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RADEMACHER FUNCTIONS IN WEIGHTED SYMMETRIC SPACES

BY

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ABSTRACT

The closed span of Rademacher functions is investigated in the weighted spaces $X(w)$, where X is a symmetric space on $[0,1]$ and w is a positive measurable function on $[0,1]$. By using the notion and properties of the Rademacher multiplicator space of a symmetric space, we give a description of the weights w for which the Rademacher orthogonal projection is bounded in $X(w)$.

1. Introduction

We recall that the Rademacher functions on $[0,1]$ are defined by $r_k(t) = \text{sign}(\sin 2^k \pi t)$ for every $t \in [0,1]$ and each $k \in \mathbb{N}$. It is well known that $\{r_k\}$ is an incomplete orthogonal system of independent random variables. This system plays a prominent role in the modern theory of Banach spaces and operators (see, e.g., [11], [12], [17] and [19]).

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A classical result of Rodin and Semenov [20] states that the sequence $\{r_k\}$ is equivalent in a symmetric space X to the unit vector basis in ℓ_2, i.e.,

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \asymp \Big(\sum_{k=1}^{\infty} |a_k|^2\Big)^{1/2}, \quad (a_k) \in \ell_2, \tag{1}$$

if and only if $G \subset X$, where G is the closure of $L_\infty[0,1]$ in the Zygmund space $\operatorname{Exp} L^2[0,1]$. When this condition is satisfied, the span $[r_k]$ of Rademacher functions is complemented in X if and only if $X \subset G'$, where the Köthe dual space G' to G coincides (with equivalence of norms) with another well-known Zygmund space $L\log^{1/2} L[0,1]$. This was proved independently by Rodin and Semenov [21] and Lindenstrauss and Tzafriri [15, Theorem 2.b.4, pp. 134–138]. Moreover, the condition $G \subset X \subset G'$ (equivalently, complementability of $[r_k]$ in X) is equivalent to the boundedness in X of the orthogonal projection

$$Pf(t) := \sum_{k=1}^{\infty} c_k(f) r_k(t), \tag{2}$$

where $c_k(f) := \int_0^1 f(u) r_k(u)\,du$, $k = 1, 2, \ldots$. The main purpose of this paper is to investigate the behaviour of Rademacher functions and of the respective projection P in the *weighted spaces* $X(w)$ consisting of all measurable functions f such that $fw \in X$ with the norm $\|f\|_{X(w)} := \|fw\|_X$. Here, X is a symmetric space on $[0,1]$ and w is a positive measurable function on $[0,1]$. We make use of the notion of the Rademacher multiplicator space $\mathcal{M}(X)$ of a symmetric space X, which originally arose from the study of vector measures and scalar functions integrable with respect to them (see [8] and [10]). For the first time a connection between the space $\mathcal{M}(X)$ and the behavior of Rademacher functions in the weighted spaces $X(w)$ was observed in [6] when proving a weighted version of inequality (1) (under more restrictive conditions in the case of L_p-spaces it was proved in [23]).

To ensure that the operator P is well defined, we have to guarantee that the Rademacher functions belong both to $X(w)$ and to its Köthe dual space $(X(w))' = X'(1/w)$. For this reason, in what follows we assume that

$$L_\infty \subset X(w) \subset L_1. \tag{3}$$

This assumption allows us to find necessary and sufficient conditions on the weight w under which the orthogonal projection P is bounded in the weighted space $X(w)$. Moreover, extending the above mentioned result of Rodin and

Semenov from [20] to the *weighted* symmetric spaces, we show that, in contrast to the symmetric spaces, the embedding $X(w) \supset G$ is a stronger condition, in general, than equivalence of the sequence of Rademacher functions in $X(w)$ to the unit vector basis in ℓ_2. In the final part of the paper, answering a question from [10], we present a concrete example of a function $f \in \mathcal{M}(L_1)$, which does not belong to the symmetric kernel of the latter space.

2. Preliminaries

Let E be a Banach function lattice on $[0,1]$, i.e., if x and y are measurable a.e. finite functions on $[0,1]$ such that $x \in E$ and $|y| \le |x|$, then $y \in E$ and $\|y\|_E \le \|x\|_E$. The *Köthe dual* of E is the Banach function lattice E' of all functions y such that $\int_0^1 |x(t)y(t)|\,dt < \infty$, for every $x \in E$, with the norm

$$\|y\|_{E'} := \sup\Big\{ \int_0^1 x(t)y(t)\,dt : x \in E, \|x\|_E \le 1\Big\};$$

E' is a subspace of the topological dual E^*. If E is separable we have $E' = E^*$. A Banach function lattice E has the *Fatou property*, if from $0 \le x_n \nearrow x$ a.e. on $[0,1]$ and $\sup_{n\in\mathbf{N}} \|x_n\|_E < \infty$ it follows that $x \in E$ and $\|x_n\|_E \nearrow \|x\|_E$.

Suppose that a Banach function lattice E satisfies $E \supset L_\infty$. By $E_\circ$ we will denote the closure of L_∞ in E. Clearly, $E_\circ$ contains the *absolutely continuous part* of E, that is, the set of all functions $x \in E$ such that $\lim_{m(A)\to 0} \|x \cdot \chi_A\|_E = 0$. Here and subsequently, m is the Lebesgue measure on $[0,1]$ and χ_A is the characteristic function of a set $A \subset [0,1]$.

Throughout the paper a *symmetric (or rearrangement invariant) space* X is a Banach space of classes of measurable functions on [0,1] such that from the conditions $y^* \le x^*$ and $x \in X$ it follows that $y \in X$ and $\|y\|_X \le \|x\|_X$. Here, x^* is the decreasing rearrangement of x, that is, the right continuous inverse of its distribution function: $n_x(\tau) = m\{t \in [0,1] : |x(t)| > \tau\}$. Functions x and y are said to be *equimeasurable* if $n_x(\tau) = n_y(\tau)$, for all $\tau > 0$. The *Köthe dual* X' is a symmetric space whenever X is symmetric. In what follows we assume that X is isometric to a subspace of its second Köthe dual $X'' := (X')'$. In particular, this holds if X is separable or it has the Fatou property. For every symmetric space X the following continuous embeddings hold: $L_\infty \subset X \subset L_1$. If X is a symmetric space, $X \ne L_\infty$, then $X_\circ$ is a separable symmetric space.

Important examples of symmetric spaces are Marcinkiewicz, Lorentz and Orlicz spaces. Let $\varphi\colon [0,1] \to [0,+\infty)$ be a *quasi-concave function*, that is, φ increases, $\varphi(t)/t$ decreases and $\varphi(0)=0$. The *Marcinkiewicz space* $M(\varphi)$ is the space of all measurable functions x on [0,1] satisfying the condition

$$\|x\|_{M(\varphi)} = \sup_{0<t\le 1} \frac{\varphi(t)}{t} \int_0^t x^*(s)\,ds < \infty.$$

If $\varphi\colon [0,1] \to [0,+\infty)$ is an increasing concave function, $\varphi(0)=0$, then the *Lorentz space* $\Lambda(\varphi)$ consists of all measurable functions x on [0,1] such that

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(s)\,d\varphi(s) < \infty.$$

For an arbitrary increasing concave function φ we have $\Lambda(\varphi)' = M(\tilde{\varphi})$ and $M(\varphi)' = \Lambda(\tilde{\varphi})$, where $\tilde{\varphi}(t) := t/\varphi(t)$ [14, Theorems II.5.2 and II.5.4].

Let M be an *Orlicz function*, that is, an increasing convex function on $[0,\infty)$ with $M(0)=0$. The norm of the *Orlicz space* L_M is defined as

$$\|x\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M\left(\frac{|x(s)|}{\lambda}\right) ds \le 1\right\}.$$

In particular, if $M(u)=u^p$, $1 \le p < \infty$, we have $L_M = L_p$ isometrically. Next, by $\|f\|_p$ we denote the norm $\|f\|_{L_p}$.

The *fundamental function* of a symmetric space X is the function $\phi_X(t) := \|\chi_{[0,t]}\|_X$. In particular, we have $\phi_{M(\varphi)}(t) = \phi_{\Lambda(\varphi)}(t) = \varphi(t)$, and $\phi_{L_M}(t) = 1/M^{-1}(1/t)$, respectively. The Marcinkiewicz $M(\varphi)$ and Lorentz $\Lambda(\varphi)$ spaces are, respectively, the largest and the smallest symmetric spaces with the fundamental function φ, that is, if the fundamental function of a symmetric space X is equal to φ, then $\Lambda(\varphi) \subset X \subset M(\varphi)$.

If ψ is a positive function defined on [0,1], then its lower and upper dilation indices are

$$\gamma_\psi := \lim_{t\to 0^+} \frac{\log\left(\sup_{0<s\le 1} \frac{\psi(st)}{\psi(s)}\right)}{\log t} \quad \text{and} \quad \delta_\psi := \lim_{t\to +\infty} \frac{\log\left(\sup_{0<s\le 1/t} \frac{\psi(st)}{\psi(s)}\right)}{\log t},$$

respectively. We always have $0 \le \gamma_\psi \le \delta_\psi \le 1$.

In the case when $\delta_\varphi < 1$, the norm in the Marcinkiewicz space $M(\varphi)$ satisfies the equivalence

$$\|x\|_{M(\varphi)} \asymp \sup_{0<t\le 1} \varphi(t) x^*(t)$$

[14, Theorem II.5.3]. Here, and throughout the paper, the notation $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ independent of all or of a part of arguments of functions (quasi-norms) A and B such that $c \cdot A \le B \le C \cdot A$.

The Orlicz spaces L_{N_p}, $p > 0$, where N_p is an Orlicz function equivalent to the function $\exp(t^p) - 1$, will be of major importance in our study. Usually these are referred to as Zygmund spaces and denoted by $\operatorname{Exp} L^p$. The fundamental function of $\operatorname{Exp} L^p$ is equivalent to the function $\varphi_p(t) = \log^{-1/p}(e/t)$. Since $N_p(u)$ increases at infinity very rapidly, $\operatorname{Exp} L^p$ coincides with the Marcinkiewicz space $M(\varphi_p)$ [16]. This, together with the equality $\delta_{\varphi_p} = 0 < 1$, gives

$$\|x\|_{\operatorname{Exp} L^p} \asymp \sup_{0<t\le 1} x^*(t) \log^{-1/p}(e/t).$$

In particular, for every $x \in \operatorname{Exp} L^p$ and $0 < t \le 1$ we have

$$x^*(t) \le C \, \|x\|_{\operatorname{Exp} L^p} \, \log^{1/p}(e/t). \tag{4}$$

Hence, for a symmetric space X, the embedding $\operatorname{Exp} L^p \subset X$ is equivalent to the condition $\log^{1/p}(e/t) \in X$.

Recall that the Rademacher functions are $r_k(t) := \operatorname{sign} \sin(2^k \pi t)$, $t \in [0,1]$, $k \ge 1$. The famous Khintchine inequality [13] states that, for every $1 \le p < \infty$, the sequence $\{r_k\}$ is equivalent in L_p to the unit vector basis in ℓ_2. As was mentioned in the introduction, Rodin and Semenov [20] extended this result to the class of symmetric spaces showing that equivalence (1) holds in a symmetric space X if and only if $G \subset X$, where $G = (\operatorname{Exp} L^2)_\circ$. Next, we will repeatedly use the Khintchine L_1-inequality from [22] with optimal constants:

$$\frac{1}{\sqrt{2}} \|(a_k)\|_{\ell_2} \le \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_1 \le \|(a_k)\|_{\ell_2}, \tag{5}$$

where $\|(a_k)\|_{\ell_2} := (\sum_{k=1}^\infty a_k^2)^{1/2}$ (next, we consider real scalars; however, all results of the paper are valid also in the complex case).

The *Rademacher multiplicator space* of a symmetric space X is the space $\mathcal{M}(X)$ of all measurable functions $f\colon [0,1] \to \mathbb{R}$ such that $f \cdot \sum_{k=1}^\infty a_k r_k \in X$, for every Rademacher sum $\sum_{k=1}^\infty a_k r_k \in X$. It is a Banach function lattice on $[0,1]$ when endowed with the norm

$$\|f\|_{\mathcal{M}(X)} = \sup\Big\{ \Big\| f \cdot \sum_{k=1}^\infty a_k r_k \Big\|_X : \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_X \le 1 \Big\}.$$

Here, $\mathcal{M}(X)$ can be viewed as the space of operators given by multiplication by a measurable function, which are bounded from the subspace $[r_k]$ in X into the whole space X.

The Rademacher multiplicator space $\mathcal{M}(X)$ was first considered in [9], where it was shown that for a broad class of classical symmetric spaces X the space $\mathcal{M}(X)$ is not symmetric. This result was extended in [3] to include all symmetric spaces such that the lower dilation index γ_{φ_X} of their fundamental function φ_X is positive. This result motivated the study of the symmetric kernel $\mathrm{Sym}\,(X)$ of the space $\mathcal{M}(X)$. The space $\mathrm{Sym}\,(X)$ consists of all functions $f \in \mathcal{M}(X)$ such that an arbitrary function g, equimeasurable with f, belongs to $\mathcal{M}(X)$ as well. The norm in $\mathrm{Sym}\,(X)$ is defined as

$$\|f\|_{\mathrm{Sym}\,(X)} = \sup \|g\|_{\mathcal{M}(X)},$$

where the supremum is taken over all g equimeasurable with f. From the definition it follows that $\mathrm{Sym}\,(X)$ is the largest symmetric space embedded into $\mathcal{M}(X)$. Moreover, if X is a symmetric space such that $X'' \supset \mathrm{Exp}\,L^2$, then

$$\|f\|_{\mathrm{Sym}\,(X)} \asymp \|f^*(t)\log^{1/2}(e/t)\|_{X''}$$

(see [5, Proposition 3.1 and Corollary 3.2]). The opposite situation is when the Rademacher multiplicator space $\mathcal{M}(X)$ is symmetric. The simplest case of this situation is when $\mathcal{M}(X) = L_\infty$. It was shown in [4] that $\mathcal{M}(X) = L_\infty$ if and only if $\log^{1/2}(e/t) \notin X_\circ$. Regarding the case when $\mathcal{M}(X)$ is a symmetric space different from L_∞, see the paper [5].

We will denote by Δ_n^k the dyadic intervals of [0,1], that is, $\Delta_n^k = [(k-1)2^{-n}, k2^{-n}]$, where $n = 0, 1, \ldots$, $k = 1, \ldots, 2^n$; we say that Δ_n^k has *rank* n. For any undefined notions we refer the reader to the monographs [7], [14], [15].

3. Rademacher sums in weighted spaces

First, we find necessary and sufficient conditions on the symmetric space X, under which there is a weight w such that the sequence of Rademacher functions spans ℓ_2 in $X(w)$. We prove the following refinement of the nontrivial part of the above mentioned Rodin–Semenov Theorem.

Proposition 3.1: *For every symmetric space X the following conditions are equivalent:*

(i) *there exists a set* $D \subset [0,1]$ *of positive measure such that*

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \cdot \chi_D \Big\|_X \le M \|(a_k)\|_{\ell_2}, \tag{6}$$

for some $M > 0$ *and arbitrary* $(a_k) \in \ell_2$*;*

(ii) $X \supset G$.

Proof. Since the implication (ii) ⇒ (i) is an immediate consequence of the fact that the sequence $\{r_k\}$ spans ℓ_2 in the space G (see [18] or [24, Theorem V.8.16]), we need to prove only that (i) implies (ii).

Assume that (6) holds. By Lebesgue's density theorem, for sufficiently large $m \in \mathbb{N}$, we can find a dyadic interval $\Delta := \Delta_m^{k_0} = [(k_0-1)2^{-m}, k_0 2^{-m}]$ such that

$$2^{-m} = m(\Delta) \ge m(\Delta \cap D) > 2^{-m-1}.$$

Let us consider the set $E = \bigcup_{k=1}^{2^m} E_m^k$, where E_m^k is obtained by translating the set $\Delta \cap D$ to the interval Δ_m^k, $k = 1, 2, \ldots, 2^m$ (in particular, $E_m^{k_0} = \Delta \cap D$). Denote $f_i = r_i \cdot \chi_E$, $i \in \mathbb{N}$. It follows easily that $|f_i(t)| \le 1$, $t \in [0,1]$, $\|f_i\|_2 \ge 1/\sqrt{2}$, and $f_i \to 0$ weakly in $L_2[0,1]$ when $i \to \infty$. Therefore, by [1, Theorem 5], the sequence $\{f_i\}_{i=1}^{\infty}$ contains a subsequence $\{f_{i_j}\}$, which is equivalent in distribution to the Rademacher system. This means that there exists a constant $C > 0$ such that

$$\begin{aligned} C^{-1} m\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j r_j(t)\Big| > Cz\Big\} &\le m\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j f_{i_j}(t)\Big| > z\Big\} \\ &\le C m\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j r_j(t)\Big| > C^{-1} z\Big\} \end{aligned}$$

for all $l \in \mathbb{N}$, $a_j \in \mathbb{R}$, and $z > 0$. Hence, by the definition of r_j and f_j, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} C^{-1} m\Big\{t \in [0,1] : &\Big|\sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t)\Big| > Cz\Big\} \\ &\le m\Big\{t \in [0,1] : \Big|\sum_{j=m+1}^{m+n} f_{i_j}(t)\chi_{\Delta}(t)\Big| > z\Big\} \\ &\le C m\Big\{t \in [0,1] : \Big|\sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t)\Big| > C^{-1} z\Big\}, \end{aligned}$$

whence

$$\Big\| \sum_{j=m+1}^{m+n} r_{i_j} \chi_{\Delta\cap D} \Big\|_X \geq \alpha \Big\| \sum_{j=m+1}^{m+n} r_j \chi_{[0,2^{-m}]} \Big\|_X, \tag{7}$$

where $\alpha > 0$ depends only on the constant C and on the space X.

Now, assume that (ii) fails, i.e., $X \not\supset G$. Then, by [4, inequality (2) in the proof of Theorem 1], there exists a constant $\beta > 0$, depending only on X, such that for every $m \geq 0$ there exists $n_0 \geq 1$ such that, if $n \geq n_0$ and Δ' is an arbitrary dyadic interval of rank m, we have

$$\Big\| \chi_{\Delta'} \sum_{i=m+1}^{m+n} r_i \Big\|_X \geq \beta \Big\| \sum_{i=1}^{n} r_i \Big\|_X.$$

From this inequality with $\Delta' = [0, 2^{-m}]$ and inequality (7) it follows that, for n large enough,

$$\Big\| \sum_{j=m+1}^{m+n} r_{i_j} \chi_D \Big\|_X \geq \Big\| \sum_{j=m+1}^{m+n} r_{i_j} \chi_{\Delta\cap D} \Big\|_X \geq \alpha\beta \Big\| \sum_{j=1}^{n} r_j \Big\|_X.$$

Combining the latter inequality together with (6) we deduce

$$\frac{1}{\sqrt{n}} \Big\| \sum_{j=1}^{n} r_j \Big\|_X \leq \frac{M}{\alpha\beta}$$

for all $n \in \mathbb{N}$ large enough. At the same time, as follows from the proof of the Rodin–Semenov Theorem in [20], the last condition is equivalent to the embedding $X \supset G$. This contradiction concludes the proof. ∎

Corollary 3.1: *Suppose X is a symmetric space. Then, $X \supset G$ if and only if there exists a weight w such that the sequence $\{r_k\}$ spans ℓ_2 in $X(w)$.*

Proof. If $\{r_k\}$ spans ℓ_2 in $X(w)$ for some weight w, we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \cdot w \Big\|_X \leq C \|(a_k)\|_{\ell_2}.$$

Since $w(t) > 0$ a.e. on $[0,1]$, there is a set $D \subset [0,1]$ of positive measure such that inequality (6) holds for some $M > 0$ and arbitrary $(a_k) \in \ell_2$. Applying Proposition 3.1, we obtain that $X \supset G$. The converse is obvious, and so the proof is completed. ∎

Corollary 3.1 shows the necessity of the condition $X \supset G$ in the following main result of this part of the paper.

Theorem 3.1: *Let X be a symmetric space such that $X \supset G$ and let a positive measurable function w on $[0,1]$ satisfy condition (3). Then we have:*

(i) *the sequence $\{r_k\}$ spans ℓ_2 in $X(w)$ if and only if $w \in \mathcal{M}(X)$, where $\mathcal{M}(X)$ is the Rademacher multiplicator space of X;*

(ii) *$X(w) \supset G$ if and only if $w \in \mathrm{Sym}\,(X)$, where $\mathrm{Sym}\,(X)$ is the symmetric kernel of $\mathcal{M}(X)$.*

Part (i) of this theorem was actually obtained in [6, p. 240]. However, for the reader's convenience we provide here its proof. But we begin with the following technical result, which will be needed to prove part (ii).

Lemma 3.1: *Let Y be a symmetric space and let w be a positive measurable function on $[0,1]$. Suppose the weighted function lattice $Y(w^*)$ contains an unbounded decreasing positive function a on $(0,1]$. Then $(Y(w))_\circ = Y_\circ(w)$.*

Proof. Since $(wa)^*(t) \le w^*(t/2)a(t/2)$, $0 < t \le 1$, [14, § II.2] and, by assumption, $w^*a \in Y$, we have $wa \in Y$. Equivalently, $a \in Y(w)$.

Let $y \in (Y(w))_\circ$. By definition, there is a sequence $\{y_k\} \subset L_\infty$ such that

$$\lim_{k\to\infty} \|y_k w - yw\|_Y = 0. \tag{8}$$

Since a decreases, for arbitrary $A \subset [0,1]$ and every (fixed) $k \in \mathbb{N}$ we have

$$\|y_k w \chi_A\|_Y \le \|y_k\|_\infty \|w^* \chi_{(0,m(A)]}\|_Y \le \frac{\|y_k\|_\infty}{a(m(A))} \|w^* a\|_Y.$$

Noting that the right hand side of this inequality tends to 0 as $m(A) \to \infty$, we get

$$\lim_{m(A)\to 0} \|y_k w \chi_A\|_Y = 0,$$

whence $y_k w \in Y_\circ$, $k \in \mathbb{N}$. Combining this with (8), we infer that $yw \in Y_\circ$ or, equivalently, $y \in Y_\circ(w)$.

To prove the opposite embedding, assume that $y \in Y_\circ(w)$. Then

$$\lim_{k\to\infty} \|y_k - yw\|_Y = 0 \tag{9}$$

for some sequence $\{y_k\} \subset L_\infty$. From the hypothesis of the lemma it follows that $Y \ne L_\infty$. Therefore, for arbitrary $A \subset [0,1]$ and each $k \in \mathbb{N}$

$$\|y_k/w \cdot \chi_A\|_{Y(w)} = \|y_k \chi_A\|_Y \to 0 \quad \text{as } m(A) \to 0.$$

Hence, $y_k/w \in (Y(w))_\circ$, $k \in \mathbb{N}$. Since $\|y_k/w - y\|_{Y(w)} = \|y_k - yw\|_Y$, from (9) it follows that $y \in (Y(w))_\circ$. ∎

Proof of Theorem 3.1. (i) Since $X \supset G$, equivalence (1) holds. At first, assume that $w \in \mathcal{M}(X)$. Then, by definition of the norm in $\mathcal{M}(X)$, we have

$$\|w\|_{\mathcal{M}(X)} \asymp \sup\Big\{\Big\|w \cdot \sum_{k=1}^{\infty} a_k r_k\Big\|_X : \|(a_k)\|_{\ell_2} \le 1\Big\}. \tag{10}$$

Therefore,

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X(w)} = \Big\|w \cdot \sum_{k=1}^{\infty} a_k r_k\Big\|_X \le C\|w\|_{\mathcal{M}(X)}\|(a_k)\|_{\ell_2}$$

for every $(a_k) \in \ell_2$. On the other hand, from embeddings (3) and inequality (5) it follows that

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X(w)} \ge c\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_1 \ge \frac{c}{\sqrt{2}}\|(a_k)\|_{\ell_2}.$$

As a result we deduce that $\{r_k\}$ spans ℓ_2 in $X(w)$.

Conversely, if

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X(w)} \asymp \|(a_k)\|_{\ell_2},$$

from (10) we obtain that $\|w\|_{\mathcal{M}(X)} < \infty$, i.e., $w \in \mathcal{M}(X)$.

(ii) Assume that $w \in \operatorname{Sym}(X)$. Then, taking into account the properties of the symmetric kernel $\operatorname{Sym}(X)$ (see Preliminaries or [5, Corollary 3.2]) we have $w^*(t)\log^{1/2}(e/t) \in X''$. Let us prove that

$$\operatorname{Exp} L_2 \subset X''(w). \tag{11}$$

Given $x \in \operatorname{Exp} L_2$, by [7, Theorem 2.7.5] there exists a measure-preserving transformation σ of $(0,1]$ such that $|x(t)| = x^*(\sigma(t))$. Applying inequality (4) and a well-known property of the rearrangement of a measurable function (see, e.g., [14, § II.2]), we have

$$\begin{aligned}(wx)^*(t) = (wx^*(\sigma))^*(t) &\le C\Big(w\log^{1/2}(e/\sigma(\cdot))\Big)^*(t)\\ &\le Cw^*(t/2)\log^{1/2}(2e/t), \qquad 0 < t \le 1.\end{aligned}$$

Therefore, $wx \in X''$ or, equivalently, $x \in X''(w)$, and (11) is proved. Hence, $G = (\operatorname{Exp} L_2)_\circ \subset (X''(w))_\circ$. Since $\log^{1/2}(e/t) \in X''(w^*)$, we can apply

Lemma 3.1, and so, by [2, Lemma 3.3],

$$G \subset (X'')_\circ(w) = X_\circ(w) \subset X(w).$$

Now, let $X(w) \supset G$. We show that $X(w^*) \supset G$. In fact, let τ be a measure-preserving transformation of $(0,1]$ such that $w(t) = w^*(\tau(t))$ [7, Theorem 2.7.5]. Suppose $x \in G$. Since $x(\tau)$ and x are equimeasurable functions, we have $x(\tau) \in G$ and $\|x(\tau)\|_G = \|x\|_G$. Therefore,

$$\|x(\tau)w^*(\tau)\|_X = \|x(\tau)w\|_X \le C\|x\|_G.$$

Then, $\|x(\tau)w^*(\tau)\|_X = \|xw^*\|_X$, because X is a symmetric space, and from the preceding inequality we infer that $\|xw^*\|_X \le C\|x\|_G$. Thus, $x \in X(w^*)$, and the embedding $X(w^*) \supset G$ is proved. Passing to the second Köthe dual spaces, we obtain $X''(w^*) \supset G'' = \operatorname{Exp} L^2$. Hence, $\log^{1/2}(e/t) \in X''(w^*)$ or, equivalently, $w \in \operatorname{Sym}(X)$ (as above, see Preliminaries or [5, Corollary 3.2]), and the proof is complete. ∎

By the Rodin–Semenov Theorem [20], the sequence $\{r_k\}$ is equivalent in a symmetric space X to the unit vector basis in ℓ_2 if and only if $X \supset G$. In contrast to that from Theorem 3.1 we immediately deduce the following result.

Corollary 3.2: *Suppose X is a symmetric space such that* $\operatorname{Sym}(X) \neq \mathcal{M}(X)$. *Then, for every $w \in \mathcal{M}(X)\backslash\operatorname{Sym}(X)$ the Rademacher functions span ℓ_2 in $X(w)$ but $X(w) \not\supset G$.*

By [3, Theorem 2.1], $\operatorname{Sym}(X) \neq \mathcal{M}(X)$ (and therefore there is $w \in \mathcal{M}(X) \setminus \operatorname{Sym}(X)$) whenever the lower dilation index of the fundamental function ϕ_X is positive. In particular, it is fulfilled for L_p-spaces, $1 \le p < \infty$. The condition $\gamma_{\phi_X} > 0$ means that the space X is situated "far" from the minimal symmetric space L_∞. Now, consider the opposite case when a symmetric space is "close" to L_∞. Then the Rademacher multiplicator space $\mathcal{M}(X)$ may be symmetric (equivalently, it coincides with its symmetric kernel). Since the space $\operatorname{Sym}(X)$ has an explicit description (see Preliminaries), in this case we are able to state a sharper result. For simplicity, let us consider only Lorentz and Marcinkiewicz spaces (for more general results of such a sort, see [5]).

Recall [5] that a function $\varphi(t)$ defined on $[0,1]$ satisfies the Δ^2-*condition* (briefly, $\varphi \in \Delta^2$) if it is nonnegative, increasing, concave, and there exists $C > 0$ such that $\varphi(t) \le C\cdot\varphi(t^2)$ for all $0 < t \le 1$. By [5, Corollary 3.5], if $\varphi \in \Delta^2$, then $\mathcal{M}(\Lambda(\varphi)) = \operatorname{Sym}(\Lambda(\varphi))$ and $\mathcal{M}(M(\varphi)) = \operatorname{Sym}(M(\varphi))$. Moreover, it is known [3, Example 2.15 and Theorem 4.1] that $\mathrm{Sym}(\Lambda(\varphi)) = \Lambda(\psi)$ (resp. $\mathrm{Sym}(M(\varphi)) = M(\psi)$), where $\psi'(t) = \varphi'(t)\log^{1/2}(e/t)$, whenever $\log^{1/2}(e/t) \in \Lambda(\varphi)$ (resp. $\log^{1/2}(e/t) \in M(\varphi)$). Therefore, we get

Corollary 3.3: *Let $\varphi \in \Delta^2$ and $\log^{1/2}(e/t) \in \Lambda(\varphi)$ (resp. $\log^{1/2}(e/t) \in M(\varphi)$). If w is a positive measurable function on $[0,1]$ satisfying condition (3), then the sequence $\{r_k\}$ is equivalent in the space $\Lambda(\varphi)(w)$ (resp. $M(\varphi)(w)$) to the unit vector basis in ℓ_2 if and only if $w \in \Lambda(\psi)$ (resp. $w \in M(\psi)$), where $\psi'(t) = \varphi'(t)\log^{1/2}(e/t)$.*

In particular, if $0 < p \le 2$, the sequence $\{r_k\}$ is equivalent in the Zygmund space $\mathrm{Exp}\, L^p(w)$ to the unit vector basis in ℓ_2 if and only if $w \in \mathrm{Exp}\, L^q$, where $q = 2p/(2-p)$ (here, we set $\mathrm{Exp}\, L^\infty = L_\infty$).

4. Rademacher orthogonal projection in weighted spaces

Here, we present necessary and sufficient conditions, under which the orthogonal projection P defined by (2) is bounded in a weighted symmetric space $X(w)$ satisfying condition (3).

Proposition 4.1: *Let E be a Banach function lattice on $[0,1]$ that is isometrically embedded into E'', $L_\infty \subset E \subset L_1$. Then the projection P defined by (2) is bounded in E if and only if there are constants C_1 and C_2 such that for all $a = (a_k) \in \ell_2$*

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_E \le C_1 \|a\|_{\ell_2} \tag{12}$$

and

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{E'} \le C_2 \|a\|_{\ell_2}. \tag{13}$$

Proof. Firstly, assume that inequalities (12) and (13) hold. Then, denoting, as above, $c_k(f) := \int_0^1 f(u) r_k(u)\,du$, $k = 1, 2, \ldots$, for every $n \in \mathbb{N}$, by (13), we have

$$\begin{aligned}\sum_{k=1}^{n} c_k(f)^2 &= \int_0^1 f(u) \sum_{k=1}^{n} c_k(f) r_k(u)\,du \\ &\le \|f\|_E \Big\| \sum_{k=1}^{n} c_k(f) r_k \Big\|_{E'} \le C_2 \|f\|_E \Big(\sum_{k=1}^{n} c_k(f)^2 \Big)^{1/2},\end{aligned}$$

whence

$$\Big(\sum_{k=1}^{\infty} c_k(f)^2\Big)^{1/2} \le C_2\|f\|_E, \quad f \in E.$$

Therefore, by (12), we obtain

$$\|Pf\|_E \le C_1\Big(\sum_{k=1}^{\infty} c_k(f)^2\Big)^{1/2} \le C_1C_2\|f\|_E$$

for all $f \in E$.

Conversely, suppose that the projection P is bounded in E. Let us consider the following sequence of finite-dimensional operators:

$$P_n f(t) := \sum_{k=1}^{n} c_k(f) r_k(t), \quad n \in \mathbb{N}.$$

Clearly, P_n is bounded in E for every $n \in \mathbb{N}$. Furthermore, by assumption, the series $\sum_{k=1}^{\infty} c_k(f) r_k$ converges in E for each $f \in E$. Therefore, by the Uniform Boundedness Principle,

$$\|P_n\|_{E\to E} \le B, \quad n \in \mathbb{N}. \tag{14}$$

Moreover, since $L_\infty \subset E \subset L_1$, then $L_\infty \subset E' \subset L_1$ as well, and hence, by the L_1-Khintchine inequality (5),

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_E \ge c\|a\|_{\ell_2} \quad \text{and} \quad \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{E'} \ge c\|a\|_{\ell_2}.$$

Therefore, for all $f \in E$, $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, we have

$$\begin{aligned}\int_0^1 f(t)\cdot\sum_{k=1}^{n} a_k r_k(t)\,dt \quad &= \quad \sum_{k=1}^{n} a_k c_k(f) \le \|a\|_{\ell_2}\Big(\sum_{k=1}^{n} c_k(f)^2\Big)^{1/2}\\ &\le \quad c^{-1}\|a\|_{\ell_2}\cdot\|P_n f\|_E \le Bc^{-1}\|a\|_{\ell_2}\cdot\|f\|_E.\end{aligned}$$

Taking the supremum over all $f \in E$, $\|f\|_E \le 1$, we get

$$\Big\|\sum_{k=1}^{n} a_k r_k\Big\|_{E'} \le Bc^{-1}\|a\|_{\ell_2}, \quad n \in \mathbb{N}.$$

Applying the latter inequality to Rademacher sums $\sum_{k=n}^{m} a_k r_k$, $1 \le n < m$, with $a = (a_k)_{k=1}^{\infty} \in \ell_2$, we deduce that the series $\sum_{k=1}^{\infty} a_k r_k$ converges in the space E' and

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{E'} \le Bc^{-1}\|a\|_{\ell_2}.$$

Thus, (13) is proved. Let us prove the corresponding inequality for E.

By the Fubini theorem and (14), for arbitrary $f \in E$, $g \in E'$ and every $n \in \mathbb{N}$ we have

$$\int_0^1 f(u) \cdot \sum_{k=1}^n c_k(g) r_k(u)\, du = \int_0^1 g(t) \cdot \sum_{k=1}^n c_k(f) r_k(t)\, dt$$
$$\leq \|P_n f\|_E \|g\|_{E'} \leq B\|f\|_E \|g\|_{E'},$$

whence

$$\Big\| \sum_{k=1}^n c_k(g) r_k \Big\|_{E'} \leq B\|g\|_{E'}, \quad n \in \mathbb{N}.$$

Applying this inequality instead of (14), as above, we get

$$\Big\| \sum_{k=1}^n a_k r_k \Big\|_{E''} \leq Bc^{-1}\|a\|_{\ell_2}.$$

Since $L_\infty \subset E$ and E is isometrically embedded into E'', from the last inequality it follows that

$$\Big\| \sum_{k=1}^n a_k r_k \Big\|_{E} \leq Bc^{-1}\|a\|_{\ell_2}$$

for all $n \in \mathbb{N}$. Hence, if $a = (a_k)_{k=1}^\infty \in \ell_2$, the series $\sum_{k=1}^\infty a_k r_k$ converges in E and

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{E} \leq Bc^{-1}\|a\|_{\ell_2}.$$

Thus, inequality (12) holds, and the proof is complete. ∎

From Proposition 4.1, Corollary 3.1 and Theorem 3.1 we obtain the following results.

Theorem 4.1: *Let a symmetric space X and a positive measurable function w on $[0,1]$ satisfy condition (3). Then, the projection P defined by (2) is bounded in $X(w)$ if and only if $G \subset X \subset G'$, $w \in \mathcal{M}(X)$ and $1/w \in \mathcal{M}(X')$.*

In particular, P is bounded in $X(w)$ whenever $w^*(t)\log^{1/2}(e/t) \in X''$ and $(1/w)^*(t)\log^{1/2}(e/t) \in X'$.

As above, the result can be somewhat refined for Lorentz and Marcinkiewicz spaces whose fundamental function satisfies the Δ^2-condition.

Corollary 4.1: *Let $\varphi \in \Delta^2$ and let w be a positive measurable function on $[0,1]$ satisfying condition (3) for $X = \Lambda(\varphi)$ (resp. $X = M(\varphi)$). Then the* *projection P defined by (2) is bounded in $\Lambda(\varphi)(w)$ (resp. $M(\varphi)(w)$) if and only if $G \subset \Lambda(\varphi) \subset G'$, $w \in \Lambda(\psi)$ and $1/w \in \mathcal{M}(M(\tilde{\varphi}))$ (resp. $G \subset M(\varphi) \subset G'$, $w \in M(\psi)$ and $1/w \in \mathcal{M}(\Lambda(\tilde{\varphi}))$), where $\psi'(t) = \varphi'(t)\log^{1/2}(e/t)$ and $\tilde{\varphi}(t) = t/\varphi(t)$.*

REMARK 4.1: It is easy to see that the orthogonal projection P is bounded in the space $X(w)$ if and only if the projection

$$P_w f(t) := \sum_{k=1}^{\infty} \int_0^1 f(s) r_k(s) \frac{ds}{w(s)} \cdot r_k(t) w(t), \quad 0 \le t \le 1$$

(on the subspace $[r_k w]$), is bounded in X.

5. Example of a function from $\mathcal{M}(L_1) \setminus \mathrm{Sym}\,(L_1)$

Answering a question from [10], we present here a concrete example of a function $f \in \mathcal{M}(L_1)$, which does not belong to the symmetric kernel $\mathrm{Sym}\,(L_1)$, that is,

$$\int_0^1 f^*(t) \log^{1/2}(e/t)\, dt = \infty.$$

Since the latter space is symmetric, it is sufficient to find a function $f \in \mathcal{M}(L_1)$, for which there exists a function $g \notin \mathcal{M}(L_1)$ equimeasurable with f. We will look for f and g in the form

$$f = \sum_{k=1}^{\infty} \alpha_k \chi_{B_k}, \quad g = \sum_{k=1}^{\infty} \alpha_k \chi_{D_k}, \tag{15}$$

where $\{B_k\}$ and $\{D_k\}$ are sequences of pairwise disjoint subsets of $[0,1]$, $m(B_k) = m(D_k)$, $\alpha_k \in \mathbb{R}$, $k = 1, 2, \ldots$. Next, we will make use of some ideas of the paper [9].

Let $n = 2^m$ with $m \in \mathbb{N}$ and let J be a subset of $\{1, 2, \ldots, 2^n\}$ with cardinality n. We define the set $A = \bigcup_{j \in J} \Delta_n^j$ associated with J (as above, Δ_n^j are the dyadic intervals of $[0,1]$). Clearly, $m(A) = n 2^{-n}$.

For arbitrary sequence $(b_i) \in \ell_2$ we have

$$\Big\| \chi_A \sum_{i=1}^{\infty} b_i r_i \Big\|_1 \le \Big\| \chi_A \sum_{i=1}^{n} b_i r_i \Big\|_1 + \| \chi_A \sum_{i=n+1}^{\infty} b_i r_i \Big\|_1. \tag{16}$$

Firstly, we estimate the tail term from the right hand side of this inequality. It is easy to see that the functions

$$\chi_A(t) \cdot \sum_{i=n+1}^{\infty} b_i r_i(t) \ \text{ and } \ \chi_{[0, n2^{-n}]}(t) \cdot \sum_{i=n+1}^{\infty} b_i r_i(t)$$

are equimeasurable on $[0,1]$ and

$$\chi_{[0,n2^{-n}]}(t)\sum_{i=n+1}^{\infty} b_i r_i(t) = \sum_{i=n+1}^{\infty} b_i r_{i+m-n}(n^{-1}2^n t), \quad 0 < t \le 1$$

(here, we set $r_j(t) = 0$ if $t \notin [0,1]$). Therefore,

$$\begin{aligned}\Big\|\chi_A \sum_{i=n+1}^{\infty} b_i r_i\Big\|_1 = \Big\|\chi_{[0,n2^{-n}]}\sum_{i=n+1}^{\infty} b_i r_i\Big\|_1 &= n2^{-n}\Big\|\sum_{i=n+1}^{\infty} b_i r_{i+m-n}\Big\|_1 \\ &\le n2^{-n}\Big(\sum_{i=n+1}^{\infty} b_i^2\Big)^{1/2}.\end{aligned} \tag{17}$$

Now, choosing a set A in a special way, estimate the first term from the right hand side of (16). Denote by ε_{ij}^n the value of the function r_i, $i = 1, 2, \dots, n$, on the interval Δ_n^j, $1 \le j \le 2^n$. Since $n = 2^m$, we can find a set $J_1(n) \subset \{1, 2, \dots, 2^n\}$, $\operatorname{card} J_1(n) = n$, such that the $n \times n$ matrix $n^{-1/2} \cdot (\varepsilon_{ij}^n)_{1\le i\le n, j\in J_1(n)}$ is orthogonal. Then, if $c_j := n^{-1/2}\sum_{i=1}^n \varepsilon_{ij}^n b_i$, $j \in J_1(n)$, we have $\|(c_j)_{j\in J_1(n)}\|_{\ell_2} = \|(b_i)_{i=1}^n\|_{\ell_2}$. Therefore, setting $B(n) := \bigcup_{j\in J_1(n)} \Delta_n^j$, we obtain

$$\begin{aligned}\Big\|\chi_{B(n)}\sum_{i=1}^{n} b_i r_i\Big\|_1 &= \Big\|\sum_{j\in J_1(n)}\Big(\sum_{i=1}^n b_i r_i\Big)\chi_{\Delta_n^j}\Big\|_1 = \Big\|\sum_{j\in J_1(n)}\sum_{i=1}^n \varepsilon_{ij}^n b_i\cdot\chi_{\Delta_n^j}\Big\|_1 \\ &= n^{1/2}\Big\|\sum_{j\in J_1(n)} c_j\chi_{\Delta_n^j}\Big\|_1 \\ &= n^{1/2}2^{-n}\sum_{j\in J_1(n)}|c_j| \le n2^{-n}\|(b_i)_{i=1}^n\|_{\ell_2}.\end{aligned}$$

Combining this inequality with (16), (17) for $A = B(n)$ and (5), by definition of the norm in the space $\mathcal{M}(L_1)$, we have

$$\|\chi_{B(n)}\|_{\mathcal{M}(L_1)} \le 2\sqrt{2}n2^{-n}. \tag{18}$$

Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers, $n_k = 2^{m_k}$, $m_k \in \mathbb{N}$, satisfying the condition

$$n_k^{1/8} \ge 2^{n_1+\cdots+n_{k-1}}, \quad k = 2, 3, \dots. \tag{19}$$

At first, we construct a sequence of sets $\{B_k\}$. Setting $J_1^1 := J_1(n_1)$ and $B_1 := B(n_1)$, in view of (18) we have

$$\|\chi_{B_1}\|_{\mathcal{M}(L_1)} \le 2\sqrt{2}n_1 2^{-n_1}.$$

.

To define B_2, we take for I_1 any interval $\Delta^j_{n_1}$ such that $j \notin J^1_1$. Now, we can choose a set $J^2_1 \subset \{1, 2, \ldots, 2^{n_1+n_2}\}$ satisfying the conditions: $\operatorname{card} J^2_1 = n_2$, $\Delta^j_{n_1+n_2} \subset I_1$ for every $j \in J^2_1$ and the $n_2 \times n_2$ matrix $n_2^{-1/2} \cdot (\varepsilon^{n_1+n_2}_{ij})_{n_1<i\le n_1+n_2, j\in J^2_1}$ is orthogonal. We set $B_2 := \bigcup_{j\in J^2_1} \Delta^j_{n_1+n_2}$. Clearly, $m(B_2) = n_2 2^{-(n_1+n_2)}$ and $B_1 \cap B_2 = \emptyset$, because of $B_2 \subset I_1$. As in the case of $B(n)$ we have

$$\begin{aligned}\Big\|\chi_{B_2}\sum_{i=1}^{n_1+n_2} b_i r_i\Big\|_1 &= \Big\|\sum_{j\in J^2_1}\Big(\sum_{i=1}^{n_1+n_2} b_i r_i\Big)\chi_{\Delta^j_{n_1+n_2}}\Big\|_1\\ &\le \Big\|\sum_{j\in J^2_1}\Big(\sum_{i=1}^{n_1} b_i r_i\Big)\chi_{\Delta^j_{n_1+n_2}}\Big\|_1 + \Big\|\sum_{j\in J^2_1}\Big(\sum_{i=n_1+1}^{n_2} b_i r_i\Big)\chi_{\Delta^j_{n_1+n_2}}\Big\|_1\\ &\le \sum_{i=1}^{n_1}|b_i|\|\chi_{B_2}\|_1 + \Big\|\sum_{j\in J^2_1}\sum_{i=n_1+1}^{n_1+n_2}\varepsilon^{n_1+n_2}_{ij} b_i\cdot\chi_{\Delta^j_{n_1+n_2}}\Big\|_1\\ &\le (n_1^{1/2}+1)n_2 2^{-(n_1+n_2)}\|(b_i)_{i=1}^{n_1+n_2}\|_{\ell_2}\\ &\le n_2 2^{-n_2}\|(b_i)_{i=1}^{n_1+n_2}\|_{\ell_2}.\end{aligned}$$

Therefore, from (16), (17) and (5) it follows that

$$\|\chi_{B_2}\|_{\mathcal{M}(L_1)} \le \sqrt{2}\Big((n_1+n_2)2^{-(n_1+n_2)} + n_2 2^{-n_2}\Big) \le 2\sqrt{2} n_2 2^{-n_2}.$$

Proceeding in the same way, we get a sequence $\{B_k\}$ of pairwise disjoint subsets of $[0,1]$ such that $m(B_k) = n_k 2^{-(n_1+\cdots+n_k)}$ and

$$\|\chi_{B_k}\|_{\mathcal{M}(L_1)} \le 2\sqrt{2} n_k 2^{-n_k}, \quad k = 1, 2, \ldots. \tag{20}$$

Now we define the sets D_k, $k = 1, 2, \ldots$. Select a set $J^1_2 \subset \{1, 2, \ldots, 2^{n_1}\}$, $\operatorname{card} J^1_2 = n_1$, such that each column of the $n_1 \times n_1$ matrix $(\varepsilon^{n_1}_{ij})_{1\le i\le n_1, j\in J^1_2}$ has exactly one entry equal to -1 and the rest are equal to 1. Setting $D_1 := \bigcup_{j\in J^1_2}\Delta^j_{n_1}$, we have $m(D_1) = n_1 2^{-n_1}$. Furthermore, from the inequality

$\|n_1^{-1/2}\sum_{i=1}^{n_1} r_i\|_1 \le 1$ (see (5)) and the definition of D_1 it follows that

$$\begin{aligned}\|\chi_{D_1}\|_{\mathcal{M}(L_1)} &\ge \Big\|\sum_{j\in J_2^1}\Big(n_1^{-1/2}\sum_{i=1}^{n_1} r_i\Big)\chi_{\Delta_{n_1}^j}\Big\|_1\\ &= \Big\|\sum_{j\in J_2^1}\Big(n_1^{-1/2}\sum_{i=1}^{n_1} \varepsilon_{ij}^{n_1}\Big)\chi_{\Delta_{n_1}^j}\Big\|_1\\ &= (n_1^{1/2}-2n_1^{-1/2})n_1 2^{-n_1} \ge \frac{1}{2}n_1^{3/2}2^{-n_1}\end{aligned}$$

if $n_1 \ge 4$.

Similarly, we can define the set D_2. Let I_2 be any interval $\Delta_{n_1}^j$ with $j \notin J_2^1$. Choose the set $J_2^2 \subset \{1,2,\dots,2^{n_1+n_2}\}$ such that $\operatorname{card} J_2^2 = n_2$, $\Delta_{n_1+n_2}^j \subset I_2$ for every $j \in J_2^2$ and each column of the $n_2 \times n_2$ matrix $(\varepsilon_{ij}^{n_1+n_2})_{n_1<i\le n_1+n_2, j\in J_2^2}$ has exactly one entry equal to -1 and the rest are equal to 1. Then, if $D_2 := \bigcup_{j\in J_2^2}\Delta_{n_1+n_2}^j$, then $m(D_2) = n_2 2^{-(n_1+n_2)}$ and $D_1 \cap D_2 = \emptyset$. Moreover, we have

$$\begin{aligned}\|\chi_{D_2}\|_{\mathcal{M}(L_1)} &\ge \Big\|\sum_{j\in J_2^2}\Big(n_2^{-1/2}\sum_{i=n_1+1}^{n_1+n_2} r_i\Big)\chi_{\Delta_{n_1+n_2}^j}\Big\|_1\\ &= \Big\|\sum_{j\in J_2^2}\Big(n_2^{-1/2}\sum_{i=n_1+1}^{n_1+n_2} \varepsilon_{ij}^{n_1+n_2}\Big)\chi_{\Delta_{n_1+n_2}^j}\Big\|_1\\ &= (n_2^{1/2}-2n_2^{-1/2})n_2 2^{-(n_1+n_2)} \ge \frac{1}{2}n_2^{3/2}2^{-(n_1+n_2)}.\end{aligned}$$

Arguing in the same way, we construct a sequence $\{D_k\}$ of pairwise disjoint subsets of $[0,1]$ such that $m(D_k) = n_k 2^{-(n_1+\cdots+n_k)}$ and

$$\|\chi_{D_k}\|_{\mathcal{M}(L_1)} \ge \frac{1}{2}n_k^{3/2}2^{-(n_1+\cdots+n_k)}, k=1,2,\dots. \tag{21}$$

Since $m(B_k) = m(D_k)$, $k = 1,2,\dots$, the functions f and g defined by (15) are equimeasurable for arbitrary $\alpha_k \in \mathbb{R}$, $k=1,2,\dots$. Setting $\alpha_k = 2^{n_k}n_k^{-5/4}$, by (20), we obtain

$$\|f\|_{\mathcal{M}(L_1)} \le \sum_{k=1}^\infty \alpha_k\|\chi_{B_k}\|_{\mathcal{M}(L_1)} \le 2\sqrt{2}\sum_{k=1}^\infty n_k^{-1/4} < \infty,$$

because of $n_k = 2^{m_k}$, $m_1 < m_2 < \cdots$. Thus, $f \in \mathcal{M}(L_1)$.

On the other hand, for every $k = 1, 2, \dots$, from (21) and (19) it follows that

$$\sup\Big\{\Big\|g\cdot\sum_{i=1}^{\infty}a_ir_i\Big\|_1 : \Big\|\sum_{i=1}^{\infty}a_ir_i\Big\|_1\le 1\Big\}\ge \alpha_k\|\chi_{D_k}\|_{\mathcal{M}(L_1)}$$
$$\ge \frac{1}{2}n_k^{1/4}2^{-(n_1+\cdots+n_{k-1})}\ge\frac{1}{2}n_k^{1/8}.$$

Hence, this supremum is infinite, and so $g \notin \mathcal{M}(L_1)$.

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