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# SOME VARIATIONS ON TVERBERG'S THEOREM

BY

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ABSTRACT

Define $T(d,r) = (d+1)(r-1)+1$. A well known theorem of Tverberg states that if $n \geq T(d,r)$, then one can partition any set of $n$ points in $\mathbb{R}^d$ into $r$ pairwise disjoint subsets whose convex hulls have a common point. The numbers $T(d,r)$ are known as Tverberg numbers. Reay added another parameter $k$ ($2 \leq k \leq r$) and asked: what is the smallest number $n$, such that every set of $n$ points in $\mathbb{R}^d$ admits an $r$-partition, in such a way that each $k$ of the convex hulls of the $r$ parts meet. Call this number $T(d,r,k)$. Reay conjectured that $T(d,r,k) = T(d,r)$ for all $d, r$ and $k$. In this paper we prove Reay's conjecture in the following cases: when $k \geq [\frac{d+3}{2}]$, and also when $d < \frac{rk}{r-k} - 1$. The conjecture also holds for the specific values $d=3, r=4, k=2$ and $d=5, r=3, k=2$.

## 1. Introduction

A well known theorem of Radon says that any set of $d+2$ or more points in $\mathbb{R}^d$ can be partitioned into two disjoint parts whose convex hulls meet. This follows easily from the fact that every set of $d+2$ points in $\mathbb{R}^d$ is affinely dependent.

The corresponding statement for partitions into more than two parts is known as Tverberg's theorem.

Theorem 1.0.1 (H. Tverberg, 1966): *Let $a_1, \dots, a_n$ be points in $\mathbb{R}^d$. If $n > (d+1)(r-1)$, then the set $N$={1,…,n} of indices can be partitioned into $r$ disjoint parts $N_1, \dots, N_r$ in such a way that the $r$ convex hulls* $\mathrm{conv}\{a_i : i \in N_j\}$ *(j=1,…,r) have a point in common.*

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(This formulation covers also the case where the points $a_1, \ldots, a_n$ are not all distinct.) Henceforth we use the abbreviation $a(N_j)$ for $\{a_i : i \in N_j\}$. The original proof (see [7]) was quite difficult. In 1981 Tverberg published another proof, much simpler than the original one (see [8]). Sarkaria [6] gave a quite accessible proof, with some algebraic flavor. It seems that the simplest proof so far is due to Roudneff [5]. See [1] §8.3 for further information.

The numbers $T(d,r) = (d+1)(r-1)+1$ are known as Tverberg numbers. The condition $n \geq T(d,r)$ in Tverberg's theorem is extremely tight. If $n < T(d,r)$, then almost surely (with the exception of a "bad set" of measure zero), for any $r$-partition $N_1, \ldots, N_r$ of the set $N = \{1, \ldots, n\}$, even the intersection of the **affine** hulls $\operatorname{aff}(a(N_j))$ $(j = 1, \ldots, r)$ is empty.

In fact, it can be shown that the "bad set" is not only of measure zero, but it is included in the set of zeroes of a polynomial. (I.e., there exists a polynomial $P$, not identically zero, in $n \cdot d$ scalar variables,

$$P(\vec{x_1}, \ldots, \vec{x_n}) = P(x_{11}, \ldots, x_{1d}, \ldots, x_{n1}, \ldots, x_{nd})$$

such that, for every $n$ points $a_1, \ldots, a_n$ satisfying $P(a_1, \ldots, a_n) \neq 0$, and for any $r$-partition $N_1, \ldots, N_r$ of $N$, we have $\bigcap_{j=1}^{r} \operatorname{aff}(a(N_j)) = \emptyset$.) (For details, see [2].)

In 1979, John R. Reay (see [4]) raised the following question: If we weaken the requirement $\bigcap_{j=1}^{r} \operatorname{conv} a(N_j) \neq \emptyset$ in Tverberg's theorem and ask only that each $k$ of the convex hulls $\operatorname{conv} a(N_j)$ $(j = 1, \ldots, r)$ intersect, where $2 \leq k \leq r$, can this be done with fewer than $T(d,r)$ points? Let us define $T(d,r,k)$ to be the smallest positive integer $n$ with the following property: for any list $a_1, \ldots, a_n$ of points in $\mathbb{R}^d$ there is an $r$-partition $N_1, \ldots, N_r$ of the set of indices $N = \{1, \ldots, n\}$, such that every $k$ of the $r$ convex hulls $\operatorname{conv} a(N_j)$ have a point in common.

The function $T(d,r,k)$ is clearly monotone non-decreasing in each of the parameters $d, r, k$, and $T(d,r,r) = T(d,r)$.

If $r > d+1$, and each $d+1$ of the convex hulls $\operatorname{conv} a(N_j)$ $(j = 1, \ldots, r)$ meet, then they all meet, by Helly's Theorem. Thus $T(d,r,k) = T(d,r)$ for $d+1 \leq k \leq r$. This reduces the interesting range of $k$ to $2 \leq k \leq \min(r-1, d)$.

Reay settled the case $d = 2$, showing that $T(2,r,2) = T(2,r)$ for all $r \geq 2$. He also showed that $T(3,3,2) = T(3,3)\,(= 9)$ and made the following bold conjecture.

Conjecture 1.0.2: *$T(d,r,k) = T(d,r)$ for all $2 \leq k \leq r$.*

The meaning of Reay's conjecture is: If $n < T(d,r)$, then there exists a set $X \subset \mathbb{R}^d$, $|X| = n$, such that for every $r$-partition of $X$ there are two parts whose convex hulls are disjoint.

We don't really believe this is true. To press our point, consider the case $d = r = 1000$. By Tverberg's theorem, a million points in $\mathbb{R}^{1000}$ can be partitioned into one thousand parts whose convex hulls have a common point. Is there a set of 999,999 points in $\mathbb{R}^{1000}$ that cannot be partitioned into 1000 parts whose convex hulls intersect just pairwise? Seems implausible. (See also the concluding remarks in Section 5.)

Nevertheless, the purpose of this paper is to establish parts of Reay's conjecture. We show, by means of suitable examples, that Reay's conjecture does hold in the following cases (Theorems 1.0.3–1.0.6):

Theorem 1.0.3: *For every dimension $d \geq 2$ and for every $r(\geq \lfloor \frac{d+3}{2} \rfloor)$,*

$$T(d, r, \left\lfloor \frac{d+3}{2} \right\rfloor) = T(d,r) = (d+1)(r-1)+1.$$

In particular, this shows that $T(3,4,3) = T(3,4) = 13$. For $d = 3$, $r = 4$, $k = 2$ we have the following:

Theorem 1.0.4: *$T(3,4,2) = T(3,4) = 13$.*

Another class of cases is covered by

Theorem 1.0.5: *For every $2 \leq k < r$ and for every dimension $d < \frac{kr}{r-k} - 1$,*

$$T(d,r,k) = T(d,r) = (d+1)(r-1)+1.$$

Therefore, if $r = 3$ and $k = 2$ then $T(d,r,k) = T(d,r)$ provided $d < 5$. The case $d = 5$ is covered by the following:

Theorem 1.0.6: *$T(5,3,2) = T(5,3) = 13$.*

In all cases, the examples are variations, specializations or perturbations of the following: $d+1$ rays that emanate from the origin and positively span $\mathbb{R}^d$, with $r-1$ points chosen on each ray.

In order to put the ranges of Theorems 1.0.3 and 1.0.5 on the same scale, we can regard $k$ as the independent variable. Theorem 1.0.3 establishes Reay's conjecture in the domain of $d$'s $d+1 \leq 2k-1$ (of which the subdomain $d+1 \leq k$

is trivial, in view of Helly's Theorem). For $d+1 \geq 2k$, Theorem 1.0.5 establishes Reay's conjecture for $k < r < \frac{d+1}{d+1-k}k$. This domain of $r$'s reduces to $k < r < 2k$ when $d+1 = 2k$. It shrinks with increasing $d$, and vanishes altogether for $d+1 \geq k(k+1)$.

The rest of the paper is organized as follows: Sections 2 and 3 are devoted to the proofs of Theorems 1.0.3 and 1.0.5 respectively. Section 4 contains a short outline of the proofs of the special cases (Theorems 1.0.4 and 1.0.6), and the last section contains some concluding remarks, pertaining in particular to the unsolved parts of Reay's Conjecture.

For detailed proofs of Theorems 1.0.4 and 1.0.6, we refer the reader to the unabridged version of this paper on the web; see [3].

## 2. Proof of Theorem 1.0.3

For the proof we will use the following (counter) example:

Let $p_0, p_1, \ldots, p_d \in \mathbb{R}^d$ be the vertices of a $d$-simplex centered at the origin, i.e., $\sum_{i=0}^{d} p_i = \underline{0}$ and each $d$ of the points $p_0, p_1, \ldots, p_d$ are linearly independent. Let $D = \{0, 1, \ldots, d\}$, and for $i \in D$ define $R_i = \{\lambda p_i : \lambda > 0\}$ (the open ray emanating from $\underline{0}$ through $p_i$).

On each ray $R_i$ we choose $r-1$ distinct points. The chosen points form a set $X \subset \mathbb{R}^d$, $|X| = (d+1)(r-1) = T(d,r) - 1$. We show that in every partition of $X$ into $r$ parts ($X = C_1 \cup \cdots \cup C_r$) there are some $j$ parts $C_{i_1}, \ldots, C_{i_j}$, $j \leq \lfloor \frac{d+3}{2} \rfloor$, whose convex hulls have empty intersection. This will show that $T(d,r,k) = T(d,r)$ for $\lfloor \frac{d+3}{2} \rfloor \leq k \leq r$. We start with some preliminaries concerning the "positive basis" $P = \{p_0, p_1, \ldots, p_d\}$ of $\mathbb{R}^d$.

### 2.1. Properties of the spanning set $P = \{p_0, p_1, \ldots, p_d\}$.

Proposition 2.1.1: *Every point $x \in \mathbb{R}^d$ has a representation*

$$x = \xi_0 p_0 + \xi_1 p_1 + \cdots + \xi_d p_d \qquad (2.1.1)$$

*where* $\min\{\xi_0, \xi_1, \ldots, \xi_d\} = 0$. *This representation is* *unique.*

*Proof.* The vectors $p_0, p_1, \ldots, p_d$ span $\mathbb{R}^d$ linearly. In fact, each $d$ of them form a linear basis of $\mathbb{R}^d$. Let $x = \sum_{i=0}^{d} \alpha_i p_i$ be some arbitrary representation of $x$ in terms of $P$. The only linear dependences among $p_0, p_1, \ldots, p_d$ are $\sum_{i=0}^{d} \lambda p_i = 0$, $\lambda \in \mathbb{R}$. Therefore, the most general representation of $x$ in terms of $P$ is $x =$

$\sum_{i=0}^{d}(\alpha_i - \lambda)p_i$, $\lambda \in \mathbb{R}$. To obtain a representation with the smallest coefficient equal 0, we must choose $\lambda = \min\{\alpha_i : i \in D\}$. ∎

We call (2.1.1) the non-negative representation of $x$ (in terms of $P$). The **support** of $x$ (with respect to $P$) is defined by

$$\operatorname{supp} x = \{i \in D : \xi_i > 0\}.$$

Simple properties of $\operatorname{supp} x$ are:

(1) $\emptyset \subseteq \operatorname{supp} x \subsetneq D$.
(2) $\operatorname{supp} x = \emptyset$ iff $x = \underline{0}$.
(3) $\operatorname{supp} p_i = \{i\}$.
(4) $\operatorname{supp} \lambda x = \operatorname{supp} x$ for $\lambda > 0$.
(5) $\operatorname{supp}(x + y) \subseteq \operatorname{supp} x \cup \operatorname{supp} y$, with equality iff $\operatorname{supp} x \cup \operatorname{supp} y \neq D$.
(6) If $x \neq \underline{0}$, then $\operatorname{supp} x \cup \operatorname{supp}(-x) = D$.

Recall that our set $X$ consists of $r - 1$ distinct points on each ray $R_i$ ($i \in D$). For a subset $C \subseteq X$, define $I(C) = \{i \in D : C \cap R_i \neq \emptyset\}$. Now make the following observations:

PROPOSITION 2.1.2: *If $C \subseteq X$ and $x \in \operatorname{conv} C$, then $\operatorname{supp} x \subseteq I(C)$. (This is obviously true also when I(C)=D.)*

When $I$ is a subset of $D$, we shall denote by $R(I)$ the union $\bigcup\{R_i : i \in I\}$.

PROPOSITION 2.1.3: *Suppose $C \subseteq X$ and $x \in \operatorname{conv} C$. If $I(C) \neq D$ then $x \in \operatorname{conv}\{C \cap R(\operatorname{supp} x)\}$.*

*Proof.* Suppose $x = \sum_{\nu=1}^{n} \gamma_\nu c_\nu$, where $c_\nu \in C$, $\gamma_\nu > 0$, $\sum_{\nu=1}^{n} \gamma_\nu = 1$. If $c_\nu = \lambda_\nu p_i \in R_i$, $\lambda_\nu > 0$, then $p_i$ will appear with a positive coefficient in the non-negative representation of $x$ in terms of $P$, and therefore $i \in \operatorname{supp} x$. Note that we have used the fact that $I(C) \neq D$. ∎

For points $a = \alpha p_i \in R_i$, $b = \beta p_i \in R_i$ ($\alpha, \beta > 0$), we say that $a$ is **lower** than $b$ (or $b$ is **higher** than $a$) on $R_i$ if $\alpha < \beta$ (or, equivalently, if $\|a\| < \|b\|$).

PROPOSITION 2.1.4: *Suppose $I \subsetneq D$. Let $C, C'$ be two finite subsets of $R(I)(= \bigcup\{R_i : i \in I\})$. If, for each $i \in I$, every point of $C \cap R_i$ is lower (on $R_i$) than every point of $C' \cap R_i$, then $\operatorname{conv} C \cap \operatorname{conv} C' = \emptyset$.*

*Proof.* Assume, w.l.o.g., that $|I|$=$d$. (We do <u>not</u> assume that $C \cap R_i \neq \emptyset$ and $C' \cap R_i \neq \emptyset$ for all $i \in I$.) For each $i \in I$ choose a point $s_i = \sigma_i p_i \in R_i$ that is higher (on $R_i$) than every point of $C \cap R_i$ and lower than every point of $C' \cap R_i$. The $d$ points $s_i$ $(i \in I)$ are linearly independent, and their affine hull $H = \operatorname{aff}\{s_i : i \in I\} \subset \mathbb{R}^d$ is a hyperplane that does not pass through the origin. Denote by $H_-, H_+$ the two open half-spaces determined by $H$, and assume $\underline{0} \in H_-$. From our assumptions it follows that $C \subset H_-$ and $C' \subset H_+$, hence $\operatorname{conv} C \cap \operatorname{conv} C' = \emptyset$. ∎

PROPOSITION 2.1.5: *Suppose $U \subsetneq D$. Let $C_1, C_2, \ldots, C_n$ $(n \geq 2)$ be subsets of $X$. Assume*

(1) $\bigcap_{\nu=1}^{n} I(C_\nu) \subseteq U$,
(2) $I(C_\nu) \subsetneq D$ *for* $\nu = 1, 2$,
(3) *for each $i \in U$, each point of $C_1 \cap R_i$ is lower (on $R_i$) than every point of $C_2 \cap R_i$.*

*Then* $\bigcap_{\nu=1}^{n} \operatorname{conv} C_\nu = \emptyset$.

*Proof.* Assume, on the contrary, that $\bigcap_{\nu=1}^{n} \operatorname{conv} C_\nu \neq \emptyset$, and suppose that $x \in \bigcap_{\nu=1}^{n} \operatorname{conv} C_\nu$. By Proposition 2.1.2 we conclude that $\operatorname{supp} x \subseteq \bigcap_{\nu=1}^{n} I(C_\nu) \subseteq U$. Applying Proposition 2.1.3 to $C_1$ and $C_2$, we find that

$$x \in \operatorname{conv}(C_\nu \cap R(U)) \quad \text{for } \nu = 1, 2. \tag{2.1.2}$$

Now invoke Proposition 2.1.4 with $C = C_1 \cap R(U)$, $C' = C_2 \cap R(U)$, and $I = U$, to conclude that $\operatorname{conv}(C_1 \cap R(U)) \cap \operatorname{conv}(C_2 \cap R(U)) = \emptyset$, which contradicts (2.1.2). ∎

2.2. COMPLETION OF THE PROOF OF THEOREM 1.0.3. Let $X \subset \mathbb{R}^d$ be the set described at the beginning of this section ($r-1$ points on each of the rays $R_0, R_1, \ldots, R_d$), and let $C_1, \ldots, C_r$ be an arbitrary partition of X into $r$ disjoint (nonempty) sets. Our aim is to apply Proposition 2.1.5 to some $n$ of the parts $C_i$, with $n$ as small as possible. We shall be able to do this with some $n \leq \left\lfloor \frac{d+3}{2} \right\rfloor$.

For $i = 1, \ldots, n$ we divide $I(C_i)$ into two disjoint sets:

$$S_i = \{j \in D : |C_i \cap R_j| = 1\},$$

$$M_i = \{j \in D : |C_i \cap R_j| > 1\},$$

and get

$$|C_i| \geq 2|M_i| + |S_i|. \tag{2.2.1}$$

Furthermore, for every subset $J$ of $D$,

$$|C_i \cap R(J)| \geq 2|M_i \cap J| + |S_i \cap J|. \tag{2.2.2}$$

Assume the parts $C_i$ are ordered in such a way that

(1) $|C_1| \leq |C_i|$ for $i = 2, 3, \dots, r$,
(2) $|C_2 \cap R(S_1)| \leq |C_i \cap R(S_1)|$ for $i = 3, 4, \dots, r$.

From condition (1) we have

$$|I(C_1)| \leq |C_1| \leq \left\lfloor \frac{1}{r}|X| \right\rfloor = \left\lfloor \frac{r-1}{r}(d+1) \right\rfloor,$$

and therefore $|I(C_1)| \leq |C_1| \leq d$.

Condition (2) yields

$$\begin{aligned} |C_2 \cap R(S_1)| &\leq \frac{1}{r-1} \sum_{i=2}^{r} |C_i \cap R(S_1)| = \frac{1}{r-1} \Big| \bigcup_{i=2}^{r} C_i \cap R(S_1) \Big| \\ &= \frac{1}{r-1} |(X \setminus C_1) \cap R(S_1)| = \frac{r-2}{r-1}|S_1| \end{aligned}$$

and therefore

$$|R(S_1) \cap C_2| < |S_1|. \tag{2.2.3}$$

This, in turn, implies $|I(C_2)| \leq d$.

Assume $|M_1| = a \, (a \geq 0)$. From (2.2.1) we obtain

$$|S_1| \leq |C_1| - 2|M_1| \leq d - 2a$$

and from (2.2.3)

$$|R(S_1) \cap C_2| \leq d - 2a - 1.$$

Next, we define the set $U$ to be plugged into Proposition 2.1.5:

The set $S_1 \cap S_2$ can be divided into two disjoint subsets:

$$U_1 = \{j \in S_1 \cap S_2 : C_1 \text{ is lower than } C_2 \text{ on } R_j\},$$

$$U_2 = \{j \in S_1 \cap S_2 : C_2 \text{ is lower than } C_1 \text{ on } R_j\}.$$

If $|U_1| \geq |U_2|$ we define $U = U_1$, otherwise we define $U = U_2$. In any case, $|U| \geq \frac{1}{2}|S_1 \cap S_2|$.

Proposition 2.2.1: *Under these notations*

$$|I(C_1) \cap I(C_2) \setminus U| \leq \left\lfloor \frac{d-1}{2} \right\rfloor.$$

*Proof.* $I(C_1) \cap I(C_2) \setminus U = \big(M_1 \cap I(C_2)\big) \cup (S_1 \cap M_2) \cup (S_1 \cap S_2 \setminus U)$ and therefore

$$\begin{aligned} |I(C_1) \cap I(C_2) \setminus U| &\leq |M_1| + |S_1 \cap M_2| + |S_1 \cap S_2 \setminus U| \\ &\leq |M_1| + |S_1 \cap M_2| + \tfrac{1}{2}|S_1 \cap S_2| \\ &\leq_{\text{by (2.2.2)}} |M_1| + \tfrac{1}{2}|R(S_1) \cap C_2| \\ &\leq a + \tfrac{1}{2}(d - 2a - 1) \\ &= \tfrac{d-1}{2}. \quad \blacksquare \end{aligned}$$

To finish the proof of Theorem 1.0.3, for each index $i \in I(C_1) \cap I(C_2) \setminus U$ we choose a set $C_q$ $(3 \leq q \leq r)$ that does not meet $R_i$, and call it $C(i)$. (Such a set does exist, since $|(X \setminus (C_1 \cup C_2)) \cap R_i| \leq r - 3$.) Note that the sets $C(i)$ $(i \in I(C_1) \cap I(C_2) \setminus U)$ are not necessarily distinct.

Under these conditions, the sets $C_1, C_2, \{C(i) : i \in I(C_1) \cap I(C_2) \setminus U\}$ satisfy the assumptions of Proposition 2.1.5 with $n \leq 2 + \left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{d+3}{2} \right\rfloor$, and therefore $\operatorname{conv} C_1 \cap \operatorname{conv} C_2 \cap \bigcap_{i \in I(C_1) \cap I(C_2) \setminus U} \operatorname{conv} C(i) = \emptyset$.

To sum it up, we have shown that for every $d \geq 2$ and $r \geq \left\lfloor \frac{d+3}{2} \right\rfloor$ there is a set $X$ of $(d+1)(r-1)$ points in $\mathbb{R}^d$, such that in any $r$-partition of $X$ there are $\left\lfloor \frac{d+3}{2} \right\rfloor$ parts whose convex hulls have empty intersection. This completes the proof of Theorem 1.0.3.

## 3. Proof of Theorem 1.0.5

For this proof we use the same counterexample as in Theorem 1.0.3, with an additional restriction. Recall that we started with a simplex, centered at the origin, with $d+1$ vertices $p_0, \ldots, p_d$. For each vertex $p_i$ we defined $R_i$ to be the open ray emanating from $\underline{0}$ through $p_i$. On each ray we chose $r-1$ points. The set of all these chosen points is denoted by $X$; $|X| = (d+1)(r-1) = T(d,r) - 1$. The additional restriction in our case is that the $r-1$ points on each ray $R_i$ $(i = 0, \ldots, d)$ be in "general position", as detailed in the next paragraph.

For a subset $M \subsetneq D$ $(D = \{0, 1, \ldots, d\})$, define an $(M, X)$-selection $S$ to be a subset of $X$, of size $|M|$, consisting of exactly one point on each ray $R_j$, $j \in M$. The set $X$ $(\subsetneq \bigcup\{R_i : i \in D\})$ is in "general position" if for any set $M \subsetneq D$, $2 \leq |M| = m \leq d$, and for every $\bar{m}$ pairwise disjoint $(M, X)$-selections $S_1, \ldots, S_{\bar{m}}$, the intersection $\bigcap_{i=1}^{\bar{m}} \operatorname{aff} S_i$ is a single point if $\bar{m} = m$, and is empty if $\bar{m} > m$. (Since the maximum possible number of pairwise disjoint $(M, X)$-selections is just $r-1$, this condition applies only to sets $M \subsetneq D$ of size $2 \leq |M| \leq \min\{r-1, d\}$.) A necessary and sufficient condition for this to happen is that if $S_i = \{\lambda_{i,j} p_j : j \in M\}$, $i = 1, \ldots, \bar{m}$ and $M = \{j_1, \ldots, j_m\}$, then

$$\det \begin{pmatrix} \lambda_{1,j_1}^{-1} & \cdots & \lambda_{1,j_m}^{-1} \\ \vdots & & \vdots \\ \lambda_{m,j_1}^{-1} & \cdots & \lambda_{m,j_m}^{-1} \end{pmatrix} \neq 0 \quad \text{if } \bar{m} = m,$$

and

$$\det \begin{pmatrix} \lambda_{1,j_1}^{-1} & \cdots & \lambda_{1,j_m}^{-1} & 1 \\ \vdots & & \vdots & \vdots \\ \lambda_{m+1,j_1}^{-1} & \cdots & \lambda_{m+1,j_m}^{-1} & 1 \end{pmatrix} \neq 0 \quad \text{if } \bar{m} = m+1.$$

We will show that if $d < \frac{rk}{r-k} - 1$, then for any $r$-partition $(C_1, \ldots, C_r)$ of $X$, some $k$ of the convex hulls $\operatorname{conv} C_1, \ldots, \operatorname{conv} C_r$ have empty intersection.

We proceed by induction on $r$. This will enable us to focus on partitions $(C_1, \ldots, C_r)$ of $X$ where each part $C_j$ misses at least one ray $R_i$.

For $r = 2$ there is nothing to prove. Now assume $r > 2$, and suppose the theorem holds for $r-1$. Let $(C_1, \ldots, C_r)$ be an $r$-partition of the set $X$ defined above. If one of the parts, say $C_r$, contains a point from each ray $R_i$, then we turn to the induction hypothesis. We delete $C_r$, define $\tilde{X} = X \setminus C_r$, and consider the $(r-1)$-partition $(C_1, \ldots, C_{r-1})$ of $\tilde{X}$. Note that $\tilde{X}$ contains at most $r-2$ points on each ray $R_i$ and is in "general position", like $X$.

If $2 \leq k < r-1$, apply the induction hypothesis: By assumption, $d < \frac{rk}{r-k} - 1$ and since $\frac{rk}{r-k} < \frac{(r-1)k}{r-1-k}$, $\tilde{X}$ satisfies the conditions of the theorem, and therefore some $k$ of the convex hulls $\operatorname{conv} C_j$ $(j = 1, \ldots, r-1)$ have empty intersection.

If $k = r-1$, then $\bigcap_{j=1}^{r-1} \operatorname{conv} C_j = \emptyset$. Indeed if $x \in \bigcap_{j=1}^{r-1} \operatorname{conv} C_j$, then $\operatorname{supp} x \subset \bigcap_{j=1}^{r-1} I(C_j)$ by Proposition 2.1.2. But $\bigcap_{j=1}^{r-1} I(C_j) = \emptyset$, since each ray $R_i$ is missed by at least one of the parts $C_1, \ldots, C_{r-1}$. The only point $x \in \mathbb{R}^d$ with $\operatorname{supp} x = \emptyset$ is the origin $\underline{0}$, but $\underline{0} \notin \operatorname{conv} C_j$ unless $I(C_j) = D$.

From now on we assume that for every $j$, $I(C_j) \subsetneq D$.

We now prove the theorem. To do this we define a (weight) function: given $k$ distinct parts (say $C_{j_1}, \ldots, C_{j_k}$) and a ray $R_i$, define

$$W((C_{j_1}, \ldots, C_{j_k}), R_i) := \begin{cases} 0 & \text{if } R_i \cap C_{j_s} = \emptyset \text{ for some } s \in \{1, \ldots, k\}, \\ 1 + \#\{s : |C_{j_s} \cap R_i| > 1\} & \text{otherwise}. \end{cases}$$

In Section 3.1 we will show that if $\bigcap_{s=1}^{k} \operatorname{conv} C_{j_s} \neq \emptyset$, then

$$\sum_{i=0}^{d} W((C_{j_1}, \ldots, C_{j_k}), R_i) \geq k. \tag{3.0.1}$$

In Section 3.2 we will show that for each $i \in D$,

$$\sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq r} W((C_{j_1}, \ldots, C_{j_k}), R_i) \leq \binom{r-1}{k}. \tag{3.0.2}$$

We use these two results to establish Theorem 1.0.5. If $\bigcap_{s=1}^{k} \operatorname{conv} C_{j_s} \neq \emptyset$ for all $1 \leq j_1 < j_2 < \cdots < j_k \leq r$, then from the inequalities (3.0.1) and (3.0.2) we conclude that

$$k\binom{r}{k} \leq \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq r} \sum_{i=0}^{d} W((C_{j_1}, \cdots, C_{j_k}), R_i) \leq (d+1)\binom{r-1}{k}. \tag{3.0.3}$$

We thus obtain

$$k\binom{r}{k} \leq (d+1)\binom{r-1}{k},$$

which is equivalent to $d \geq \frac{rk}{r-k} - 1$, and the theorem follows.

3.1. A lower bound for the weight function $W$. Let $\{C_j\}_{j \in J}$ ($J \subset \{1, \ldots, r\}$, $|J| = k$) be a collection of $k$ parts. We aim to show that if $\bigcap\{\operatorname{conv} C_j : j \in J\} \neq \emptyset$ then $\sum_{i=0}^{d} W(\{C_j\}_{j \in J}, R_i\} \geq k$.

For the weight $W(\{C_j\}_{j \in J}, R_i)$ to be positive, each of the parts $C_j$ ($j \in J$) must meet the ray $R_i$. We say that $R_i$ is a **common** ray (for the given collection) if $R_i \cap C_j \neq \emptyset$ for all $j \in J$. For convenience, we define $I(J) = \bigcap_{j \in J} I(C_j)$ to be the set of indices of the common rays. Then the union of the common rays is just $R(I(J))$. Proposition 3.1.1 below says that the intersection of the convex hulls $\bigcap\{\operatorname{conv} C_j : j \in J\}$ depends only on the intersections of the parts $C_j (j \in J)$ with the common rays.

Proposition 3.1.1: *For $J \subset \{1, \ldots, r\}$, if $I(C_j) \subsetneq D$ for every $j \in J$ then*

$$\bigcap_{j \in J} \operatorname{conv} C_j = \bigcap_{j \in J} \operatorname{conv}(C_j \cap R(I(J))).$$

*Proof.* The r.h.s. is clearly a subset of the l.h.s. We show that the l.h.s. is included in the r.h.s. as follows: suppose $x \in \bigcap_{j \in J} \operatorname{conv} C_j$. Then, by Proposition

2.1.2, $\operatorname{supp} x \subset I(J)$. By Proposition 2.1.3 it follows that $x \in \operatorname{conv}(C_j \cap R(I(J)))$ for all $j \in J$. ∎

Proposition 3.1.2: *Given a set of $k$ parts, $\{C_j\}_{j \in J}(J \subset \{1, \ldots, r\}, |J| = k)$, if $|I(J)| = m < k$, and if each of the common rays $R_i$ ($i \in I(J)$) contains exactly one point of each of the $C_j$-s, then $\bigcap_{j \in J} \operatorname{conv} C_j = \emptyset$.*

*Proof.* Since $|I(J)| = m$, we have $m$ common rays spanning an $m$-dimensional linear space. Each of the sets $C_j \cap R(I(J))$ consists of $m$ linearly independent points and therefore spans a hyperplane in that space. By the definition of "general position" (see above), we have

$$\bigcap_{j \in J} \operatorname{aff}(C_j \cap R(I(J))) = \emptyset. \qquad ∎$$

The following is a natural generalization of the last proposition:

Proposition 3.1.3: *Given a set of $k$ parts, $\{C_j\}_{j \in J}$ ($J \subset \{1, \ldots, r\}$, $|J| = k$), suppose $|I(J)| = m$ and denote by $t$ the number of parts among the $C_j$'s that contain more than one point on at least one of the common rays. Under these conventions, $m < k - t$ implies $\bigcap_{j \in J} \operatorname{conv} C_j = \emptyset$.*

*Proof.* Divide $J$ into two subsets $S, T$ as follows:

$j \in S$ if $C_j$ meets each ray $R_i$ ($i \in I(J)$) in a single point.

$j \in T$ if $C_j$ meets at least one ray $R_i$ ($i \in I(J)$) in more than one point.

Then $|T| = t, |S| = k - t$.

By Proposition 3.1.1,

$$\begin{aligned} \bigcap_{j \in J} \operatorname{conv} C_j &= \bigcap_{j \in J} \operatorname{conv}(C_j \cap R(I(J))) \\ &\subseteq \bigcap_{j \in S} \operatorname{conv}(C_j \cap R(I(J))) \\ &\subseteq \bigcap_{j \in S} \operatorname{aff}(C_j \cap R(I(J))). \end{aligned}$$

The last expression is the intersection of $k - t (> m)$ hyperplanes in the $m$-dimensional space spanned by $R(I(J))$, which is empty due to the "general position" of $X$. ∎

Proposition 3.1.3 implies inequality (3.0.1). Indeed, from Proposition 3.1.3 it follows that if $J = \{j_1, \ldots, j_k\}$ and $\bigcap_{j \in J} \operatorname{conv} C_j \neq \emptyset$, then $m + t \geq k$. But the l.h.s. of (3.0.1) is just $m + \#\{(i,s) : i \in I(J),\ 1 \le s \le k$ and $|C_{j_s} \cap R_i| > 1\}$, which is $\ge m + t$.

3.2. An upper bound for the weight function $W$. The weight of a ray $R_i$ is defined as

$$W(R_i) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le r} W((C_{j_1}, \ldots, C_{j_k}), R_i).$$

We will show that $W(R_i)$ is maximal when each point in $R_i$ belongs to a different part $C_j$, i.e., when $|C_j \bigcap R_i| \le 1$ for all $j$. In that case it is clear that $W(R_i) = \binom{r-1}{k}$.

Now assume $|C_j \cap R_i| > 1$ for some $j$, say $|C_1 \cap R_i| > 1$. Since $|X \cap R_i| = r - 1$, there is another part, say $C_2$, that does not meet $R_i$ at all. Choose one point $x \in C_1 \cap R_i$, and change the given partition $\mathcal{C} = (C_1, \ldots, C_r)$ into $\mathcal{C}' = (C_1', \ldots, C_r')$ as follows:

$$C_1' = C_1 \smallsetminus \{x\},\ C_2' = C_2 \cup \{x\},\ C_j' = C_j \text{ for } 3 \le j \le r.$$

This change will increase the value of $W(R_i)$, or leave it unaffected. In fact, if $|C_1 \cap R_i| > 2$, then

$$W(\{C_j' : j \in J\}, R_i) \ge W((C_j : j \in J\}, R_i)$$

for all $k$-subsets $J \subset D$. If $|C_1 \cap R_i| = 2$, define

$$P_i = \{j \in \{1, \ldots, r\} :\ C_j \cap R_i \ne \emptyset\}$$

and note that

$$W(\{C_j' : j \in J\}, R_i) = W(\{C_j : j \in J\}, R_i) - 1$$

iff $J \subseteq P_i$ ($|J| = k$) and $1 \in J$. This happens exactly $\binom{|P_i|-1}{k-1}$ times. On the other hand,

$$W(\{C_j' : j \in J\}, R_i) \ge W(\{C_j : j \in J\}, R_i) + 1 \ \ (= 1)$$

iff $2 \in J$ ($|J| = k$) and $J \smallsetminus \{2\} \subset P_i$. This happens exactly $\binom{|P_i|}{k-1}$ times. For all other $k$-sets $J \subset D$ there is no change at all. Since $\binom{|P_i|}{k-1} - \binom{|P_i|-1}{k-1} = \binom{|P_i|-1}{k-2} \ge 0$, the total change in $W(R_i)$ is non-negative.

We can repeat this operation until all $r-1$ points of $X \cap R_i$ belong to different parts $C_j$, in which case $W(R_i) = \binom{r-1}{k}$. Thus initially $W(R_i) \le \binom{r-1}{k}$, as claimed in (3.0.2).

If initially $|P_i| < r-1$, then in the last step of the process described above $|P_i|$ increases from $r-2$ to $r-1$. In that step $W(R_i)$ increases by $\binom{|P_i|-1}{k-2} = \binom{r-3}{k-2}$, which is strictly positive, since $0 \leq k-2 \leq r-3$. This shows that $W(R_i) = \binom{r-1}{k}$ iff the $r-1$ points of $X \cap R_i$ belong to $r-1$ different parts $C_j$.

Remark 3.2.1: In case $d+1 = \frac{rk}{r-k}$ (or, equivalently, $k\binom{r}{k} = (d+1)\binom{r-1}{k}$) we can repeat the arguments of the proof of Theorem 1.0.5 and find that if $\mathcal{C} = (C_1, \ldots, C_r)$ is an $r$-partition of $X$, and each $k$ of the convex hulls $\operatorname{conv} C_j$ have a point in common, then inequality (3.0.3) holds. (This is true if we assume that no part $C_j$ visits all $d+1$ rays $R_0, \ldots, R_d$. But if one part visits all rays, some $k$ of the convex hulls of the remaining $r-1$ parts have empty intersection. This is shown in detail in the earlier part of the proof of Theorem 1.0.5.) Since $k\binom{r}{k} = (d+1)\binom{r-1}{k}$, both inequalities in (3.0.3) must hold as equalities. In view of (3.0.2), this implies that

$$W(R_i) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq r} W((C_{j_1} \ldots, C_{j_k}), R_i) = \binom{r-1}{k} \quad \text{for } i = 0, 1, \ldots, d.$$

This, in turn, implies that each ray $R_i$ carries $r-1$ points of $X$ that belong to $r-1$ different parts. One can easily deduce that for each $k$ distinct parts $C_{j_1}, \ldots, C_{j_k}$, there are exactly $k$ rays $R_i$ that intersect each of these parts, and therefore the convex hulls $\operatorname{conv} C_{j_1}, \ldots, \operatorname{conv} C_{j_k}$ intersect in a single point (because of the requirement that the points of $X$ be in "general position"). In these cases there is some hope to transform $X$ by a small perturbation into a "bad" set $X'$, such that in any $r$-partition of $X'$ there are some $k$ parts whose convex hulls have empty intersection. In the next section we shall outline how to do this in the two special cases $d = 3, r = 4, k = 2$ and $d = 5, r = 3, k = 2$.

## 4. Two special cases

To prove Theorem 1.0.4 ($T(3,4,2) = T(3,4) = 13$) we must produce a set $X''$ of 12 points in $\mathbb{R}^3$, such that every 4-partition $(C_0'', C_1'', C_2'', C_3'')$ of $X''$ contains two parts with disjoint convex hulls.

We start with the usual construction: four rays $R_0, R_1, R_2, R_3$ that emanate from the origin and together positively span $\mathbb{R}^3$, and a set $X$ that consists of three points chosen on each ray in "general position", as specified in the beginning of the proof of Theorem 1.0.5 above; $|X| = 12$.

Call a 4-partition of $X$ (or of another set $X'$) "bad" if some two of the parts have disjoint convex hulls. A partition is "good" if the convex hulls of each two parts have a point in common.

In view of Remark 3.2.1 above, with $d = 3, r = 4, k = 2$, in a "good" partition of $X$ each part misses one ray completely, and contains just one point fron each of the remaining three rays. For $i = 0, 1, 2, 3$, denote by $C_i$ the part that misses the ray $R_i$. Two distinct parts $C_i, C_j$ have exactly two common rays, $R_k$ and $R_l$ ($\{i, j, k, l\} = \{0, 1, 2, 3\}$), with $C_i$ higher than $C_j$ on $R_k$, and $C_j$ higher than $C_i$ on $R_l$. It follows easily that each part $C_i$ must contain the highest point on one ray, the lowest point on another ray and the middle point on a third ray.

As a matter of fact, there are exactly 18 good partitions $(C_0, C_1, C_2, C_3)$ of $X$, three for each of the six possible assignments of the points of $X \cap R_0$ to $C_1, C_2$ and $C_3$. In each of these partitions, the sets $\operatorname{conv} C_i$ $(i = 0, 1, 2, 3)$ are triangles. Each two of these triangles have just one point in common, and this common point lies in the relative interior of an edge in both triangles.

A suitable small perturbation of the highest point on $R_0$, followed by a suitable small perturbation of the lowest point on $R_3$, will separate at least two of the triangles $\operatorname{conv} C_i$ in each of the 18 "good" partitions. (Note that a "bad" partition remains "bad" if we apply a sufficiently small perturbation to the points of $X$.)

Let us now pass to the proof of Theorem 1.0.6 ($T(5, 3, 2) = T(5, 3) = 13$). Here we are looking for a set $Y'''$ of 12 points in $\mathbb{R}^5$, such that in every 3-partition $(C_1''', C_2''', C_3''')$ of $Y'''$, some two parts have disjoint convex hulls.

We start, as usual, with six rays $R_0, R_1, \ldots, R_5$ that emanate from the origin and together positively span $\mathbb{R}^5$, and a set $Y$ that consists of two points chosen on each ray; $|Y| = 12$.

By Remark 3.2.1, a 3-partition $(C_1''', C_2''', C_3''')$ of $Y'''$ is "good" (i.e., the convex hulls of each two parts meet) iff every two distinct parts $C_i, C_j$ have two common rays $R_k$ and $R_l$, with $C_i$ higher than $C_j$ on $R_k$ and $C_j$ higher than $C_i$ on $R_l$. We can use $R_0$ to name the three parts. (The part that contains the higher point on $R_0$ is "red", the part that contains the lower point on $R_0$ is "yellow", and the remaining part is "blue".) This leaves $120\,(= 5!)$ "good" ways to split the points on the remaining five rays between red, yellow and blue. Thus there are altogether exactly 120 "good" partitions of $Y$. In each such partition the convex hulls of each part form a tetrahedron (= 3-simplex), and each two tetrahedra touch at a single point that lies in the relative interior of an edge in both tetrahedra.

We apply to $Y$ a sequence of three small perturbations, first to the higher point on $R_0$, then to the higher point on $R_5$, and finally to the higher point on $R_2$. As a result, we strictly separate at least two of the three monochromatic tetrahedra in each of the 120 "good" partitions of $Y$. In other words, after the three perturbations we arrive at a set $Y'''$ of 12 points in $\mathbb{R}^5$, for which all 3-partitions are "bad", hence $T(5,3,2) = 13$. The full details of both constructions, along with their proofs, can be found in [3] and will appear in a forthcoming paper.

## 5. Conclusion

This paper is devoted to the proof of parts of Reay's conjecture ($T(d,r,k) = T(d,r)$ for $2 \leq k \leq \min(d, r-1)$). The meaning of this conjecture (for specified values of $d, r$ and $k$) is just this: there is a subset $X$ of $\mathbb{R}^d$, $|X| = T(d,r) - 1$ ($= (d+1)(r-1)$), such that in **every** $r$-partition of $X$ ($X = C_1 \cup \cdots \cup C_r$) there are some $k$ parts whose convex hulls have empty intersection. The conjecture is meaningful for all triples $(d, r, k)$ of values that satisfy $2 \leq k \leq d$ and $k < r$. We prove the conjecture whenever $k + 1 \leq d + 1 \leq 2k - 1$ (Theorem 1.0.3). When $2k \leq d + 1 < k(k + 1)$ we prove the conjecture for $k < r < \frac{d+1}{d+1-k}k$ (see the comment following the statement of Theorem 1.0.6), and also in the two special cases $(d, r, k) = (3, 4, 2)$ and $(d, r, k) = (5, 3, 2)$. In all cases, the set $X$ is a variation, specialization or perturbation of the same example: $d + 1$ rays that emanate from the origin and positively span $\mathbb{R}^d$, with $r - 1$ points chosen on each ray.

Unfortunately, we were unable to disprove Reay's conjecture for any admissible triple $(d, r, k)$. It is conceivable, though, that $T(d, r, k) < T(d, r)$ holds for any given values of $r$ and $k$ ($2 \leq k < r$), provided $d$ is large enough. In particular, one might try to show that $T(d, 3, 2) < T(d, 3) = 2d + 3$ from some $d$ onward (maybe already for $d \geq 6$).

Note that claims concerning $T(d, r, 2)$ ($k = 2$) are actually statements about Radon partitions. Radon partitions are much better understood and easier to handle than Tverberg $k$-partitions for $k \geq 3$.

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