ISRAEL JOURNAL OF MATHEMATICS **219** (2017), 479–505
DOI: 10.1007/s11856-017-1472-y

INVARIANT MEASURES FOR SOLVABLE GROUPS AND DIOPHANTINE APPROXIMATION

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ABSTRACT

We show that if $\mathcal{L}$ is a line in the plane containing a badly approximable vector, then almost every point in $\mathcal{L}$ does not admit an improvement in Dirichlet's theorem. Our proof relies on a measure classification result for certain measures invariant under a nonabelian two-dimensional group on the homogeneous space $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. Using the measure classification theorem, we reprove a result of Shah about planar nondegenerate curves (which are not necessarily analytic), and prove analogous results for the framework of Diophantine approximation with weights. We also show that there are line segments in $\mathbb{R}^3$ which do contain badly approximable points, and for which all points do admit an improvement in Dirichlet's theorem.

* The authors are supported by NSFC (11201388), NSFC (11271278), BSF grant 2010428, and ERC starter grant DLGAPS 279893.
Received April 25, 2015 and in revised form March 6, 2016

1. Introduction

A classical result in Diophantine approximation is Dirichlet's theorem which asserts that for any $\mathbf{v}\in\mathbb{R}^n$ and any $Q\geq 1$ there are $q\in\mathbb{N}$ and $\mathbf{p}\in\mathbb{Z}^n$ such that

$$\|q\mathbf{v}-\mathbf{p}\| < \frac{1}{Q^{1/n}} \quad \text{and} \quad q \leq Q.$$

The norm used here and throughout this paper is the sup-norm on $\mathbb{R}^n$. Let $\sigma \in (0,1)$. Following Davenport and Schmidt [5], we say that $\mathbf{v}$ admits a **σ-improvement in Dirichlet's theorem**, and write $\mathbf{v} \in \mathrm{DI}(\sigma)$, if for all sufficiently large Q, there are $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^n$ such that

$$\|q\mathbf{v}-\mathbf{p}\| < \frac{\sigma}{Q^{1/n}} \quad \text{and} \quad q < \sigma Q.$$

Finally we say that $\mathbf{v}$ admits **no improvement in Dirichlet's theorem** if $\mathbf{v} \notin \bigcup_{\sigma<1} \mathrm{DI}(\sigma)$. It is known that almost every $\mathbf{v} \in \mathbb{R}^n$ (with respect to Lebesgue measure) admits no improvement in Dirichlet's theorem. It is an interesting problem to decide, given a measure μ on $\mathbb{R}^n$, whether μ-a.e. $\mathbf{v}$ admits no improvement in Dirichlet's theorem. See [5, 9] for some results and questions in this direction.

In a recent breakthrough, Shah [14] showed that if μ is the length measure on an analytic curve in $\mathbb{R}^n$, which is not contained in any affine hyperplane, then μ-a.e. $\mathbf{v}$ admits no improvement in Dirichlet's theorem. For certain fractal measures μ in $\mathbb{R}^2$, the same conclusion is obtained in [16] and [17]. These works leave open the question of measures which are length measures on lines. In this direction, Kleinbock [7] showed that for any line $\mathcal{L}$ which is not contained in $\mathrm{DI}(\sigma_0)$ for some $\sigma_0 > 0$, for almost every $\mathbf{v} \in \mathcal{L}$ (w.r.t. length measure on $\mathcal{L}$), there is $\sigma = \sigma(\mathbf{v})$ such that $\mathbf{v} \notin \mathrm{DI}(\sigma)$. Our first result strengthens this conclusion under a stronger hypothesis, for planar lines. Recall that $\mathbf{v}$ is called **badly approximable** if there is $c > 0$ such that for any $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^n$, $\|q\mathbf{v}-\mathbf{p}\| \geq \frac{c}{q^{1/n}}$. It was shown by Davenport and Schmidt [5] that if $\mathbf{v}$ is badly approximable then it admits an improvement in Dirichlet's theorem.

THEOREM 1.1: *Suppose that a line $\mathcal{L}$ in $\mathbb{R}^2$ contains a badly approximable vector. Then almost every element of $\mathcal{L}$ (w.r.t. length measure) admits an improvement in Dirichlet's theorem.*

Another question raised by Shah's work is to what extent one can relax the hypothesis of the analyticity of the curve. A map $\varphi : [0,1] \to \mathbb{R}^n$ is called

nondegenerate if it is n times continuously differentiable, and for almost every s, the Wronskian determinant of $\varphi'(s)$ does not vanish (i.e., the vectors $\varphi'(s), \varphi''(s), \ldots, \varphi^{(n)}(s)$ are linearly independent in $\mathbb{R}^n$); in the case of planar curves, this simply means that the curvature of φ does not vanish at s. It is clear that analytic curves not contained in affine hyperplanes are nondegenerate, and one may expect that the conclusion of Shah's theorem holds under this weaker hypothesis. This was proved by Shah in the case $n = 2$ (unpublished) by adapting the method of [14]. We obtain a simpler proof. That is we show:

THEOREM 1.2: *Let $\varphi : [0,1] \to \mathbb{R}^2$ be a nondegenerate curve. Then for almost every $s \in [0,1]$ (with respect to Lebesgue measure), $\varphi(s)$ admits no improvement in Dirichlet's theorem.*

A similar proof of Theorem 1.2 was obtained independently by Manfred Einsiedler (also unpublished).

Our proofs rely on results in homogeneous dynamics. Before stating them we introduce some notation, to be used in §1–§4. Let $G := \mathrm{SL}_3(\mathbb{R})$, $\Gamma := \mathrm{SL}_3(\mathbb{Z})$, $X := G/\Gamma$, so that X is the space of unimodular lattices in $\mathbb{R}^3$. This is a space on which any subgroup of G acts by left-translations preserving the G-invariant Borel probability measure m induced by Haar measure on G. For $\mathbf{v} = (v_1, v_2)^{\mathrm{tr}} \in \mathbb{R}^2$, $t \in \mathbb{R}$ and $\mathbf{r} = (r_1, r_2) \in \mathbb{R}^2_{>0}$ with $r_1 + r_2 = 1$, we set

$$(1.1) \qquad f_t^{(\mathbf{r})} := \begin{pmatrix} e^{r_1 t} & 0 & 0 \\ 0 & e^{r_2 t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad u(v_1, v_2) := u(\mathbf{v}) := \begin{pmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and let $\bar{u} = \pi \circ u$, where $\pi : G \to G/\Gamma$ is the natural quotient map. Theorem 1.1 follows from:

THEOREM 1.3: *Let $x_0 \in X$, $a, b \in \mathbb{R}$ and let $I, J \subset \mathbb{R}$ be bounded intervals, and suppose there is a compact $K \subset X$ such that*

$$(1.2) \qquad \text{for all } t \geq 0 \text{ there is } s_t \in J \text{ with } f_t^{(\mathbf{r})} u(s_t, as_t + b)x_0 \in K.$$

Let ν be a probability measure on I which is absolutely continuous with respect to Lebesgue measure. Then for any $\psi \in C_c(X)$ one has

$$\frac{1}{T}\int_0^T \int_I \psi(f_t^{(\mathbf{r})} u(s, as+b)x_0)\, d\nu(s)\, dt \to_{T\to\infty} \int_X \psi\, dm;$$

that is, $\frac{1}{T}\int_0^T (f_t^{(\mathbf{r})})_*\bar{\nu}\, dt \to_{T\to\infty} m$ *in the weak-* topology on Borel probability measures on* X*, where* $\bar{\nu}$ *is the image of* ν *under the map* $s \mapsto u(s, as+b)x_0$.

Similarly, Theorem 1.2 follows from:

Theorem 1.4: *Let* $\varphi : [0,1] \to \mathbb{R}^2$ *be a nondegenerate curve. Then for any* $\psi \in C_c(X)$ *and any probability measure* ν *on* $[0,1]$ *which is absolutely continuous with respect to Lebesgue measure, one has*

$$\frac{1}{T}\int_0^T \int_0^1 \psi(f_t^{(\mathbf{r})}\bar{u}(\varphi(s)))\, d\nu(s)\, dt \to_{T\to\infty} \int_X \psi\, dm.$$

Theorems 1.3 and 1.4 in turn follow from the following measure classification result:

Theorem 1.5: *Let* U *(resp.* F*) be a one-parameter unipotent (resp. diagonalizable) subgroup of* G*. Suppose that* U *is normalized by* F*,* FU *is nonabelian and* F *does not fix any nonzero vector of* $\mathbb{R}^3$*. Then the action of* FU *on* X *is uniquely ergodic, i.e.,* m *is the only* FU*-invariant probability measure on* X*.*

Our method of proof allows a generalization to 'Diophantine approximation with weights', which we now describe. Let $\mathbf{r} = (r_1, r_2)^{\mathrm{tr}}$ be as above. Following [6] we say that $\mathbf{v} \in \mathbb{R}^2$ is **badly approximable w.r.t. weights r** if there is $c > 0$ such that for all $q \in \mathbb{N}$, all $\mathbf{p} \in \mathbb{Z}^2$, we have

$$\max_{i=1,2} |qv_i - p_i|^{1/r_i} \geq \frac{c}{q}.$$

Also, following [9] we say that $\mathbf{v}$ admits **no improvement in Dirichlet's theorem w.r.t. weights r** if there does not exist $\sigma \in (0,1)$ such that for all sufficiently large Q, there is a solution $q \in \mathbb{N}$, $\mathbf{p} \in \mathbb{Z}^2$ to the inequalities

$$\max_{i=1,2} |qv_i - p_i| < \frac{\sigma}{Q^{r_i}},\ q < \sigma Q.$$

We show:

Theorem 1.6: *For any* $\mathbf{r}$ *as above, the following hold:*

(i) *Suppose* $\mathcal{L}$ *is a line in* $\mathbb{R}^2$ *which contains one point which is badly approximable w.r.t. weights* $\mathbf{r}$*. Then almost every* $\mathbf{v} \in \mathcal{L}$ *(w.r.t. the length measure on* $\mathcal{L}$*) admits no improvement in Dirichlet's theorem w.r.t. weights* $\mathbf{r}$*.*

(ii) *Let* $\varphi : [0,1] \to \mathbb{R}^2$ *be a nondegenerate curve. Then for almost every* $s \in [0,1]$ *(w.r.t. Lebesgue measure),* $\varphi(s)$ *admits no improvement in Dirichlet's theorem w.r.t. weights* $\mathbf{r}$.

Theorem 1.6(ii) was proved for nondegenerate analytic curves in $\mathbb{R}^n$, in [15]. The hypothesis of Theorem 1.1 and 1.6(i) can be verified in many cases. In light of recent work of Badziahin–Velani [2] and An–Beresnevich–Velani [1], we obtain:

Corollary 1.7: *Suppose that* $\mathcal{L}$ *is a line in* $\mathbb{R}^2$ *given by the equation* $\mathrm{y} = a\mathrm{x}+b$ *where* $a \neq 0$. *If*

$$\liminf_{q\to\infty} |q|^{\frac{1}{r}-\varepsilon} \min_{\mathbf{p}\in\mathbb{Z}^2} \|q(a,b)-\mathbf{p}\| > 0 \quad \text{where } r = \min\{r_1, r_2\} \tag{1.3}$$

for some $\varepsilon > 0$, *then almost every* $\mathbf{v} \in \mathcal{L}$ *admits no improvement in Dirichlet's theorem w.r.t. weights* $\mathbf{r}$. *Moreover, the same conclusion holds if* $a \in \mathbb{Q}$ *and* (1.3) *holds for* $\varepsilon = 0$.

Indeed, [2, 1] showed that under the hypotheses of Corollary 1.7, $\mathcal{L}$ contains a badly approximable vector, so Theorem 1.6 applies.

In §5 we give several examples showing the necessity of the hypotheses in our theorems. In particular, we show in Theorem 5.1 that the analog of Theorem 1.1 fails in dimension $n = 3$.

Acknowledgements. We are grateful to Jinpeng An, Manfred Einsiedler, Dmitry Kleinbock and Elon Lindenstrauss for helpful discussions.

2. Invariant measure for solvable groups

In this section we prove Theorem 1.5. As we will show in §5, it is not possible to relax the hypotheses of the theorem.

Let the notation be as in the statement of Theorem 1.5, and let $F = \{f_t : t \in \mathbb{R}\}$ where $t \mapsto f_t$ is a group homomorphism from $\mathbb{R} \to F$. Let μ be an FU-invariant Borel probability measure on X. Our goal is to show that $\mu = m$, and we can assume with no loss of generality that μ is ergodic for the action of FU.

We can decompose μ into its U-ergodic components. That is we write

$$\mu = \int_X m_x \, d\mu(x)$$

where each m_x is U-invariant and ergodic. According to Ratner's measure classification theorem [13], for every x there is a closed connected subgroup $H = H_x$ such that $\overline{Ux} = Hx$ and m_x is the unique H-invariant measure on Hx induced by the Haar measure on H. Also, since μ is F-invariant, by the Poincaré recurrence theorem, for almost every x and m_x-a.e. y, the orbit Fy is recurrent in both positive and negative times, i.e., there are $t_n \to +\infty$ and $t'_n \to -\infty$ such that

$$f_{t_n} y \to y \quad \text{and} \quad f_{t'_n} y \to y. \tag{2.1}$$

We will need the following result:

Theorem 2.1 (Mozes): *There exists a closed subgroup H of G generated by one-parameter unipotent subgroups and containing U such that the following hold:*

(i) *For μ-almost every $x \in X$ we have $H_x = H$.*
(ii) *The group H is normalized by F and conjugation by F preserves the Haar measure of H.*

Theorem 2.1 was proved in [12] but not stated explicitly; it is stated as [11, Main Theorem] and reproved in a more general context.

Let $\{h_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of G. We say that $\{h_t x : t \geq 0\}$ (respectively $\{h_t x : t \leq 0\}$) is **divergent** if for any compact $K \subset X$ there is t_0 such that for all $t > t_0$ (resp. all $t < t_0$), $h_t x \notin K$. We will need the following well-known fact:

Proposition 2.2: *If $\rho : G \to \mathrm{GL}(V)$ is a representation defined over $\mathbb{Q}$, and $v \in V(\mathbb{Q}) \smallsetminus \{0\}$ such that $\rho(h_t g)v \to_{t \to +\infty} 0$, then $\{h_t \pi(g) : t \geq 0\}$ is divergent. The analogous statement replacing $+\infty$ with $-\infty$ and $t \geq 0$ with $t \leq 0$ also holds.*

Proof. This follows from a standard bounded denominators argument; see, e.g., [18, Prop. 3.1]. ∎

We let E_{ij} be the matrix whose matrix coefficient in the ith row and jth column is 1, and 0 elsewhere. Set

$$U_{ij} := \{\exp(sE_{ij}) : s \in \mathbb{R}\}. \tag{2.2}$$

Let $U^+ := \langle U_{12}, U_{13}, U_{23} \rangle$ be the upper triangular unipotent group. We will need the following:

Proposition 2.3: *Let $x \in X$ such that U^+x is closed. Then for any one-parameter subgroup $\{h_t\}$ of the diagonal group, at least one of the two trajectories $\{h_t x : t \geq 0\}$, $\{h_t x : t \leq 0\}$ is divergent.*

Proof. First suppose that x is the point corresponding to the identity coset Γ, that is, $x = \pi(e)$ where e is the identity element of G. There is a natural action of G on $\mathbb{R}^3$ by linear transformations and a corresponding induced action on the second exterior power $\bigwedge^2 \mathbb{R}^3$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{R}^3$ and let $\mathbf{v}_{12} := \mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^3$. The vectors $\mathbf{e}_1, \mathbf{v}_{12}$ are eigenvectors for the diagonal group, and we let χ_1, χ_2 be the corresponding characters. That is, if $a = \operatorname{diag}(e^s, e^t, e^{-(s+t)})$, then

$$a\mathbf{e}_1 = \chi_1(a)\mathbf{e}_1, \quad \text{where } \chi_1(a) = e^s$$

and

$$a\mathbf{v}_{12} = \chi_2(a)\mathbf{v}_{12}, \quad \text{where } \chi_2(a) = e^{s+t}.$$

For any one-parameter diagonal subgroup $\{h_t\}$, at least one of the two restrictions $\chi_i|_{h_t}$, $i = 1, 2$ is not trivial. This implies that $h_t\mathbf{e}_1 \to 0$ or $h_t\mathbf{v}_{12} \to 0$ as t tends to either $+\infty$ or $-\infty$, and we apply Proposition 2.2.

Now suppose that $x = \pi(g)$ for some $g \in G$. For definiteness, assume that $h_t\mathbf{e}_1 \to_{t\to+\infty} 0$ (if not, replace $\mathbf{e}_1$ by $\mathbf{v}_{12}$ or $+\infty$ by $-\infty$). Since closed orbits for unipotent groups are of finite volume, $g^{-1}U^+g \cap \Gamma$ is a lattice in U^+. Therefore the group $g^{-1}U^+g$ is defined over $\mathbb{Q}$. So both the normalizers of U^+ and $g^{-1}U^+g$ are minimal $\mathbb{Q}$-parabolic subgroups of G, and hence are conjugate over $\mathbb{Q}$. This implies that there exists $g_0 \in \mathrm{SL}_3(\mathbb{Q})$ such that

$$g^{-1}U^+g = g_0^{-1}U^+g_0.$$

It follows that $ng_0 = g$ where $n \in N_G(U^+)$. Note that both $\mathbf{e}_1$ and $\mathbf{v}_{12}$ are eigenvectors for the upper triangular group $N_G(U^+)$, so we write $n\mathbf{e}_1 = c\mathbf{e}_1$ for some $c \in \mathbb{R}$. Therefore we have

$$h_t g g_0^{-1}\mathbf{e}_1 = h_t n\mathbf{e}_1 = ch_t\mathbf{e}_1 \to 0.$$

Since $g_0 \in \mathrm{SL}_3(\mathbb{Q})$, $g_0^{-1}\mathbf{e}_1$ is a $\mathbb{Q}$-vector. Applying again Proposition 2.2 (with $g_0^{-1}\mathbf{e}_1$ instead of $\mathbf{e}_1$) we see that the trajectory $\{h_t x\}$ is divergent. ∎

We remark that Proposition 2.3 can be generalized to $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, for any $n \geq 3$, with a similar argument.

Let $H_0 \cong \mathrm{SL}_2(\mathbb{R})$ denote the subgroup of G generated by U_{12} and U_{21}. We will need a similar fact for H_0.

Proposition 2.4: *Let $x \in X$ such that $H_0 x$ is closed, and let $\{h_t\}$ be a one-parameter subgroup of the group of diagonal matrices which is not contained in H_0. Then $\{h_t x : t \geq 0\}$ and $\{h_t x : t \leq 0\}$ are both divergent.*

Proof. First suppose that $x = \pi(e)$ and consider the vector $\mathbf{v}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^3$ of the previous proof, along with the vector $\mathbf{e}_3$. For any one-parameter group $\{h_t\}$ not contained in H_0, possibly after switching the roles of $+\infty$ and $-\infty$, we have $h_t \mathbf{e}_3 \to_{t \to +\infty} 0$ and $h_t \mathbf{v}_{12} \to_{t \to -\infty} 0$. Therefore the claim follows from Proposition 2.2.

Now assume that $x = \pi(g)$ for some $g \in G$. The group H_0 is the intersection of the stabilizers of the vectors $\mathbf{v}_{12}, \mathbf{e}_3$ in the two representations $W_1 := \bigwedge^2 \mathbb{R}^3$, $W_2 := \mathbb{R}^3$. Moreover, the pair $(\mathbf{v}_{12}, \mathbf{e}_3)$ represents the unique splitting of $\mathbb{R}^3$ into a direct sum decomposition of a 2-dimensional and 1-dimensional space which is left invariant by H_0. Consider the group $H' := g^{-1} H_0 g$ and the pair of vectors

$$\mathbf{w}_1 := g^{-1} \mathbf{v}_{12} \in W_1, \quad \mathbf{w}_2 := g^{-1} \mathbf{e}_3 \in W_2.$$

This pair represents the unique splitting into a direct sum decomposition as above, which is H' invariant. Also, since $H_0 x$ is closed, it is of finite volume and $H' \cap \Gamma$ is a lattice in H'. This implies that H' is defined over $\mathbb{Q}$.

Now let $\iota : \mathbb{C} \to \mathbb{C}$ be any field automorphism. The map ι acts on G (by its action on matrix entries) and on W_1, W_2 (by its action on vector coefficients) in a compatible way, and $\iota(H') = H'$ since H' is defined over $\mathbb{Q}$. This implies that the pair $(\iota(\mathbf{w}_1), \iota(\mathbf{w}_2))$ also represents the unique splitting $\iota(H')$-invariant decomposition of $\mathbb{R}^3$ into a 2- and 1-dimensional subspace. Since the dimensions of these two subspaces are different, ι also preserves each subspace in this splitting, that is, ι preserves $\mathbf{w}_1, \mathbf{w}_2$ up to multiplication by scalars. Since this is true for any field automorphism ι, the subspaces represented by $\mathbf{w}_1, \mathbf{w}_2$ are $\mathbb{Q}$-subspaces of $\mathbb{R}^3$, and hence $\mathbf{w}_1$, $\mathbf{w}_2$ are scalar multiples of $\mathbb{Q}$-vectors in W_1, W_2 respectively. We have

$$h_t g \mathbf{w}_2 = h_t \mathbf{e}_3 \to_{t \to +\infty} 0, \quad h_t g \mathbf{w}_1 = h_t \mathbf{w}_1 \to_{t \to -\infty} 0.$$

Thus the claim follows using Proposition 2.2 with scalar multiples of $\mathbf{w}_1, \mathbf{w}_2$. ∎

Proof of Theorem 1.5. Let F and U be as in the statement of the theorem, and for an FU-invariant ergodic measure μ, let H be as in Theorem 2.1. We will prove Theorem 1.5 by showing $H = G$, and to this end we will assume by contradiction that $H \neq G$, consider various possibilities for the triple (F, U, H), and derive a contradiction in each case.

Let $\mathfrak{h}, \mathfrak{u}$ denote respectively the Lie algebras of H and U. The key observation is the following. Since conjugation by f_1 preserves the volume of H and the adjoint action of f_1 on $\mathfrak{u} \subset \mathfrak{h}$ is nontrivial, $\mathfrak{h}$ must contain eigenvectors of $\mathrm{Ad}(f_1)$ with both positive and negative eigenvalues.

The group of automorphisms of G is generated by inner automorphisms (conjugation) and the automorphism $g \mapsto (g^{-1})^{\mathrm{tr}}$. With no loss of generality we can apply an automorphism of G and a reparametrization of F to the triple (F, U, H) to assume:

(1) The conjugation of f_1 expands U. (Since U is one-dimensional and is acted on nontrivially by f_t, this can be ensured by re-parameterizing f_t if necessary.)
(2) f_1 has two positive eigenvalues. (Since f_t does not preserve a vector in $\mathbb{R}^3$, this can be ensured by applying an outer automorphism of G if necessary.)
(3) $f_t = \mathrm{diag}(e^t, e^{at}, e^{bt})$ w here $1 \geq a > 0 > b$, $a + b = -1$. (This can be ensured by applying an inner automorphism of G preserving the diagonal group, and reparameterizing $f_t \mapsto f_{ct}$ for some $c > 0$.) It follows that U is contained in the upper triangular group U^+.
(4) The subgroup $H \cap U^-$, where U^- is the lower triangular unipotent subgroup $\langle U_{21}, U_{31}, U_{32}\rangle$, contains a nontrivial group N (whose Lie algebra is denoted by $\mathfrak{n}$) such that F normalizes N and acts on its Lie algebra by a strict contraction (since the action of F on H preserves Haar measure on H so there must be a subgroup which is contracted).

Suppose first that $a = 1$, so that $b = -2$. In this case the centralizer Z of F is a copy of $\mathrm{GL}_2(\mathbb{R})$ embedded as

$$Z = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \tag{2.3}$$

and we can further simplify our problem by conjugating by elements of Z. We decompose $\mathfrak{g}$ into eigenspaces for $\mathrm{Ad}(f_1)$, writing $\mathfrak{g} = V^+ \oplus V^- \oplus V^0$, where

$$V^+ := \mathrm{span}(E_{13}, E_{23}), \quad V^- := \mathrm{span}(E_{31}, E_{32}), \quad V^0 := \mathfrak{z}$$

(where $\mathfrak{z}$ is the Lie algebra of Z, and this is the decomposition into eigenspaces of $\mathrm{Ad}(f_1)$ with eigenvalues $e^3, e^{-3}, 1$ respectively). Since conjugation by F preserves Haar measure on H, if $\mathfrak{h}$ contains V^+ it also contains V^-. Since V^+ and V^- generate $\mathfrak{g}$ as a Lie algebra, this is impossible, so

$$\mathfrak{h} \cap V^+ = \mathfrak{u}, \quad \mathfrak{h} \cap V^- = \mathfrak{n}. \tag{2.4}$$

A direct computation in the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ shows that Z acts transitively on nonzero elements of V^+ and also acts transitively on nonzero elements of V^-. Moreover, when acting on $\mathfrak{g} \oplus \mathfrak{g}$ via $\mathrm{Ad} \oplus \mathrm{Ad}$, there is an element of Z which maps $\mathfrak{u}$ to $\mathrm{span}(E_{13})$ and maps $\mathfrak{n}$ to either $\mathrm{span}(E_{31})$ or $\mathrm{span}(E_{32})$. With no loss of generality we apply such a conjugation, and treat first the case that

$$\mathfrak{u} = \mathrm{span}(E_{13}), \quad \mathfrak{n} = \mathrm{span}(E_{32}). \tag{2.5}$$

Then H contains the group U_0 generated by U_{13}, U_{32}, which is 3-dimensional with Lie algebra $\mathfrak{u}_0 := \mathrm{span}(E_{13}, E_{32}, E_{12})$. There is no proper Lie subalgebra of $\mathfrak{g}$ which is $\mathrm{Ad}(f_1)$-invariant, satisfies (2.4), and properly contains $\mathfrak{u}_0$. This implies that $H = U_0$. But U_0 is a conjugate of U^+, by a conjugation which leaves F inside the group of diagonal matrices. By applying such a conjugation we obtain a contradiction to Proposition 2.3 and (2.1).

We now continue with the assumption $a = 1$ and assume that (2.5) does not hold, so that (after conjugating by an element of Z)

$$\mathfrak{u} = \mathrm{span}(E_{13}), \quad \mathfrak{n} = \mathrm{span}(E_{31}). \tag{2.6}$$

Then H contains the group $H_0 \cong \mathrm{SL}_2(\mathbb{R})$ whose Lie algebra is generated by $\mathfrak{u}$ and $\mathfrak{n}$, and $F \not\subset H_0$. By Proposition 2.4 and (2.1) we cannot have $H = H_0$. So $H_0 \subsetneq H$ and, since the group generated by F and H_0 contains the full diagonal group, H is invariant under conjugation by all elements of the diagonal group. Therefore H must contain at least one other eigenspace U_{ij} not contained in H_0. By (2.4), H contains one of U_{12}, U_{21}. However, H_0 and any one of these two groups generate a group which contains one of U_{23}, U_{32} and (2.4) cannot hold.

Finally, suppose $a < 1$ so that the three eigenvalues of f_1 are distinct. In this case E_{12}, E_{13} and E_{23} belong to different eigenspaces of $\mathrm{Ad}(f_1)$, with corresponding eigenvalues $e^{1-a}, e^{1-b}, e^{a-b}$. The equations $a + b = -1, 0 < a < 1$ imply that these eigenvalues are distinct:

$$e^{1-b} > e^{a-b} > e^{1-a}.$$

Moreover, the product of the eigenvalues that correspond to eigenspaces belonging to $\mathfrak{h}$ is 1, since conjugation by elements of F preserves the Haar measure on H. We consider the possibilities for H. If $\dim H = 3$, then H is either generated by a pair U_{ij}, U_{ji}, or is conjugate to U^+. In the former case, up to a conjugation by a matrix preserving the diagonal group, H coincides with the group H_0 considered above. But this leads to a contradiction via (2.1) and Proposition 2.4. In the latter case, we also get a contradiction by combining (2.1) and Proposition 2.3. If $\dim H \geq 4$, then H contains at least two expanding or two contracting eigenvalues. It is easy to check that (up to re-indexing) H contains U_{13}, U_{21}, U_{32}, and these groups generate G, which is impossible. ∎

3. Equidistribution of a line segment

The aim of this section is to prove Theorems 1.3, 1.1 and 1.6 (i). We first assume the notation and assumptions in Theorem 1.3, in particular $f_t^{(\mathbf{r})}$ and u are as in (1.1), and $\bar{\nu}$ is the image of ν under $s \mapsto u(s, as+b)x_0$. That is,

$$\int_X \psi \, d\bar{\nu} = \int_{\mathbb{R}} \psi(u(s, as+b)x_0) \, d\nu(s) \tag{3.1}$$

for every $\psi \in C_c(X)$. Sometimes we need to treat the cases where $r_1 = r_2$ and $r_1 \neq r_2$ separately, so we let $f_t := f_t^{(1/2,1/2)}$ to emphasize that we are in the former case. First we show that there is no escape of mass.

Lemma 3.1: *Let μ be a weak-* accumulation point of*

$$\frac{1}{T} \int_0^T (f_t^{(\mathbf{r})})_* \bar{\nu} \, dt \quad \text{as } T \to \infty. \tag{3.2}$$

Then $\mu(X) = 1$.

Proof. It suffices to show that for each $\varepsilon > 0$ there is a compact $K_0 \subset X$ such that for all large enough t,

$$\nu(\{s \in I : f_t^{(\mathbf{r})} u(s, as+b)x_0 \notin K_0\}) < \varepsilon. \tag{3.3}$$

Since ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, we can write $d\nu(s) = h(s)ds$ where h is a non-negative measurable function on I with $\int_I h(s)ds = 1$. Given $\varepsilon > 0$, let R be large enough so that

$$\int_{I_R} h(s)\, ds < \frac{\varepsilon}{2}, \quad \text{where } I_R := \{s \in I : h(s) \geq R\}. \tag{3.4}$$

We will show below that we can find a compact $K_0 \subset X$ such that for all sufficiently large t,

$$\frac{|\{s \in I : f_t^{(\mathbf{r})} u(s, as+b)x_0 \notin K_0\}|}{|I|} < \frac{\varepsilon}{2R} \tag{3.5}$$

(where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}$). Given such a set K_0, we can establish (3.3) by noting that the subset contained in I_R contributes at most $\frac{\varepsilon}{2}$ by (3.4), and the subset contained in $I \smallsetminus I_R$ contributes at most $\frac{\varepsilon}{2}$ since on $I \smallsetminus I_R$, h is bounded above by R. So it remains to prove (3.5).

Using (1.2), let $K \subset X$ be a compact subset such that for each t, there is $s_t \in J$ with $f_t^{(\mathbf{r})} u(s_t, as_t + b)x_0 \in K$. We choose $c > 0$ so that $I \cup J \subset [-c, c]$. Multiplying matrices, one sees that

$$\begin{aligned} &f_t^{(\mathbf{r})} u(s, as+b)x_0 \\ &\quad = u(e^{(r_1+1)t}(s - s_t), ae^{(r_2+1)t}(s - s_t)) f_t^{(\mathbf{r})} u(s_t, as_t + b)x_0. \end{aligned} \tag{3.6}$$

By assumption (1.2), $f_t^{(\mathbf{r})} u(s_t, as_t + b)x_0 \in K$ where $K \subset X$ is a compact set. It follows from [4, Theorem 6.1] that, given $\varepsilon > 0$, there exists a compact subset K_0 of X such that for every $x \in K$ and every $t \geq 0$ one has

$$|\{s \in [-c, c] : u(e^{(r_1+1)t}(s - s_t), ae^{(r_2+1)t}(s - s_t))x \notin K_0\}| < \Big(\frac{\varepsilon |I|}{4cR}\Big) 2c. \tag{3.7}$$

Combining (3.6) with (3.7) gives (3.5). ∎

Next we show unipotent invariance.

Lemma 3.2: *Any weak-* limit of* (3.2) *is invariant under some one-dimensional unipotent subgroup U of G normalized by $\{f_t^{(\mathbf{r})} : t \in \mathbb{R}\}$.*

Proof. To simplify the notation we let

$$\ell : \mathbb{R} \to \mathbb{R}^2, \quad \ell(s) := (s, as+b)^{\mathrm{tr}}.$$

We first prove that in the case $r_1 = r_2$, any limit measure of (3.2) is invariant under $U = \{u(s, as) : s \in \mathbb{R}\}$. It suffices to show that for any $\tilde{s} \in \mathbb{R}$,

$$\lim_{t\to\infty} (f_t)_*\bar{\nu} - (u(\tilde{s}, a\tilde{s})f_t)_*\bar{\nu} = 0. \tag{3.8}$$

Let $h \in L^1(\mathbb{R})$ be a nonnegative function such that $d\nu(s) = h(s)ds$, and let $\psi \in C_c(X)$. We have

$$\begin{aligned}\int_X \psi\, d[(f_t)_*\bar{\nu} - (u(\tilde{s}, a\tilde{s})f_t)_*\bar{\nu}] \\ &= \int_{\mathbb{R}} [\psi(f_t u(\ell(s))x_0) - \psi(u(\tilde{s}, a\tilde{s})f_t u(\ell(s))x_0)]h(s)\, ds \\ &= \int_{\mathbb{R}} [\psi(f_t u(\ell(s))x_0) - \psi(f_t u(\ell(s + e^{-3t/2}\tilde{s}))x_0)]h(s)\, ds.\end{aligned}$$

By continuity of ψ, the integrand converges pointwise to 0 as $t \to \infty$. Since $h \in L^1(\mathbb{R})$ and ψ is bounded, using the dominated convergence theorem we see that the limit is zero. This implies (3.8).

If $r_1 > r_2$ we show that any limit measure is invariant under

$$U_{13} := \{u(s, 0) : s \in \mathbb{R}\}.$$

It suffices to show that for any $\tilde{s} \in \mathbb{R}$,

$$\lim_{t\to\infty} (f_t^{(\mathbf{r})})_*\bar{\nu} - (u(\tilde{s}, 0)f_t^{(\mathbf{r})})_*\bar{\nu} = 0. \tag{3.9}$$

Let ψ, h be as above; set $s' := s + e^{-(1+r_1)t}\tilde{s}$ and compute as follows:

$$\begin{aligned}\int_X \psi\, d[(f_t^{(\mathbf{r})})_*\bar{\nu} - (u(\tilde{s}, 0)f_t^{(\mathbf{r})})_*\bar{\nu}] \\ &= \int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s))x_0) - \psi(u(\tilde{s}, 0)f_t^{(\mathbf{r})} u(\ell(s))x_0)]\, d\nu(s) \\ &= \int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s))x_0) - \psi(f_t^{(\mathbf{r})} u(e^{-(1+r_1)t}\tilde{s}, 0)u(\ell(s))x_0)]\, d\nu(s) \\ &= \int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s))x_0) - \psi(f_t^{(\mathbf{r})} u(\ell(s'))x_0)]\, d\nu(s) \\ &\quad + \int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s'))x_0) - \psi(f_t^{(\mathbf{r})} u(0, -ae^{-(1+r_1)t})u(\ell(s'))x_0)]d\nu(s).\end{aligned}$$

By a change of variables, the absolute value of the first summand in this integral is bounded above by $2 \sup |\psi| \int_{\mathbb{R}} |h(s) - h(s')|ds$, which tends to zero as $t \to +\infty$ since $s' \to s$ and the regular representation of $\mathbb{R}$ on L^1 is continuous.

To bound the second summand we argue as follows:

$$\int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s'))x_0) - \psi(f_t^{(\mathbf{r})} u(0, ae^{-(1+r_1)t}) u(\ell(s'))x_0)]\, d\nu(s)$$
$$= \int_{\mathbb{R}} [\psi(f_t^{(\mathbf{r})} u(\ell(s'))x_0) - \psi(u(0, as_0 e^{(r_2-r_1)t}) f_t^{(\mathbf{r})} u(\ell(s'))x_0)]\, d\nu(s),$$

and this tends to zero by the uniform continuity of ψ and the dominated convergence theorem. Hence $(f_t^{(\mathbf{r})})_*\bar{\nu} - (\exp(s_0 E_{13}) f_t^{(\mathbf{r})})_*\bar{\nu} \to_{t\to\infty} 0$. Since μ is a sequential limit as $T \to \infty$, we see that μ is U_{13}-invariant, as required.

Finally, we consider the case where $r_1 < r_2$. If $a \neq 0$, then a similar argument as for the case where $r_1 > r_2$ implies the invariance for U_{23}. If $a = 0$, then the argument for the case where $r_1 = r_2$ goes through and shows that the limit measure is invariant under U_{13}. ■

Proposition 3.3: *Let λ be a probability measure on $\mathbb{R}^2$. Suppose that*

$$\frac{1}{T}\int_0^T (f_t^{(\mathbf{r})}\bar{u})_*\lambda\, dt \to_{T\to\infty} m. \tag{3.10}$$

Then for λ-almost every $\mathbf{v} \in \mathbb{R}^2$, $\{f_t^{(\mathbf{r})}\bar{u}(\mathbf{v}) : t \geq 0\}$ is dense in X, and in particular, $\mathbf{v}$ admits no improvement in Dirichlet's theorem w.r.t. weights $\mathbf{r}$.

Proof. According to [9, Prop. 2.1], if $\{f_t^{(\mathbf{r})}\bar{u}(\mathbf{v}) : t \geq 0\}$ is dense in X, then $\mathbf{v}$ admits no improvement in Dirichlet's theorem w.r.t. weights $\mathbf{r}$, so it suffices to prove the first assertion. Suppose by contradiction that

$$\lambda(\{\mathbf{v} : \{f_t^{(\mathbf{r})}\bar{u}(\mathbf{v}) : t \geq 0\} \text{ is not dense}\}) > 0.$$

Let $\{U_1, U_2, \ldots\}$ be a countable collection of open subsets of X which form a basis for the topology of X. Then for some i,

$$\lambda(A) > 0, \quad \text{where } A := \{\mathbf{v} : \forall t \geq 0,\ f_t^{(\mathbf{r})}\bar{u}(\mathbf{v}) \notin U_i\}.$$

Let λ_0 be the (normalized) restriction of λ to A, let λ_1 be the (normalized) restriction of λ to the complement of A, and choose a sequence $\{T_n\}$ with $T_n \to \infty$ such that

$$\mu_0 := \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} (f_t^{(\mathbf{r})}\bar{u})_*\lambda_0\, dt$$

exists. Then μ_0 gives zero mass to U_i. In view of (3.10), the limit

$$\mu_1 = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} (f_t^{(\mathbf{r})}\bar{u})_*\lambda_1\, dt$$

also exists, and m is a convex combination of μ_0 and μ_1 with weights $\lambda(A)$, $1 - \lambda(A)$. Both measures μ_0, μ_1 are invariant under $\{f_t^{(\mathbf{r})}\}$, and since m is ergodic, $m = \mu_0 = \mu_1$. This contradicts the fact that $\mu_0(U_i) = 0$. ∎

Proof of Theorem 1.3. Let μ be a weak-* limit of (3.2). Then μ is invariant under the one-parameter diagonal subgroup $F := \{f_t^{(\mathbf{r})} : t \in \mathbb{R}\}$. It follows from Lemma 3.2 that μ is also invariant under some one-parameter unipotent group U normalized by F. Lemma 3.1 implies that μ is a probability measure. Therefore $\mu = m$ according to Theorem 1.5. Since μ is an arbitrary weak-* limit as $T \to \infty$, the conclusion follows. ∎

Proof of Theorem 1.1 and 1.6(i). We only prove the latter since the former is a special case. By switching the roles of x and y there is no loss of generality in assuming that $\mathcal{L}$ is not vertical, i.e., it is given by an equation of the form $s \mapsto \ell(s) := (s, as + b)$ for some $a, b \in \mathbb{R}$. Let $\tilde{s} \in \mathbb{R}$ such that $\ell(\tilde{s})$ is badly approximable w.r.t. weights $\mathbf{r}$. According to Dani's correspondence [3], and its generalization to the framework of approximation with weights [6], there is a compact $K \subset X$ such that $f_t^{(\mathbf{r})}\bar{u}(\ell(\tilde{s})) \in K$ for all $t \geq 0$. That is, (1.2) is satisfied. Now the conclusion is immediate from Theorem 1.3 and Proposition 3.3. ∎

4. Equidistribution of a nondegenerate curve

The goal of this section is to prove Theorems 1.2, 1.4 and 1.6(ii). Our argument uses many ideas of Shah [14, 15] but is made significantly simpler by the extra averaging with respect to t, appearing in Proposition 3.3.

Let the notation be as in Theorem 1.4. We write $f_t = f_t^{(1/2,1/2)}$ and $\varphi = (\varphi_1, \varphi_2)$ where each φ_i is a C^2 function on $[0,1]$. Without loss of generality we further assume that $r_1 \geq r_2$. We claim that $\varphi_1'(s) \neq 0$ for a.e. s; indeed, set

$$A := \{s \in [0,1] : \varphi_1'(s) = 0\}$$

and let A' denote the set of Lebesgue density points of A. Then A and A' have the same Lebesgue measure, and by Rolle's theorem, for $s \in A'$,

$$\varphi_1'(s) = \varphi_1''(s) = 0.$$

Thus the Wronskian determinant of φ' vanishes on A', so by nondegeneracy A and A' must have measure zero.

It follows that there exists a countable collection $\mathcal{I}$ of closed intervals such that

- $\cup_{\mathcal{I}} I$ has full measure in $[0,1]$ and $I_1 \cap I_2$ contains at most one point for distinct $I_1, I_2 \in \mathcal{I}$.
- $\varphi_1'(s) \neq 0$ for every $s \in \bigcup_{I\in\mathcal{I}} I^\circ$ (where I° is the interior of I).

Therefore it suffices to prove Theorem 1.4 for each closed interval properly contained in some $I \in \mathcal{I}$, replacing ν with the restriction of ν to this closed interval. So we assume without loss of generality that $\varphi_1'(s) \neq 0$ for every $s \in [0,1]$.

There exists a continuously differentiable function $M : [0,1] \to \mathrm{SL}_2(\mathbb{R})$ such that $M(s)\varphi'(s) = \mathbf{e}_1$. We define the map

$$z : [0,1] \to \mathrm{SL}_3(\mathbb{R}) \quad \text{by } z(s) = \begin{pmatrix} M(s) & 0 \\ 0 & 1 \end{pmatrix}.$$

Let ν_φ be the probability measure on X defined by

$$\int_X \psi \, d\nu_\varphi = \int \psi(z(s)\bar{u}(\varphi(s))) \, d\nu(s) \tag{4.1}$$

for every $\psi \in C_c(X)$. We set

$$\nu_{\mathbf{r}} := \begin{cases} \nu_\varphi & \text{if } r_1 = r_2, \\ (\bar{u})_*\nu & \text{if } r_1 > r_2. \end{cases}$$

Lemma 4.1: *Any weak-* limit of*

$$\frac{1}{T}\int_0^T (f_t^{(\mathbf{r})})_*\nu_{\mathbf{r}} \, dt \quad \text{as } T \to \infty \tag{4.2}$$

is invariant under the group $U_{13} = \{u(s,0) : s \in \mathbb{R}\}$.

Proof. In the case where $r_1 = r_2$ it suffices to prove that for any $\psi \in C_c(X)$, any $\varepsilon > 0$, and any $\tilde{s} \in \mathbb{R}$,

$$\left| \int_0^1 [\psi(f_t z(s)\bar{u}(\varphi(s))) - \psi(u(\tilde{s},0) f_t z(s)\bar{u}(\varphi(s)))] \, d\nu(s) \right| < \varepsilon \tag{4.3}$$

provided that t is sufficiently large.

We fix a C^2 extension of φ on $[-1,2]$. On the one hand, a change of variables, the boundedness of ψ, and the continuity of the regular representation imply that

$$\int_0^1 |\psi(f_t z(s)\bar{u}(\varphi(s))) - \psi(f_t z(s)\bar{u}(\varphi(s+\tilde{s}e^{-3t/2})))|\, d\nu(s) \to_{t\to\infty} 0.$$

On the other hand, since φ is a C^2-function on a compact interval,

$$\varphi(s+\tilde{s}e^{-3t/2}) = \varphi(s) + \tilde{s}e^{-3t/2}\varphi'(s) + O(e^{-3t}) \quad \text{as } t \to +\infty,$$

where the implicit constant in the error term is independent of s. Therefore

$$\begin{aligned} & f_t z(s)\bar{u}(\varphi(s+\tilde{s}e^{-3t/2})) \\ &= f_t z(s) u[\varphi(s) + \tilde{s}e^{-3t/2}\varphi'(s) + O(e^{-3t})]\pi(e) \\ &= [f_t z(s) u(\tilde{s}e^{-3t/2}\varphi'(s) + O(e^{-3t}))(f_t z(s))^{-1}][f(t)z(s)\bar{u}(\varphi(s))] \\ &= u(\tilde{s}E_{13} + O(e^{-3t/2})) f_t z(s)\bar{u}(\varphi(s)) \\ &= u(O(e^{-3t/2}))u(\tilde{s},0) f_t z(s)\bar{u}(\varphi(s)). \end{aligned} \tag{4.4}$$

By uniform continuity of ψ, this implies that

$$\int_0^1 \psi(f_t z(s)\bar{u}(\varphi(s+\tilde{s}e^{-3t/2})))\, d\nu(s) \to \int_0^1 \psi(u(\tilde{s},0) f_t z(s)\bar{u}(\varphi(s)))\, d\nu(s)$$

as $t \to +\infty$. Now (4.3) follows for all large enough t.

In the case where $r_1 > r_2$ it suffices to show that for any $\psi \in C_c(X)$, any $\varepsilon > 0$, and any $\tilde{s} \in \mathbb{R}$,

$$\left| \int_0^1 [\psi(f_t^{(\mathbf{r})}\bar{u}(\varphi(s))) - \psi(u(\tilde{s},0) f_t^{(\mathbf{r})}\bar{u}(\varphi(s)))]\, d\nu(s) \right| < \varepsilon \tag{4.5}$$

provided that t is sufficiently large.

We first prove (4.5) for $d\nu = ds$. Let $N_t = [\delta e^{(1+r_1)t}] \in \mathbb{N}$ where

$$\delta = \varepsilon(16\|\psi\|_{\sup}\|1/\varphi_1'\|_{\sup})^{-1}. \tag{4.6}$$

Here

$$\|\psi\|_{\sup} := \sup_{x\in X} |\psi(x)| \quad \text{and} \quad \|1/\varphi_1'\|_{\sup} = \sup_{s\in[0,1]} |1/\varphi_1'(s)|.$$

In what follows we always assume t is large so that $N_t > 1$. We partition $I = \bigcup_{k=1}^{N_t} I_k$ where $I_k = [s_k, s_{k+1}]$ and $s_{k+1} - s_k = 1/N_t$. Let

$$\ell_k(s) = \varphi(s_k) + (s - s_k)\varphi'(s_k).$$

Then for all $s \in I_k$ we have

$$\varphi(s) = \ell_k(s) + O(N_t^{-2})$$

and, arguing as in (4.4),

$$f_t^{(\mathbf{r})}\bar{u}(\ell_k(s)) = u(O(N_t^{-1}))f_t^{(\mathbf{r})}\bar{u}(\varphi(s)).$$

Therefore for t sufficiently large we have

$$\left| \int_0^1 \psi(f_t^{(\mathbf{r})}\bar{u}(\varphi(s)))\, ds - \sum_{k=1}^{N_t} \int_{I_k} \psi(f_t^{(\mathbf{r})}\bar{u}(\ell_k(s)))ds \right| \leq \frac{\varepsilon}{4}.$$

The same holds for $\psi(u(\tilde{s})\cdot)$ in place of ψ. Therefore to prove (4.5) it suffices to show that for t sufficiently large

$$\sum_{k=1}^{N_t} \int_{I_k} |\psi(f_t^{(\mathbf{r})}\bar{u}(\ell_k(s))) - \psi(u(\tilde{s},0)f_t^{(\mathbf{r})}\bar{u}(\ell_k(s)))|\, ds < \frac{\varepsilon}{2}. \tag{4.7}$$

For $1 \leq k \leq N_t$ let $\tilde{s}_k = \tilde{s}e^{-(1+r_1)t)}\varphi_1'(s_k)^{-1}$. We have

$$\begin{aligned} u(\tilde{s},0)f_t^{(\mathbf{r})}\bar{u}(\ell_k(s)) =& f_t^{(\mathbf{r})}u(0,-\tilde{s}_k\varphi_2'(s_k))\bar{u}(\ell_k(s+\tilde{s}_k)) \\ =& u(0,-\tilde{s}_k e^{(1+r_2)t})f_t^{(\mathbf{r})}\bar{u}(\ell_k(s+\tilde{s}_k)). \end{aligned} \tag{4.8}$$

By the dominated convergence theorem and (4.8), to prove (4.7) it suffices to show that for t sufficiently large

$$\sum_{k=1}^{N_t} \int_{I_k} |\psi(f_t^{(\mathbf{r})}\bar{u}(\ell_k(s))) - \psi(f_t^{(\mathbf{r})}\bar{u}(\ell_k(s)+\tilde{s}_k))|\, ds < \frac{\varepsilon}{4}. \tag{4.9}$$

The left-hand side of (4.9) is

$$\leq N_t(2\|\psi\|_{\sup}\tilde{s}e^{-(1+r_1)t}\|1/\varphi_1'\|_{\sup}) \leq \varepsilon/4$$

by (4.6) as required.

Now we turn to the proof of (4.5) for general ν. We write $\nu = h(s)\,ds$ for some nonnegative function h on $[0,1]$. The case for $\nu = ds$ implies the case where h is a characteristic function of open subsets. By approximating functions in L^1 norm we get the results for characteristic functions and finally for any h. ∎

Lemma 4.2: *Any weak-* limit of* (4.2) *is a probability measure.*

Proof. Since $z([0,1])$ is relatively compact, it suffices to prove no escape of mass replacing $\nu_{\mathbf{r}}$ by $(\bar{u})_*\nu$. As in the proof of Lemma 3.1, we can reduce the problem to the case that ν is the measure ds; then one uses [8, Proposition 2.3]. ∎

Lemma 4.3: *We have*

$$\frac{1}{T}\int_0^T (f_t^{(\mathbf{r})})_*\nu_{\mathbf{r}}\,dt \to_{T\to\infty} m.$$

Proof. Let μ be a weak-* limit of (4.2). It is easy to see that μ is invariant under $F := \{f_t^{(\mathbf{r})} : t \in \mathbb{R}\}$. It follows from Lemma 4.1 that μ is invariant under the group U_{13}. In view of Lemma 4.2 the measure μ is a probability measure. Therefore Theorem 1.5 implies that $\mu = m$. Since μ is an arbitrary weak-* limit, the conclusion follows. ∎

Proof of Theorem 1.4. If $r_1 \neq r_2$, then the conclusion is contained in Lemma 4.3. Now we prove the case where $r_1 = r_2 = 1/2$. It suffices to show that given $\psi \in C_c(X)$ and $\varepsilon > 0$ one has

$$\left|\frac{1}{T}\int_0^T\int_0^1 \psi(f_t\bar{u}(\varphi(s)))\,d\nu(s)dt - \int_X \psi\,dm\right| < \varepsilon \tag{4.10}$$

for T sufficiently large. We first divide $[0,1]$ into finitely many closed intervals $\{I_k : 1 \le k \le N\}$ such that for any points $s, \tilde{s} \in I_k$ and any $x \in X$ one has

$$|\psi(z(\tilde{s})^{-1}z(s)x) - \psi(x)| < \frac{\varepsilon}{2}. \tag{4.11}$$

Let s_k be the left endpoint of the interval I_k. Since the matrices $z(s)$ commute with f_t, we have

$$\begin{aligned}&\frac{1}{T}\int_0^T\int_0^1 \psi(f_t\bar{u}(\varphi(s)))\,d\nu(s)dt\\ &\quad=\sum_{k=1}^N \frac{1}{T}\int_0^T\int_{I_k} \psi(z(s)^{-1}z(s_k)z(s_k)^{-1}f_t z(s)\bar{u}(\varphi(s)))\,d\nu(s)dt.\end{aligned} \tag{4.12}$$

In view of (4.11) and (4.12), to prove (4.10) it suffices to show that for T sufficiently large

$$\left|\frac{1}{T}\int_0^T\int_{I_k} \psi(z(s_k)^{-1}f_t z(s)\bar{u}(\varphi(s)))\,d\nu(s)dt - |I_k|\int_X \psi(z(s_k)^{-1}x)\,dm\right| < \frac{\varepsilon}{2}.$$

This follows from Lemma 4.3 applied to the function $x \mapsto \psi(z(s_k)x)$. ∎

Proof of Theorem 1.2 and 1.6(ii). Follows from Theorem 1.4 and Proposition 3.3. ∎

5. Some examples

In this section we give some examples which explain the necessity of conditions which appear in our theorems.

5.1. Examples for Theorem 1.5. All of the conditions of Theorem 1.5 are necessary for its validity. The following examples illustrate two of them which are not obvious to see.

First we show that the assumption that F has no nonzero invariant vectors in $\mathbb{R}^3$ is necessary. We can embed $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ into G so that it induces an embedding of

$$Y = (\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2)/(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)$$

into X. An example of such an embedding is the map τ which sends $(g, \mathbf{v})$ to $\left(\begin{smallmatrix} g & \mathbf{v} \\ 0 & 1 \end{smallmatrix}\right)$ where $g \in \mathrm{SL}_2(\mathbb{R})$ and $\mathbf{v} \in \mathbb{R}^{\mathbf{2}}$. Let μ_1 be the standard probability measure on Y induced by the haar measure on $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ and let μ be its image under the map above. Then μ is clearly invariant under the group

$$F' := \tau(F)$$

and also under

$$U' := \tau(\{(I_2, (s, 0)^{\mathrm{tr}}) : s \in \mathbb{R}\}),$$

where I_2 is the identity in $\mathrm{SL}_2(\mathbb{R})$. Then F' normalizes U', $F'U'$ is not abelian, and the conclusion of Theorem 1.5 does not hold, as the existence of μ shows.

In fact there are $F'U'$-invariant ergodic measures on X which are not even homogeneous. Indeed, it is well known that there are uncountably many F' invariant and ergodic nonhomogeneous probability measures on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. For each such measure ν, integrating along the fiber of $Y \to \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ constructs a measure ν' on Y which is not homogeneous. The image of any such measure under τ will be a measure on X which is $F'U'$-invariant and not homogeneous.

Next we show that the theorem is not true for $X_4 := \mathrm{SL}_4(\mathbb{R})/\mathrm{SL}_4(\mathbb{Z})$. We are grateful to Elon Lindenstrauss for pointing out this example, which relies on some results of [10]. Let

$$H' := \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \subset \mathrm{SL}_4(\mathbb{R}). \tag{5.1}$$

In [10] it was shown, using number fields of degree 4 containing subfields of degree 2, how to find $x \in X_4$ such that $H'x$ is closed and admits a finite H'-invariant measure m'. Let

$$F := \{\mathrm{diag}(e^{3t}, e^t, e^{-t}, e^{-3t})\} \quad \text{and} \quad U := U_{12}.$$

Then clearly F, U satisfy the conditions of Theorem 1.5, and m' is FU-invariant but not $\mathrm{SL}_4(\mathbb{R})$-invariant.

5.2. Example for Theorem 1.1. The goal of this subsection is to show that Theorem 1.1 does not extend to $n = 3$. In fact, we prove:

Theorem 5.1: *There is a line segment $\mathcal{L} \subset \mathbb{R}^3$ which contains a badly approximable vector, such that every point in $\mathcal{L}$ admits an improvement in Dirichlet's theorem.*

The proof is an elaboration on the construction in §5.1, and also uses a result of Hajós, which we now state. For a permutation σ of $\{1, \dots, n\}$, let U_σ^+ denote the group generated by $\{U_{\sigma(i)\sigma(j)} : i < j\}$; that is, the conjugate of the upper triangular group by the permutation matrix corresponding to σ.

Theorem 5.2 (Hajós): *Let X_n be the space of unimodular lattices in $\mathbb{R}^n$ and let $\Lambda \in X$ such that Λ contains no nonzero points in the interior of the unit cube. Then there is σ such that $\Lambda \in U_\sigma^+ \mathbb{Z}^n$.*

Note that each of the orbits $U_\sigma^+ \mathbb{Z}^n$ is compact; thus, recalling that $\|\cdot\|$ denotes the sup-norm, if we set

$$K_\varepsilon := \{\Lambda \in X_n : \forall v \in \Lambda \smallsetminus \{0\}, \|v\| \geq \varepsilon\}$$

then Theorem 5.2 says that K_1 is a finite union of compact orbits of the groups U_σ^+.

We will also need [9, Prop. 2.1]. We extend the notation (1.1) and (2.2) to arbitrary dimension $n \geq 2$ in the obvious way.

Proposition 5.3: *The vector $\mathbf{v} \in \mathbb{R}^n$ admits no improvement in Dirichlet's theorem if and only if there is $t_n \to \infty$ such that $\lim_{n\to\infty} f_{t_n} \bar{u}(\mathbf{v})$ exists and belongs to K_1.*

Let $G = \mathrm{SL}_4(\mathbb{R})$, $X = X_4$, $H = H'$ as in (5.1) and $\pi : G \to X$ be the natural quotient map. In [10] it was shown that there are $x \in X$ for which Hx is a closed orbit of finite volume. We will need the following well-known strengthening:

Proposition 5.4: *There is a dense set of $x \in X$ such that Hx is closed of finite volume, and $\{f_t x : t \geq 0\}$ is bounded.*

Proof. As shown in [10], there are $x_0 \in X$ for which Hx_0 is closed and Ax_0 is compact, where A is the group of diagonal matrices in G. Thus x_0 clearly satisfies the required conclusions. Now write $x_0 = \pi(g_0)$ and let $g \in G(\mathbb{Q})$, $x := \pi(g_0 g)$. The set of such x is dense since $G(\mathbb{Q})$ is dense in G, and we claim that x also satisfies the required conclusions; equivalently, if we set $\Gamma = \mathrm{SL}_4(\mathbb{Z})$, $\Gamma' := g\Gamma g^{-1}$, that $Hg_0\Gamma'$ and $\{f_t g_0 \Gamma' : t \geq 0\}$ are bounded in G/Γ'. Since g is in the commensurator of Γ, there is a finite-index subgroup Γ_0 of Γ such that the maps $\tau_1 : G/\Gamma_0 \to G/\Gamma$, $\tau_2 : G/\Gamma_0 \to G/\Gamma'$ are G-equivariant and proper. Since $x \in \tau_2(\tau_1^{-1}(x_0))$, the conclusion follows. ∎

Proof of Theorem 5.1. Let

$$P := \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} \subset G.$$

Then

$$P = \{p \in G : \{f_t p f_{-t} : t \geq 0\} \text{ is bounded in } G\}. \tag{5.2}$$

This implies that if $p \in P$ and $x \in X$ then, for $t \geq 0$, the distance between $f_t px$ and $f_t x$ is bounded (independently of t). Also let

$$Q := \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong \mathrm{GL}_3(\mathbb{R}) \subset G.$$

There is a projection $q : P \to Q$ obtained by identifying Q with the quotient of P by its unipotent radical, or more concretely, by replacing the $(41), (42), (43)$ matrix entries by 0. A simple calculation in matrix conjugation shows that for all $p \in P$,

$$q(p) = \lim_{t \to +\infty} f_t p f_{-t}. \tag{5.3}$$

Let

$$U = \{u(\mathbf{v}) : \mathbf{v} \in \mathbb{R}^3\} = \langle U_{14}, U_{24}, U_{34} \rangle \cong \mathbb{R}^3.$$

Then the set PU is open and dense in G. Let

$$\mathcal{D} := \{g \in PU : H\pi(g) \text{ is closed}, \{f_t\pi(g) : t \geq 0\} \text{ bounded}\}.$$

According to Proposition 5.4, $\mathcal{D}$ is dense in PU. Let

$$g = pu(\mathbf{v}_0) \in PU$$

for some $\mathbf{v}_0 \in \mathbb{R}^3$ and $p \in P$. If $g \in \mathcal{D}$, then (5.2) implies that $\{f_t\pi(g) : t \geq 0\}$ and $\{f_t\pi(u(\mathbf{v}_0)) : t \geq 0\}$ are both bounded and hence $\mathbf{v}_0$ is badly approximable. Now define $u_s = \exp(sE_{34}) \in H \cap U$ and consider the formula

$$u_s p = p(s)^{-1}\widetilde{u}(s). \tag{5.4}$$

Note that $p(s), \widetilde{u}(s)$ depend on p and hence on g, but we omit this dependence to simplify notation.

We will show that there is $g \in \mathcal{D}$, and an open interval I containing 0 such that:

(i) For all $s \in I$, (5.4) has unique solutions $p(s) \in P$, $\widetilde{u}(s) \in U$.

(ii) There is $\mathbf{w} \in \mathbb{R}^3 \smallsetminus \{0\}$ such that $\widetilde{u}(s) = u(\tau(s)\mathbf{w})$, where $\tau(s)$ is a non-constant rational function of s; that is $\mathcal{L}_0 = \{u^{-1} \circ \widetilde{u}(s) : s \in I\}$ is a smooth parameterization of a line segment in $\mathbb{R}^3$.

(iii) For any $s \in I \smallsetminus \{0\}$, $K_1 \cap q(s)Hx = \varnothing$, where

$$q(s) := q(p(s)).$$

For any $s \in I$ such that $K_1 \cap q(s)Hx = \varnothing$, there is no $t_n \to \infty$ for which the sequence $(f_{t_n}\widetilde{u}(s)\bar{u}(\mathbf{v}_0))_{n\in\mathbb{N}}$ converges to an element of K_1.

First we explain why the theorem follows from (i)–(iv). Consider

$$\mathcal{L} := \mathbf{v}_0 + \mathcal{L}_0 = \{\ell(s) : s \in I\}, \quad \text{where } \ell(s) := \mathbf{v}_0 + \tau(s)\mathbf{w}.$$

According to (i), (ii) this is a nontrivial line segment in $\mathbb{R}^3$, and we need to show that $\ell(s)$ admits an improvement in Dirichlet's theorem for every $s \in I$. For $s = 0$, this follows from the fact that $\ell(0) = \mathbf{v}_0$ is badly approximable using [5]. By (iii), for all $s \in I \smallsetminus \{0\}$ we have $K_1 \cap q(s)Hx = \varnothing$. Then, according to (iv), for such points we have

$$\bar{u}(\ell(s)) = u(\tau(s)\mathbf{w})\bar{u}(\mathbf{v}_0) = \widetilde{u}(s)\bar{u}(\mathbf{v}_0),$$

and so, according to Proposition 5.3, $\ell(s)$ admits an improvement in Dirichlet's theorem.

We turn to the proof of (i)–(iv). In view of Proposition 5.4 it suffices to show that there exists a nonempty open subset of PU such that any element g in the intersection of $\mathcal{D}$ and this open subset satisfies (i)–(iv) for some interval I.

Let p_{ij} denote the matrix entries of p. Then we have

$$u_s p = \begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 \\ p_{21} & p_{22} & p_{23} & 0 \\ p_{31}+sp_{41} & p_{32}+sp_{42} & p_{33}+sp_{43} & sp_{44} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}.$$

The top left 3×3 block of a product $p(s)^{-1}\widetilde{u}(s)$ is the same as that of $p(s)^{-1}$. It follows that

$$q(s) = \begin{pmatrix} a(s) & 0 \\ 0 & b(s) \end{pmatrix} \quad \text{with } a(s) = b(s) \begin{pmatrix} a_{11}(s) & a_{12}(s) & a_{13} \\ a_{21}(s) & a_{22}(s) & a_{23} \\ a_{31}(s) & a_{32}(s) & a_{33} \end{pmatrix},$$

where $b(s)^{-1}$ is the determinant of the top left 3×3 matrix of $u_s p$, $a_{i1}(s), a_{i2}(s)$ are affine functions of s and a_{i3} are constants. Also

$$\tilde{u}(s) = a(0,0,sp_{44})^{\mathrm{tr}} = sb(s)p_{44}(a_{13},a_{23},a_{33})^{\mathrm{tr}}.$$

It follows that for any element of PU there exists an interval I of $\mathbb{R}$ such that (i) and (ii) hold.

For any σ let $\mathfrak{u}_\sigma^+$ denote the Lie algebra of U_σ^+ and let $\mathfrak{h}$ denote the Lie algebra of H. We claim that the set $\mathcal{S}$ of elements $g \in PU$ such that

$$\text{for any } \sigma,\ q'(0)q(0)^{-1} \notin \mathfrak{u}_\sigma^+ + \mathrm{Ad}(q(0))(\mathfrak{h}) \tag{5.5}$$

is a nonempty open subset. Assume the claim; then there exists $g \in \mathcal{D}$ such that (5.5) holds. Recall that

$$K_1 = \bigcup_\sigma U_\sigma^+ \mathbb{Z}^n,$$

that is, a finite union of compact 6-dimensional manifolds, each of which is a U_σ^+-orbit. Also the orbit $H\pi(g)$ is a 7-dimensional manifold, and $q(s)H\pi(g)$ is thus a closed $q(s)Hq(s)^{-1}$-orbit. If $q(0)H\pi(g)$ intersects K_1 at a point x, then (5.5) implies that the application of $q(s)$ for small nonzero s maps a neighborhood of x in $q(0)H\pi(g)$ away from K_1. Since K_1 is compact, $q(s)H\pi(g)$ and K_1 are disjoint, and (iii) follows. By (5.4), $\widetilde{u}(s)\bar{u}(\mathbf{v}_0) = \widetilde{u}(s)p^{-1}\pi(g) = p(s)u_s\pi(g)$.

If $t_n \to \infty$ and the sequence $(f_{t_n} p(s) u_s \pi(g))_{n\geq 1}$ converges, then by (5.3),

$$\begin{aligned}\lim_{n\to\infty} f_{t_n} p(s) u_s \pi(g) &= \lim_{n\to\infty} f_{t_n} p(s) f_{-t_n} f_{t_n} u_s \pi(g) \\ &= \lim_{n\to\infty} q(s) f_{t_n} u_s x \in q(s) H \pi(g).\end{aligned}$$

Thus (iv) follows from (iii).

It remains to prove the claim. It is easy to see that the set $\mathcal{S}$ is open. So we only need to show that it is nonempty. We will show that there exists $g \in \mathcal{S}$ such that p is equal to

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 1 \end{pmatrix},$$

for an appropriate choice of x, y, z. Expressing $q(s)^{-1}$ using (5.4), and taking the derivative with respect to s in the equation

$$q(s)q(s)^{-1} = e,$$

yields

$$q'(0)q(0)^{-1} = \begin{pmatrix} x & y & z & 0 \\ x & y & z & 0 \\ -x & -y & -z & 0 \\ 0 & 0 & 0 & z-x-y \end{pmatrix}. \tag{5.6}$$

Computing explicitly the adjoint representation for p we obtain

$$\mathrm{Ad}(q(0)) \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} = \begin{pmatrix} a & b & -a-b+e & f \\ c & d & -c-d+e & f \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}. \tag{5.7}$$

That is, an element of $\mathrm{Ad}(q(0))(\mathfrak{h})$ can be written as the right-hand side of (5.6), for an appropriate choice of a, b, c, d, e, f, g, h (with $a + d + e + h = 0$).

We will show that for each σ, the failure of (5.5) leads to a nontrivial linear relation among the x, y, z. So taking x, y, z which do not solve these finitely many linear relations forces (5.5). For instance, if $E_{31} \notin \mathfrak{u}_\sigma^+$, then examining the (31) entry in (5.6) and (5.7) leads to $x = 0$. Similarly $E_{32} \notin \mathfrak{u}_\sigma^+$ leads to $y = 0$. For a more interesting case consider the case when both E_{12}, E_{13} do not belong to $\mathfrak{u}_\sigma^+$. From two of the diagonal entries in (5.6), (5.7) we obtain $a = x$, $e = -z$. From the (12) entry we obtain $b = y$, and from the (13) entry we find $-a-b+e=z$. We have four linear equations for the three variables a, b, e, and they only have a solution when $0 = x+y+2z$. This is the sought-for linear relation.

By similar arguments one deals with the case when both E_{21}, E_{23} are not in $\mathfrak{u}_\sigma^+$, and since for each σ one of the two elements E_{12}, E_{21} is contained in $\mathfrak{u}_\sigma^+$, these cases cover all possibilities. This concludes the proof. ∎

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