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RINGS WITH LINEARLY ORDERED RIGHT ANNIHILATORS

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ABSTRACT

We introduce the class of *lineal rings,* defined by the property that the lattice of right annihilators is linearly ordered. We obtain results on the structure of these rings, their ideals, and important radicals; for instance, we show that the lower and upper nilradicals of these rings coincide. We also obtain an affirmative answer to the Köthe Conjecture for this class of rings. We study the relationships between lineal rings, distributive rings, Bézout rings, strongly prime rings, and Armendariz rings. In particular, we show that lineal rings need not be Armendariz, but they fall not far short.

1. Introduction

A good deal of the structure of a ring can often be determined from the lattice structure of its right (or left) ideals. Some preeminent cases include the theories of noetherian rings, von Neumann regular rings, local rings, Goldie dimension, 2-firs, uniserial rings, and rings whose right ideals form a distributive lattice.

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In this paper we introduce *lineal rings*, which are characterized by a natural order condition within the right ideal lattice, namely that the annihilator right ideals are linearly ordered. These rings subsume several diverse classes of rings; nevertheless, they turn out to have a rich structure theory, and they enable us to extend some useful results known to hold for narrower classes of rings.

Throughout the present paper all rings are associative, and, apart from Theorem 4.1(i), all rings contain 1. Given a ring R, the right (resp. left) annihilator of a subset $A \subseteq R$ is denoted by $\mathrm{ann}_r^R(A)$ (resp. $\mathrm{ann}_\ell^R(A)$), and the right (resp. left) annihilator of an element $a \in R$ is denoted by $\mathrm{ann}_r^R(a)$ (resp. $\mathrm{ann}_\ell^R(a)$). *Right annihilators* (or *annihilator right ideals*) of R are sets of the form $\mathrm{ann}_r^R(A)$ with $A \subseteq R$. Whenever we say two right ideals of a ring are *comparable* or *incomparable,* we will always mean with respect to inclusion.

Lineal rings are those rings in which any two right annihilators are comparable. Lineal rings are an obvious generalization of *right uniserial rings* (also called *right chain rings*, cf. [6]), that is, rings in which any two right ideals are comparable. Lineal rings also generalize the broader class of *right annelidan rings*, introduced in [22] as those rings in which any right annihilator is comparable with every right ideal.

Further examples of lineal rings are presented in Section 2, where we also establish some basic properties of these rings. In particular, we show that there is no need to distinguish between "right lineal" and "left lineal" (Theorem 2.1), and that the characteristic of a lineal ring is either 0 or a prime power (Proposition 2.7).

In Section 3 we concentrate on zero-divisors of lineal rings and the relationships between lineal rings, *right distributive rings* (i.e. rings whose right ideals lattices are distributive), and *right Bézout rings* (i.e. rings whose finitely generated right ideals are principal). We prove that in any lineal ring the right zero-divisors form a right ideal (Proposition 3.1), and that this property characterizes lineal rings among right distributive rings (Proposition 3.2), as well as among right Bézout rings (Proposition 3.3). We also discuss relationships between reduced rings, domains, right strongly prime rings, and left strongly prime rings within the class of lineal rings (Proposition 3.7 and Example 3.9).

In Section 4 we focus on nilpotent elements and nilradicals of lineal rings. We prove in Theorem 4.1 that for any lineal ring R the set of nilpotent elements of R is a (nonunital) subring of R, which is equal to the sum of its own nilpotent ideals. We also prove that in a lineal ring several standard nilradicals coincide, so that, as with a commutative ring, one can speak of *the* nilradical of a lineal ring. We conclude that the Köthe conjecture has an affirmative answer for the class of lineal rings.

In Section 5 we prove that a polynomial ring $R[x]$ is lineal if and only if the ring R is lineal and Armendariz (Theorem 5.2). This result leads one to ask whether every lineal ring is Armendariz. The answer to the question is "no" (Example 5.10), which dashes any hope that the lineal condition is inherited by polynomial rings. This counterexample stands in marked contrast to [22, Theorem 6.1], which states that every right annelidan ring is Armendariz. There are some partial positive results for lineal rings, however. If a lineal ring R contains infinitely many central elements whose differences are regular, then R is Armendariz (Corollary 5.7). In particular, for any lineal ring R of characteristic 0, the ring R modulo its torsion ideal is Armendariz (Corollary 5.9). Surprisingly, although a lineal ring need not be Armendariz, it is always "quadratically Armendariz" (Theorem 5.3).

Given a ring R, the Jacobson radical of R is denoted by $\mathrm{rad}(R)$, and the set of right (resp. left) zero-divisors of R by $\mathrm{RZD}(R)$ (resp. $\mathrm{LZD}(R)$), i.e. $\mathrm{RZD}(R) = \{a \in R\colon\ \mathrm{ann}_\ell^R(a) \neq 0\}$ and $\mathrm{LZD}(R) = \{a \in R\colon\ \mathrm{ann}_r^R(a) \neq 0\}$. An ideal $\mathfrak{p} \subsetneqq R$ is *prime* if $a, b \in R \setminus \mathfrak{p} \Rightarrow aRb \not\subseteq \mathfrak{p}$. A one-sided ideal $\mathfrak{p} \subsetneqq R$ is *completely prime* if $a, b \in R \setminus \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$. The set of positive integers is denoted by $\mathbb{N}$.

All other ring-theoretic terminology and notation will be standard, pursuant to the usage in [19] and [20].

2. Basic properties and some examples of lineal rings

Ordered by inclusion, the set of right annihilators in a ring has the structure of a complete lattice—albeit not, in general, a sublattice of the right ideal lattice under the natural embedding. With reference to the lattice, we introduce the class of lineal rings.

Definition: A ring R is called *lineal* if its right annihilator lattice is linearly ordered, that is, for any subsets $A, B \subseteq R$ we have $\mathrm{ann}_r^R(A) \subseteq \mathrm{ann}_r^R(B)$ or $\mathrm{ann}_r^R(B) \subseteq \mathrm{ann}_r^R(A)$.

There is no need to distinguish between "right lineal" and "left lineal," by the following theorem.

THEOREM 2.1: *For any ring R, the following conditions are equivalent:*

(i) *The ring R is lineal.*
(i′) *The opposite ring R^{op} is lineal.*
(ii) *For any $a, b \in R$ we have $\mathrm{ann}_r^R(a) \subseteq \mathrm{ann}_r^R(b)$ or $\mathrm{ann}_r^R(b) \subseteq \mathrm{ann}_r^R(a)$.*
(ii′) *For any $a, b \in R$ we have $\mathrm{ann}_\ell^R(a) \subseteq \mathrm{ann}_\ell^R(b)$ or $\mathrm{ann}_\ell^R(b) \subseteq \mathrm{ann}_\ell^R(a)$.*

Proof. By symmetry, it suffices to prove (ii′) $\Rightarrow$ (i). Assume (i) fails, so there exist subsets $A, B \subseteq R$ such that $\mathrm{ann}_r^R(A)$ and $\mathrm{ann}_r^R(B)$ are incomparable. Choose $x \in \mathrm{ann}_r^R(A) \setminus \mathrm{ann}_r^R(B)$ and $y \in \mathrm{ann}_r^R(B) \setminus \mathrm{ann}_r^R(A)$. Then $\mathrm{ann}_\ell^R(x)$ and $\mathrm{ann}_\ell^R(y)$ are incomparable, so (ii′) fails. ∎

The lineal condition on rings is a generalization of the annelidan condition, introduced and studied in [22]. A ring R is said to be *right annelidan* if any right annihilator in R is comparable with every right ideal of R, that is,

$$\mathrm{ann}_r^R(A) \subseteq I \quad \text{or} \quad I \subseteq \mathrm{ann}_r^R(A) \quad \text{for any subset } A \subseteq R \text{ and right ideal } I \text{ of } R.$$

The classes of lineal and annelidan rings generalize the class of uniserial rings (also known as chain rings or valuation rings), which date back to the number-theoretic origins of commutative ring theory and in recent years have found considerable application in connection with coding theory. The strength of the annelidan condition allows one to obtain certain desirable results not possible for lineal rings (notably vis-à-vis the Armendariz property). Nevertheless, the lineal condition is, from a certain perspective, more natural, being a purely lattice-theoretic condition on the right annihilators of a ring.

The following proposition might be regarded as the "magic square criterion" for a ring to be lineal. We omit the easy proof.

PROPOSITION 2.2: *A ring R is lineal if and only if every 2 by 2 matrix over R whose row products equal 0 has some diagonal product equal to 0, i.e. whenever*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$$

satisfies $ab = cd = 0$ we have $ad = 0$ or $cb = 0$.

A consequence of Proposition 2.2 is as follows.

COROLLARY 2.3: *If R is a lineal ring, then 0 and 1 are the only idempotents of R.*

Proof. Apply Proposition 2.2 with $a = d = e$ and $b = c = 1 - e$, where e is an idempotent. ∎

Next we characterize those rings R for which the factor ring $R[x]/(x^n)$ is lineal, where x is an indeterminate and (x^n) denotes the ideal of $R[x]$ generated by x^n.

PROPOSITION 2.4: *Let R be a ring, and let $n \geqslant 2$ be an integer. Then the ring $R[x]/(x^n)$ is lineal if and only if R is a domain.*

Proof. We can think of the ring $T = R[x]/(x^n)$ as the set of polynomials over R of the form $f = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, with usual addition and with multiplication subject to the rule $x^n = 0$.

Assuming R is a domain, it is easy to see that if $f = \sum_{i=0}^{n-1} a_i x^i \in T$ is nonzero and k is minimal for $a_k \neq 0$, then $\operatorname{ann}_r^T(f) = x^{n-k}T$. Hence

$$(0) = x^n T \subsetneqq x^{n-1}T \subsetneqq x^{n-2}T \subsetneqq \cdots \subsetneqq x^2T \subsetneqq xT \subsetneqq T$$

are the only annihilator right ideals of T, which shows that T is lineal.

If R is not a domain, then $ab = 0$ for some nonzero elements of R, in which case $\operatorname{ann}_r^T(x)$ and $\operatorname{ann}_r^T(a)$ are incomparable, so T is not lineal. ∎

To construct more examples of lineal rings, we extend the concept of a lineal ring to modules. A module M over a ring R is said to be a *lineal module* if the set $\{\operatorname{ann}^R(m) : m \in M\}$ is linearly ordered.

Given a ring R and an (R, R)-bimodule M, the *trivial extension of R by M*, in the literature often denoted by $R \propto M$, is the ring whose underlying additive group is $R \oplus M$ and with multiplication given by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

PROPOSITION 2.5: *Let R be a ring and M an (R, R)-bimodule. Consider the following two conditions:*

(i) *R is a domain, and the modules ${}_RM$ and M_R are both lineal.*
(ii) *The trivial extension $R \propto M$ is lineal.*

Then

$$\text{(i)} \Longrightarrow \text{(ii)}.$$

If the modules ${}_RM$ and M_R are both faithful, then

$$\text{(i)} \Longleftrightarrow \text{(ii)}.$$

Proof. Set $T = R \propto M$. If R is a domain and $(r,m) \in T$, then

$$\text{(1)} \qquad \operatorname{ann}_r^T((r,m)) = \begin{cases} 0 \oplus \operatorname{ann}_r^M(r) & \text{if } r \neq 0, \\ \operatorname{ann}_r^R(m) \oplus M & \text{if } r = 0, \end{cases}$$

where $\operatorname{ann}_r^M(r) = \{k \in M\colon\ rk = 0\}$. Furthermore, the set $\{\operatorname{ann}_r^M(r)\colon\ r \in R\}$ is linearly ordered if and only if the set $\{\operatorname{ann}_\ell^R(m)\colon\ m \in M\}$ is linearly ordered. Hence (i) implies (ii).

Now assume that the bimodule M is faithful on both sides. If R is not a domain, then $ab = 0$ for some nonzero elements $a, b \in R$, in which case $\operatorname{ann}_r^T((a,0))$ and $\operatorname{ann}_r^T(0 \oplus M)$ are incomparable, so T is not lineal. If R is a domain, and ${}_RM$ or M_R is not a lineal module, then Equation (1) implies that T is not lineal. Thus, if ${}_RM$ and M_R are both faithful, and (i) fails, then (ii) fails. ∎

By Proposition 2.5, if R is a domain and M is a uniserial, faithful (R,R)-bimodule, then the ring $R \propto M$ is lineal. Another consequence of Proposition 2.5 is the following characterization of domains via lineal rings and trivial extensions of the form $R \propto R$. Note that the characterization follows also from Proposition 2.4, as the ring $R \propto R$ is isomorphic to the factor ring $R[x]/(x^2)$.

Corollary 2.6: *A ring R is a domain if and only if the trivial extension $R \propto R$ is lineal.*

Various types of rings are excluded from the class of lineal rings. For example, by Corollary 2.3 a lineal ring cannot contain nontrivial idempotents; therefore, full matrix rings (and various subrings thereof) and Dedekind-infinite rings are never lineal, and the maximal right quotient ring of a lineal ring need not be lineal.

Below we show that there are also some restrictions on the characteristic of a lineal ring. For a ring R and any $r \in R$, we write $o(r)$ to denote the order of r in the additive group of the ring R. The *torsion ideal* $T(R)$ of a ring R consists of elements $r \in R$ whose order $o(r)$ is finite. Recall that the *characteristic* of a ring R is the order $o(1)$, provided it is finite; otherwise, the characteristic of R is defined to be 0.

Proposition 2.7: *Let R be a lineal ring.*

(i) *The torsion ideal $T(R)$ is a p-subgroup of the additive group of the ring R.*
(ii) *If R has characteristic $n \neq 0$, then n is a prime power.*
(iii) *If R has characteristic 0, then $T(R)^2 = (0)$.*

Proof. To prove (i), consider any elements $x, y \in T(R)$. Applying Proposition 2.2 to

$$\begin{bmatrix} o(x) & x \\ o(y) & y \end{bmatrix} \in \mathbb{M}_2(R),$$

we infer that $o(y)$ divides $o(x)$ or $o(x)$ divides $o(y)$. Hence only one prime p can exist for which the p-primary component of $T(R)$ is nontrivial. Since a torsion abelian group is the direct sum of its p-primary components, (i) follows.

Part (ii) is an immediate consequence of (i).

To prove (iii), assume R has characteristic 0. Given any $x, y \in T(R)$, apply Proposition 2.2 to

$$\begin{bmatrix} x & o(x) \\ o(y) & y \end{bmatrix} \in \mathbb{M}_2(R).$$

Since R has characteristic 0, we cannot have $o(y)o(x) = 0$ in R; therefore, Proposition 2.2 implies $xy = 0$. Thus, $T(R)^2 = (0)$. ∎

Clearly, a factor ring of a lineal ring need not be lineal. As the following proposition shows, however, the factor ring of a lineal ring R modulo its torsion ideal $T(R)$ is always lineal. A ring R is said to be *torsion-free* if $T(R) = (0)$.

Proposition 2.8: *If R is a lineal ring of characteristic 0, then the factor ring $R/T(R)$ is lineal and torsion-free.*

Proof. The result follows from the observation that for any ring R, if $a, b \in R$ and $\text{ann}_r^R(a) \subseteq \text{ann}_r^R(b)$, then in the factor ring $\overline{R} = R/T(R)$ we have $\text{ann}_r^{\overline{R}}(\overline{a}) \subseteq \text{ann}_r^{\overline{R}}(\overline{b})$, where $\overline{a}$ (resp. $\overline{b}$) is the image of a (resp. b) in $\overline{R}$. ∎

Needless to say, Proposition 2.8 is of interest only in the case where $(0) \subsetneqq T(R) \subsetneqq R$. The same is true of the parallel result [22, Proposition 2.4], on passage of the annelidan condition from R to $R/T(R)$. Thus, it behooves us to observe that $(0) \subsetneqq T(R) \subsetneqq R$ can actually occur when R is a lineal ring—even when R is a right annelidan ring. A lineal example can be obtained by taking the trivial extension $R = \mathbb{Z} \propto (\mathbb{Z}/q\mathbb{Z})$ where q is a prime power (see

Proposition 2.5). Unfortunately, this ring is not right or left annelidan. An example with R not only annelidan but uniserial follows.

Example 2.9: (*There exists a commutative uniserial ring R with* $(0) \subsetneqq T(R) \subsetneqq R$.) Let $p \in \mathbb{N}$ be a prime number, and let $\mathbb{Z}_{(p)}$ denote the localization of the ring of integers at the prime ideal (p). Let S be the following subring of the power series ring $\mathbb{Q}[[x]]$:

$$S = \mathbb{Z}_{(p)} + x \cdot \mathbb{Q}[[x]].$$

Then S is a commutative uniserial ring. The factor ring $R = S/xS$ is a commutative uniserial ring whose torsion ideal is neither (0) nor R.

Here $T(R)$ is isomorphic as an additive group to the Prüfer p-group. This is no accident. It is easy to see that for any right annelidan ring R satisfying $(0) \subsetneqq T(R) \subsetneqq R$, as an additive group $T(R)$ is divisible.

3. Zero-divisors and prime ideals

In general, neither left zero-divisors nor right zero-divisors of a ring form one-sided ideals of the ring. However, for lineal rings we have the following result.

Proposition 3.1: *If R is a lineal ring, then the set* $\mathrm{LZD}(R)$ *is a completely prime left ideal of R and the set* $\mathrm{RZD}(R)$ *is a completely prime right ideal of R.*

Proof. Since $\mathrm{LZD}(R) = \bigcup_{a \in R \setminus \{0\}} \mathrm{ann}_\ell^R(a)$ is the union of a chain of left ideals, it is a left ideal, which is clearly completely prime. Analogously one shows that $\mathrm{RZD}(R)$ is a completely prime right ideal of R. ∎

By the above proposition, if a ring R is lineal, then $\mathrm{RZD}(R)$ is a right ideal of R. The requirement that $\mathrm{RZD}(R)$ be a right ideal of R actually *characterizes* lineal rings among right distributive rings as well as right Bézout rings. Indeed, in the case of right distributive rings we have the following proposition, which follows directly from results of [24]. Recall that a ring R is said to be *right distributive* if the lattice of right ideals of R is distributive, that is, $(A + B) \cap C = (A \cap C) + (B \cap C)$ for any right ideals A, B, C of R. Among commutative domains, the (right) distributive rings are precisely the Prüfer rings [17]. Right distributive noncommutative rings and modules were first studied by W. Stephenson in [28] and subsequently by numerous authors (e.g. [7, 9, 11, 29, 30]).

Background information on uniform dimension can be found in [19, §6]. The following proposition follows from results of Stephenson [28] and the second author [24].

Proposition 3.2: *Let R be a right distributive ring. Then:*

(i) *The following conditions are equivalent:*
  (1) *R is lineal.*
  (2) $\mathrm{RZD}(R)$ *is a right ideal of R.*
  (3) *R has right uniform dimension* 1.

(ii) *If R is prime, then R is lineal.*

Proof. The equivalence of conditions (1) and (3) of part (i) was first proved by W. Stephenson in [28, p. 300, Corollary 4]; the equivalence of all three conditions in part (i) was proved by the second author in [24, Proposition 2.2]. For part (ii), see [24, Corollary 2.4]. ∎

Recall that a ring R is said to be a *right Bézout ring* if all finitely generated right ideals of R are principal, that is, for any $a, b \in R$ there exists $c \in R$ such that $aR + bR = cR$. For lineal right Bézout rings we have the following result.

Proposition 3.3: *Let R be a right Bézout ring. Then:*

(i) *The following conditions are equivalent:*
  (1) *R is lineal.*
  (2) $\mathrm{RZD}(R)$ *is a right ideal of R.*

(ii) *If R is lineal, then R has right uniform dimension* 1.

Proof. (i): (1) $\Rightarrow$ (2) follows from Proposition 3.1. To prove (2) $\Rightarrow$ (1), assume $\mathrm{RZD}(R)$ is a right ideal of R. To prove R is lineal we will apply Proposition 2.2. Thus, suppose we have a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$$

such that $ab = cd = 0$. We have to show that $ad = 0$ or $cb = 0$. Since R is right Bézout, $bR + dR = eR$ for some $e \in R$. Hence there exist $x, y, z, t \in R$ with

$$b = ex, \qquad d = ey, \qquad \text{and} \qquad e = bz + dt.$$

It follows that $e(1 - xz - yt) = 0$. If $e = 0$ then $b = d = 0$, and we are done by Proposition 2.2. If $e \neq 0$ then $1 - xz - yt \in \mathrm{RZD}(R)$. Since $\mathrm{RZD}(R)$ is a right ideal of R, we have $x \notin \mathrm{RZD}(R)$ or $y \notin \mathrm{RZD}(R)$. If $x \notin \mathrm{RZD}(R)$, then from $0 = ab = aex$ we get $ae = 0$, which implies $ad = aey = 0$. Similarly, $y \notin \mathrm{RZD}(R)$ implies $cb = 0$. In any case $ad = 0$ or $cb = 0$; therefore, R is lineal.

(ii): Suppose $aR \cap bR = \{0\}$ for some $a, b \in R$. Since R is right Bézout, there exists $c \in R$ with $aR \oplus bR = cR$. Choose $x, y, z, t \in R$ such that

$$a = cx, \qquad b = cy, \qquad \text{and} \qquad c = az + bt.$$

From $aR \cap bR = \{0\}$ it follows that

$$a(1 - zx) = btx = 0 \qquad \text{and} \qquad b(1 - ty) = azy = 0.$$

Since R is lineal, $\mathrm{ann}_\ell^R(x) \subseteq \mathrm{ann}_\ell^R(y)$ or $\mathrm{ann}_\ell^R(y) \subseteq \mathrm{ann}_\ell^R(x)$. Without loss of generality, we can assume $\mathrm{ann}_\ell^R(x) \subseteq \mathrm{ann}_\ell^R(y)$. Then

$$btx = 0 \qquad \Longrightarrow \qquad bty = 0 \qquad \Longrightarrow \qquad b = cy = (az + bt)y = azy = 0.$$

Therefore R has right uniform dimension 1. ∎

By Proposition 3.1, for any lineal ring R the set of left zero-divisors $\mathrm{LZD}(R)$ is a left ideal of R and the set of right zero-divisors $\mathrm{RZD}(R)$ is a right ideal of R, but as we will show in Example 3.9, neither $\mathrm{LZD}(R)$ nor $\mathrm{RZD}(R)$ need be an ideal of R. This contrasts with the situation for right annelidan rings, in which $\mathrm{LZD}(R)$ and $\mathrm{RZD}(R)$ are ideals [22, Theorem 3.1]. For a lineal ring R, it makes sense to study the largest ideals of R contained in $\mathrm{LZD}(R)$ and $\mathrm{RZD}(R)$. In Theorem 3.5 we will see that these two ideals are, respectively, right strongly prime and left strongly prime.

A ring is said to be *right* (resp. *left*) *strongly prime* if every nonzero ideal contains a finite subset whose right (resp. left) annihilator is zero. For example, any prime one-sided Goldie ring is left and right strongly prime. Strongly prime rings were first studied, under a different name (viz. *absolutely torsion-free rings*), by R. A. Rubin in [27], as a noncommutative generalization of commutative domains characterized via kernel functors on module categories. Strongly prime rings were independently discovered by D. Handelman and J. Lawrence, who developed various fundamental properties and constructed interesting examples in [13]. Strongly prime rings were pivotal in K. R. Goodearl and Handelman's classification of simple self-injective rings in [12]. Left and right strongly prime rings are also of interest in radical theory, where they are used to define the left and right Groenewald–Heyman radicals.

An ideal I of a ring R is *right* (resp. *left*) *strongly prime* if the factor ring R/I is right (resp. left) strongly prime. Completely prime ideals and maximal ideals are left and right strongly prime, and a left or right strongly prime ideal is prime.

If $\mathfrak{A}$ is a one-sided ideal of a ring R, let $\operatorname{core}(\mathfrak{A})$ denote the largest ideal of R contained in $\mathfrak{A}$. Although a number of authors have studied the relationship between the completely prime and strongly prime conditions, the following basic lemma does not seem to be on record.

Lemma 3.4: *If $\mathfrak{A}$ is a completely prime left (resp. right) ideal of a ring R, then $\operatorname{core}(\mathfrak{A})$ is a right (resp. left) strongly prime ideal of R.*

Proof. Assume that $\mathfrak{A}$ is a completely prime left ideal of R, and put

$$I = \operatorname{core}(\mathfrak{A}) = \{x \in R\colon\ xR \subseteq \mathfrak{A}\}.$$

Let $\overline{R} = R/I$, and for any $a \in R$ let $\overline{a} = a + I \in \overline{R}$. To show that $\overline{R}$ is right strongly prime, let $\overline{K}$ be any nonzero ideal of $\overline{R}$. Then $\overline{K} = K/I$ for some ideal K of R with $I \subsetneqq K$, whence $K \nsubseteq \mathfrak{A}$. Choose any $b \in K \setminus \mathfrak{A}$. We claim that $\operatorname{ann}_r^{\overline{R}}(\overline{b}) = \{\overline{0}\}$. For suppose $\overline{c} \in \operatorname{ann}_r^{\overline{R}}(\overline{b})$; then $bc \in I$, and thus $bcR \subseteq \mathfrak{A}$. Since $\mathfrak{A}$ is completely prime and $b \notin \mathfrak{A}$, we have $cR \subseteq \mathfrak{A}$. Hence $c \in I$, i.e. $\overline{c} = \overline{0}$, so I is a right strongly prime ideal. The opposite case follows by symmetry. ∎

As promised, we now show that for a lineal ring R, the largest ideals of R contained in $\operatorname{LZD}(R)$ and $\operatorname{RZD}(R)$ are, respectively, right strongly prime and left strongly prime. The last two points of the following result are parallel to Proposition 2.8, establishing that certain factor rings of a lineal ring are lineal.

Theorem 3.5: *Let R be a lineal ring.*

- (i) *The ideal* $\operatorname{core}(\operatorname{LZD}(R))$ *is right strongly prime.*
- (i′) *The ideal* $\operatorname{core}(\operatorname{RZD}(R))$ *is left strongly prime.*
- (ii) *If* $\operatorname{LZD}(R)$ *is not an ideal of* R, *then* $\operatorname{rad}(R) \subsetneqq \operatorname{LZD}(R)$.
- (ii′) *If* $\operatorname{RZD}(R)$ *is not an ideal of* R, *then* $\operatorname{rad}(R) \subsetneqq \operatorname{RZD}(R)$.
- (iii) *The factor ring* $R/\operatorname{core}(\operatorname{LZD}(R))$ *is a right strongly prime lineal ring.*
- (iii′) *The factor ring* $R/\operatorname{core}(\operatorname{RZD}(R))$ *is a left strongly prime lineal ring.*

Proof. (i): Combine Proposition 3.1 and Lemma 3.4.

(ii): A left ideal that is closed under right multiplication by units is closed under right multiplication by elements of the Jacobson radical; therefore, by Proposition 3.1, $\mathrm{LZD}(R)\cdot\mathrm{rad}(R)\subseteq\mathrm{core}(\mathrm{LZD}(R))$. From (i) we infer that $\mathrm{LZD}(R)\subseteq\mathrm{core}(\mathrm{LZD}(R))$ or $\mathrm{rad}(R)\subseteq\mathrm{core}(\mathrm{LZD}(R))$. If $\mathrm{LZD}(R)$ is not an ideal then the first case cannot occur, and (ii) follows.

(iii): By (i), we only need to show that the factor ring $\overline{R}=R/\mathrm{core}(\mathrm{LZD}(R))$ is lineal. For any $a\in R$ let $\overline{a}=a+\mathrm{core}(\mathrm{LZD}(R))\in\overline{R}$. By Theorem 2.1, to prove that $\overline{R}$ is lineal, it is enough to show for any $a,b\in R$ that $\mathrm{ann}_r^R(a)\subseteq\mathrm{ann}_r^R(b)$ implies $\mathrm{ann}_r^{\overline{R}}(\overline{a})\subseteq\mathrm{ann}_r^{\overline{R}}(\overline{b})$. Let $\overline{t}\in\mathrm{ann}_r^{\overline{R}}(\overline{a})$. Then $at\in\mathrm{core}(\mathrm{LZD}(R))$, whence $atR\subseteq\mathrm{LZD}(R)$. Since $\mathrm{ann}_r^R(a)\subseteq\mathrm{ann}_r^R(b)$, it follows that $btR\subseteq\mathrm{LZD}(R)$, and thus $bt\in\mathrm{core}(\mathrm{LZD}(R))$. Hence $\overline{t}\in\mathrm{ann}_r^{\overline{R}}(\overline{b})$, which proves that $\mathrm{ann}_r^{\overline{R}}(\overline{a})\subseteq\mathrm{ann}_r^{\overline{R}}(\overline{b})$.

(i′), (ii′), (iii′): Symmetry. ∎

Within the class of lineal rings, right and left strongly prime rings can be characterized as follows.

Proposition 3.6: *Let R be a lineal ring. Then R is right strongly prime if and only if* $\mathrm{core}(\mathrm{LZD}(R))=(0)$; *$R$ is left strongly prime if and only if* $\mathrm{core}(\mathrm{RZD}(R))=(0)$.

Proof. By symmetry, it suffices to prove the first statement. The "if" part follows from Theorem 3.5(i). To prove the "only if" part, suppose R is lineal and right strongly prime. If $0\neq a\in\mathrm{core}(\mathrm{LZD}(R))$, then the ideal RaR contains a finite subset with zero right annihilator, and since R is lineal, RaR contains a single element b with zero right annihilator. But $b\in\mathrm{LZD}(R)$, a contradiction. ∎

For any ring R, we write $\mathfrak{N}(R)$ to denote the set of nilpotent elements of R. Recall that a ring is *reduced* if it contains no nonzero nilpotent elements.

Proposition 3.7: *For a ring R, consider the following conditions:*

(i) *R is reduced.*
(ii) *R is a domain.*
(iii) *R is right strongly prime.*
(iv) *R is left strongly prime.*

If R is lineal, then (i) $\Leftrightarrow$ (ii). *If, in addition,* $\mathfrak{N}(R)\subseteq\mathrm{rad}(R)$ *or* $\mathrm{rad}(R)\neq(0)$, *then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. The implications (ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iv) are true for any ring R.

Henceforth assume R is lineal. If a and b are elements of a reduced ring, $ab = 0 \Leftrightarrow ba = 0$. Thus, to prove (i) $\Rightarrow$ (ii), assume that R is reduced and apply Proposition 2.2 to

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \in \mathbb{M}_2(R).$$

Now assume that $\mathfrak{N}(R) \subseteq \mathrm{rad}(R)$ or $\mathrm{rad}(R) \neq (0)$. If (iii) holds, then by Proposition 3.6 we have $\mathrm{core}(\mathrm{LZD}(R)) = (0)$. If $\mathrm{LZD}(R) = \mathrm{core}(\mathrm{LZD}(R))$, then R is a domain. If $\mathrm{LZD}(R) \neq \mathrm{core}(\mathrm{LZD}(R))$, then it follows from Theorem 3.5(ii) that $\mathrm{rad}(R) \subsetneqq \mathrm{LZD}(R)$, which implies $\mathrm{rad}(R) = (0)$, and consequently $\mathfrak{N}(R) \subseteq \mathrm{rad}(R) = (0)$. So in this case R is reduced. In either case we obtain (iii) $\Rightarrow$ (i). By symmetry, (iv) $\Rightarrow$ (i). ∎

The condition $\mathfrak{N}(R) \subseteq \mathrm{rad}(R)$ in Proposition 3.7 is satisfied, for instance, by any *right quasi-duo ring*, that is, a ring whose maximal right ideals are two-sided ideals (e.g. see [31, Lemma 2.3]). Since right distributive rings are right quasi-duo (see [28, p. 293, Corollary 4]), and any right or left strongly prime ring is prime, by combining Propositions 3.2(ii) and 3.7 we recover the following known characterization of right strongly prime rings among right distributive rings.

Corollary 3.8 ([25, Proposition 5]): *For any right distributive ring R, the following conditions are equivalent:*

(i) *R is right strongly prime.*
(ii) *R is left strongly prime.*
(iii) *R is a domain.*

The following example shows that conditions (iii) and (iv) of Proposition 3.7 are genuinely weaker than conditions (i) and (ii) for lineal rings, though all four conditions are equivalent for right annelidan rings, as shown in Corollary 3.10.

Example 3.9: (*A primitive, strongly prime, lineal ring need not be reduced.*) Let k be a field, let F be the free algebra $F = k\langle x, y\rangle$, and let R be the factor ring $R = F/Fx^2F$. (We will continue to write x and y for their images in R.)

In [10, Example 9.3] it is shown that the right annihilators in R are precisely $\{0\}$, xR, and R. Thus, R is lineal. Obviously R is not reduced. It is observed in

[10, Example 9.3] (and also follows from [15, Example 1]) that R is semiprime. In fact, as we will now show, R satisfies two much stronger conditions: it is strongly prime and primitive.

Note that

$$\operatorname{core}(\mathrm{RZD}(R)) = \operatorname{core}(xR) = (0),$$

since no nonzero subset of xR is closed under left multiplication by y. By Proposition 3.6, R is left strongly prime; by symmetry, R is also right strongly prime.

To see that R is left primitive, let V be a countably-infinite-dimensional k-vector space with basis $\{v_1, v_2, v_3, \ldots\}$. Define a left action of F on V by

$$xv_n = \begin{cases} 0 & \text{if } n \in \{2^{k+1} : \ k \in \mathbb{N}\} \\ v_{2^{n+1}} & \text{otherwise} \end{cases}$$

and

$$yv_n = \begin{cases} v_{n-1} & \text{if } n \geqslant 2 \\ 0 & \text{if } n = 1, \end{cases}$$

extended linearly. Since $x^2 \in \operatorname{ann}_\ell^F(V)$, this action induces a left R-module structure on V.

Assume, for a contradiction, that some element $\sum_i c_i w_i \in R$ annihilates V, where $c_i \in k \setminus \{0\}$ for each i, and each w_i is a word in x and y containing no two adjacent x's. Order the words w_i as follows: a word is larger than another if it contains a larger number of x's, and for two words with the same number of x's, each of the form $y^{m_{n+1}} x y^{m_n} x y^{m_{n-1}} x \cdots y^{m_1} x y^{m_0}$, the larger word is the one for which $(m_0, m_1, \ldots, m_{n+1})$ is smaller under the natural lexicographic order. Then for some n, the maximal w_i (with respect to this ordering) will carry v_n to a basis element with larger subscript than will any other w_i. Hence v_n is not annihilated by $\sum_i c_i w_i$, a contradiction. This proves that V is a faithful left R-module.

Fix any nonzero $v \in V$, say, $v = \sum_{i=1}^m c_i v_i$ with each $c_i \in k$ and $c_m \neq 0$. Since $c_m^{-1} y^{m-1} v = v_1$, we have $v_1 \in Rv$. For every $n \in \mathbb{N}$ there exist some $s, t \in \mathbb{N}$ such that $y^s (yx)^t v_1 = v_n$. Consequently, $Rv = V$. This proves that V is a simple left R-module.

Thus, R is left primitive. By symmetry, R is also right primitive.

It is well known that a right strongly prime ring need not be left strongly prime, and vice versa (see [13, Example 1]). However, by Corollary 3.8, for any right distributive ring R the right strongly prime condition and the left strongly prime condition are both equivalent to R being a domain. For right annelidan rings we can get even more.

Corollary 3.10: *For a right annelidan ring R, the following conditions are equivalent:*

(i) *R is reduced.*
(ii) *R is a domain.*
(iii) *R is right strongly prime.*
(iv) *R is left strongly prime.*

Proof. Since R is right annelidan, R is lineal. Hence, by Proposition 3.7, to prove the result it suffices to show that $\mathfrak{N}(R) \subseteq \mathrm{rad}(R)$. Suppose $\mathfrak{N}(R) \nsubseteq \mathrm{rad}(R)$. Then for some $t \in \mathfrak{N}(R)$ and some maximal right ideal $\mathfrak{m}$ of R we have $t \notin \mathfrak{m}$. Choose $n \in \mathbb{N}$ minimal such that $t^n = 0$ (note that $n \geqslant 2$). Since $t \in \mathrm{ann}_r^R(t^{n-1}) \setminus \mathfrak{m}$ and R is right annelidan, it follows that $\mathfrak{m} \subsetneqq \mathrm{ann}_r^R(t^{n-1})$. Hence $\mathrm{ann}_r^R(t^{n-1}) = R$ and thus $t^{n-1} = 0$, contradicting the minimal choice of n. ∎

We close this section with two results on semiprime lineal rings. Recall that a ring is said to be *right duo* if every right ideal of the ring is a two-sided ideal.

Proposition 3.11: *Let R be a lineal ring.*

(i) *R is semiprime if and only if R is prime.*
(ii) *If R is semiprime and right duo, then R is a right Ore domain.*

Proof. To prove (i), note that if I and J are ideals of a semiprime ring R, then $IJ = (0)$ implies $JI = (0)$. If, in addition, R is lineal, it follows that $I^2 = (0)$ or $J^2 = (0)$, so $I = (0)$ or $J = (0)$, proving (i).

To prove (ii), assume the ring R is lineal, right duo, and semiprime. Any semiprime right duo ring is reduced; thus, by Proposition 3.7, R is a domain, and (ii) follows. ∎

Corollary 3.12: *Suppose R is a lineal, semiprime ring that has the ascending chain condition on right annihilators of elements. If R is right distributive or right Bézout, then R is a right Ore domain.*

Proof. By Proposition 3.2(i) (in the right distributive case) or Proposition 3.3(ii) (in the right Bézout case), R has right uniform dimension 1. Hence $\mathrm{LZD}(R)$ coincides with the right singular ideal of R. Since any semiprime ring with the ascending chain condition on right annihilators of elements is right nonsingular (see [19, Corollary (7.19)]), it follows that R is a domain. Having right uniform dimension 1, R is a right Ore domain. ∎

4. Nilpotent elements, nilradicals, and the Köthe conjecture

Among the major unsolved problems in noncommutative ring theory is the Köthe conjecture, which posits that a ring with no nonzero nil ideals has no nonzero nil one-sided ideals. (See [26] for discussion and context.) We will presently show that the Köthe conjecture has an affirmative answer in the special case of lineal rings.

Given a ring R, the set of nilpotent elements of R will continue to be denoted by $\mathfrak{N}(R)$. The upper nilradical of R will be denoted by $\mathrm{Nil}^*(R)$, the Baer lower nilradical (i.e. the prime radical) of R by $\mathrm{Nil}_*(R)$, the Levitzki nil radical by $\mathrm{L}(R)$, and the sum of all nil one-sided ideals of R by $\mathrm{A}(R)$. For any ring R we have $\mathrm{Nil}_*(R) \subseteq \mathrm{L}(R) \subseteq \mathrm{Nil}^*(R) \subseteq \mathrm{A}(R)$; furthermore, the set $\mathrm{A}(R)$ is an ideal of R (see [26, §3]). The Köthe conjecture is equivalent to the statement that $\mathrm{A}(R)$ is always nil, i.e. $\mathrm{Nil}^*(R) = \mathrm{A}(R)$ for every ring R.

Given a ring R, with or without unity, let $\mathrm{N}_1(R)$ denote the sum of all nilpotent ideals of R. Recall that $\mathrm{N}_1(R)$ is the first in a transfinite ascending chain of ideals whose union equals the prime radical, and in general it may require an arbitrarily large ordinal before the chain stabilizes (see [1, §IV]). However, as we now show, if R is lineal, then the chain stabilizes at the first step on $\mathrm{A}(R)$, proving the Köthe conjecture for such rings.

Theorem 4.1: *Let R be a lineal ring.*

(i) *The set $\mathfrak{N}(R)$ is a nonunital subring of R, and $\mathfrak{N}(R) = \mathrm{N}_1(\mathfrak{N}(R))$.*
(ii) *Any finitely generated nil one-sided ideal of R is nilpotent.*
(iii) *We have $\mathrm{N}_1(R) = \mathrm{Nil}_*(R) = \mathrm{L}(R) = \mathrm{Nil}^*(R) = \mathrm{A}(R)$.*

Thus, the Köthe conjecture has an affirmative answer for the class of lineal rings.

Proof. Observe first that for any $a, b, c \in R$ and $n \in \mathbb{N}$,

$$(ab = 0 \quad \text{and} \quad acb \neq 0) \Rightarrow ac^n b \neq 0. \tag{2}$$

Indeed, if there is a minimal $n \in \mathbb{N}$ for which (2) fails, then applying Proposition 2.2 with

$$\begin{bmatrix} a & b \\ ac^{n-1} & cb \end{bmatrix}$$

yields a contradiction.

Clearly (2) implies that if $t \in R$ is nilpotent and $a, b \in R$ satisfy $ab = 0$, then $atb = 0$. From this it follows immediately that for any integer $n \geqslant 2$ and for any $a_1, a_2, \ldots, a_n \in R$ and $t_1, t_2, \ldots, t_{n-1} \in \mathfrak{N}(R) \cup \{1\}$, we have

$$a_1 a_2 \cdots a_n = 0 \Rightarrow a_1 t_1 a_2 t_2 \cdots a_{t-1} t_{n-1} a_n = 0. \tag{3}$$

Now we are ready to prove (i). Let $a, b \in \mathfrak{N}(R)$, with $a^n = b^n = 0$. As a consequence of (3) we obtain $(a + b)^{2n-1} = 0 = (ab)^n$, and thus $\mathfrak{N}(R)$ is a nonunital subring of R. Furthermore, if I is the ideal of $\mathfrak{N}(R)$ generated by a, then $I^n = 0$. This proves (i).

To prove (ii), let J be a nil right or left ideal of R generated by elements $a_1, a_2, \ldots, a_m \in R$. Each a_i lies in $\mathrm{A}(R)$, and by (i) the ideal of $\mathrm{A}(R)$ generated by a_i is nilpotent. By the Andrunakievich Lemma [3, p. 186, Лемма 4], the ideal of R generated by a_i is nilpotent. Thus J is contained in a finite sum of nilpotent ideals, so J is nilpotent.

The equation in (iii), and a fortiori the conclusion of the theorem, follows from (ii). ∎

The "finitely generated" hypothesis is needed in Theorem 4.1(ii); indeed, there are examples of commutative uniserial rings containing ideals that are nil but not nilpotent. For right annelidan rings satisfying certain mild chain conditions, Theorem 4.1(i) can be strengthened to the conclusion that $\mathfrak{N}(R)$ is actually an ideal: see [22, Theorem 3.5]. Nevertheless, $\mathfrak{N}(R)$ need not even be a one-sided ideal for R right annelidan, or even uniserial. Examples can be obtained as prime factor rings of the exceptional rank one chain domains constructed by H. H. Brungs and N. I. Dubrovin in [8].

5. Lineal rings and Armendariz rings

A ring R with the property that

$$(4) \qquad \left(\sum_i a_i x^i\right)\left(\sum_j b_j x^j\right) = 0 \text{ in } R[x] \qquad \Longrightarrow \qquad a_i b_j = 0 \quad \forall\, i, j$$

is said to be *Armendariz*. The terminology was chosen to honor E. P. Armendariz, who noted in [4] that all reduced rings satisfy condition (4). Various interesting properties and constructions of Armendariz rings can be found in [2, 10, 14, 16, 18, 21, 23]. In particular, Y. Hirano's result [14, Proposition 3.1] affords an important reformulation of the Armendariz condition. Given a ring R, put

$$\mathcal{L}_{\mathrm{ann}_r}(R) = \{\mathrm{ann}_r^R(S)\colon \ S \text{ is a nonempty subset of } R\}.$$

Under the inclusion ordering, $\mathcal{L}_{\mathrm{ann}_r}(R)$ is a lattice, with join and meet given by $\mathfrak{A} \wedge \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}$ and $\mathfrak{A} \vee \mathfrak{B} = \mathrm{ann}_r^R(\mathrm{ann}_\ell^R(\mathfrak{A} \cup \mathfrak{B}))$. For R to be lineal means, of course, that this lattice is a chain. For any ring R, there is a natural lattice monomorphism

$$\mathcal{L}_{\mathrm{ann}_r}(R) \to \mathcal{L}_{\mathrm{ann}_r}(R[x])$$

given by $\mathrm{ann}_r^R(S) \mapsto \mathrm{ann}_r^{R[x]}(S[x])$ for every nonempty subset $S \subseteq R$. The following proposition is a special case of [23, Theorem 3.4].

Proposition 5.1: *A ring R is Armendariz if and only if the natural map $\mathcal{L}_{\mathrm{ann}_r}(R) \to \mathcal{L}_{\mathrm{ann}_r}(R[x])$ is a lattice isomorphism.*

From this it follows immediately that if a ring R is lineal and Armendariz, then the polynomial ring $R[x]$ is lineal. Below we prove the converse, obtaining a characterization of polynomial rings that are lineal, even for polynomials in any set of commuting indeterminates.

Theorem 5.2: *Let R be a ring and let X be any nonempty set of commuting indeterminates over R. Then the following conditions are equivalent:*

(i) *$R[X]$ is lineal.*
(ii) *$R[X]$ is lineal and Armendariz.*
(iii) *R is lineal and Armendariz.*

Proof. (iii) ⇒ (ii): Assume R is lineal and Armendariz. Then $R[X]$ is Armendariz by [2, Corollary 3], so we need only show that $R[X]$ is lineal. Note that for any $x \in X$, the subring $R[x]$ of $R[X]$ is Armendariz, and by Proposition 5.1, $R[x]$ is lineal as well. By induction, it follows that for any finite subset $S \subseteq X$, the ring $R[S]$ is lineal and Armendariz. It is clear from Proposition 2.2 that a direct limit of a directed system of lineal rings is lineal; therefore, $R[X]$ is lineal.

(i) $\Rightarrow$ (iii): Assume $R[X]$ is lineal. Clearly, so is R. Suppose R is not Armendariz. Then there exist polynomials

$$f(x) = a_0 + a_1x + \cdots + a_kx^k, \qquad g(x) = b_0 + b_1x + \cdots + b_mx^m$$

in $R[x]$ such that $f(x)g(x) = 0$, but not all a_ib_j equal 0. Choose the two polynomials so as to minimize $k + m$. Being a subring of $R[X]$, the ring $R[x]$ is lineal. Applying Proposition 2.2 to

$$\begin{bmatrix} a_0 & b_0 \\ f(x) & g(x) \end{bmatrix} \in \mathbb{M}_2(R[x]),$$

we infer that

$$(5) \qquad a_0g(x) = 0, \qquad \text{whence} \qquad \left(\sum_{i=1}^{k} a_ix^{i-1}\right) g(x) = 0,$$

or

$$(6) \qquad f(x)b_0 = 0, \qquad \text{whence} \qquad f(x) \left(\sum_{i=1}^{m-1} b_ix^{i-1}\right) = 0.$$

By the minimal choice of $f(x)$ and $g(x)$, either (5) or (6) implies $a_ib_j = 0$ for all i and j, a contradiction. ∎

Theorem 5.2 raises the obvious question whether every lineal ring is Armendariz. The answer, as we will see in Example 5.10, is “no”; consequently, by Theorem 5.2, the polynomial ring over a lineal ring need not be lineal. In this context, it is noteworthy that every one-sided annelidan ring is Armendariz [22, Theorem 6.1], and therefore a polynomial ring over a one-sided annelidan ring must be lineal. This gives some indication how much stronger than *lineal* the annelidan condition is. A similar comparison can be drawn between Proposition 3.7 and Corollary 3.10.

Before constructing a counterexample to show that lineal rings need not be Armendariz, let us prove some positive results.

THEOREM 5.3: *Every lineal ring R is "quadratically Armendariz," that is to say, if in $R[x]$ we have*

$$(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = 0, \tag{7}$$

then $a_ib_j = 0$ for all $i, j \in \{0, 1, 2\}$. In particular, every lineal ring R is "linearly Armendariz" (in the terminology of [10]*).*

Proof. Since the ring R is lineal, the right annihilators $\mathrm{ann}_r^R(a_0)$, $\mathrm{ann}_r^R(a_1)$, and $\mathrm{ann}_r^R(a_2)$ are linearly ordered. If $\mathrm{ann}_r^R(a_0) \subseteq \mathrm{ann}_r^R(a_1)$ and $\mathrm{ann}_r^R(a_0) \subseteq \mathrm{ann}_r^R(a_2)$, then (7) clearly implies $a_ib_j = 0$ for all $i, j \in \{0, 1, 2\}$. We are also done in the case where $\mathrm{ann}_r^R(a_2) \subseteq \mathrm{ann}_r^R(a_0)$ and $\mathrm{ann}_r^R(a_2) \subseteq \mathrm{ann}_r^R(a_1)$. This leaves only the case where $\mathrm{ann}_r^R(a_1) \subseteq \mathrm{ann}_r^R(a_0)$ and $\mathrm{ann}_r^R(a_1) \subseteq \mathrm{ann}_r^R(a_2)$. Since Equation (7) is equivalent to the equation

$$(a_2 + a_1x + a_0x^2)(b_2 + b_1x + b_0x^2) = 0,$$

we can assume without loss of generality that

$$\mathrm{ann}_r^R(a_1) \subseteq \mathrm{ann}_r^R(a_0) \subseteq \mathrm{ann}_r^R(a_2). \tag{8}$$

In particular, since $a_0b_0 = 0$ by (7), it follows from (8) that

$$a_2b_0 = 0. \tag{9}$$

Equation (7) implies $a_0b_0 = 0$ and $(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 0$. Applying Proposition 2.2 to

$$\begin{bmatrix} a_0 & b_0 \\ a_0 + a_1 + a_2 & b_0 + b_1 + b_2 \end{bmatrix} \in \mathbb{M}_2(R),$$

we infer that $a_0(b_1+b_2)=0$ or $(a_1+a_2)b_0=0$. Now

$$\begin{aligned}
&a_0(b_1+b_2)=0 \quad \text{or} \quad (a_1+a_2)b_0=0 \\
&\quad\Rightarrow\quad a_0(b_1+b_2)=0 \quad \text{or} \quad a_1b_0=0 && \text{(by (9))}\\
&\quad\Rightarrow\quad a_2(b_1+b_2)=0 \quad \text{or} \quad a_1b_0=0 && \text{(by (8))}\\
&\quad\Rightarrow\quad a_2b_1=0 \quad \text{or} \quad a_0b_1=0 && \text{(by (7))}\\
&\quad\Rightarrow\quad a_2b_1=0 && \text{(by (8))}\\
&\quad\Rightarrow\quad a_1b_2=0 && \text{(by (7))}\\
&\quad\Rightarrow\quad a_0b_2=0 && \text{(by (8))}\\
&\quad\Rightarrow\quad a_1b_1=0 && \text{(by (7) and (9))}\\
&\quad\Rightarrow\quad a_0b_1=0 && \text{(by (8))}\\
&\quad\Rightarrow\quad a_1b_0=0 && \text{(by (7)).}
\end{aligned}$$

Hence $a_ib_j=0$ for all $i,j\in\{0,1,2\}$. ∎

For a positive integer n, we call a ring R an *n-Armendariz ring* if the Armendariz condition (4) is satisfied for polynomials of degree at most n. Thus, in this terminology, all lineal rings are 2-Armendariz by Theorem 5.3. As we will see in Example 5.10, a lineal ring need not be "cubically Armendariz" and so in general lineal rings are not n-Armendariz for $n\geqslant 3$. Nevertheless, for any n, by the following theorem, a lineal ring R is n-Armendariz provided R contains $2n+1$ central elements whose differences are regular elements of R.

Theorem 5.4: *Let R be a lineal ring and n a positive integer. If R is an algebra over a commutative ring D, and there exist elements $d_1,d_2,\dots,d_{2n+1}\in D$ such that $\operatorname{ann}^R(d_i-d_j)=\{0\}$ whenever $1\leqslant i<j\leqslant 2n+1$, then R is n-Armendariz.*

The proof of Theorem 5.4 is based on the following observation, which extends to modules a well-known result in linear algebra on homogeneous Vandermonde systems of linear equations.

Lemma 5.5: *Let R be a ring, M a right R-module, and n a positive integer. Assume the elements $r_0,r_1,\dots,r_n\in R$ satisfy the following two conditions:*

(i) $r_ir_j=r_jr_i$ *whenever* $2\leqslant i\leqslant n$ *and* $0\leqslant j\leqslant n$,
(ii) $\operatorname{ann}_\ell^M(r_i-r_j)=\{0\}$ *whenever* $0\leqslant i<j\leqslant n$.

Then for any $m_0, m_1, \ldots, m_n \in M$, *if*

$$m_0 + m_1 r_i + m_2 r_i^2 + \cdots + m_n r_i^n = 0 \qquad \text{for every } i \in \{0, 1, \ldots, n\}, \tag{10}$$

then $m_0 = m_1 = \cdots = m_n = 0$.

Proof. We induct on n. The case $n = 1$ is obvious. Assume that $n \geqslant 2$ and the result is true for $n-1$. Suppose $m_0, m_1, \ldots, m_n \in M$ satisfy (10). Then for every $i \in \{0, 1, \ldots, n-1\}$ we have

$$m_1(r_i - r_n) + m_2(r_i^2 - r_n^2) + \cdots + m_n(r_i^n - r_n^n) = 0. \tag{11}$$

Since $n \geqslant 2$, condition (i) implies that for every $k \in \mathbb{N}$ we have

$$r_i^k - r_n^k = \left(\sum_{j=0}^{k-1} r_i^j r_n^{k-1-j} \right) (r_i - r_n).$$

Hence for any $i \in \{0, 1, \ldots, n-1\}$ Equation (11) can be rewritten as

$$\left(\sum_{k=1}^{n} m_k \left(\sum_{j=0}^{k-1} r_i^j r_n^{k-1-j} \right) \right) (r_i - r_n) = 0,$$

so condition (ii) implies that for any $i \in \{0, 1, \ldots, n-1\}$ we have

$$\sum_{k=1}^{n} m_k \left(\sum_{j=0}^{k-1} r_i^j r_n^{k-1-j} \right) = 0. \tag{12}$$

For each $i \in \{0, 1, \ldots, n-1\}$, put

$$v_i = m_{i+1} + m_{i+2} r_n + m_{i+3} r_n^2 + \cdots + m_n r_n^{n-i-1};$$

then Equation (12) becomes

$$v_0 + v_1 r_i + v_2 r_i^2 + v_3 r_i^3 + \cdots + v_{n-1} r_i^{n-1} = 0.$$

By inductive hypothesis, $v_{n-1} = v_{n-2} = \cdots = v_1 = v_0 = 0$, which implies that $m_n = m_{n-1} = \cdots = m_2 = m_1 = 0$. From Equation (10) we obtain $m_0 = 0$ as well, which completes the proof. ∎

Now we are in a position to prove Theorem 5.4.

Proof of Theorem 5.4. Suppose R is not n-Armendariz. Then there exist

$$f(x) = a_0 + a_1 x + \cdots + a_k x^k \in R[x], \qquad g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$$

where $k \leqslant n$ and $m \leqslant n$ and $f(x)g(x) = 0$, but not all $a_i b_j$ equal 0. Choose $f(x)$ and $g(x)$ so as to minimize $k + m$.

For every $i \in \{1, 2, \ldots, 2n+1\}$ we have $f(d_i)g(d_i) = 0$; thus, we can apply Proposition 2.2 to

$$\begin{bmatrix} a_0 & b_0 \\ f(d_i) & g(d_i) \end{bmatrix},$$

obtaining $2n+1$ pairs of alternatives:

$$a_0 g(d_i) = 0 \qquad \text{or} \qquad f(d_i)b_0 = 0.$$

There exists a subset $I \subseteq \{1, 2, \ldots, 2n+1\}$ such that

$$|I| = m+1 \quad \text{and} \quad a_0 g(d_i) = 0 \quad \text{for every } i \in I,$$

or

$$|I| = k+1 \quad \text{and} \quad f(d_i)b_0 = 0 \quad \text{for every } i \in I.$$

Now Lemma 5.5 implies that $a_0 g(x) = 0$ or $f(x)b_0 = 0$. But then $f(x)$ can be replaced by $\sum_{i=1}^{k} a_i x^{i-1}$ or $g(x)$ can be replaced by $\sum_{j=1}^{m} b_j x^{j-1}$, contradicting the minimal choice of $f(x)$ and $g(x)$. ∎

Corollary 5.6: *Let R be a lineal ring and n a positive integer. If R contains central elements $r_1, r_2, \ldots, r_{2n+1}$ such that $\mathrm{ann}_\ell^R(r_i - r_j) = \{0\}$ whenever $1 \leqslant i < j \leqslant 2n+1$, then R is n-Armendariz.*

Proof. Let D be the subring of R generated by $r_1, r_2, \ldots, r_{2n+1}$, and apply Theorem 5.4. ∎

Corollary 5.7: *Let R be a lineal ring in which there exist infinitely many central elements $r_1, r_2, r_3, \ldots$ such that $\mathrm{ann}_\ell^R(r_i - r_j) = \{0\}$ whenever $i \neq j$. Then R is Armendariz.*

As a consequence of Corollary 5.7, we obtain the full Armendariz condition for some broad classes of lineal rings.

Corollary 5.8: *Let R be a lineal ring that is an algebra over an infinite field. Then R is Armendariz.*

Corollary 5.9: *If R is a lineal ring whose additive group is torsion-free, then R is Armendariz. In particular, if R is a lineal ring of characteristic 0, and $T(R)$ is the torsion ideal of R, then the torsion-free, lineal factor ring $R/T(R)$ is Armendariz.*

Proof. Apply Proposition 2.8 and Corollary 5.7. ∎

We conclude this paper with an example of a lineal ring that is not Armendariz. By Theorem 5.2, this shows that the lineal condition is not inherited by polynomial rings.

Example 5.10: (*A lineal ring need not be Armendariz.*) Define the free algebra $R_0 = \mathbb{F}_2\langle a, b, c, d\rangle$, and let I be the ideal of R_0 generated by the following set of elements:

$$\begin{array}{llll} a^2, & ab+ba, & ab+ac, & ac+b^2+ca, \\ ad, & bc+cb+da, & bd+c^2+db, & bd+cd, \\ cd+dc, & d^2, & xyz \text{ for all } x,y,z \in \{a,b,c,d\}. & \end{array} \tag{13}$$

Let $R = R_0/I$. Abusing notation a bit, we will write a, b, c, d for the images of these elements in R.

With respect to the lexicographic order on length 2 words in $\{a, b, c, d\}$, the last summand of every element in (13) is a linear combination of smaller words modulo I. As a straightforward consequence of the Diamond Lemma [5, Theorem 1.2], every element $s \in R$ can be written uniquely as

$$s = k_0 + k_1a + k_2b + k_3c + k_4d + k_5ab + k_6b^2 + k_7bc + k_8bd + k_9cb + k_{10}c^2 \tag{14}$$

where $k_0, k_1, \dots, k_{10} \in \mathbb{F}_2$.

It is evident that in $R[x]$ we have

$$\left(a + bx + cx^2 + dx^3\right)^2 = 0,$$

and thus R is not an Armendariz ring (e.g. $ab \neq 0$). Yet R is lineal, as we will now show.

Note that R is a finite local ring whose maximal ideal $\mathfrak{m} = aR + bR + cR + dR$ satisfies $\mathfrak{m}^3 = (0)$. If $s \in R\backslash\mathfrak{m}$ then $\operatorname{ann}_r^R(s) = \{0\}$. If $s = 0$ then $\operatorname{ann}_r^R(s) = R$. If $s \in \mathfrak{m}^2\backslash\{0\}$ then $\operatorname{ann}_r^R(s) = \mathfrak{m}$. In each of these cases, $\operatorname{ann}_r^R(s)$ is comparable with every right ideal of R. We are left with the case where $s \in \mathfrak{m} \setminus \mathfrak{m}^2$ (i.e. in Equation (14) we have $k_0 = 0$ and $k_i = 1$ for at least one $i \in \{1, 2, 3, 4\}$). Then $\mathfrak{m}^2 \subseteq \operatorname{ann}_r^R(s) \subseteq \mathfrak{m}$, and thus $\operatorname{ann}_r^R(s) = \operatorname{ann}_r^R(k_1a + k_2b + k_3c + k_4d) + \mathfrak{m}^2$. We assume without loss of generality that $s = k_1a + k_2b + k_3c + k_4d$, and direct calculation yields

$$\operatorname{ann}_r^R(s) = \begin{cases} \mathbb{F}_2 a + \mathbb{F}_2(b+c) + \mathbb{F}_2 d + \mathfrak{m}^2 & \text{if } s \in \{a\} \\ \mathbb{F}_2(a+b+c) + \mathbb{F}_2 d + \mathfrak{m}^2 & \text{if } s \in \{b+c+d,\ a+b+c+d\} \\ \mathbb{F}_2 d + \mathfrak{m}^2 & \text{if } s \in \{d,\ a+d,\ b+c,\ a+b+c\} \\ \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

Since $\mathfrak{m}^2 \subseteq \mathbb{F}_2 d + \mathfrak{m}^2 \subseteq \mathbb{F}_2(a+b+c) + \mathbb{F}_2 d + \mathfrak{m}^2 \subseteq \mathbb{F}_2 a + \mathbb{F}_2(b+c) + \mathbb{F}_2 d + \mathfrak{m}^2$, we conclude that R is lineal.

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