that σ is regular. Then the A-linear filtration $\{\mathscr{I}_k^0\}_k$ on $\mathfrak{S}_\infty^0$ is $k[[K_0(p)]]$-stable and the Γ, U^+ actions are trivial on the subquotients*

$$\mathscr{I}_k^0/\big(\mathscr{I}_{k+(p-2)}^{\geq 3} + \mathscr{I}_{k+(p-2)-h}^{\leq 2}\big).$$

In particular, the filtration $\{\mathscr{I}_k^\bullet\}_k$ defines the $k[[K_0(p)]]$-radical filtration on $\mathfrak{S}_\infty^1$ and on $\mathfrak{S}_\infty^{\geq 3}$; the same result holds true for $\mathfrak{S}_\infty^0$ if σ is further assumed to be regular.

As the action of Γ, U_0^+ is continuous and the projection maps pr_{n+1} are $k[[K_0(p)]]$-equivariant we deduce from the expressions (34), (35) above that it is enough to prove Proposition 7.1 for an arbitrary finite level $\mathfrak{S}_{n+1}^\bullet$.

We first consider the case $n = 1$.

Lemma 7.2: *Assume that σ is a regular Serre weight. The A-linear filtration $\mathscr{I}_{k,2}^0$ on $\mathfrak{S}_2^0$ is formed by $k[[K_0(p)]]$-modules. Moreover for any $k \geq 0$ the Γ,*

U^+*-action is trivial on the quotients*

$$\mathscr{I}^0_{k,2}/\mathscr{I}^0_{k+(p-2)-h,2}.$$

Proof. Recall that $\mathcal{K}er_2$ is a $k[[K_0(p)]]$-submodule of S_2 and $\mathscr{I}^0_{k,2} = I_{k,2}$ for $k \geq 1$ (where $I_{k,2}$ are the $k[[K_0(p)]]$-submodules defining the radical filtration on $\mathcal{K}er_2$, cf. §6). It is therefore enough, by means of Lemmas 6.4 and 6.6, to show that for any $g \in \Gamma, \mathrm{U}^+$ we have $(g-1)\underline{X}^{\underline{r}} \in \mathfrak{m}^{p-2-h}\mathcal{K}er_2$. Writing

$$(g-1)\underline{X}^{\underline{r}} = \sum_s X_s^{r_s+1} P_s(\underline{X})$$

for some $P_s(\underline{X}) \in A$, we have $ord(X_s^{r_s+1}P_s(\underline{X})) \geq p-2+|\underline{r}|$ by Corollary 4.9 hence $ord(P_s(\underline{X})) \geq p-3+|\underline{r}|-r_s$. The result follows, recalling that $f \geq 2$ and $r_i \geq 1$ for all i. ∎

Proof of Proposition 7.1 in the finite case. Fix $n \geq 2$ and consider the module $\mathfrak{S}^\bullet_{n+1} \subseteq S_{n+1}(\sigma)$. We note that $\mathscr{I}^\bullet_{k,n+1}$ is the image, inside $\mathfrak{S}^\bullet_{n+1}$, of the A-module $\mathfrak{m}^k \otimes_A \mathcal{M}^\bullet_{n+1}$:

$$\mathscr{I}^\bullet_{k,n+1} = \langle e_{2(j+1)+\bullet,i}, e_\bullet,\ j=0,\dots,\frac{n-1-\bullet}{2}, i=0,\dots,f-1\rangle_{\mathfrak{m}^kA}.$$

Fix a couple $(j_0,i_0) \in \{0,\dots,\frac{n-1-\bullet}{2}\} \times \{0,\dots,f-1\}$, an f-tuple $\underline{l} \in \mathbf{N}^f$ of length $|l| = k$; consider the element

$$\underline{X}^{\underline{l}} e_{2(j_0+1)+\bullet,i_0} \qquad (\text{resp. } \underline{X}^{\underline{l}} e_\bullet).$$

As $\ker(\mathfrak{S}^\bullet_{n+1} \twoheadrightarrow \mathfrak{S}^\bullet_{2j_0+\bullet})$ is generated (over A) by the elements $e_{2(j+1)+\bullet,i}$ for $j_0 \leq j \leq \frac{n-1-\bullet}{2}$, $i=0,\dots,f-1$ we can write

$$(g-1)\cdot \underline{X}^{\underline{l}} e_{2(j_0+1)+\bullet,i_0} \quad = \quad \sum_{i=0}^{f-1}\sum_{j\geq j_0} P_{j,i}(\underline{X}) e_{2(j+1)+\bullet,i}$$

(resp. $(g-1)\cdot\underline{X}^{\underline{l}} e_\bullet = \sum_{i=0}^{f-1}\sum_{j\geq 0} P_{j,i}(\underline{X})e_{2(j+1)+\bullet,i} + P_\bullet(\underline{X})e_\bullet$) and, by Corollary 4.8, we deduce that

$$ord(P_{j,i}(\underline{X})e_{2(j+1)+\bullet,i}) \geq k+(p-2)+ord(e_{2(j_0+1)+\bullet,i_0})$$

for all i and $j \geq j_0$
(resp. $ord(P_\bullet(\underline{X})e_\bullet) \geq k+(p-2)+ord(e_\bullet)$ and $ord(P_{j,i}(\underline{X})e_{2(j+1)+\bullet,i}) \geq k+(p-2)+ord(e_\bullet)$ for all $j \geq 0$, i).

Thanks to Lemma 7.2 and Proposition 6.1 it is now enough to prove that

$$ord(e_{2(j_0+1)+\bullet,i_0}) \geq ord(e_{2(j+1)+\bullet,i})$$

for any $j > j_0$ and any i (resp. to prove that $ord(e_\bullet) \geq ord(e_{2(j+1)+\bullet,i})$ for any $j \geq 1-\bullet$ and any i). Recalling the valuation of the elements $e_{2(j+1)+\bullet,i}$, $e_\bullet$ (cf. Definition 3.5) we are left to prove the inequality

$$\sum_{l=0}^{f-1}\sum_{s=2j_0+1+\bullet}^{n-1}(-1)^{s+\bullet}(r_{l+s}+1)p^s \geq p^{2j+\bullet}(r_{i+2j+\bullet}+1)+\sum_{l=0}^{f-1}\sum_{s=2j+1+\bullet}^{n-1}(-1)^{s+\bullet}(r_{l+s}+1)p^s$$

for all $j > j_0$ and all $i = 0,\dots,f-1$ (resp.

$$\sum_{l=0}^{f-1}\left(\sum_{s=0}^{n-1}(-1)^{s+\bullet}(r_{l+s}+1)p^s-\delta_{\bullet,0}\right) \geq p^{2j+\bullet}(r_{i+2j+\bullet}+1)+\sum_{l=0}^{f-1}\sum_{s=2j+1+\bullet}^{n-1}(-1)^{s+\bullet}(r_{l+s}+1)p^s$$

for all $j \geq 1-\bullet$ and all $i = 0,\dots,f-1$). By a simple manipulation we are reduced to prove that

$$p^m(r_{i+m}+1) \leq \sum_{l=0}^{f-1} p^m(r_{l+m}+1) - p^{m-1}(r_{l+m-1}+1)$$

where $m \overset{\text{def}}{=} 2j+\bullet \geq 1$. This is trivially true if $f \geq 3$ or $f = 2$ and σ is weakly regular. ∎

Thanks to Proposition 7.1 we obtain the isotypical components appearing in the cosocle of the universal module $\mathfrak{S}^\bullet_\infty$:

Corollary 7.3: *Assume that σ is a regular Serre weight. Then*

$$cosoc_{k[[I]]}(\mathfrak{S}^0_\infty) = V(\chi_{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V\left(\chi_{\underline{r}}\det^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}\right),$$

$$cosoc_{k[[I]]}(\mathfrak{S}^1_\infty) = V\left(\chi_{\underline{r}}\det^{-\underline{r}}\right) \oplus \bigoplus_{i=0}^{f-1} V\left(\chi_{\underline{r}}\det^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}\right),$$

where

$$\dim(V(\chi_{-\underline{r}})) = \dim(V(\chi_{\underline{r}}\det^{-\underline{r}})) = 1,$$

$$\dim\left(V\left(\chi_{\underline{r}}\det^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}\right)\right) = \begin{cases} \infty & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } F \neq \mathbf{Q}_p, \\ 0 & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } F = \mathbf{Q}_p. \end{cases}$$

Proof. It is an immediate consequence of Proposition 7.1. ∎

7.2. Torsion properties of the universal module. In this section we prove that $\mathfrak{S}^{\bullet}_{\infty}$ is A-torsion free and, given any elements $e, e' \in \mathscr{G}^{\bullet}_{\infty}$, the natural morphism $A\cdot e \oplus A \cdot e' \to \mathfrak{S}^{\bullet}_{\infty}$ factors through a rank one quotient of $A\cdot e\oplus A\cdot e'$. Recall that $\sigma = \sigma_{\underline{r}}$ is a fixed regular Serre weight.

We start from the following elementary observation:

Lemma 7.4: *Let $\bullet \in \{0,1\}$. Then, for any $l \in \{0,\dots,f-1\}$ and any $j \in \mathbf{N}$, we have*

$$\lim_{\substack{n\to\infty \\ n\in 2\mathbf{N}+1+\bullet}} \sum_{s=2(j+1)+1+\bullet}^{n} (-1)^{s+1+\bullet} p^s (r_{l+s}+1) = +\infty.$$

Proof. It is an elementary computation. ∎

One first property of the universal module is that, for any choice of generators $e, e' \in \mathscr{G}^{\bullet}_{\infty}$, the natural morphism $A \cdot e \oplus A \cdot e' \to \mathfrak{S}^{\bullet}_{\infty}$ has a nonzero kernel.

Proposition 7.5: *Let $\bullet \in \{0,1\}$ and fix two elements $e, e' \in \mathscr{G}^{\bullet}_{\infty}$ with $e \neq e'$. The natural morphism*

$$\begin{array}{rcl} A\cdot e\oplus A\cdot e' & \to & \mathfrak{S}^{\bullet}_{\infty} \\ \big(P(\underline{X}), P'(\underline{X})\big) & \mapsto & P(\underline{X})e + P'(\underline{X})e' \end{array}$$

has a nonzero kernel.

Proof. The proof is elementary and we only consider the case $\bullet = 0$ (the other is similar).

By the construction of $\mathfrak{S}^0_\infty$ it is enough to prove that for $n \in 2\mathbf{N}+1$, $n >> 0$, we have a commutative diagram

$$\begin{tikzcd}
A\cdot e\oplus A\cdot e' \arrow[rr] \arrow[dr] \arrow[dd, equal] & & \mathfrak{S}^0_{n+3} \arrow[dd] \\
& Q \arrow[ur] \arrow[dd, equal] & \\
A\cdot e\oplus A\cdot e' \arrow[rr] \arrow[dr] & & \mathfrak{S}^0_{n+1} \\
& Q \arrow[ur] &
\end{tikzcd}$$

where Q is an appropriate A-module of rank at most 1.

There exists $n \in 2\mathbf{N}+1$, $n >> 0$ such that the maps $Ae \to \mathfrak{S}^0_{n+1}$, $Ae' \to \mathfrak{S}^0_{n+1}$ are both nonzero. Since $\mathfrak{S}^0_{n+1} \hookrightarrow S_{n+1}$ and the latter is a quotient of A, we deduce that there exist two monomials $P(\underline{X})$, $P'(\underline{X}) \in A$ such that $P(\underline{X})e \neq 0 \neq P'(\underline{X})e'$ and $P(\underline{X})e + P'(\underline{X})e' = 0$ in $\mathfrak{S}^0_{n+1}$. We can therefore set $Q \overset{\text{def}}{=} (Ae \oplus Ae')/\langle P(\underline{X})e, P'(\underline{X})e'\rangle_A$.

It is now enough to show that we have $P(\underline{X})e+P'(\underline{X})e' = 0$ in $\mathfrak{S}^0_{n+3}$ and this is clear since (the image of) e, e' are monomials of S_{n+3}, hence $P(\underline{X})e$, $P'(\underline{X})e'$ are monomials of S_{n+3} which maps to nonzero elements in S_{n+2} via the natural projection $S_{n+3} \twoheadrightarrow S_{n+2}$ (and, by construction, their image in S_{n+2} belongs to the image of S_{n+1} in S_{n+2}). ∎

On the other hand, we can prove that $\mathfrak{S}^\bullet_\infty$ is A-torsion free:

Proposition 7.6: *Let σ be a regular Serre weight. The associated universal modules $\mathfrak{S}^0_\infty$, $\mathfrak{S}^1_\infty$ are torsion free as A-modules.*

Proof. Recall that for $n \in 2\mathbf{N}+1+\bullet$ we write $pr_{n+1} : \mathfrak{S}^\bullet_\infty \twoheadrightarrow \mathfrak{S}_{n+1}$ for the natural projection.

Assume the statement is false. Then there are non-zero elements $P(\underline{X}) \in A$, $x \in \mathfrak{S}^\bullet_\infty$ such that $P(\underline{X}) \cdot x = 0$ in $\mathfrak{S}^\bullet_\infty$. Since $x \neq 0$ there is $n_0 \in 2\mathbf{N}+1+\bullet$ such that $pr_{n_0+1}(x) \neq 0$ in $\mathfrak{S}_{n_0+1}$. Write

$$m_i \overset{\text{def}}{=} \mathrm{ord}_{X_i}(P(\underline{X})), \qquad m_i' \overset{\text{def}}{=} \mathrm{ord}_{X_i}(pr_{n_0+1}(x))$$

where $\mathrm{ord}_{X_i}(*)$ denotes the order in the direction X_i of an element $* \in S_{n+1}$.

An immediate induction, together with the definition of the transition morphisms $\mathfrak{S}^\bullet_{n+1} \twoheadrightarrow \mathfrak{S}^\bullet_{n-1}$, gives

$$\begin{aligned}\mathrm{ord}_{X_i}(pr_{n+1}(x)) &= m_i' + \sum_{j=0}^{n-n_0-1} p^{n_0+j}(-1)^{j+\bullet}(r_{i+n_0+j}+1)\\ &= m_i' + \sum_{j=n_0}^{n-1} p^{j}(-1)^{j+\bullet}(r_{i+j}+1)\end{aligned}$$

for all $n \in 2\mathbf{N}+1+\bullet$, $n \geq n_0$. Hence, by Lemma 7.4, we deduce that for any $i \in \{0,\ldots,f-1\}$ there exists $n_i \in 2\mathbf{N}+1+\bullet$, $n_i >> 0$ such that:

$$m_i + m_i' + \sum_{j=n_0}^{n_i-1} p^{j}(-1)^{j+\bullet}(r_{i+j}+1) < p^{n_i}(r_{i+n_i}+1); \tag{36}$$

in particular, there exists $N \in 2\mathbf{N}+1+\bullet$, $N >> n_0$ such that (36) holds for all $i \in \{0,\ldots,f-1\}$ with n_i replaced by N.

We can thus find a suitable lift $y \in A$ of $pr_{N+1}(x)$ via the morphism $A \twoheadrightarrow S_{N+1}$ such that

$$\mathrm{ord}_{X_i}(P(\underline{X})y) < p^N(r_{i+N}+1)$$

for all $i \in \{0,\ldots,f-1\}$ and this means precisely that $P(\underline{X})y$ maps to a nonzero element via $A \twoheadrightarrow S_{N+1}$, against the hypothesis that $P(\underline{X})x = 0$ in $\mathfrak{S}^\bullet_\infty$. ∎

We deduce, from Propositions 7.5 and 7.6, the following result on the torsion properties of the $k[[K_0(p)]]$-module $\mathfrak{S}^\bullet_\infty$:

Proposition 7.7: *Let $x \in \mathfrak{S}^\bullet_\infty$ be a nonzero element, lying in the image of the natural morphism*

$$\bigoplus_{e \in \mathscr{G}^\bullet_\infty} A\cdot e \to \mathfrak{S}^\bullet_\infty.$$

Then $\mathfrak{S}^\bullet_\infty/\langle x\rangle_A$ has a natural structure of profinite A-module and the torsion submodule $Tor\big(\mathfrak{S}^\bullet_\infty/\langle x\rangle_A\big)$ is dense in $\mathfrak{S}^\bullet_\infty/\langle x\rangle_A$ for the natural profinite topology.

Proof. Since A is compact it is clear that $\langle x\rangle_A$ is a closed submodule of $\mathfrak{S}^\bullet_\infty$. By Proposition 7.5 we deduce that the image of the natural morphism

$$\bigoplus_{e \in \mathscr{G}^\bullet_\infty} A\cdot e \to \mathfrak{S}^\bullet_\infty$$

is a rank one, dense A-submodule of $\mathfrak{S}^\bullet_\infty$. The result follows. ∎

7.3. The case $F = \mathbf{Q}_p$. The aim of this section is to describe explicitly the $k[[K_0(p)]]$-module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ in Galois theoretical terms when $F = \mathbf{Q}_p$.

Let $\mathbf{Q}_{p^2}$ be the quadratic unramified extension of $\mathbf{Q}_p$. We fix an embedding $\mathbf{Q}_{p^2} \overset{\iota}{\hookrightarrow} E$ and, for $j \in \{0,1\}$, we write $\tau_j \overset{\text{def}}{=} \iota \circ \mathrm{Frob}_{\mathbf{Q}_{p^2}}^j$ where $\mathrm{Frob}_{\mathbf{Q}_{p^2}}$ is the absolute Frobenius on $\mathbf{Q}_{p^2}$. With this choice, we can define the fundamental Serre character ω_2 of niveau 2 associated to the residual embedding $\mathbf{F}_{p^2} \hookrightarrow k$. For $n \in \{1, \dots, p\}$ we write $\mathrm{ind}(\omega_2^n)$ for the unique (absolutely) irreducible 2-dimensional representation of $G_{\mathbf{Q}_p}$ whose restriction to the inertia subgroup $I_{\mathbf{Q}_p}$ is isomorphic to $\omega_2^n \oplus \omega_2^{pn}$ and whose determinant is ω^n (where ω is the mod-p cyclotomic character).

If $0 \leq r \leq p-1$ the Galois representation $\mathrm{ind}(\omega_2^{r+1})$ corresponds to the supersingular representation $\pi(\sigma_r, 0)$ and the aim of this section is to show that the $\mathscr{F}$-module $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is isomorphic to the "mod-p Wach module" associated to the dual of $\mathrm{ind}(\omega_2^{r+1})$.

Recall that for $\bullet \in \{0,1\}$ we have defined (cf. Definition 3.5) the elements $e_{2(j+1)+\bullet,i}$; as $F = \mathbf{Q}_p$ we omit the subscript i in what follows. Then $\mathfrak{S}_\infty^\bullet$ is easily seen to be generated over A by the only element $e_\bullet$.

Fix $n \in 2\mathbf{N}+1$. Using the definition of the elements e_0, e_2 one verifies that $e_2 = X \cdot e_0$ in $\mathfrak{S}_{n+1}^0$. Since $n \in 2\mathbf{N}+1$ is arbitrary we deduce that $\mathfrak{S}_\infty^{\geq 1}$ is the submodule of $\mathfrak{S}_\infty^0$ generated (over A) by the element $e_2 = X \cdot e_0$; a completely analogous argument shows that $\mathfrak{S}_\infty^{\geq 2}$ is the submodule of $\mathfrak{S}_\infty^1$ generated (over A) by the element $e_3 = X^{r+1} \cdot e_1$.

We turn our attention to the action of the Frobenius. By Proposition 5.11 we have

$$\begin{aligned} \mathscr{F}(e_2) &= e_3, \\ \mathscr{F}(e_3) &= e_4 = X^{(p-1)(r+1)} e_2. \end{aligned} \tag{37}$$

As usual, the equality $e_4 = X^{(p-1)(r+1)} e_2$ is verified in any quotient $\mathfrak{S}_{n+1}^\bullet$ (with $n \geq 3$) using the explicit description of the elements e_4, e_2 given in Proposition 3.7.

We leave to the reader the task to verify that $\mathbf{T}(k_F)$ acts on e_2, e_3 by the character $\left(\chi_r \mathfrak{a}\right)^\vee$ so that, by Proposition 7.1 we deduce the $\mathbf{Z}_p^\times$ action:

$$\gamma e_{2+\bullet} \equiv \overline{\gamma} e_{2+\bullet} + (X^{p-1}) \cdot e_{2+\bullet} \qquad \mathrm{mod}\ X^{p-1}$$

for $\bullet \in \{0,1\}$.

Let $V_{r+1}(0)$ be the irreducible crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge–Tate weights $(0,-(r+1))$ and whose trace of Frobenius equals zero. We claim that $\mathfrak{S}_{\infty}^{\geq 1}\oplus\mathfrak{S}_{\infty}^{\geq 2}$ is isomorphic, as a Frobenius module, to the mod-p reduction of the Wach module associated to $V_{r+1}(0)$.We decided to include a self contained argument of this well-known result and we invite the reader to refer to [Ber] for the general theory of the Wach modules (cf. also [BLZ] or [Dou]).

Let $0\leq r\leq p-1$ and $\mathrm{N}_{r+1}(0)$ be the rank two φ-module over $\mathscr{O}[[X]]$ whose Frobenius action is characterized by

$$[\varphi(n_1),\varphi(n_2)]=[n_1,n_2]\begin{bmatrix} 0 & 1\\ \mathfrak{q}^{r+1} & 0\end{bmatrix} \tag{38}$$

where (n_1,n_2) is a $\mathscr{O}[[X]]$-basis for $\mathrm{N}_{r+1}(0)$ and $\mathfrak{q}\overset{\mathrm{def}}{=}\frac{(1+X)^p-1}{X}\in\mathscr{O}[[X]]$. By the work of [BLZ] (Proposition 3.1.3), there exists a $\mathscr{O}[[X]]$-semilinear, φ-equivariant $\mathbf{Z}_p^{\times}$-action on $\mathrm{N}_{r+1}(0)$, which is trivial modulo $X\mathrm{N}_{r+1}(0)$; this gives rise to a well-defined structure of the Wach module on $\mathrm{N}_{r+1}(0)$.

The module $\mathrm{N}_{r+1}(0)$ is endowed with a filtration (cf. [Ber], Théorème III.4.4)

$$\mathrm{Fil}^j\big(\mathrm{N}_{r+1}(0)\big)\overset{\mathrm{def}}{=}\big\{x\in\mathrm{N}_{r+1}(0),\ \phi(x)\in\mathfrak{q}^j\mathrm{N}_{r+1}(0)\big\}$$

and one sees that

$$\mathrm{Fil}^j\big(\mathrm{N}_{r+1}(0)/X\mathrm{N}_{r+1}(0)\big)=\begin{cases}\mathscr{O}n_1\oplus\mathscr{O}n_2 & \text{if} j\leq 0\\ \mathscr{O}n_1 & \text{if} 1\leq j\leq r+1\\ 0 & \text{if} j\geq r+2\end{cases} \tag{39}$$

(cf. [BLZ], proof of Proposition 3.2.4).

By [Ber], Proposition III.4.2 and Corollaire III.4.5 we have an isomorphism of filtered φ-modules over E:

$$E\otimes_{\mathscr{O}}\big(\mathrm{N}_{r+1}(0)/X\mathrm{N}_{r+1}(0)\big)\xrightarrow{\sim}\mathbf{D}_{cris}(V_{r+1}(0))$$

for an appropriate crystalline representation $V_{r+1}(0)$ with Hodge–Tate weights $\{0,-(r+1)\}$.

Lemma 7.8: *In the previous hypotheses, we have an isomorphism of crystalline representations* $V_{r+1}(0)\cong\mathrm{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}}\chi_{(0,1)}^{r+1}$*, where* $\chi_{(0,1)}$ *is the crystalline character of* $G_{\mathbf{Q}_{p^2}}$ *with labelled Hodge–Tate weights* $-(0,1)$ *and such that* $\chi_{(0,1)}(p)=1$.

Proof. By equations (38) and (39) we have a complete description of the filtered φ-module $E \otimes_{\mathscr{O}} \big(\mathrm{N}_{r+1}(0)/X\mathrm{N}_{r+1}(0)\big)$ (note that $\mathfrak{q}^{r+1}n_2 \equiv p^{r+1}n_2 \bmod X\mathrm{N}_{r+1}(0)$). The result follows then from Breuil [Bre03b] Propositions 3.1.2 and 3.1.1.

Alternatively, we can prove the Lemma using the theory of Wach modules, as we outline in the following lines.

One easily sees by (39) that the filtered module $E \otimes_{\mathscr{O}_E} \big(\mathrm{N}_{r+1}(0)/XN_{r+1}(0)\big)$ has no nonzero, φ-admissible proper submodules, and hence $V_{r+1}(0)$ is irreducible. Let $T_{r+1}(0)$ be the $\mathscr{O}_E$-lattice of $V_{r+1}(0)$ corresponding to $\mathrm{N}_{r+1}(0)$ via the equivalence of [Ber] Proposition III.4.2.

By results of [Dou], §2 we can describe the $G_{\mathbf{Q}_{p^2}}$-restriction of $T_{r+1}(0)$ in terms of Wach modules over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathscr{O}_E[[X]]$. Recall the natural isomorphism of rings

$$\begin{array}{rcl} \mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathscr{O}[[X]] & \xrightarrow{\sim} & \mathscr{O}[[X]] \oplus \mathscr{O}[[X]] \\ x \otimes P(X) & \longmapsto & (\tau_0(x)P(X), \tau_1(x)P(X)). \end{array}$$

By [Dou], Propositions 2.5 and 2.6, the Wach module over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathscr{O}_E[[X]]$ associated to $T_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}}$ is obtained by extension of scalars from $\mathrm{N}_{r+1}(0)$. In particular, its Frobenius action is defined by

$$[\varphi(n_1), \varphi(n_2)] = [n_1, n_2] \begin{bmatrix} (0,0) & (1,1) \\ (\mathfrak{q}^{r+1}, \mathfrak{q}^{r+1}) & (0,0) \end{bmatrix},$$

and the matrix equality

$$\begin{bmatrix} (1,0) & (0,1) \\ (0,1) & (1,0) \end{bmatrix} \begin{bmatrix} (0,0) & (1,1) \\ (\mathfrak{q}^{r+1}, \mathfrak{q}^{r+1}) & (0,0) \end{bmatrix} = \begin{bmatrix} (1,\mathfrak{q}^{r+1}) & (0,0) \\ (0,0) & (\mathfrak{q}^{r+1},1) \end{bmatrix} \varphi\left(\begin{bmatrix} (1,0) & (0,1) \\ (0,1) & (1,0) \end{bmatrix} \right)$$

shows that we have an isomorphism of Wach modules

$$\mathbf{N}\big(T_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}}\big) \xrightarrow{\sim} \mathbf{N}_{(0,r+1),1} \oplus \mathbf{N}_{(r+1,0),1}$$

where $\mathbf{N}_{(0,r+1),1}$ (resp. $\mathbf{N}_{(r+1,0),1}$) are the rank one Wach modules over $\mathbf{Z}_{p^2} \otimes_{\mathbf{Z}_p} \mathscr{O}_E[[X]]$ whose Frobenius action is characterized, on appropriate generators η_0, η_1, by $\varphi(\eta_0) = (1, \mathfrak{q}^{r+1})\eta_0$ (resp. $\varphi(\eta_1) = (\mathfrak{q}^{r+1}, 1)\eta_1$, cf. also [Dou], §3.1).

By [Dou], Proposition 3.5 et seq., we have

$$E \otimes_{\mathscr{O}} \mathbf{N}_{(0,r+1),1} = \mathbf{D}_{cris}(\chi_{(0,1)}^{r+1}), \qquad E \otimes_{\mathscr{O}} \mathbf{N}_{(r+1,0),1} = \mathbf{D}_{cris}(\chi_{(1,0)}^{r+1})$$

where $\chi_{(1,0)}$ (resp. $\chi_{(0,1)}$) is the crystalline character of labelled Hodge–Tate weights $-(1,0)$ (resp. $-(0,1)$) such that $\chi_{(0,1)}(p) = 1 = \chi_{(1,0)}(p)$. It follows that $V_{r+1}(0)|_{G_{\mathbf{Q}_{p^2}}} \cong \chi_{(1,0)}^{r+1} \oplus \chi_{(0,1)}^{r+1}$ and we deduce

$$V_{r+1}(0) \cong \mathrm{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \chi_{0,1}^{r+1}$$

as $V_{r+1}(0)$ is irreducible. ∎

The mod-p reduction of the crystalline character $\chi_{(0,1)}$ is deduced from [Dou], Lemma 6.2:

$$\overline{\chi}_{(0,1)} \cong \omega_2^{-1},$$

hence the mod-p reduction of the crystalline representation $V_{r+1}(0)$ is given by

$$\overline{V}_{r+1}(0) \cong \big(\mathrm{ind}(\omega_2^{r+1})\big)^*$$

(note that $\overline{V}_{r+1}(0)$ is irreducible as $r \leq p-1$), and we define the mod-p Wach module:

$$\mathbf{N}\big(\big(\mathrm{ind}(\omega_2^{r+1})\big)^*\big) \stackrel{\mathrm{def}}{=} \mathrm{N}_{r+1}(0) \otimes_{\mathscr{O}} k.$$

Since $\mathfrak{q} n_1 \otimes 1 = X^{p-1} n_1 \otimes 1$ in $\mathrm{N}_{r+1}(0) \otimes_{\mathscr{O}} k$, by comparing equations (38) and (37) we deduce:

Proposition 7.9: *Let $F = \mathbf{Q}_p$ and $\sigma = \sigma_r$ for $r \in \{0, \ldots, p-1\}$. We have an isomorphism of φ-modules over A:*

$$\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2} \cong \mathbf{N}\big(\big(\mathrm{ind}(\omega_2^{(r+1)})\big)^*\big),$$

where $\mathfrak{S}_\infty^{\geq 1} \oplus \mathfrak{S}_\infty^{\geq 2}$ is the $k[[K_0(p)]]$-module associated to the supersingular representation $\pi(\sigma, 0)$.

8. A note on principal and special series

We give here a glimpse of the previous constructions when considering tamely ramified principal series. The arguments are now much simpler; we invite the reader to refer to [Mo1], §5 for the omitted details.

Recall that the tamely ramified principal series for $\mathbf{GL}_2(F)$ are described (up to a twist by a smooth character) by the parabolic induction

$$\pi_{\underline{r},\mu} \stackrel{\mathrm{def}}{=} \mathrm{ind}_{\mathbf{B}(F)}^{\mathbf{GL}_2(F)}(\mathrm{un}_\mu \otimes \omega_f^r \mathrm{un}_{\mu^{-1}}) \tag{40}$$

where $\mu \in \overline{k}^{\times}$, un_μ is the unramified character of $F^\times$ verifying $\mathrm{un}_\mu(\varpi) = \mu$, $\underline{r} \in \{0, \dots, p-1\}^f$ and ω_f is a choice of a Serre fundamental character of level f.

It is known by the work of Barthel and Livné [BL94] that the principal series (40) is absolutely irreducible if either $|\underline{r}| \notin \{0, q\}$ or $|\underline{r}| \in \{0, q\}$ and $\mu \notin \{1, -1\}$.

On the other hand, if $|\underline{r}| \in \{0, q\}$ and $\mu \in \{1, -1\}$ we have a short exact sequence

$$0 \to 1 \to \mathrm{ind}_{\mathbf{B}(F)}^{\mathbf{GL}_2(F)} 1 \to \mathrm{St} \to 0$$

where St denotes the Steinberg representation for $\mathbf{GL}_2(F)$ (which is absolutely irreducible).

Since $\mathbf{B}(F)\backslash\mathbf{GL}_2(F)$ is compact, we have the following K-equivariant isomorphism:

$$\left(\mathrm{ind}_{\mathbf{B}(F)}^{\mathbf{GL}_2(F)}(\mathrm{un}_\mu \otimes \omega^{\underline{r}} \mathrm{un}_{\mu^{-1}})\right)|_K \cong \mathrm{ind}_{K_0(\varpi^\infty)}^{K} \chi_{\underline{r}}^s \cong \varinjlim_{n \geq 1} \left(\mathrm{ind}_{K_0(\varpi^{n+1})}^{K} \chi_{\underline{r}}^s\right) \tag{41}$$

where the transition morphisms for the co-limit in the RHS are obtained inducing the natural monomorphisms of $K_0(\varpi^n)$-representations

$$\chi_{\underline{r}}^s \hookrightarrow \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \chi_{\underline{r}}^s \tag{42}$$

(which is unique up to a scalar).

To the tamely ramified principal series $\pi_{\underline{r},\mu}$ we associate the $K_0(\varpi)$-subrepresentation

$$R_\infty^- \overset{\mathrm{def}}{=} \mathrm{ind}_{K_0(\varpi^\infty)}^{K_0(\varpi)} \chi_{\underline{r}}^s.$$

The representation R_∞^- controls the representation theoretic behavior of principal and special series representations for $\mathbf{GL}_2(F)$:

Proposition 8.1: *Let $\pi_{\underline{r},\mu}$ be a tamely ramified principal series and let R_∞^- be the associated $K_0(\varpi)$ submodule. We have a K-equivariant isomorphism*

$$\pi_{\underline{r},\mu}|_K \cong \mathrm{ind}_{K_0(p)}^{K} R_\infty^-.$$

If N denotes the normalizer in $\mathbf{GL}_2(F)$ of the standard Iwahori subgroup, we have a N-equivariant isomorphism

$$\pi_{\underline{r},\mu}|_N \cong R_\infty^- \oplus \left(R_\infty^-\right)^s$$

where the action of α on the RHS is given by the involution

$$\begin{array}{ccc} R_\infty^- & \longrightarrow & \left(R_\infty^-\right)^s \\ v & \longmapsto & \mu v. \end{array}$$

In particular, the Steinberg representation fits in the following exact sequences:

$$0 \to 1 \to \mathrm{ind}_{K_0(\varpi)}^K R_\infty^- \to St|_K \to 0,$$
$$0 \to 1 \to R_\infty^- \oplus \left(R_\infty^-\right)^s \to St|_N \to 0.$$

Proof. The assertions on the K-structure of $\pi_{\underline{r},\mu}$ follow from the isomorphism (41) and the formal properties of the compact induction functor.

The assertions on the N-structure of $\pi_{\underline{r},\mu}$ can be checked directly using the Mackey decomposition

$$(\pi_{\underline{r},\mu})|_{K_0(\varpi)} \cong \left(\mathrm{ind}_{K_0(\varpi^\infty)}^{K_0(\varpi)} \chi_{\underline{r}}^s\right) \oplus \left(\mathrm{ind}_{K_0(\varpi)\cap\overline{\mathbf{B}}(F)}^{K_0(\varpi)} \chi_{\underline{r}}\right) \tag{43}$$

and noticing that α normalizes $K_0(\varpi)$ (hence the $K_0(\varpi)$-equivariant isomorphism between the direct summands in the RHS of (43), once we endow one of them with the conjugate action of $K_0(\varpi)$). ∎

Assume now that F is unramified over $\mathbf{Q}_p$.

We define $\mathfrak{S}_\infty$ to be the Pontryagin dual of R_∞^-. In other words, $\mathfrak{S}_\infty$ is the Verma module

$$\mathfrak{S}_\infty = k[[K_0(p)]] \otimes_{k[[K_0(p^\infty)]]} (\chi_{\underline{r}}^s)^\vee$$

so that the Iwahori decomposition and (41) yield

$$\mathfrak{S}_\infty = \varprojlim_{n\geq 1} A/\left\langle X_i^{p^n}, i = 0, \ldots, f-1\right\rangle$$

where the morphisms defining the projective system are the natural projections (and respect the $k[[K_0(p)]]$-module structures).

By Proposition 4.9 we deduce that the A-linear filtration on $\mathfrak{S}_\infty$ induced by powers of the maximal ideal $\mathfrak{m} \triangleleft A$ is $k[[K_0(p)]]$-stable and the Γ, U^+-action is trivial on the quotients

$$\mathfrak{m}^k/\mathfrak{m}^{k+(p-2)}.$$

Acknowledgments. This paper has been written while the author was a post-doctoral fellow at the Fields Institute and the University of Toronto. He would like to thank these institutions for their generosity, the excellent working conditions and the exciting mathematical environment provided by the thematic program on Galois representation in Winter 2012.

The author wish to express his gratitude to Konstantin Ardakov, Pierre Colmez, Peter Schneider for their interest in this work. He is especially thankful to Benjamin Schraen for sharing a preliminary version of [Sch], whose viewpoint has been crucial to the development of this paper. This work would not have been possible without the constant interest, encouragements and precious help of Florian Herzig. Also, the author is very thankful to Makis Dousmanis for his patient explanation of the theory of Wach modules and to Gabriel Dospinescu for his very many and excellent remarks and critics.

Finally, he express his deepest gratitude to the anonymous referee. His meticulous work, his careful reading, his pertinent remarks helped the author to correct embarrassing inaccuracies, and enormously improve the readability of this paper.

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