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ON THE CONGRUENCE SUBGROUP PROBLEM FOR BRANCH GROUPS

BY

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ABSTRACT

We answer a question of Bartholdi, Siegenthaler and Zalesskii, showing that the congruence subgroup problem for branch groups is independent of the branch action on a tree. We prove that the congruence topology of a branch group is determined by the group, specifically, by its structure graph, an object first introduced by Wilson. We also give a more natural definition of this graph.

1. Introduction

Groups acting on rooted trees have been the subject of intense study over the past few decades after the appearance in the 1980s of examples with exotic properties (e.g., finitely generated infinite torsion groups, groups of intermediate word growth, amenable but not elementary amenable groups, etc.). Several attempts were made at the time to round up these examples into one class of groups. One of these led to the definition of branch groups ([1]), which also arise in the classification of just infinite groups ([13]).

For a sequence $(m_n)_{n\geq 0}$ of integers $m_n \geq 2$, the **rooted tree of type** (m_n) is a tree T with a distinguished vertex v_0, called the **root**, of valency m_0 and

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such that every vertex at distance $n \geq 1$ from v_0 has valency $m_n + 1$ (where the distance of a vertex from v_0 is the number of edges in the unique path from that vertex to v_0). The set of all vertices at distance n from v_0 is the nth layer of T, denoted by V_n. We picture T with v_0 at the top and with m_n edges descending from each vertex in V_n, so we call the vertices below a given v the **descendants** of v. Each vertex $v \in V_r$ is the root of a subtree T_v of type $(m_n)_{n \geq r}$.

Let G be a group acting faithfully on T fixing v_0. For each vertex v, the **rigid stabilizer of** v is the subgroup $\mathrm{rist}_G(v)$ of elements of G which fix every vertex outside T_v. For each $n \geq 0$, the direct product $\mathrm{rist}_G(n) = \langle \mathrm{rist}_G(v) \mid v \in V_n \rangle$ is the **rigid stabilizer of the nth layer**. We call the faithful action of G on T a **branch action** if the following holds for all $n \geq 0$:

(i) G acts transitively on V_n;
(ii) $\mathrm{rist}_G(n)$ has finite index in G.

We say that G is a **branch group** if there exists a branch action of G on some tree T.

Since branch groups have such specific actions on rooted trees, it is natural to wonder what the action tells us about the subgroup structure of the group. Consider, for each $n \geq 0$, the kernel $\mathrm{Stab}_G(n)$ of the action of G on V_n; can we "detect" every finite index subgroup of G (or, equivalently, every finite quotient) by looking at the finite quotients $G/\mathrm{Stab}_G(n)$? In other words, does every finite index subgroup of G contain some $\mathrm{Stab}_G(n)$? We can rephrase this question in terms of profinite completions. Taking the subgroups $\{\mathrm{Stab}_G(n) \mid n \geq 0\}$ as a neighbourhood basis for the identity gives a topology on G—the congruence topology—and the completion $\overline{G}$ of G with respect to this topology is a profinite group called the **congruence completion** of G. As G acts faithfully on T we have $\bigcap_n \mathrm{Stab}_G(n) = 1$, so G embeds in $\overline{G}$. A fortiori, G is residually finite, so it also embeds in its profinite completion $\widehat{G}$ which maps onto $\overline{G}$. Asking whether each finite index subgroup of G contains some stabilizer $\mathrm{Stab}_G(n)$ is tantamount to asking whether the map $\widehat{G} \to \overline{G}$ is injective. The congruence subgroup problem asks us to compute the congruence kernel C of this map, which measures the deviation from a positive answer.

Since a branch group G has another obvious family of finite index subgroups, namely $\{\mathrm{rist}_G(n) \mid n \geq 0\}$, we may ask the same question for this family. Let $\widetilde{G}$ denote the **branch completion** of G, that is, the completion of G with respect to the branch topology, which is generated by taking $\{\mathrm{rist}_G(n) \mid n \geq 0\}$

as a neighbourhood basis of the identity. Then, as above, there is a surjective homomorphism $\widehat{G} \to \widetilde{G}$ and we are asked to determine the **branch kernel** B of this map. Note that the branch topology is stronger than the congruence topology so that we may also ask about the kernel R of the map $\widetilde{G} \to \overline{G}$, the **rigid kernel**.

The term "congruence" is used by analogy with the classical congruence subgroup problem for $\mathrm{SL}_n(\mathbb{Z})$ (solved in [3, 9]), from which these questions take inspiration. There are now several generalizations of this problem: for instance, a now classical generalization in the context of algebraic groups (see [10] and references therein), and a more recent one in the context of automorphisms of free groups F_n, with the kernels of the action on finite quotients of F_n playing the role of our subgroups $\mathrm{Stab}_G(n)$ (see [4]).

The problem of determining the congruence, branch and rigid kernels for a branch group G was first posed in [2], where the authors also ask a "preliminary question of great importance":

Question: Do any of the kernels depend on the branch action of G?

In other words, is the nature of these kernels a property of the branch action or of the group?

We provide a full answer to the above question by proving the following:

THEOREM 1: *Let G have two branch actions on trees. Then the congruence kernels with respect to these actions coincide.*

THEOREM 2: *Let G have two branch actions on trees. Then the branch kernels with respect to these actions coincide.*

The above immediately imply that the rigid kernels of a branch group with respect to any two branch actions are naturally isomorphic.

We will deduce these theorems from a more powerful observation, namely that the congruence and branch topologies of a branch group G can be defined in purely group-theoretic terms, with no reference to a branch action, using the structure graph $\mathcal{B}$ of G. This graph depends only on the subgroup structure of G and is related to every tree on which G acts as a branch group. As we shall see in Section 3 (Proposition 3.1), if G acts on T as a branch group then T embeds G-equivariantly in $\mathcal{B}$, where the action of G on $\mathcal{B}$ is that induced by conjugation on subgroups. Further, the image of T is coinitial in $\mathcal{B}$ (Proposition 3.2). Once we have analogues of $\mathrm{Stab}_G(n)$ and $\mathrm{rist}_G(n)$ for the action of G on $\mathcal{B}$, the above-mentioned results are the key to showing that all congruence and branch topologies induced by a branch action coincide; they all agree with the topologies with respect to $\mathcal{B}$.

In Section 2 we define the structure graph and the larger structure lattice of a branch group. These very useful objects were first introduced in [12] and [13] and used to analyse just infinite groups. They were also used in [8] to characterize branch groups in purely group-theoretic terms. In those settings, they are defined as quotients of the lattice of subnormal subgroups of a branch group. Here we give a more direct description by examining the subgroups with finitely many conjugates.

2. The structure lattice and structure graph

Notation: We write $H \leq_{\mathrm{f}} G$ and $H \trianglelefteq_{\mathrm{f}} G$ to indicate, respectively, that H is a finite index subgroup of G and that H is a normal finite index subgroup of G. We also use the standard notation $\mathrm{N_G}(H)$ (resp. $\mathrm{C_G}(H)$) for the normalizer (resp. centralizer) of a subgroup H in G. Furthermore, H^G will denote the subgroup generated by all conjugates of H by G. Throughout the rest of the paper, G will denote a branch group.

Subgroups of branch groups are subject to several constraints. The proof of [7, Lemma 2] shows that branch groups have no non-trivial virtually abelian normal subgroups and the following is obtained in [6, Theorem 4]:

Theorem 2.1: *Suppose that G is a branch group acting on a tree T and let $K \trianglelefteq G$ with $K \neq 1$. Then K contains the derived subgroup $\mathrm{rist}_G(n)'$ of $\mathrm{rist}_G(n)$ for some integer n.*

Thus all branch groups are just non-(virtually abelian); that is, they are not virtually abelian but all of their proper quotients are.

Let $L(G)$ be the collection of all subgroups of G which have finitely many conjugates (in other words, whose normalizer has finite index). If $H, K \in L(G)$, then clearly $H \cap K, \langle H, K\rangle \in L(G)$. Thus $L(G)$ forms a lattice with respect to subgroup inclusion, with $H \cap K$ and $\langle H, K\rangle$ respectively the meet and join of two elements H, K. Lemma 2.2 of [5] shows that $L(G)$ contains no non-trivial virtually soluble subgroups. We will use this without further comment in the remainder of the paper.

This allows us to prove the following, which is a generalization of [8, Theorem 8.3.1] and [13, Lemma 4.3].

Proposition 2.2: *Let $H, K \in L(G)$ with $K \trianglelefteq H$ and H/K virtually nilpotent. Then* $\mathrm{C_G}(K) = \mathrm{C_G}(H)$.

Proof. First we claim that if A is a subgroup with finitely many conjugates in a group Γ and A is virtually nilpotent then so is A^Γ. Let N be the normal core of $\mathrm{N}_\Gamma(A)$, so that $A_0 := A \cap N$ is virtually nilpotent and normal in N. By Fitting's theorem ([11, 5.2.8]), A_0 has a unique maximal nilpotent normal subgroup, B say, which is normal in N. The finitely many Γ-conjugates of B are also nilpotent and normal in N; thus B^Γ is nilpotent, again by Fitting's theorem. It remains to show that B^Γ has finite index in A^Γ. Since A/B is finite so is the quotient $(AB^\Gamma)/B^\Gamma$ and this has finitely many conjugates in Γ/B^Γ because A has finitely many in Γ. Therefore the quotient $(A^\Gamma B^\Gamma)/B^\Gamma \cong A^\Gamma/B^\Gamma$ is finite by Dicman's lemma (see [11, 14.5.7]) and our claim is proved.

Suppose that $H, K \in L(G)$ with H/K virtually nilpotent and write $C := \mathrm{C_G}(K) \in L(G)$. We will prove the proposition for $H_1 := H^C$, K and deduce the result for H, K from this. To see that H_1/K is virtually nilpotent, note that for each $c \in C$ the conjugate $(H/K)^c = H^c/K$ is isomorphic to H/K. There are finitely many of these C-conjugates, as $H \in L(G)$, so it follows from the claim that H_1/K is virtually nilpotent. Now, $C \cap K \in L(G)$ is abelian, hence trivial, and we have

$$C \cap H_1 = (C \cap H_1)/(C \cap K) \cong K(C \cap H_1)/K \leq H_1/K.$$

Thus $C \cap H_1 \in L(G)$ is virtually nilpotent and therefore trivial. Note that C is normalized by H_1, since $K \trianglelefteq H_1$, whence $[C, H_1] \leq C$. As H_1 is also normalized by C we have $[C, H_1] \leq C \cap H_1 = 1$ and therefore $C \leq \mathrm{C_G}(H_1)$. The proof is complete as $K \leq H \leq H_1$ implies that $\mathrm{C_G}(H_1) \leq \mathrm{C_G}(H) \leq C$. ∎

Notation: For $H, K \in L(G)$, we write $K \leq_{\mathrm{va}} H$ (respectively, $K \trianglelefteq_{\mathrm{va}} H$) if $K \leq H$ (resp. $K \trianglelefteq H$) and K contains the derived group of a finite index subgroup of H. Note that if $K \trianglelefteq_{\mathrm{va}} H$ then H/K is virtually abelian.

Proposition 2.2 has the following consequences.

Lemma 2.3: *Let $H_1, H_2 \in L(G)$. Then $H_1 \cap H_2 = 1$ if and only if $[H_1, H_2] = 1$.*

Proof. If $[H_1, H_2] = 1$ then $H_1 \cap H_2 \in L(G)$ is abelian and therefore trivial.

For the converse, let N be the normal core of the intersection $\mathrm{N_G}(H_1) \cap \mathrm{N_G}(H_2)$. Thus $N \trianglelefteq_{\mathrm{f}} G$ normalizes H_1 and H_2. For $i = 1, 2$, let $K_i = H_i \cap N$. Then $K_i \trianglelefteq_{\mathrm{f}} H_i$ and $K_i \in L(G)$. Now, since $K_i \trianglelefteq K_1K_2$, we have $[K_1, K_2] \leq K_1 \cap K_2 \leq H_1 \cap H_2 = 1$. Therefore, applying Proposition 2.2 to $K_i \leq_{\mathrm{f}} H_i$, we obtain $\mathrm{C_G}(H_1) = \mathrm{C_G}(K_1) \leq \mathrm{C_G}(K_2) = \mathrm{C_G}(H_2)$, and vice-versa, so $[H_1, H_2] = 1$. ∎

Lemma 2.4: *For every $H \in L(G)$ we have $\langle H, \mathrm{C_G}(H)\rangle = H \times \mathrm{C_G}(H) \leq_{\mathrm{va}} G$.*

Proof. Write $C = \mathrm{C_G}(H)$ and note that $\langle H, C\rangle = H \times C$ by Lemma 2.3. If the normal core N of H is non-trivial then $N \trianglelefteq_{\mathrm{va}} G$, so $H \leq_{\mathrm{va}} G$ and hence $H \times C \leq_{\mathrm{va}} G$. Suppose then that $N = 1$ and let $V \in L(G)$ be the intersection of a maximal number of conjugates of H such that $1 < V \leq H$. If W is a conjugate of V which is not contained in H, then $1 \leq W \cap H \leq H$ is the intersection of one more conjugate of H than V. Therefore $W \cap H = 1$, by the choice of V and $W \leq C$ by Lemma 2.3. This implies that $H \times C$ contains all conjugates of V; in particular, it contains their product $V^G \trianglelefteq_{\mathrm{va}} G$. Thus $H \times C \leq_{\mathrm{va}} G$, as required. ∎

Lemma 2.5: *Let $H, K \in L(G)$. The following are equivalent:*

(i) $H \cap K \leq_{\mathrm{va}} H, K$;
(ii) $\mathrm{C_G}(H) = \mathrm{C_G}(K)$;
(iii) *there exists $D \in L(G)$ such that $H \times D \leq_{\mathrm{va}} G$ and $K \times D \leq_{\mathrm{va}} G$.*

Proof. If (i) holds, we have $A' \leq H \cap K$ for some $A \leq_{\mathrm{f}} H$. Let N and L be, respectively, the normal cores of A and $H \cap K$ in H. Then $N \trianglelefteq_{\mathrm{f}} H$ and $N' \trianglelefteq L \trianglelefteq H$, that is $L \trianglelefteq_{\mathrm{va}} H$. Thus $\mathrm{C_G}(L) = \mathrm{C_G}(H)$ by Proposition 2.2, but then $L \leq K \leq H$ implies that $\mathrm{C_G}(H \cap K) = \mathrm{C_G}(L) = \mathrm{C_G}(H)$. Repeating the procedure with H replaced by K yields $\mathrm{C_G}(H \cap K) = \mathrm{C_G}(K) = \mathrm{C_G}(H)$. This immediately implies (iii) by Lemma 2.4.

Suppose that (iii) is true; so $G_0' \leq H \times D$ for some $G_0 \leq_{\mathrm{f}} G$. Then $G_0 \cap K \leq_{\mathrm{f}} K$ and, since $D \cap K = 1$, we have $(G_0 \cap K)' \leq (H \times D) \cap K = H \cap K$, that is, $H \cap K \leq_{\mathrm{va}} K$. The same argument with H and K swapped gives $H \cap K \leq_{\mathrm{va}} H$. ∎

The structure lattice. For $H, K \in L(G)$, write $K \sim H$ if any of the equivalent conditions of Lemma 2.5 holds. It is immediate that $\sim$ is an equivalence relation on $L(G)$. We show that $\sim$ is a congruence on $L(G)$.

Proposition 2.6: *Let $H_1, H_2, K_1, K_2 \in L(G)$ with $H_1 \sim H_2$ and $K_1 \sim K_2$. Then $H_1 \cap K_1 \sim H_2 \cap K_2$ and $\langle H_1, K_1\rangle \sim \langle H_2, K_2\rangle$.*

Proof. For $i = 1, 2$ we have $H_1 \cap H_2 \cap K_i \leq_{\mathrm{va}} H_i \cap K_i$ and $K_1 \cap K_2 \cap H_i \leq_{\mathrm{va}} H_i \cap K_i$ so that $(H_1 \cap H_2 \cap K_i) \cap (K_1 \cap K_2 \cap H_i) \leq_{\mathrm{va}} H_i \cap K_i$. Thus $H_1 \cap H_2 \sim K_1 \cap K_2$.

To show that the join operation is respected, write

$$M := \langle H_1 \cap H_2, K_1 \cap K_2\rangle \in L(G).$$

Then $\mathrm{C_G}(M)$ centralizes H_i and K_i for $i = 1, 2$ so that Lemma 2.3 yields $\mathrm{C_G}(M) \cap H_i = 1$ and $\mathrm{C_G}(M) \cap K_i = 1$. Therefore

$$\langle \mathrm{C_G}(M), H_i, K_i\rangle = \mathrm{C_G}(M) \times \langle H_i, K_i\rangle \geq \mathrm{C_G}(M) \times M$$

and, since $\mathrm{C_G}(M) \times M \leq_{\mathrm{va}} G$ (by Lemma 2.4), we have $\mathrm{C_G}(M) \times \langle H_i, K_i\rangle \leq_{\mathrm{va}} G$ for $i = 1, 2$. ∎

By the above, the join and meet of two equivalence classes $[H], [K]$ of elements $H, K \in L(G)$ is well defined by

$$[H] \vee [K] = [\langle H, K\rangle] \quad \text{and} \quad [H] \wedge [K] = [H \cup K],$$

and the quotient $\mathcal{L} = L(G)/\sim$ is again a lattice with respect to these operations and the natural partial order inherited from $L(G)$:

$$[K] \leq [H] \quad \text{if and only if } [K \cap H] = [K].$$

This quotient $\mathcal{L}$ is the **structure lattice** of G.

Note that $[G]$ and $[1] = \{1\}$ are, respectively, the greatest and least elements of $\mathcal{L}$. Observe also that $[H]^g = [H^g]$ for each $H \in L(G)$ and $g \in G$. Thus the action of G on its subgroups by conjugation induces a well-defined action on $\mathcal{L}$. It is shown in [8] that $\mathcal{L}$ is a Boolean lattice (it is uniquely complemented and distributive), but we do not require this fact here.

The structure graph. An element B of $L(G)$ is **basal** if $\langle B^G\rangle$ is the direct product of the finitely many conjugates of B; in particular, if $B^g \neq B$ then $B^g \cap B = 1$. Examples of basal subgroups include $\mathrm{rist}_G(v)$ for every vertex v of a tree on which G acts as a branch group.

Basal subgroups have the following useful properties.

Lemma 2.7 ([13]): *Let B, B_1, B_2 be basal subgroups of G. Then*

(i) *$B_1 \cap B_2$ is basal;*
(ii) *if $[B_1] \leq [B_2]$ then $\mathrm{N_G}(B_1) \leq \mathrm{N_G}(B_2)$;*
(iii) *$\mathrm{N_G}(B)$ is the stabilizer of $[B]$ under the action of G by conjugation;*
(iv) *$\bigcap(\mathrm{N_G}(B) \mid B$ is basal$) = 1$.*

Proof. (i) Suppose that $(B_1 \cap B_2)^g = B_1^g \cap B_2^g \neq B_1 \cap B_2$ for some $g \in G$. Then either $B_1^g \neq B_1$ or $B_2^g \neq B_2$. In each case we have $(B_1 \cap B_2)^g \cap (B_1 \cap B_2) = 1$.

(ii) Let $g \in \mathrm{N_G}(B_1)$. Then $1 \neq [B_1]^g = [B_1] \leq [B_2]^g \wedge [B_2] = [B_2^g \cap B_2]$ so $B_2^g = B_2$ and $g \in \mathrm{N_G}(B_2)$, as required.

(iii) This follows from the argument in the previous part.

(iv) Note that $\mathrm{N_G}([\mathrm{rist}_G(v)]) = \mathrm{Stab}_G(v)$ for every $v \in T$, where T is a tree on which G acts as a branch group. The claim follows from the observation that every $\mathrm{rist}_G(v)$ is basal and $\bigcap(\mathrm{Stab}_G(v) \mid v \in T) = 1$. ∎

The **structure graph** $\mathcal{B}$ of G has as vertices all non-trivial elements $[B] \in \mathcal{L}$ such that B is basal. Two elements $[A] \leq [B]$ of $\mathcal{B}$ are joined by an edge if $[A]$ is maximal subject to this inequality. Again, the conjugation action of G on its basal subgroups induces an action on $\mathcal{B}$ by graph automorphisms.

3. The congruence and branch topologies

Notation: Since in this section we will deal with different branch actions of the same group G, we will write $\mathrm{Stab}_\rho(n)$ and $\mathrm{rist}_\rho(v)$ for the stabilizer of the nth layer and the rigid stabilizer of vertex v with respect to a given branch action $\rho : G \to \mathrm{Aut}(T)$. We shall omit the subscript when there is no risk of confusion.

Proposition 3.1 ([8]): *If G acts as a branch group on a tree T, then there is an order-preserving G-equivariant embedding $\phi : T \to \mathcal{B}$ defined by $\phi : v \mapsto [\mathrm{rist}(v)]$.*

Proof. We have already seen that ϕ is G-equivariant as

$$\phi(v)^g = [\mathrm{rist}(v)]^g = [\mathrm{rist}(v)^g] = [\mathrm{rist}(v^g)] = \phi(v^g).$$

That ϕ is order-preserving is also clear since v is a descendant of w if and only if $\mathrm{rist}(v) \leq \mathrm{rist}(w)$.

To see that ϕ is injective, let $\mathrm{rist}(v) \sim \mathrm{rist}(w)$. If v and w were incomparable vertices, then $\mathrm{rist}(v) \cap \mathrm{rist}(w) = 1$ would imply that $\mathrm{rist}(v)$ and $\mathrm{rist}(w)$ are virtually abelian, a contradiction. Thus v and w are comparable, say $v \leq w$.

Suppose for a contradiction that $v \neq w$; then there is a vertex $v_2 \neq v$ in the same layer as v such that $v_2 \leq w$ (a 'sibling' of v). Since $v, v_2 \leq w$ we have $\mathrm{rist}(v) \times \mathrm{rist}(v_2) \leq \mathrm{rist}(w)$ and $\mathrm{rist}(v) \leq_{\mathrm{va}} \mathrm{rist}(w)$. It then follows that $\mathrm{rist}(v) \leq_{\mathrm{va}} (\mathrm{rist}(v) \times \mathrm{rist}(v_2))$ and that $\mathrm{rist}(v_2)$ is virtually abelian. This gives the desired contradiction, so $v = w$, as required. ∎

Proposition 3.2 ([8]): *Let G act as a branch group on T and let $1 \neq B \in L(G)$. Then there exists some $v \in T$ such that $[\mathrm{rist}(v)] \leq [B]$.*

Proof. Since B is non-trivial it contains a non-trivial element g which moves some vertex v. Let $w = v^g$ and N be the normal core of the normalizer of B. Then $\mathrm{rist}(v) \cap N \trianglelefteq_{\mathrm{f}} \mathrm{rist}(v)$. Let $h, k \in \mathrm{rist}(v) \cap N$. Note that $h^g \in \mathrm{rist}(v^g) = \mathrm{rist}(w)$; so h^g and k commute (because $[\mathrm{rist}(w), \mathrm{rist}(v)] = 1$). Then $[[h, g], k] \in B$ since h, k normalize B and we have

$$[h, k] = h^{-1}k^{-1}hk = h^{-1}h^g k^{-1}(h^{-1})^g hk = [h^{-1}h^g, k] = [[h, g], k].$$

Thus $(\mathrm{rist}(v) \cap N)' \leq B$ and $(\mathrm{rist}(v) \cap N)' \sim \mathrm{rist}(v)$ yields the result. ∎

The above imply that the structure graph "contains" all possible branch actions of G. It is then reasonable to define the congruence and branch topologies with respect to the action of G on $\mathcal{B}$.

When the structure graph is itself a tree, then G acts on it with a branch action and all other trees on which G acts as a branch group are obtained from the structure graph by "deleting layers". This was proved in [7], where a sufficient condition for the structure graph to be a tree is given. Examples of branch groups satisfying that condition include Grigorchuk's first group, the Gupta–Sidki p-groups and the Hanoi tower group. A necessary and sufficient condition for the structure graph to be a tree is given in [8].

The congruence topology. In order to define the congruence topology with respect to the action of G on $\mathcal{B}$, we must find analogues of the level stabilizers $\mathrm{Stab}_\rho(n)$ for a branch action ρ of G. As G acts level-transitively, the obvious analogue in $\mathcal{B}$ of a layer is an orbit b^G where $b = [B] \in \mathcal{B}$ and B is basal. Recall from Lemma 2.7 that $\mathrm{Stab}([B]) = \mathrm{N_G}(B)$, so the analogue of a level stabilizer $\mathrm{Stab}_\rho(n)$ is an **orbit stabilizer**

$$\mathrm{Stab}_{\mathcal{B}}(b^G) := \bigcap(\mathrm{N_G}(B^g) \mid g \in G)$$

for $b = [B] \in \mathcal{B}$. Since each basal subgroup has only finitely many conjugates, these orbit stabilizers have finite index in G. Furthermore, the intersection of all orbit stabilizers is trivial. We take the family $\{\mathrm{Stab}_{\mathcal{B}}(b^G) \mid b \in \mathcal{B}\}$ as a neighbourhood basis of the identity to define the **congruence topology** of G with respect to the action of G on $\mathcal{B}$. The completion of G with respect to this topology is a profinite group $\overline{G}_{\mathcal{B}}$ onto which the profinite completion $\widehat{G}$ maps. We denote the kernel of this map by $C_{\mathcal{B}}$. In the following theorem we prove that for any branch action ρ of G, the topologies induced by $\{\mathrm{Stab}_\rho(n) \mid n \geq 0\}$ and by $\{\mathrm{Stab}_{\mathcal{B}}(b^G) \mid b \in \mathcal{B}\}$ are equal. Thus the congruence kernels with respect to each of them coincide and Theorem 1 as stated in the introduction follows.

THEOREM 3.3: *For any branch action $\rho : G \to \mathrm{Aut}(T)$, denote by C_ρ the congruence kernel with respect to this action. Then $C_\rho = C_{\mathcal{B}}$.*

Proof. We must show that for every $n \geq 0$ there is some $b \in \mathcal{B}$ such that $\mathrm{Stab}_\rho(n) \geq \mathrm{Stab}_{\mathcal{B}}(b^G)$ and conversely, that for every $b \in \mathcal{B}$ there is some $n \geq 0$ such that $\mathrm{Stab}_{\mathcal{B}}(b^G) \geq \mathrm{Stab}_\rho(n)$. However, by Proposition 3.1, the nth layer V_n of the tree corresponds to an orbit of basal subgroups so that the former statement holds trivially. It therefore suffices to show the latter statement.

For a given $b = [B] \in \mathcal{B}$, Proposition 3.2 gives a vertex v of T such that $[\mathrm{rist}_\rho(v)] \leq b$. Suppose that v is in the nth layer of T and let $x \in \mathrm{Stab}_\rho(n)$. Note that $\mathrm{Stab}_\rho(n)$ is the pointwise stabilizer of the G-orbit of $[\mathrm{rist}_\rho(v)]$ by Proposition 3.1. Then

$$1 \neq [\mathrm{rist}_\rho(v^x)] = [\mathrm{rist}_\rho(v)] \leq b^x \wedge b.$$

Since B is basal this implies that $B^x = B$ (so $b^x = b$). Thus the normal subgroup $\mathrm{Stab}_\rho(n)$ fixes all elements of b^G and we conclude that $\mathrm{Stab}_\rho(n) \leq \mathrm{Stab}_{\mathcal{B}}(b^G)$, as required. ∎

THE BRANCH TOPOLOGY. As with the congruence kernel, Theorem 2 will follow from the fact that the branch kernel with respect to a branch action is equal to the analogous object for the action on the structure graph. We must therefore define the branch topology with respect to $\mathcal{B}$. To do this we generalize the notion of a rigid stabilizer. For a non-trivial basal subgroup A of a branch group, define its **rigid normalizer** $\mathrm{R_G}(A)$ by

$$\mathrm{R_G}(A) := \bigcap (\mathrm{N_G}(B) \mid B \text{ is basal and } A \cap B = 1).$$

We will use the following properties of rigid normalizers. These were proved in [8] but we include a shorter argument for the reader's convenience.

Proposition 3.4: *Let A, A_1, A_2 be basal subgroups and v a vertex of a tree on which G acts as a branch group. Then*

(i) $A \leq \mathrm{R_G}(A) \leq \mathrm{N_G}(A)$;

(ii) *if* $[A_1] \leq [A_2]$ *then* $\mathrm{R_G}(A_1) \leq \mathrm{R_G}(A_2)$ (*in particular,* $\mathrm{R_G}(A_1) = \mathrm{R_G}(A_2)$ *if* $[A_1] = [A_2]$);

(iii) $\mathrm{R_G}(A)$ *is basal and the unique maximal element of* $[A]$*;*

(iv) $\mathrm{R_G}(\mathrm{rist}(v)) = \mathrm{rist}(v)$.

Proof. (i) Let B be a basal subgroup with $A \cap B = 1$. Then $[A, B] = 1$ by Lemma 2.3 and so $A \leq \mathrm{N_G}(B)$; thus $A \leq \mathrm{R_G}(A)$. If $g \in \mathrm{R_G}(A)$, then g normalizes each of the conjugates of A distinct from A as they are basal and have trivial intersection with A. Therefore g must normalize A itself.

(ii) Since $[A_1] \leq [A_2]$, there is some finite index subgroup K of A_1 such that $K' \leq A_1 \cap A_2$. Now, for a basal subgroup B with $B \cap A_2 = 1$, we have

$$(K \cap B)' \leq K' \cap B \leq A_1 \cap A_2 \cap B \leq A_2 \cap B = 1$$

and $B \cap K \leq_{\mathrm{f}} B \cap A_1$. That is to say, $B \cap A_1$ is virtually abelian and so $B \cap A_1 = 1$. Hence every $g \in \mathrm{R_G}(A_1)$ normalizes B and $\mathrm{R_G}(A_1) \leq \mathrm{R_G}(A_2)$, as required.

(iii) We first show that $\mathrm{R_G}(A) \sim A$ using Lemma 2.5(iii). Let D denote the product of the finitely many conjugates of A that are distinct from A; then $A^G = A \times D \trianglelefteq_{\mathrm{va}} G$. Since $A \leq \mathrm{R_G}(A)$, it suffices to show that $\mathrm{R_G}(A) \cap D = 1$. Let $x \in \mathrm{R_G}(A) \cap D$ and B be a basal subgroup. If $B \cap A = 1$ then $x \in \mathrm{R_G}(A)$ normalizes B. If $B \cap A \neq 1$ then $B \cap A \leq A$ is basal and is centralized by $x \in D \leq \mathrm{C_G}(A)$; hence $x \in \mathrm{N_G}(B \cap A) \leq \mathrm{N_G}(B)$ by Lemma 2.7. Thus $x \in \bigcap(\mathrm{N_G}(B) \mid B \text{ is basal}) = 1$.

To see that $\mathrm{R_G}(A)$ is basal, suppose that $\mathrm{R_G}(A)^g \neq \mathrm{R_G}(A)$ for some $g \in G$. Then, as $\mathrm{R_G}(A)^g = \mathrm{R_G}(A^g)$, we have $A \neq A^g$ by (ii) so that $A^g \cap A = 1$ and $[A^g, A] = 1$ by Lemma 2.3. Now, $\mathrm{R_G}(A) \sim A$, so $\mathrm{C_G}(\mathrm{R_G}(A)) = \mathrm{C_G}(A)$ by Lemma 2.5 which implies that $[A^g, \mathrm{R_G}(A)] = 1$. Similarly, $\mathrm{R_G}(A)^g \sim A^g$ implies that $[\mathrm{R_G}(A)^g, \mathrm{R_G}(A)] = 1$, yielding $\mathrm{R_G}(A)^g \cap \mathrm{R_G}(A) = 1$.

For any $H \in [A]$ and any basal subgroup B such that $B \cap A = 1$ we have $[B, A] = 1$, so $B \leq \mathrm{C_G}(A) = \mathrm{C_G}(H)$. This means that $[B, H] = 1$, in particular, $H \leq \mathrm{N_G}(B)$. Hence $H \leq \mathrm{R_G}(A)$.

(iv) By part (i), it suffices to show that $\mathrm{R_G}(\mathrm{rist}(v)) \leq \mathrm{rist}(v)$. Let $g \in \mathrm{R_G}(\mathrm{rist}(v))$ and $w \in T \setminus T_v$. If w is incomparable with v, then

$$\mathrm{rist}(w) \cap \mathrm{rist}(v) = 1$$

and so g fixes w. If $v \leq w$, then $\mathrm{rist}(v) \leq \mathrm{rist}(w)$ and

$$\mathrm{R_G}(\mathrm{rist}(v)) \leq \mathrm{N_G}(\mathrm{rist}(v)) \leq \mathrm{N_G}(\mathrm{rist}(w))$$

by Lemma 2.7. Thus $w = w^g$ and the claim follows. ∎

By analogy with the rigid stabilizers of layers, the **rigid normalizer of an orbit** $a^G = [A]^G$ in $\mathcal{B}$ is defined to be $\mathrm{R_G}(a^G) := \mathrm{R_G}(A)^G$. That these rigid normalizers of orbits have finite index in G follows from Proposition 3.2. It also follows that the intersection of all of them is trivial. We may therefore take $\{\mathrm{R_G}(a^G) \mid a \in \mathcal{B}\}$ as a neighbourhood basis of the identity to generate the **branch topology** of G with respect to the action of G on $\mathcal{B}$. The completion $\widetilde{G}_{\mathcal{B}}$ with respect to this topology is a profinite group in which G embeds. We denote by $B_{\mathcal{B}}$ the kernel of the map $\widehat{G} \to \widetilde{G}_{\mathcal{B}}$. In our last theorem we prove that for any branch action ρ of G, the topologies induced by $\{\mathrm{rist}_\rho(n) \mid n \geq 0\}$ and by $\{\mathrm{R_G}(a^G) \mid a \in \mathcal{B}\}$ are equal; consequently, the branch kernels with respect to each of them coincide and Theorem 2 as stated in the introduction follows.

Theorem 3.5: *For a branch action $\rho : G \to \mathrm{Aut}(T)$, denote by B_ρ the branch kernel with respect to this action. Then $B_\rho = B_{\mathcal{B}}$.*

Proof. As with the proof for the congruence kernels, we must show that for every $n \geq 0$ there is some $b \in \mathcal{B}$ such that $\mathrm{rist}_\rho(n) \geq \mathrm{R_G}(b^G)$ and that for every $b \in \mathcal{B}$ there is some $n \geq 0$ such that $\mathrm{R_G}(b^G) \geq \mathrm{rist}_\rho(n)$. The former claim follows from Proposition 3.4(iv) since all rigid stabilizers of vertices are basal. It therefore suffices to prove the latter claim. Given $b \in \mathcal{B}$, Proposition 3.2 yields that $[\mathrm{rist}_\rho(v)] \leq b$ for some $v \in T$. Suppose that $v \in V_n \subset T$. We then have $\mathrm{rist}_\rho(v) = \mathrm{R_G}(\mathrm{rist}_\rho(v)) \leq \mathrm{R_G}(b)$ by Proposition 3.4, and the transitivity of G on each of the layers of T implies that $\mathrm{rist}_\rho(n) \leq \mathrm{R_G}(b^G)$, as required. ∎

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