

THE STRUCTURE OF THE MITCHELL ORDER—I

BY

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ABSTRACT

We isolate here a wide class of well-founded orders called tame orders, and show that each such order of cardinality at most κ can be realized as the Mitchell order on a measurable cardinal κ , from a consistency assumption weaker than $o(\kappa) = \kappa^+$.

1. Introduction

This paper is the first of a two-part study on the possible structure of the Mitchell order. In this first paper, we identify a large class of well-founded orders with some appealing properties, and prove that each of its members can be realized as $\triangleleft(\kappa)$ —the Mitchell order on the set of normal measures on κ . In [15] Mitchell introduced the following relation: Given two normal measures U, W , we write $U \triangleleft W$ to denote that $U \in M_W \cong \text{Ult}(V, W)$. Mitchell proved that \triangleleft is a well-founded order now known as the Mitchell ordering. The Mitchell ordering and its extension to arbitrary extenders have become a major tool in the study of large cardinals, with important applications to consistency results and inner model theory. Given a cardinal κ , we write $o(\kappa)$ to denote the rank of the well-founded order $\triangleleft(\kappa)$. The research on the possible structure on the Mitchell order $\triangleleft(\kappa)$ is closely related to the question of its possible size, namely, the number of normal measures on κ : The first results by Kunen [10] and by

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Kunen and Paris [11] showed that this number can take the extremal values of 1 and κ^{++} (in a model of GCH) respectively. Soon after, Mitchell [15] [16] showed that this size can be any cardinal λ between 1 and κ^{++} , under the large cardinal assumption and in a model of $o(\kappa) = \lambda$. Baldwin [2] showed that for $\lambda < \kappa$ and from stronger large cardinal assumptions, κ can also be the first measurable cardinal. Apter–Cummings–Hamkins [1] proved that there can be κ^+ normal measures on κ from the minimal assumption of a single measurable cardinal; for $\lambda < \kappa^+$, Leaning [12] reduced the large cardinal assumption from $o(\kappa) = \lambda$ to an assumption weaker than $o(\kappa) = 2$. The question of the possible number of normal measures on κ was finally resolved by Friedman and Magidor in [7], where it is shown that κ can carry any number of normal measures $1 \leq \lambda \leq \kappa^{++}$ from the minimal assumption. The Friedman–Magidor poset will be extensively used in this paper and the subsequent part II.

Further results were obtained on the possible structure of the Mitchell order: Mitchell [15] and Baldwin [2] showed that from some large cardinal assumptions, every well-order and pre-well-order (respectively) can be isomorphic to $\triangleleft(\kappa)$ at some κ . Cummings [5],[6] and Witzany [18] studied the \triangleleft ordering in various generic extensions, and showed that $\triangleleft(\kappa)$ can have a rich structure. Cummings constructed models where $\triangleleft(\kappa)$ embeds every order from a specific family of orders we call tame. Witzany showed that in a Kunen–Paris extension of a Mitchell model $L[\mathcal{U}]$, with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, every well-founded order of cardinality $\leq \kappa^+$ embeds into $\triangleleft(\kappa)$. However, the general question of the possible structure of $\triangleleft(\kappa)$ has remained open.

In this paper and the subsequent part II ([4]) we gradually develop a series of techniques by which we obtain an increasing variety of possible \triangleleft structures from increasingly large cardinal assumptions: In this paper, we develop a technique for realizing a wide family of well-founded orders called tame orders from assumptions weaker than the existence of a measurable cardinal κ with $o(\kappa) = \kappa^+$. In part II ([4]) we increase our large cardinal assumption slightly above the existence of a sharp to a strong cardinal 0^\sharp , and show that every well-founded order can be consistently realized as $\triangleleft(\kappa)$ on a measurable cardinal κ .

The forcing constructions in both papers obey the following guidelines:

1. The ground model $V = \mathcal{K}(V)$ is a core model, presented as an extender model $L[E]$ (as in [17]) or a Mitchell model $L[\mathcal{U}]$ (see [16] or [14]).
2. An intermediate forcing extension $V' = V[G']$ is introduced, to serve as an intermediate ground for a final \triangleleft structure. Our goal is to make $\triangleleft(\kappa)^{V'}$ as rich

as possible (relative to the large cardinal assumption) while ensuring that the normal measures on κ are separated by sets. We say that the normal measures on κ are separated when one can assign each normal measure U on κ a set $X_U \in U$ which does not belong to any distinct normal measure $U' \neq U$.

3. In a final last extension, we restrict $\triangleleft(\kappa)^{V'}$ to any chosen $\mathcal{W} \subset \triangleleft(\kappa)^{V'}$ of cardinality $|\mathcal{W}| \leq \kappa$. We refer to this last forcing as a final cut. The final cut relies on the fact that the normal measures in V' are separated by sets.

The orders at the center of this paper are tame orders. For every ordinal λ , we define an order $(R_\lambda, <_{R_\lambda})$ by $R_\lambda = \{(\alpha, \beta) \in \lambda^2 \mid \alpha \leq \beta\}$, and $(\alpha, \beta) <_{R_\lambda} (\alpha', \beta') \iff \beta < \alpha'$. In Section 2, we introduce tame orders and show that up to a simple operation called reduction, every tame order $(S, <_S)$ embeds in some $(R_\lambda, <_{R_\lambda})$. The rest of the paper is largely devoted to realizing R_λ using $\triangleleft(\kappa)$. The following observation relates R_λ to the generalized Mitchell order in V : Suppose that $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a \triangleleft -increasing sequence, and let t be the map defined by

$$t(\alpha, \beta) = \begin{cases} U_\alpha \times U_\beta & \text{if } \alpha < \beta, \\ U_\alpha & \text{if } \alpha = \beta. \end{cases}$$

Then t defines an isomorphism of $(R_\lambda, <_{R_\lambda})$ with a set of ultrafilters in V , ordered by \triangleleft . Here $U_\alpha \times U_\beta = \{X \subseteq \kappa^2 \mid \{\nu \mid \{\mu \mid (\nu, \mu) \in X\} \in U_\beta\} \in U_\alpha\}$. The purpose of the main forcing is to reduce each $U_\alpha \times U_\beta$ to a normal measure on κ to construct an intermediate model V' where $\triangleleft(\kappa)^{V'}$ embeds R_λ . This is done by forcing with a Magidor iteration of one-point Prikry forcings \mathcal{P}^1 . The iteration introduces an almost injective function $d : \kappa \rightarrow \kappa$ and ultrafilters $U_{(\alpha, \beta)}^1$, so that the map $\nu \mapsto (\nu, d^{-1}(\nu))$ defines an isomorphism of $U_{(\alpha, \beta)}^1$ with an extension of $U_\alpha \times U_\beta$. However, there is a problem with forcing directly over V . The one-point Prikry forcing at stage $\nu < \kappa$ is based on a normal measure $U_{\nu, \alpha}$ on ν (where $\alpha < o(\nu)$). For each $\beta < o(\kappa)$, we get that the choice $\nu \mapsto U_{\nu, \alpha}$ determines a unique measure U_α modulo U_β ($\alpha < \beta$), and it follows that we cannot form a normal projection of $U_\alpha \times U_\beta$ for more than a single α . This problem is solved by first forcing with a Friedman–Magidor poset \mathcal{P}^0 . The forcing \mathcal{P}^0 splits each U_β into κ many \triangleleft -equivalent extensions. The different extensions allow us to simultaneously deal with $U_\alpha \times U_\alpha$ for every $\alpha < \beta$.

Sections 4 through 6 provide an analysis of the normal measures $U_{(\alpha, \beta)}^1$ in a $\mathcal{P}^0 * \mathcal{P}^1$ generic extension V' . It is shown that $\triangleleft(\kappa)^{V'}$ embeds R_λ and

that the normal measures are separated by sets. The analysis of the measures $U_{(\alpha,\beta)}^1$ focuses on the iterated ultrapowers obtained from the restriction of $i_{U_{(\alpha,\beta)}^1} : V' \rightarrow \text{Ult}(V', U_{(\alpha,\beta)}^1)$ to $V = \mathcal{K}(V')$. Finally, in Section 7, we introduce the final cut iteration and apply it to V' to remove unwanted measures without changing the \triangleleft structure on the rest. This construction is used to prove the main result in this paper, Theorem 7.5.

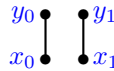
The notations in this paper obey the following conventions: A pair $(S, <_S)$ will be called an **order**, if $<_S \subset S \times S$ is a relation which defines a partial order (anti-symmetric and transitive relation) on S . When there is no danger of confusion, we will use S to denote the entire order $(S, <_S)$. By a **suborder** of $(S, <_S)$ we mean the restriction of $(S, <_S)$ to a subset $X \subset S$, and denote it by $(X, <_S \upharpoonright X)$. We use the Jerusalem convention for the forcing order, in which $p \geq q$ means that p is stronger than q . Thus the trivial condition of \mathcal{P} will be denoted by $0_{\mathcal{P}}$. A name of a set x in a generic extension will be denoted by \dot{x} , and a canonical name for an element x in the ground model V will be denoted by \check{x} . In certain cases we will write $V^{\mathcal{P}}$ to denote a generic extension of V by a generic filter of \mathcal{P} .

2. Tame orders

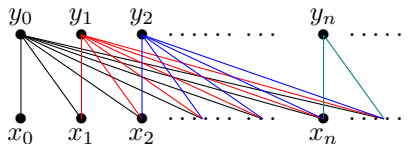
We define a family of orders called Tame Orders which is closely related to the orders R_λ , $\lambda \in \text{On}$, introduced above. The relation between tame orders and the orders R_λ is given in Proposition 2.10 and makes use of the notions of reduced orders (Definition 2.3) and tame ranks (Definition 2.9).

Definition 2.1:

- (1) $(R_{2,2}, <_{R_{2,2}})$ is an order on a set of four elements $R_{2,2} = \{x_0, x_1, y_0, y_1\}$, defined by $<_{R_{2,2}} = \{(x_0, y_0), (x_1, y_1)\}$.



- (2) $(S_{\omega,2}, <_{S_{\omega,2}})$ is an order on a disjoint union of two countable sets $S_{\omega,2} = \{x_n\}_{n < \omega} \uplus \{y_n\}_{n < \omega}$, defined by $<_{S_{\omega,2}} = \{(x_m, y_n) \mid m \geq n\}$.



Let $(S, <_S)$ and $(R, <_R)$ be two orders. An injection $\pi : S \rightarrow R$ is an **embedding** of $(S, <_S)$ into $(R, <_R)$ if it is compatible and incompatibility preserving. We say that $(R, <_R)$ embeds $(S, <_S)$.

Definition 2.2 (Tame Orders): An order $(S, <_S)$ is **tame** if it does not embed $R_{2,2}$ nor $S_{\omega,2}$.

The rest of this section is devoted to describing how tame orders relate to the orders R_λ , $\lambda \in \text{On}$.

Definition 2.3: Let $(S, <_S)$ be an order.

- (1) For $x \in S$ let $d(x) = \{z \in S \mid z <_S x\}$ and $u(x) = \{z \in S \mid x <_S z\}$.
- (2) Let \sim_S be the equivalence relation on S , defined by $x \sim_S y$ if and only if $(d(x), u(x)) = (d(y), u(y))$.
- (3) $(S, <_S)$ is **reduced** if and only if there are no distinct \sim_S equivalent elements in S .
- (4) For any $(S, <_S)$ let $([S], <_{[S]}) = (S, <_S) / \sim_S$ be the induced order on \sim_S equivalent classes. $([S], <_{[S]})$ is clearly reduced.

The main result in this section (Proposition 2.10) shows that a well-founded reduced order $(S, <_S)$ is tame if and only if it embeds in R_λ for some $\lambda < |S|^+$.

LEMMA 2.4: Let $(S, <_S)$ be an order. The following are equivalent:

- (1) $(S, <_S)$ does not embed $R_{2,2}$.
- (2) For every $x, x' \in S$, the sets $u(x), u(x')$ are \subseteq -comparable.
- (3) For every $x, x' \in S$, the sets $d(x), d(x')$ are \subseteq -comparable.

Proof. In $R_{2,2}$, the sets $u(x_0), u(x_1)$ are \subseteq -incomparable as $y_0 \in u(x_0) \setminus u(x_1)$ and $y_1 \in u(x_1) \setminus u(x_0)$. If $\pi : R_{2,2} \rightarrow S$ is an embedding it follows that $u(\pi(x_0)), u(\pi(x_1))$ are \subseteq -incomparable. Therefore 2 implies 1. The fact that $d(y_0)$ and $d(y_1)$ are \subseteq -incomparable is similarly used to show that 3 implies 1.

Suppose now that $(S, <_S)$ does not embed $R_{2,2}$, and let $x, x' \in S$. If $u(x)$ and $u(x')$ were \subseteq -incomparable, there would be some y, y' so that

- (1) $x <_S y$, $x' \not<_S y$, and
- (2) $x' <_S y'$, $x \not<_S y'$.

This is impossible as it would imply that $<_S \upharpoonright \{x, x', y, y'\}$ is isomorphic to $R_{2,2}$. It follows that 1 implies 2. The proof that 1 implies 3 is similar. ■

Definition 2.5: Let $(S, <_S)$ be an order.

- (1) For every $x \in S$ define $\text{cu}(x) = S \setminus u(x) = \{y \in S \mid x \not<_S y\}$.
- (2) Define $D(S) = \{d(x) \mid x \in S\}$ and $\text{CU}(S) = \{\text{cu}(x) \mid x \in S\}$.

For $x, y \in S$, $u(x) \supseteq u(y)$ if and only if $\text{cu}(x) \subseteq \text{cu}(y)$, hence

$$(\text{CU}(S), \subseteq) \cong (U(S), \supseteq),$$

where $U(S) = \{u(x) \mid x \in S\}$. By Lemma 2.4, if $(S, <_S)$ does not embed $R_{2,2}$ then $(D(S), \subseteq)$, $(\text{CU}(S), \subseteq)$ are linear.

LEMMA 2.6: *The following are equivalent for a well-founded order $(S, <_S)$ which does not embed $R_{2,2}$:*

- (1) $(S, <_S)$ does not embed $(S_{\omega,2}, <_{S_{\omega,2}})$.
- (2) $(D(S), \subseteq)$ is well-founded.
- (3) $(\text{CU}(S), \subseteq)$ is well-founded.

Proof. In $S_{\omega,2}$, the sequence $\langle d(y_n) \mid n < \omega \rangle$ is \subseteq -strictly decreasing as witnessed by $\langle x_n \mid n < \omega \rangle$. Suppose that $\pi : S_{\omega,2} \rightarrow S$ is an embedding of $(S_{\omega,2}, <_{S_{\omega,2}})$ into $(S, <_S)$. Then $\pi(x_n) \in d(\pi(y_n)) \setminus d(\pi(y_m))$ for every $n < m < \omega$. But $D(\pi(y_n)), D(\pi(y_m))$ are \subseteq -comparable as $(S, <_S)$ does not embed $R_{2,2}$, so it must be that $d(\pi(y_m)) \subsetneq d(\pi(y_n))$. Therefore $\langle d(\pi(y_n)) \mid n < \omega \rangle$ is \subseteq -strictly decreasing. This shows 2 implies 1. Similarly, the fact that in $S_{\omega,2}$, $\langle \text{cu}(x_n) \mid n < \omega \rangle$ is a \subseteq strictly decreasing sequence, is used to show that 3 implies 1.

Suppose now that $(S, <_S)$ is a well-founded order which does not embed $R_{2,2}$ and fails to satisfy 2. Let $\langle y_n \mid n < \omega \rangle$ be a sequence of distinct elements in S such that $\langle d(y_n) \mid n < \omega \rangle$ is \subseteq -strictly decreasing. Since S is well-founded we may assume that $y_m \not<_S y_n$ for every $n < m$. Furthermore, we cannot have $y_n <_S y_m$ as it would imply that $y_n \in d(y_n)$. It follows that the elements in $\langle y_n \mid n < \omega \rangle$ are $<_S$ pairwise incomparable.

Next, for each $n < \omega$ pick $x_n \in d(y_n) \setminus d(y_{n+1})$, thus $x_n \in d(y_n) \setminus d(y_m)$ for every $n < m$. We can thin out the sequence $\langle x_n \mid n < \omega \rangle$ to get an infinite subsequence so that $x_m \not<_S x_n$ whenever $m > n$ and x_n, x_m are members of the subsequence. For simplicity let us assume that $x_m \not<_S x_n$ for every $n < m$. We claim that also $x_n \not<_S x_m$ for every $n < m$. For this, note that $d(y_m), d(x_m)$ are \subseteq compatible and $x_m \in d(y_m) \setminus d(x_m)$, so $d(x_m) \subseteq d(y_m)$. Therefore if $x_n <_S x_m$, $x_n \in d(x_m) \subseteq d(y_m)$ which contradicts our choice of x_n . We

conclude that the elements in the sequence $\langle x_n \mid n < \omega \rangle$ are also $<_S$ pairwise incomparable.

We claim that $<_S \upharpoonright (\{x_n\}_{n < \omega} \uplus \{y_n\}_{n < \omega})$ is isomorphic to $S_{\omega,2}$. It remains to show that the sets $\{x_n\}_{n < \omega}, \{y_n\}_{n < \omega}$ are disjoint. To this end, we have that $x_n \neq y_n$ for all $n < \omega$, as $x_n \in d(y_n)$. Also, if $m \neq n$ then $x_n \neq y_m$, as otherwise y_n, y_m would be $<_S$ comparable in contradiction to the above. It follows that 1 implies 2. Using a similar argument one can show that 1 implies 3. ■

Definition 2.7: Let $(S, <_S)$ be an order.

- (1) Define $\mathfrak{D}(S)$ to be the completion of $D(S)$ under \subseteq increasing sequences, namely $d \in \mathfrak{D}(S) \setminus D(S)$ if and only if $d = \cup C$ for some $C \subset D(S)$ which is \subsetneq -downward closed, i.e., for all $d_1, d_2 \in D(S)$, if $d_2 \in C$ and $d_1 \subseteq d_2$ then $d_1 \in C$.
- (2) For every $x \in S$ let
 - $<_{\mathfrak{D}}(x) = \{d \in \mathfrak{D}(S) \mid d \subsetneq d(x)\}$, and
 - $<_{\text{CU}}(x) = \{\text{cu}(y) \in \text{CU}(S) \mid \text{cu}(y) \subsetneq \text{cu}(x)\}$.

Remarks 2.8:

- (1) Note that $D(S) \subset \mathfrak{D}(S)$. Indeed for every $x \in S$, $d(x) = \cup C_x$, where $C_x = \{d \in D(S) \mid d \subseteq d(x)\}$ is \subseteq -downward closed.
- (2) The elements $d \in \mathfrak{D}(S) \setminus D(S)$ are \subseteq -limits of $D(S)$. Therefore, if $d \in \mathfrak{D}(S)$ has a \subseteq immediate successor $d^+ \in \mathfrak{D}(S)$, then d^+ is not a \subseteq -limit and therefore $d^+ \in D(S)$ is of the form $d^+ = d(z)$ for some $z \in S$.
- (3) If $(S, <_S)$ is a well order which does not embed $R_{2,2}$ or $S_{\omega,2}$, then $(D(S), \subseteq)$ and $(\text{CU}(S), \subseteq)$ are well orders. Since $\mathfrak{D}(S)$ introduces only \subseteq -limits to $D(S)$, it follows that $(\mathfrak{D}(S), \subseteq)$ is a well order.
- (4) For every $x \in S$, the sets $<_{\mathfrak{D}}(x), <_{\text{CU}}(x)$ are \subseteq initial segments of $\mathfrak{D}(S), \text{CU}(S)$ respectively. In particular, $(<_{\mathfrak{D}}(x), \subseteq)$ and $(<_{\text{CU}}(x), \subseteq)$ are well orders.

Definition 2.9: Let $(S, <_S)$ be a tame order. We define the **tame rank** of $(S, <_S)$ to be the ordertype of the well-ordered set $(\text{CU}(S), \subset)$, and denote it by $\text{Trank}(S, <_S)$

It is not difficult to see that for every tame order $(S, <_S)$,

$$\text{rank}(S, <_S) \leq \text{Trank}(S, <_S) < |S|^+.$$

PROPOSITION 2.10: *A well-founded reduced order $(S, <_S)$ is tame if and only if it embeds in some R_λ . Moreover, $\lambda = \text{Trank}(S, <_S) < |S|^+$ is the minimal embedding ordinal.*

Proof of Proposition 2.10. Let $\lambda \in \text{On}$ and $X \subseteq R_\lambda$. The order $(X, <_{R_\lambda} \upharpoonright X)$ is clearly well-founded and reduced. To show that it does not embed $R_{2,2}, S_{\omega,2}$, it is sufficient to check that R_λ does not embed these orders:

- (1) $(R_\lambda$ does not embed $R_{2,2})$

Let $(a_0, b_0), (A_0, B_0), (a_1, b_1), (A_1, B_1)$ be four elements in R_λ and suppose that $(a_i, b_i) <_{R_\lambda} (A_i, B_i)$ for $i \in \{0, 1\}$ (i.e., they satisfy all $R_{2,2}$ relations). We claim that $<_{R_\lambda}$ must satisfy an additional relation which is not compatible with $R_{2,2}$. Indeed if $i \in \{0, 1\}$ satisfies $A_i = \max(A_0, A_1)$ then $b_0, b_1 < A_i$. Hence both $(a_0, b_0), (a_1, b_1)$ are $<_{R_\lambda}$ smaller than (A_i, B_i) .

- (2) $(R_\lambda$ does not embed $S_{\omega,2})$

Let $X = \{(m_n^x, M_n^x) \mid n < \omega\} \cup \{(m_n^y, M_n^y) \mid n < \omega\} \subset R_\lambda$. Let $\pi : S_{\omega,2} \rightarrow R_\lambda$ be defined by $\pi(x_n) = (m_n^x, M_n^x)$ and $\pi(y_n) = (m_n^y, M_n^y)$, $n < \omega$. We claim that π cannot be an embedding of $(S_{\omega,2}, <_{S_{\omega,2}})$ into $(R_\lambda, <_{R_\lambda})$. Otherwise setting $m^* = \min(\{m_n^y \mid n < \omega\}) < \lambda$ and $n^* = \min(\{n < \omega \mid m_n^y = m^*\})$, we get that $\pi(x_{n^*}) <_{R_\lambda} \pi(y_{n^*})$, i.e., $M_{n^*}^x < m_{n^*}^y = m^*$. It follows that for every $n > n^*$, $M_n^x < m^* \leq m_n^y$ thus $\pi(x_{n^*}) <_{R_\lambda} \pi(y_n)$. But this is incompatible with $<_{S_{\omega,2}}$.

It follows that $(R_\lambda, <_{R_\lambda})$ is tame, and it is easy to see that $\text{Trank}(R_\lambda) = \lambda$. Therefore if $(S, <_S)$ is tame and embeds in R_λ then $\text{Trank}(S, <_S) \leq \lambda$.

Next, suppose that $(S, <_S)$ is a reduced well-founded order which does not embed $R_{2,2}$ or $S_{\omega,2}$, and let $\lambda = \text{Trank}(S, <_S)$. Define functions $m, M : S \rightarrow \lambda$ by

$$m(x) = \text{otp}(<_{\mathfrak{D}}(x), \sqsubseteq) \quad \text{and} \quad M(x) = \text{otp}(<_{\text{CU}}(x), \sqsubseteq).$$

We claim that the map $\pi : S \rightarrow R_\lambda$, defined by $\pi(x) = (m(x), M(x))$, is an embedding of $(S, <_S)$ into $(R_\lambda, <_{R_\lambda})$.

We have that $(S, <_S)$ is reduced, therefore for every distinct $x, y \in S$, $(d(x), \text{cu}(x)) \neq (d(y), \text{cu}(y))$. We get that one of $<_{\mathfrak{D}}(x), <_{\mathfrak{D}}(y)$ is a \sqsubseteq -strict initial segment of the other, or one of $<_{\text{CU}}(x), <_{\text{CU}}(y)$ is a \sqsubseteq -strict initial segment of the other. Hence $m(x) = \text{otp}(<_{\mathfrak{D}}(x), \sqsubseteq) \neq \text{otp}(<_{\mathfrak{D}}(y), \sqsubseteq) = m(y)$, or $M(x) = \text{otp}(<_{\text{CU}}(x), \sqsubseteq) \neq \text{otp}(<_{\text{CU}}(y), \sqsubseteq) = M(y)$. Therefore π is injective.

The next three claims show that π is order preserving:

CLAIM 1: For every $x \in S$ there exists a \subseteq -order preserving injection $f : <_{\mathfrak{D}}(x) \rightarrow <_{\text{CU}}(x)$, thus $m(x) \leq M(x)$.

Let $d \in <_{\mathfrak{D}}(x)$ and define $f(d)$ as follows: Let $d^+ \in \mathfrak{D}(S)$ be the \subseteq immediate successor of d , i.e., $d \subsetneq d^+ \subseteq d(x)$. Pick an element $y_d \in d^+ \setminus d$ and set $f(d) = \text{cu}(y_d)$.

To show that $f(d)$ belongs to $<_{\text{CU}}(x)$, note that $y_d \in d^+ \subset d(x)$, so $y_d <_S x$ and $x \in \text{cu}(x) \setminus \text{cu}(y_d)$. As $\text{cu}(x)$ and $\text{cu}(y_d)$ are \subseteq comparable it must mean that $\text{cu}(y_d) \subsetneq \text{cu}(x)$.

As pointed out in Remarks 2.8, we have that $d^+ = d(z)$ for some $z \in S$. Therefore $y_d <_S z$. Let $d' \in <_{\mathfrak{D}}(x)$ so that $d \subsetneq d'$; $d^+ = d(z) \subset d' \subsetneq (d')^+$ implies that $y_{d'} \not<_S z$ and therefore $z \in \text{cu}(y_{d'}) \setminus \text{cu}(y_d)$, so

$$f(d) = \text{cu}(y_d) \subsetneq \text{cu}(y_{d'}) = f(d').$$

Hence f is \subseteq -order preserving.

CLAIM 2: For every $x <_S y$ there is a \subseteq -order preserving injection $g : <_{\text{CU}}(x) \rightarrow <_{\mathfrak{D}}(y)$, witnessing that $M(x) < m(y)$.

Let $\text{cu}(z) \in <_{\text{CU}}(x)$ and define $g(\text{cu}(z))$ as follows: Let $z^+ \in S$ so that $\text{cu}(z^+)$ is the \subseteq immediate successor of $\text{cu}(z)$. Let

$$\Gamma_{\text{cu}(z)} = \{d(w) \mid w \in \text{cu}(z^+) \setminus \text{cu}(z)\}$$

and let $g(\text{cu}(z)) \in \Gamma_{\text{cu}(z)}$ be a \subseteq minimal set in Γ_z (it is actually unique). Also pick $w_{\text{cu}(z)} \in \text{cu}(z^+) \setminus \text{cu}(z)$ so that $g(\text{cu}(z)) = d(w_{\text{cu}(z)})$. We get that $w_{\text{cu}(z)} \not<_S x$ as $w_{\text{cu}(z)} \in \text{cu}(z^+) \subseteq \text{cu}(x)$, and $z <_S w_{\text{cu}(z)}$ as $w \notin \text{cu}(z)$.

For every $w' \in S$, if $w' <_S w_{\text{cu}(z)}$ then $w' <_S y$ as otherwise $<_S \uparrow \{w_{\text{cu}(z)}, w', x, y\}$ would be isomorphic to $R_{2,2}$. It follows that $g(\text{cu}(z)) = d(w_{\text{cu}(z)}) \subseteq d(y)$. Moreover, $d(w_{\text{cu}(z)}) \subsetneq d(y)$ since $x \in d(y) \setminus d(w_{\text{cu}(z)})$. This shows that $g(\text{cu}(z)) \in <_{\mathfrak{D}}(y)$.

Let $z' \in S$ so that $\text{cu}(z) \subsetneq \text{cu}(z') \in <_{\text{CU}}(x)$. We have $w_{\text{cu}(z)} \in \text{cu}(z^+) \subseteq \text{cu}(z')$, i.e., $z' \not<_S w_{\text{cu}(z)}$ and therefore $z' \in d(w_{\text{cu}(z')}) \setminus d(w_{\text{cu}(z)})$. It follows that $g(\text{cu}(z)) = d(w_{\text{cu}(z)}) \subsetneq d(w_{\text{cu}(z')}) = g(\text{cu}(z'))$. Therefore g is \subseteq -order preserving.

Suppose that $\text{otp}(<_{\mathfrak{D}}(y), \subseteq) = \rho + n$ where ρ is a limit ordinal and $n < \omega$. Let $\langle d_i \mid i < \rho + n \rangle$ be a \subseteq -continuous increasing enumeration of $<_{\mathfrak{D}}(y)$. In order to prove $M(x) < m(y)$ it is sufficient to verify that $d_\rho = \bigcup_{i < \rho} d_i \notin \text{rng}(g)$. We consider the following three cases which address the identity of d_ρ :

- (1) If $\rho = 0$ then $d_\rho = \emptyset$, as $\emptyset = d(t) \in \langle_D (x)$ for every \langle_S minimal element $t \langle_S x$. We saw that $x \in d(w_{\text{cu}(z)}) = g(\text{cu}(z))$ for every $\text{cu}(z) \in \langle_{\text{CU}} (x)$, therefore $g(\text{cu}(z)) \neq \emptyset$ for every $\text{cu}(z) \in \text{dom}(g)$.
- (2) If $d_\rho \in \mathfrak{D}(S) \setminus D(S)$ then $d_\rho \notin \text{rng}(g)$, since $g(\text{cu}(z)) = d(w_{\text{cu}(z)}) \in D(S)$ for every $\text{cu}(z) \in \langle_{\text{CU}} (x)$.
- (3) Suppose that $d_\rho = d(w)$ for some $w \in S$. Recall that for every $\text{cu}(z) \in \langle_{\text{CU}} (x)$, $g(\text{cu}(z))$ is a \subseteq minimal set in

$$\Gamma_{\text{cu}(z)} = \{d(w) \mid w \in \text{cu}(z^+) \setminus \text{cu}(z)\}.$$

Therefore to show $d_\rho \notin \text{rng}(g)$, it is sufficient to verify d_ρ is not \subseteq -minimal in $\Gamma_{\text{cu}(z)}$ for any $\text{cu}(z) \in \langle_{\text{CU}} (x)$. For this, note that $d_\rho = d(w) \in \Gamma_{\text{cu}(z)}$ implies $z \in d_\rho = \bigcup_{i < \rho} d_i$. Let $i < \rho$ be a successor ordinal such that $z \in d_i = d(w_i)$. Since $z^+ \notin d_\rho$ and $d(w_i) \subseteq d_\rho$, we get that $w_i \in \text{cu}(z^+) \setminus \text{cu}(z)$, witnesses that d_ρ is not \subseteq -minimal.

CLAIM 3: For every $x \not\langle_S y$ there is a \subseteq -preserving injection $h : \langle_{\mathfrak{D}}(y) \rightarrow \langle_{\text{CU}} (x)$, thus $m(y) \leq M(x)$.

Let $d \in \langle_{\mathfrak{D}} (y)$ and define $h(d)$ as follows: Let $d^+ \in \mathfrak{D}(S)$ be the \subseteq immediate successor of d , i.e., $d \subsetneq d^+ \subseteq d(y)$. Pick an element $w_d \in d^+ \setminus d$ and set $h(d) = \text{cu}(w_d)$. Since $\text{cu}(w_D), \text{cu}(y)$ are \subseteq comparable, and $y \in \text{cu}(x) \setminus \text{cu}(w_d)$ (as $x \not\langle_S y$), we get $h(d) = \text{cu}(w_d) \subsetneq \text{cu}(x)$, i.e., $h(d) \in \langle_{\text{cu}} (x)$. Let $d' \in \langle_{\mathfrak{D}} (y)$ so that $d \subsetneq d'$; $d^+ = d(z)$ for some $z \in S$. We have $d(z) \subset d'$ and $w_{d'} \notin d'$, so $z \in \text{cu}(w_{d'}) \setminus \text{cu}(w_d)$ and thus $h(d) = \text{cu}(w_d) \subsetneq \text{cu}(w_{d'}) = h(d')$. Therefore h is \subsetneq -order preserving. ■

Remark 2.11:

- (1) Definition 2.2 (of tame orders) is slightly different from the author’s original (equivalent) definition, where S is tame, if it does not embed $R_{2,2}$ and the linear ordered sets $(\mathfrak{D}(S), \subset)$, $(\text{CU}(S), \subset)$ are well-orders. The author would like to thank the referee for pointing out that the last property is equivalent to the fact that S does not embed $S_{\omega,2}$ as well.
- (2) In [5], James Cummings constructed a model in which $\triangleleft(\kappa)$ is divided into blocks $\{M(\alpha, \beta) \mid \alpha < o(\kappa), \beta \in (\alpha, o(\kappa)) \cup \{\infty\}\}$. The blocks determine the \triangleleft structure in this model, where for every $U' \in M(\alpha', \beta')$ and $U \in M(\alpha, \beta)$, $U' \triangleleft U$ if and only if $\beta' \leq \alpha$. It is not difficult to see that this order is tame.

3. The posets \mathcal{P}^0 and \mathcal{P}^1

The purpose of this section is to introduce the main poset $\mathcal{P} = \mathcal{P}^0 * \mathcal{P}^1$, comprised of \mathcal{P}^0 , a Friedman–Magidor forcing, introduced in [7], and of \mathcal{P}^1 , a Magidor iteration of Prikry-type forcings. The Magidor iteration of Prikry-type forcings was introduced by Magidor in [13] (see [9] for an extensive survey). The definitions of \mathcal{P}^0 and \mathcal{P}^1 rely on certain parameters, chosen relative to $\lambda = o^V(\kappa)$. To simplify the presentation we restrict the presentation of $\mathcal{P}^0, \mathcal{P}^1$ in this section to when $\lambda \leq \kappa$. The more general case, $\lambda < \kappa^+$, will be treated in Section 7.

Suppose that the ground model is a Mitchell model, $V = L[\mathcal{U}]$, so that $\mathcal{U} = \langle U_{\nu, \tau} \mid \nu \leq \kappa, \tau < o(\alpha) \rangle$ is a coherent sequence of normal measures. When $\nu = \kappa$ we write U_τ to denote $U_{\kappa, \tau}$, for every $\tau < \lambda$. For every $\tau < \lambda$ let $\Delta_\tau = \{\nu < \kappa \mid o(\nu) = \tau\}$. The sets $\{\Delta_\tau \mid \tau < \lambda\}$ are pairwise disjoint as $\lambda \leq \kappa$. For every $\alpha < \lambda$, let $j_\alpha : V \rightarrow M_\alpha \cong \text{Ult}(V, U_\alpha)$ be the induced ultrapower embedding. Therefore $U_\alpha \in M_\beta$ if and only if $\alpha < \beta < \lambda$. Let $j_\alpha^{M_\beta} : M_\beta \rightarrow M_{\alpha, \beta} \cong \text{Ult}(M_\beta, U_\alpha)$, and $j_{\alpha, \beta} = j_\alpha^{M_\beta} \circ j_\beta : V \rightarrow M_{\alpha, \beta}$. Now $j_{\alpha, \beta}$ is known to be equivalent to the ultrapower embedding induced by the product measure $U_\alpha \times U_\beta$; $j_{\alpha, \beta}$ can also be formed using a normal iteration. Taking $j_\alpha(U_\beta) \in M_\alpha$, if $i_\beta^{M_\alpha} : M_\alpha \rightarrow M_{\alpha, \beta} \cong \text{Ult}(M_\alpha, j_\alpha(U_\beta))$ denote the induced ultrapower embedding of M_α by $j_\alpha(U_\beta)$; then $j_{\alpha, \beta} = i_\beta^{M_\alpha} \circ j_\alpha$. The iteration is normal since the sequence of critical points $\langle \kappa_0, \kappa_1 \rangle = \langle \kappa, j_\alpha(\kappa) \rangle$ is increasing. The fact that $j_\beta(\kappa) > \kappa$ is inaccessible in M_β implies that $j_{\alpha, \beta}(\kappa) = j_\alpha^{M_\beta}(j_\beta(\kappa)) = j_\beta(\kappa)$.

3.1. THE POSET \mathcal{P}^0 . The forcing \mathcal{P}^0 used here was introduced by Friedman and Magidor in [7]. Now $\mathcal{P}^0 = \mathcal{P}_{\kappa+1}^0 = \langle \mathcal{P}_\nu^0, \mathcal{Q}_\nu^0 \mid \nu \leq \kappa \rangle$ is a non-stationary support iteration. Conditions $p \in \mathcal{P}_\nu^0$ are denoted by $p = \langle p_\mu \mid \mu < \nu \rangle$. Non-stationary support means that for every limit $\nu \leq \kappa$, every $p \in \mathcal{P}_\nu^0$ belongs to the inverse limit of the posets $\langle \mathcal{P}_\mu^0 \mid \mu < \nu \rangle$, with the restriction that if ν is inaccessible then the set of $\mu < \nu$ such that p_μ is non-trivial is a non-stationary subset of ν . For every $\nu \leq \kappa$, if ν is non-inaccessible then $\Vdash_{\mathcal{P}_\nu^0} \mathcal{Q}_\nu^0 = \emptyset$, otherwise $\Vdash_{\mathcal{P}_\nu^0} \mathcal{Q}_\nu^0 = \text{Sacks}_{\lambda(\nu)}(\nu) * \text{Code}(\nu)$, where for every $\nu < \kappa$ we set

$$\lambda(\nu) = \begin{cases} \lambda & \text{if } \lambda < \kappa, \\ \nu & \text{if } \lambda = \kappa. \end{cases}$$

Conditions in $\text{Sacks}_\nu(\nu)$ are trees $T \subset {}^{<\nu}\nu$ for which there is a closed unbounded $C \subset \nu$ so that whenever $s \in T$, if $\text{len}(s) \in C$ then $s \frown \langle i \rangle \in T$ for all $i < \lambda(\text{len}(s))$. $\text{Code}(\nu)$ is a coding poset which adds a club set to ν^+ coding the $\text{Sacks}_{\lambda(\nu)}(\nu)$ generic function $s_\nu : \nu \rightarrow \nu$ by destroying stationary subsets of $\text{Cof}(\nu) \cap \nu^+$ in V (see [7]). Let $G^0 \subset \mathcal{P}^0$ be a generic filter over V , and for every $\nu \leq \kappa$ let $G^0(\mathcal{Q}_\nu^0) = \bigcup \{ \dot{p}_{\nu G^0 \upharpoonright \nu} \mid p \in G^0 \}$ be the induced \mathcal{Q}_ν^0 -generic over $V[G^0 \upharpoonright \nu]$. For every non-trivial stage $\nu \leq \kappa$ in the iteration \mathcal{P}^0 , let $s_\nu : \nu \rightarrow \lambda(\nu)$ denote the generic Sacks function specified by $G^0(\mathcal{Q}_\nu^0)$. For every $\eta < \lambda$ define

$$\Delta(\eta) = \{ \nu < \kappa \mid s_\nu = s_\kappa \upharpoonright \nu, \text{ and } s_\kappa(\nu) = \eta \}.$$

The sets in $\{ \Delta(\eta) \mid \eta < \lambda \}$ are clearly pairwise disjoint. According to Friedman and Magidor [7], $V[G^0]$ satisfies the following properties:

Fact 3.1:

- (1) $V[G^0]$ agrees with V on all cardinals and cofinalities.
- (2) For every normal measure U on κ and $\eta < \lambda$, there is a unique normal measure, $U(\eta) \in V[G^0]$ containing $U \cup \{ \Delta(\eta) \}$. Furthermore, these are the only normal measures on κ in $V[G^0]$.
- (3) For every $\eta < \lambda$, let $j_{U(\eta)} : V[G^0] \rightarrow M_{U(\eta)}^0 \cong \text{Ult}(V[G^0], U(\eta))$ be the induced ultrapower embedding of $V[G^0]$ by $U(\eta)$. We have:
 - (a) $j_{U(\eta)} \upharpoonright V = j_U : V \rightarrow M_U \cong \text{Ult}(V, U)$ is the induced ultrapower embedding of V by U .
 - (b) $M_{U(\eta)}^0 = M_U[G_{U(\eta)}^0]$ where $G_{U(\eta)}^0 \subset j_U(\mathcal{P}^0)$ is M_U generic.
 - (c) $G_{U(\eta)}^0 \upharpoonright \kappa + 1 = G^0$.
 - (d) $\bigcup j_U \text{``}(G^0 \upharpoonright \kappa)$ completely determines $G_{U(\eta)}^0 \upharpoonright j_U(\kappa) \subset j_U(\mathcal{P}^0) \upharpoonright j_U(\kappa)$.
In particular, $G_{U(\eta)}^0 \upharpoonright j_U(\kappa)$ is independent of $\eta < \lambda$.
 - (e) For every $p \in G^0$ let $j_U(p)(\eta)$ be the condition obtained from $j_U(p)$ by reducing its Sacks tree at level κ , $T = (j_U(p))_\kappa$, to the set of functions $s \in \text{Lev}_{\kappa+1}(T)$ which satisfy $s(\kappa) = \eta$. For every dense open set $D \subset j_U(\mathcal{P}^0)$, there exists $p \in G^0$ so that $j_U(p)(\eta) \in D$.

Definition 3.2:

- (1) Let $U_{(\alpha, \eta)}^0$ denote the $V[G^0]$ extension $U(\eta)$ described above, for $U = U_\alpha$.
- (2) Let $j_{(\alpha, \eta)}^0 : V[G^0] \rightarrow M_\alpha[G_\alpha^0(\eta)]$ denote the ultrapower embedding of $V[G^0]$ by $U_{(\alpha, \eta)}^0$.

We write $G_\alpha^0(\eta) = j_{(\alpha,\eta)}^0(G^0)$ to denote the $j_\alpha(\mathcal{P}^0)$ generic filter $G_{U(\eta)}^0$ described above with $U = U_\alpha$. It is clear that $j_{(\alpha,\eta)}^0 \upharpoonright V = j_\alpha$.

Definition 3.3:

- (1) For every $\alpha < o(\kappa)$ let $\Delta_\alpha = \{\nu < \kappa \mid o(\nu) = \alpha\}$.
- (2) For $\alpha < o(\kappa)$ and $\eta < \lambda$ let $\Delta_\alpha(\eta) = \Delta(\eta) \cap \Delta_\alpha$.

Based on the above definition, it immediately follows that:

- (1) $\{\Delta_\alpha(\eta) \mid \alpha, \eta < \lambda\}$ are pairwise disjoint.
- (2) $\Delta_\alpha(\eta) \in U_{(\alpha,\eta)}^0$ for every $\alpha, \eta < \lambda$.

We can easily describe the iterated ultrapowers using $U_{(\alpha,\eta)}^0$, $\alpha, \eta < \lambda$. According to [7], $j_{\alpha,\beta} G^0$ determines a unique generic filter in $j_{\alpha,\beta}(\mathcal{P}^0)$ up to the following values:

- (1) a tuning fork at the value $s_{j_\alpha(\kappa)}(\kappa) < \lambda$, and
- (2) a tuning fork at the value $s_{j_{\alpha,\beta}(\kappa)}(j_\alpha(\kappa)) < j_\alpha(\lambda)$.

For every $\eta_\alpha < \lambda$ and $\eta_\beta < j_\alpha(\lambda)$ there exists a unique extension of $j_{\alpha,\beta}$ to an embedding of $V[G^0]$, which determines $s_{j_\alpha(\kappa)}(\kappa) = \eta_\alpha$ and $s_{j_{\alpha,\beta}(\kappa)}(j_\alpha(\kappa)) = \eta_\beta$. In this paper we are only interested in values $\eta_\alpha, \eta_\beta < \lambda$.

Notations 3.4:

- (1) Denote $V[G^0]$ by V^0 .
- (2) For every $\alpha < o(\kappa) = \lambda$ and $\eta < \lambda$, let $j_{(\alpha,\eta)}^0 : V^0 \rightarrow M_{(\alpha,\eta)}^0$ denote the ultrapower induced embedding

$$j_{(\alpha,\eta)}^0 : V[G^0] \rightarrow M_\alpha[G_\alpha^0(\eta)] \cong \text{Ult}(V^0, U_{(\alpha,\beta)}^0).$$

- (3) For every $\eta_\alpha, \eta_\beta < \lambda$, let

$$i_{(\beta,\eta_\beta)}^{0,M(\alpha,\eta_\alpha)} : M_{(\alpha,\eta_\alpha)}^0 \rightarrow M_{(\alpha,\eta_\alpha),(\beta,\eta_\beta)}^0 \cong \text{Ult}(M_{(\alpha,\eta_\alpha)}^0, j_{(\alpha,\eta_\alpha)}^0(U_{(\beta,\eta_\beta)}^0))$$

denote the ultrapower of $M_{(\alpha,\eta_\alpha)}^0$ by $j_{(\alpha,\eta_\alpha)}^0(U_{(\beta,\eta_\beta)}^0)$, and let

$$j_{(\alpha,\eta_\alpha),(\beta,\eta_\beta)}^0 = i_{(\beta,\eta_\beta)}^{0,M(\alpha,\eta_\alpha)} \circ j_{(\alpha,\eta_\alpha)}^0 : V[G^0] \rightarrow M_{(\alpha,\eta_\alpha),(\beta,\eta_\beta)}^0.$$

The following summarizes the connections between iterated ultrapowers of V and V^0 :

- (1) $j_{(\alpha,\eta_\alpha)}^0 \upharpoonright V = j_\alpha$,
- (2) $i_{(\beta,\eta_\beta)}^{0,M(\alpha,\eta_\alpha)} \upharpoonright M_\alpha = i_\beta^{M_\alpha}$,
- (3) $j_{(\alpha,\eta_\alpha),(\beta,\eta_\beta)}^0 \upharpoonright V = j_{\alpha,\beta}$,

$$(4) \ s_{j_\alpha(\kappa)}^{M_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)}^0}(\kappa) = \eta_\alpha,^1$$

$$(5) \ s_{j_{\alpha, \beta}(\kappa)}^{M_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)}^0}(j_\alpha(\kappa)) = \eta_\beta.$$

Suppose that $\alpha < o(\kappa) = \lambda$ and $\eta < \lambda$. The definition of $U_{(\alpha, \eta)}^0$ requires knowledge of U_α and G^0 . For every $\beta > \alpha$ and $\eta' < \lambda$, $U_\alpha \in M_\beta \subset M_{(\beta, \eta')}^0$ and $G^0 = G_\beta^0(\eta') \upharpoonright (\kappa + 1) \in M_{(\beta, \eta')}^0$. We conclude the following:

COROLLARY 3.5: *For every $\alpha < \beta < o(\kappa)$ and $\eta, \eta' < \lambda$, $U_{(\alpha, \eta)}^0 \triangleleft U_{(\beta, \eta')}^0$.²*

The description of the normal measures on κ in a \mathcal{P}^0 extension applies to all measurable cardinals $\nu < \kappa$: Let

$$\vec{U}_\nu = \langle U_{\nu, \alpha} \mid \alpha < o(\nu) \rangle$$

be an \triangleleft -increasing sequence of the normal measures on ν in the coherent sequence $\mathcal{U} \in V = L[\mathcal{U}]$. Each normal measure $U_{\nu, \alpha} \in V$ extends to $\lambda(\nu)$ normal measures in $V[G^0 \upharpoonright (\nu + 1)]$, denoted $\{U_{\nu, (\alpha, \eta)}^0 \mid \eta < \lambda(\nu)\}$. No measures on ν are added or removed by the rest of the iteration since $\mathcal{P}^0 \setminus (\nu + 2)$ is $2^{(2^\nu)}$ -distributive. Furthermore, we get that $U_{\nu, (\alpha, \eta)}^0 \triangleleft U_{\nu, (\beta, \eta')}^0$ for every $\alpha < \beta < o(\nu)$ and $\eta, \eta' < \lambda(\nu)$.

3.2. THE POSET \mathcal{P}^1 . The poset $\mathcal{P}^1 = \langle \mathcal{P}_\nu^1, \dot{Q}_\nu^1 \mid \nu < \kappa \rangle$ is a Magidor iteration of one-point Prikry forcings. See [9] for a comprehensive survey of Magidor iteration of Prikry-type forcings. One-point Prikry forcing is a simplified version of the well known Prikry forcing. The one-point version at a measurable cardinal ν chooses a single (indiscernible) ordinal $d(\nu) < \nu$, instead of a cofinal ω sequence.

Let U be a normal measure on ν . The one-point Prikry forcing $Q(U)$ consists of elements $p \in U \cup \nu$. For every $q, q' \in Q(U)$ we set:

(1) $q \geq_{Q(U_\nu)}^* q'$ (i.e., q is a direct extension of q') if and only if either $q, q' \in U$ and $q \subset q'$, or $q, q' \in \nu$ and $q = q'$.

(2) $q \geq_{Q(U_\nu)} q'$ if and only if $q \geq_{Q(U_\nu)}^* q'$ or $q \in \nu, q' \in U$, and $q \in q'$.

$(Q(U_\nu), \geq_{Q(U_\nu)}, \geq_{Q(U_\nu)}^*)$ is a Prikry-type forcing notion ([9]).

Let us describe the way \mathcal{P}^1 is formed from certain one-point Prikry forcings. Let $\nu \leq \kappa$ be a measurable cardinal in V . Suppose \mathcal{P}_ν^1 has been defined

¹ Here $s_\nu(\mu)$ denotes the value of the generic ν -Sacks at μ .

² It is not difficult to verify that $U_{(\alpha, \eta)}^0 \triangleleft U_{(\beta, \eta')}^0$ if and only if $\alpha < \beta$.

and $\vec{U}_\nu^1 = \langle U_{\nu,(\alpha,\beta)}^1 \mid \beta < o(\nu), \alpha < \lambda(\nu) \rangle$ is a given sequence of normal measures on ν in a \mathcal{P}_ν^1 generic extension of V^0 .

Definition 3.6 (recipe for \mathcal{Q}_ν^1): Let $\alpha, \beta < \lambda$ be the unique ordinals so that $\nu \in \Delta_\alpha(\beta)$ (i.e., $o(\nu) = \alpha$ and $s_\kappa(\nu) = \beta$). Define \mathcal{Q}_ν^1 by

$$\mathcal{Q}_\nu^1 = \begin{cases} Q(U_{\nu,(\alpha,\beta)}^1) & \text{if } \beta < \alpha, \\ 0\text{---the trivial forcing} & \text{otherwise.} \end{cases}$$

To complete the definition of \mathcal{Q}_ν^1 , we need to define the normal measures in $\vec{U}_\nu^1 = \langle U_{\nu,(\alpha,\beta)}^1 \mid \beta < o(\nu), \alpha < \lambda(\nu) \rangle$. These are given in Definitions 3.8 and 3.10. Note, however, that Definition 3.8 is the only one which applies to \mathcal{Q}_ν^1 .

To simplify the notations, let us assume that $\nu = \kappa$ and use the abbreviations $U_{(\alpha,\beta)}^0$ for $U_{\kappa,(\alpha,\beta)}^0$, and $U_{(\alpha,\beta)}^1$ for $U_{\kappa,(\alpha,\beta)}^1$. Our definitions make use of the embeddings $j_{(\alpha,\eta)}^0$, $i_{(\beta,\eta_\beta)}^{0,M(\alpha,\eta_\alpha)}$, and $j_{(\alpha,\eta_\alpha),(\beta,\eta_\beta)}^0$, introduced in Notations 3.4 above.

Definition 3.7 (Prikrý function):

- (1) Let $\Delta' = \{ \nu \in \Delta \mid 0_{\mathcal{P}_\nu^1} \Vdash \mathcal{Q}_\nu^1 \text{ is not trivial} \}$.
- (2) Let $\dot{d} : \Delta' \rightarrow \kappa$, be the \mathcal{P}^1 name for the **generic Prikrý function**, so that for every V^0 generic filter $G^1 \subset \mathcal{P}^1$, $d(\nu) < \nu$ is the \mathcal{Q}_ν^1 generic point given by G^1 .

Let $G^1 \subset \mathcal{P}^1$ be generic over V^0 .

Definition 3.8 ($U_{(\alpha,\beta)}^1$ for $\alpha \geq \beta$): Suppose that $\alpha \geq \beta$, where $\beta < o(\kappa)$ and $\alpha < \lambda(\kappa) = \lambda$. Let X be a subset of κ in $V^0[G^1]$ and \dot{X} be a \mathcal{P}^1 name of X . Set $X \in U_{(\alpha,\beta)}^1$ if and only if there are $p \in G^1$ and $q \geq^* j_{(\beta,\alpha)}^0(p) \setminus \kappa$ so that $p \hat{\wedge} q \geq^* j_{(\beta,\alpha)}^0(p)$ is a condition in $j_{(\beta,\alpha)}^0(\mathcal{P}^1)$ and

$$(1) \quad p \hat{\wedge} q \Vdash_{j_{(\beta,\alpha)}^0(\mathcal{P}^1)} \check{\kappa} \in j_{(\beta,\alpha)}^0(\dot{X}).$$

Definition 3.8 implies that $U_{(\alpha,\beta)}^1$ extends $U_{(\beta,\alpha)}^0$ in V^0 . In particular, $\Delta_\beta(\alpha) \in U_{(\alpha,\beta)}^1$. It is not difficult to verify that $U_{(\alpha,\beta)}^1$ is a normal measure on κ . For proof see [9] or the description of U_0^* in [3].³

³ To verify the normality of $U_{(\alpha,\beta)}^1$, note that $\kappa \in j_{(\beta,\alpha)}^0(\Delta_\beta(\alpha))$, so by Definition 3.6, stage κ of $j_{(\beta,\alpha)}^0(\mathcal{P}^1)$ is trivial.

Remark 3.9: Suppose that $X \in U^1_{(\alpha,\beta)}$, $\alpha \geq \beta$, and let $p \in G^1$ and $q \geq^* j^0_{(\beta,\alpha)}(p) \setminus \kappa$ as in Definition 3.8 above. Essentially, one can assimilate q into $j^0_{(\beta,\alpha)}(p)$, and use it to produce a simpler characterization for the sets in $U^1_{(\alpha,\beta)}$: Let Q be a function representing q in $M^0_{(\beta,\alpha)}$ where $Q(\alpha) \geq^* p \setminus \alpha$ for every $\alpha < \kappa$. Let $t \geq^* p$ be the condition obtained from p by reducing each p_ν , $\nu \notin \text{supp}(p)$ to $t_\nu = p_\nu \cap \Delta_{\alpha < \nu} Q(\alpha)_\nu$. It follows that for every $\alpha < \kappa$, $t^{-\alpha} \geq^* Q(\alpha)$, where $t^{-\alpha}$ is the condition obtained from t by replacing each set t_ν , $\nu > \alpha$, with $t_\nu \setminus \alpha + 1$.

By a standard density argument it follows that for every $X \in U^1_{(\alpha,\beta)}$, $\alpha \geq \beta$, there is some $p \in G^1$ so that

$$j^0_{(\beta,\alpha)}(p)^{-\kappa} \Vdash \check{\kappa} \in j^0_{(\beta,\alpha)}(\dot{X}).$$

We proceed to define $U^1_{(\alpha,\beta)}$ when $\alpha < \beta$. We first introduce the following auxiliary definitions.

Definition 3.10 ($k^0_{\alpha,\beta}$ and $p^{+(\mu,\nu)}$):

- (1) For $\alpha < \beta < \lambda = o(\kappa)$, let $k^0_{\alpha,\beta} : V^0 \rightarrow N^0_{\alpha,\beta}$ denote the iterated ultrapower $j^0_{(\alpha,\beta),(\beta,\alpha)} : V^0 \rightarrow M^0_{(\alpha,\beta),(\beta,\alpha)}$ (introduced in Notations 3.4).
- (2) For every condition $p \in \mathcal{P}^1$, $\nu < \kappa$ so that $p \upharpoonright \nu \Vdash \dot{p}_\nu \in Q(U^*_\nu)$, and $\mu < \nu$, let $p^{+(\mu,\nu)}$ denote the condition obtained from \dot{p} by replacing \dot{p}_ν with $\{\check{\mu}\}$, i.e., $p^{+(\mu,\nu)} \Vdash \check{\mu} = \dot{d}(\check{\nu})$.

Note that $p^{+(\mu,\nu)}$ is not necessarily an extension of p . If $p \upharpoonright \nu \Vdash \check{\mu} \in \dot{p}_\nu$ then $p^{+(\mu,\nu)}$ is an extension of p .

Definition 3.11 ($U^1_{(\alpha,\beta)}$ for $\alpha < \beta$): Let $\alpha < \beta < \lambda$. In $V^0[G^1]$ define $U^1_{(\alpha,\beta)}$ to be the set of all $X = \dot{X}_{G^1} \subseteq \kappa$ for which there are $p \in G^1$ and $q \geq^* k^0_{\alpha,\beta}(p) \setminus \kappa$ such that $(p \frown q)^{+(\kappa, j^0_{(\alpha,\beta)}(\kappa))} \geq p \frown q$, and

$$(2) \quad (p \frown q)^{+(\kappa, j^0_{(\alpha,\beta)}(\kappa))} \Vdash \check{\kappa} \in k^0_{\alpha,\beta}(\dot{X}).$$

For proof that $U^1_{(\alpha,\beta)}$ is a normal measure on κ , see [3].⁴ It follows that $U^1_{(\alpha,\beta)}$ extends $U^0_{(\alpha,\beta)}$ and, in particular, $\Delta_\alpha(\beta) \in U^1_{(\alpha,\beta)}$.

Remark 3.12: Similar to Remark 3.9, one can show that for every $X \in U^1_{(\alpha,\beta)}$ there exists a condition $p \in G^1$ so that

$$k^0_{\alpha,\beta}(p)^{+(\kappa, j^0_{(\beta,\gamma)}(\kappa)) - \kappa - j^0_{(\beta,\gamma)}(\kappa)} \Vdash \check{\kappa} \in k^0_{\alpha,\beta}(\dot{X}).$$

See [3] for further details.

⁴ See the proof for the normality of U^1_κ .

3.3. SEPARATION BY SETS. Let $G^1 \subset \mathcal{P}^1$ be a generic filter over $V^0 = V[G^0]$. Let us denote $V^0[G^1]$ by V^1 . The normal measures $\{U^1_{(\alpha,\beta)} \mid \alpha \leq \beta < \lambda\}$ will be used to realize R_λ . As mentioned in the introduction, we would like the normal measure on κ in V^1 to be separated by sets.

Definition 3.13 ($\Gamma, X_{(\alpha,\beta)}$): Define sets in V^1 :

- (1) $\Gamma = d^{\Delta'}$, the set of Prikry generic points.
- (2) For every $\alpha, \beta < \lambda$,

$$X_{(\alpha,\beta)} = \begin{cases} \Delta_\beta(\alpha) \setminus \Gamma & \text{if } \alpha \geq \beta, \\ \Delta_\alpha(\beta) \cap \Gamma & \text{if } \alpha < \beta. \end{cases}$$

A simple inspection of Definitions 3.8 and 3.11 shows that $X_{(\alpha,\beta)} \in U^1_{(\alpha,\beta)}$ for every $\alpha, \beta < \lambda$.

COROLLARY 3.14: *The sets in $\{X_{(\alpha,\beta)} \mid \alpha, \beta < \lambda\}$ are pairwise disjoint, and $X_{(\alpha,\beta)} \in U^1_{(\alpha,\beta)}$ for all $\alpha, \beta < \lambda$. In particular, the measures in $\{U^1_{(\alpha,\beta)} \mid \alpha, \beta < \lambda\}$ are separated by sets.*

4. The restriction $j^1_{(\alpha,\beta)} \upharpoonright V$

For every $\alpha, \beta < \lambda$, let $j^1_{(\alpha,\beta)} : V^1 \rightarrow M^1_{(\alpha,\beta)} \cong \text{Ult}(V^1, U^1_{(\alpha,\beta)})$. The purpose of this section is to describe the restriction $j^1_{(\alpha,\beta)} \upharpoonright V^0$ as an iterated ultrapower of the measures in V^0 . For every $\alpha, \beta < \lambda$, we first define an iterated ultrapower T^0 resulting in an embedding $\pi^0_{\alpha,\beta} : V^0 \rightarrow Z^0_{\alpha,\beta}$. The main proposition in this section (Proposition 4.3) states that $\pi^0_{\alpha,\beta} = j^1_{(\alpha,\beta)} \upharpoonright V^0$. While the definition of $\pi^0_{\alpha,\beta}$ makes the statement natural, its proof requires preliminary technical results. For simplicity, we assume that the ground model V is a Mitchell model $V = L[\mathcal{U}]$ with $V = \mathcal{K}(V)$, where \mathcal{U} is the coherent sequence of normal measures and $o(\kappa) = \lambda$. In particular, V does not contain an overlapping extender. Let $\alpha, \beta < o(\kappa) = \lambda$. Our ground model assumption $V = \mathcal{K}(V) = L[\mathcal{U}]$ and the fact $V^1 = V[G^0 * G^1]$ imply the following:

- (1) $V = \mathcal{K}(V^1)$,
- (2) $j^1_{(\alpha,\beta)} \upharpoonright V : V \rightarrow Z_{\alpha,\beta}$ is an iterated ultrapower of V ,
- (3) if $j^1_{(\alpha,\beta)} : V^1 \rightarrow M^1_{\alpha,\beta}$ then $M^1_{\alpha,\beta} = Z_{\alpha,\beta}[G^0_{\alpha,\beta} * G^1_{\alpha,\beta}]$, where $G^0_{\alpha,\beta} * G^1_{\alpha,\beta} \subset j^1_{(\alpha,\beta)}(\mathcal{P}^0 * \mathcal{P}^1)$ is generic over $Z_{\alpha,\beta}$.

We refer to [19] for these results. The definition of $\pi_{\alpha,\beta}^0$ for $\alpha, \beta < \lambda$ makes use of the ultrapower embedding $j_{(\alpha,\beta)}^0 : V^0 \rightarrow M_{(\alpha,\beta)}^0$ defined in Definition 3.2, and the iterated ultrapower embedding $k_{\alpha,\beta}^0 : V^0 \rightarrow N_{\alpha,\beta}^0$ defined in Definition 3.10. Let $\vec{\Delta} = \langle \Delta_\alpha(\eta) \mid \alpha < o(\kappa), \eta < \lambda \rangle$.

Definition 4.1 ($\pi_{\alpha,\beta}^0$): Here $\pi_{\alpha,\beta}^0$ results from a linear iteration

$$T^0 = \langle Z_i^0, \sigma_{i,j}^0 \mid 0 \leq i < j \leq \theta \rangle,$$

with critical points $\nu_i = \text{cp}(\sigma_{i,i+1}^0)$ of length θ . Here Z_i^0 are the intermediate models (iterands) of the iteration, and $\sigma_{i,j}^0 : Z_i^0 \rightarrow Z_j^0$ are the connecting iterations. For every $i < \theta$ we denote the image of the i -th critical point ν_i , $\sigma_{i,i+1}^0(\nu_i)$, by ν_i^1 . We set $Z_0^0 = V^0$, $\sigma_{0,0}^0 = \text{id}_{Z_0^0}$, $Z_1^0 = N_{\alpha,\beta}^0$, and

$$\sigma_{0,1}^0 = \begin{cases} j_{(\beta,\alpha)}^0 & \text{if } \beta \leq \alpha, \\ k_{\alpha,\beta}^0 & \text{if } \alpha \geq \beta. \end{cases}$$

We define $\nu_0 = \kappa$, and set $\nu_0^1 = j_{(\alpha,\beta)}^0(\kappa) < \sigma_{0,1}^0(\kappa)$ if $\alpha < \beta$, and leave ν_0^1 undefined otherwise.

SUCCESSOR STAGE: Suppose that $T^0 \upharpoonright i$ has been defined up to stage $1 \leq i < \theta$; define Z_{i+1}^0 and $\sigma_{i,i+1}^0$ as follows: Let ν_i^* be the supremum of $\{\nu_j \mid j < i\}$ (the set of critical points in $T^0 \upharpoonright i$), and take ν_i to be the minimal ordinal $\nu \geq \nu_i^*$ which satisfies:

- (1) the forcing of $\sigma_{0,i}^0(\mathcal{P}^1)$ at stage ν is not trivial, i.e., $\nu \in \sigma_{0,i}^0(\Delta')$, and
- (2) ν does not belong to $\sigma_{0,i}^0$ “ $\{\nu_j^1 \mid j < i\}$ ”.

These two requirements imply that the critical points of the iteration T^0 are strictly increasing (i.e., the iteration is normal). Since $\nu_i \in \sigma_{0,i}^0(\Delta')$, then there are unique $\beta_i < \alpha_i$ so that $\nu_i \in \sigma^0(\Delta')_{\alpha_i}(\beta_i)$. We define

$$\sigma_{i,i+1}^0 = j_{\nu_i,(\beta_i,\alpha_i)}^0 : Z_i^0 \rightarrow Z_{i+1}^0$$

and set $\nu_i^1 = j_{\nu_i,(\beta_i,\alpha_i)}^0(\nu_i)$.

LIMIT STAGE: If $\delta < \theta$ is a limit ordinal, then we take Z_δ^0 to be the direct limit of $T^0 \upharpoonright \delta$.

The iteration terminates at stage θ , when $\sigma_{0,\theta}^0(\Delta') \subset \{\nu_i^1 \mid i < \theta\} \cup \nu_\theta^*$. Note that

$$\sigma_{0,1}^0(\kappa) = \begin{cases} j_{(\beta,\alpha)}^0(\kappa) & \text{if } \beta \leq \alpha, \\ k_{\alpha,\beta}^0(\kappa) & \text{if } \alpha \geq \beta. \end{cases}$$

By induction on $i < \theta$, it is not difficult to verify that $\sigma_{0,1}^0(\kappa)$ is a fixed point of $\sigma_{1,i}^0$. It follows that the iteration must terminate after at most $\sigma_{0,1}^0(\kappa)$ many steps as each $\nu_i < \sigma_{1,i}^0(\sigma_{0,1}^0(\kappa)) = \sigma_{0,1}^0(\kappa)$, and the iteration is normal.

The following facts summarize the main properties of $\pi_{\alpha,\beta}^0$, and can be easily proved by induction on $1 \leq i < \theta$.

COROLLARY 4.2:

(1) For every $\alpha \geq \beta$ we have:

- $\sigma_{0,1}^0 = j_{(\beta,\alpha)}^0$, $\nu_0 = \kappa$, and ν_0^1 is not defined,
- for every $1 \leq i \leq \theta$ both $\kappa = \nu_0$ and $j_{(\beta,\alpha)}^0(\kappa)$ are not moved by $\sigma_{1,i}^0$, and $\nu_i \in (\kappa, j_{(\beta,\alpha)}^0(\kappa))$,
- $\sigma_{0,i}^0(\Delta') \cap [\kappa, \nu_i) = \{\nu_j^1 \mid j < i\}$ for every $1 \leq i < \theta$.

(2) For every $\alpha < \beta$ we have:

- $\sigma_{0,1}^0 = k_{\alpha,\beta}^0$, $\nu_0 = \kappa$, $\nu_0^1 = j_{(\alpha,\beta)}^0 < k_{\alpha,\beta}^0(\kappa)$;
- for every $1 \leq i \leq \theta$, neither ν_0 , ν_0^1 , nor $k_{\alpha,\beta}^0(\kappa)$ are moved by $\sigma_{1,i}^0$; furthermore, each critical point ν_i either belongs to (ν_0, ν_0^1) or $(\nu_0^1, k_{\alpha,\beta}^0(\kappa))$;
- for every $i < \theta$ with $\nu_i \in (\nu_0, \nu_0^1)$ we have

$$\sigma_{1,i}^0(\Delta') \cap [\kappa, \nu_i) = \{\nu_j^1 \mid 1 \leq j < i\};$$

- for every $i < \theta$ with $\nu_i \in (\nu_0^1, k_{\alpha,\beta}^0(\kappa))$,

$$\sigma_{1,i}^0(\Delta') \cap [\kappa, \nu_i) = \{\nu_j^1 \mid 0 \leq j < i\}.$$

PROPOSITION 4.3: For every $\alpha, \beta < \lambda$, $\pi_{\alpha,\beta}^0$ is the restriction of

$$j_{(\alpha,\beta)}^1 : V^1 \rightarrow M_{(\alpha,\beta)}^1 \cong \text{Ult}(V^1, U_{(\alpha,\beta)}^1)$$

to V^0 .

4.1. STRUCTURAL RESULTS FOR DENSE OPEN SETS IN \mathcal{P}^1 . In this section we prove several preliminary results, which will be used in the proof of Proposition 4.3. We focus on a specific family of finite subiterations of T^0 named structural iterations for $U_{(\alpha,\beta)}^1$, and use them to describe a criterion for meeting dense open sets in $\pi_{\alpha,\beta}^0(\mathcal{P}^1)$.

Definition 4.4 (structural function, and structural extension): We define by induction on $n < \omega$ a **structural function** f of degree n , avoiding $b \subset \kappa$. For $n = 0$, a structural function of degree 0 is the trivial function $f^0 = \emptyset$. A

function $f = f^{n+1}$ is a structural function of degree $n + 1$, avoiding b , if there is a unique ordinal $\nu_f < \kappa$, and a $\mathcal{P}_{\nu_f}^1$ name \dot{X}_f so that the following holds:

- (1) $\nu_f \in \Delta' \setminus b$.
- (2) $0_{\mathcal{P}_{\nu_f}^1} \Vdash \dot{X}_f \in \dot{U}_{\nu_f}^*$.
- (3) $\text{dom}(f)$ is the set of all $\mathcal{P}_{\nu_f}^1$ names for ordinals in \dot{X}_f .
- (4) For every name $\tau \in \text{dom}(f)$, $f(\tau)$ is a structural function g of degree n avoiding b , and $\nu_g < \nu_f$.

We say that f is a structural function if there exists some $n < \omega$ so that f is a structural function of degree n .

Let p be a condition, and f be a structural function avoiding $\text{supp}(p)$. We say that a condition q is a **structural extension** of p by f if the following holds:

- (1) If f has degree 0, then q is a structural extension of p by f if $q \geq^* p$.
- (2) If f has degree $n + 1$, then q is a structural extension of p by f if there are $r \geq^* p \upharpoonright \nu_f$ and $\tau \in \text{dom}(f)$ so that $r \Vdash \tau \in p_{\nu_f}$, and q is a structural extension of $r \frown (p \setminus \nu_f)^{+(\tau, \nu_f)}$ by (the degree n structural function) $f(\tau)$. Note that $r \frown (p \setminus \nu_f)^{+(\tau, \nu_f)} \geq p$.

LEMMA 4.5: *For every open dense set $D \subset \mathcal{P}^1$ and $p \in \mathcal{P}^1$ there exists a structural function f , avoiding $\text{supp}(p)$, so that every structural extension of p by f has a direct extension in D .*

Proof. We prove by induction on $\nu \leq \kappa$ that the above holds for every dense open set $D \subset \mathcal{P}_\nu^1$ and $p \in \mathcal{P}_\nu^1$. Suppose the claim holds for every dense open set $D \subset \mathcal{P}_\nu^1$ and $p \in \mathcal{P}_\nu^1$. We have $\mathcal{P}_{\nu+1}^1 = \mathcal{P}_\nu^1 * \dot{Q}_\nu$. If $\nu \notin \Delta'$, then \dot{Q}_ν is trivial and there is nothing to prove. If $\nu \in \Delta'$, then $\dot{Q}_\nu = Q(\dot{U}_\nu^*)$. Let $D \subset \mathcal{P}_{\nu+1}^1$ be an open dense set, and $p = p \upharpoonright \nu \frown p_\nu \in \mathcal{P}_{\nu+1}^1$. If $\nu \in \text{supp}(p)$, then the forcing \mathcal{P}_ν^1 over p is equivalent to \mathcal{P}_ν^1 .

For every $\nu \notin \text{supp}(p)$ the name p_ν is a \mathcal{P}_ν^1 name of a set in U_ν^* . For every $G_\nu \subset \mathcal{P}_\nu^1$ generic, the set $D(G_\nu) = \{(q_\nu)_{G_\nu} \mid q \in D\}$ is a dense open set in $Q(U_\nu^*)$. It follows that there is some $Y_\nu \subset (p_\nu)_{G_\nu}$ with $Y_\nu \in U_\nu^*$ so $\mu \in D(G_\nu)$ for every $\mu \in Y_\nu$. Let \dot{Y}_ν be a name for Y_ν in \mathcal{P}_ν^1 . For every name τ of an ordinal in \dot{Y}_ν , the set $D_\tau = \{q \geq p \upharpoonright \nu \mid q \frown \langle \tau \rangle \in D\}$ is dense open in \mathcal{P}_ν^1 . The inductive assumption guarantees that there is some $n(\tau) < \omega$ and a structural function $f(\tau)$ of degree $n(\tau)$, such that every structural extension of $p \upharpoonright \nu$ by $f(\tau)$ has a direct extension in D_τ . For every $n < \omega$, let \dot{X}_ν^n be the \mathcal{P}_ν^1 name of

the set $\{\tau \in Y_\nu \mid n(\tau) = n\}$, and let σ_n^0 be the \mathcal{P}_ν^1 statement

$$\sigma_n^0 : \dot{X}_\nu^n \in U_\nu^*.$$

Since \mathcal{P}_ν^1 satisfies the Prikry condition, there exists a unique $n < \omega$ and some $r \geq^* p \upharpoonright \nu$ so that $r \Vdash \dot{X}_\nu^n \in \dot{U}_\nu^*$. Let us denote X_ν^n by X_ν . We conclude that for every name of an ordinal in X_ν , τ , there is a structural tree $f(\tau)$ of degree n such that every $f(\tau)$ structural extension of $r^{+(\tau, \nu)}$ has a direct extension in D . It follows that the function f mapping every such name τ to $f(\tau)$ is a structural function of degree $n + 1$, and the claim of the Lemma holds with respect to p, f .

Let $\delta \leq \kappa$ be a limit ordinal, and suppose that the claim holds in every \mathcal{P}_ν^1 for $\nu < \delta$. Fix some $p \in \mathcal{P}_\delta^1$ and a dense open $D \subset \mathcal{P}_\delta^1$. In order to prove the result it is sufficient to show that for some $\nu < \delta$ and a \mathcal{P}_ν^1 name \dot{t} , $p \upharpoonright \nu \Vdash \dot{t} \geq^* p \setminus \nu$ and $D_{\dot{t}} = \{r \geq p \upharpoonright \nu \mid r \frown \dot{t} \in D\} \subset \mathcal{P}_\nu^1$ is dense open. Suppose otherwise, and let us construct a direct extension of p , $p^* = \langle p_\nu^* \mid \nu < \delta \rangle$ so that for every $\nu < \delta$, $p^* \upharpoonright \nu \Vdash \sigma_\nu^0$ where

$$(3) \quad \sigma_\nu^0 : \forall t \geq^* (p \setminus \nu). t \notin D(\dot{G}_\nu).$$

Note that the existence of p^* would contradict the fact that D is dense. Suppose $p^* \upharpoonright \nu = \langle p_\mu^* \mid \mu < \nu \rangle$ has been defined and satisfies (3). Fix a \mathcal{P}_ν^1 generic G_ν with $p^* \upharpoonright \nu \frown p \setminus \nu \in G_\nu$, and consider the forcing

$$\mathcal{P}^1 \setminus \nu = \mathcal{Q}_{\nu+1}^1 * \mathcal{P}^1 \setminus (\nu + 1).$$

Since $\mathcal{Q}_{\nu+1}^1$ satisfies the Prikry property, there exists some $r_\nu \geq^* (p_\nu)_{G_\nu}$ which decides $\sigma_{\nu+1}^0$.⁵ If $r_\nu \Vdash \neg \sigma_{\nu+1}^0$, there would be $q_{>\nu} \geq^* p \setminus (\nu + 1)$ and $r_\nu^* \geq^* r_\nu$ so that $r_\nu^* \frown q_{>\nu} \in D(G_\nu)$. This is impossible as $r_\nu^* \frown q_{>\nu} \geq^* p \setminus \nu$ while σ_ν^0 holds in $V[G_\nu]$ as $p^* \upharpoonright \nu \in G_\nu$.

We conclude that r_ν forces $\sigma_{\nu+1}^0$ in $V[G_\nu]$. Back in V , let p_ν^* be a \mathcal{P}_ν^1 name for r_ν , so $p^* \upharpoonright \nu \Vdash p_\nu^* \geq^* p_\nu$, and $p^* \upharpoonright \nu \frown p_\nu^* \Vdash \sigma_{\nu+1}^0$.

Let $\delta' < \delta$ be a limit ordinal. Suppose $p^* \upharpoonright \delta' = \langle p_\mu^* \mid \mu < \delta' \rangle$ has been constructed, and let us show that $p^* \upharpoonright \delta' \Vdash \sigma_{\delta'}^0$. Otherwise, there would be conditions, $t \in \mathcal{P}^1 \setminus \delta'$ and $r \geq p^* \upharpoonright \delta'$, so that $p^* \upharpoonright \delta' \Vdash t \geq^* p \setminus \delta'$ and $r \Vdash t \in D(\dot{G}_{\delta'})$. Thus there exists some $r' \geq r$ so that $r' \frown t \in D$. As δ' is a limit ordinal and $\text{supp}(r')$ is finite, it follows there is some $\nu < \delta'$ with $r' \setminus \nu \geq^* (p^* \upharpoonright \delta' \setminus \nu)$. Let $s = r' \upharpoonright \nu$; we get $(r' \setminus \nu) \frown t \geq^* p \setminus \nu$ and $s \Vdash (r' \setminus \nu) \frown t \in D(\dot{G}_\nu)$. This is absurd as $s \geq p^* \upharpoonright \nu$ and must force σ_ν^0 . ■

⁵ Considered as a \mathcal{Q}_ν^1 -statement.

Definition 4.6 (structural iteration and compatible conditions): For $\alpha, \beta < \lambda$ a **structural iteration** for $U^1_{(\alpha, \beta)}$ is a finite iterated ultrapower

$$\vec{M} = \langle M_m, j_{k,m} \mid k < m \leq n \rangle$$

of length $n < \omega$, satisfying the following properties:

$$M_0 = V^0,$$

$$j_{0,1} = \begin{cases} j^0_{(\beta, \alpha)} : V^0 \rightarrow M^0_{(\beta, \alpha)} & \text{if } \beta \leq \alpha, \\ k^0_{\alpha, \beta} : V^0 \rightarrow M^0_{(\alpha, \beta), (\beta, \alpha)} & \text{if } \alpha > \beta. \end{cases}$$

Define $\nu_0 = \kappa$, set $\nu^1_0 = j^0_{(\alpha, \beta)}(\kappa) < k^0_{\alpha, \beta}(\kappa)$ if $\alpha < \beta$, and leave ν^1_0 undefined otherwise. For every $1 \leq k < n$, suppose that $\vec{M} \upharpoonright k + 1 = \langle M_m, j_{i,m} \mid i < m \leq k \rangle$ and $\langle \nu_i, \nu^1_i \mid i < k \rangle$ have been defined. Then there is an ordinal $\nu_k < j_{0,k}(\kappa)$ so $\nu_k \in j_{0,k}(\Delta') \setminus (\kappa \cup \{\nu^1_i \mid i < k\})$ and unique α_k, β_k with $\nu \in j_{0,k}(\Delta)_{\alpha_k}(\beta_k)$.⁶

- (1) $j_{k,k+1} : M_k \rightarrow M_{k+1} \cong \text{Ult}(M_k, U^0_{\nu_k, (\beta_k, \alpha_k)})$,
- (2) $\nu^1_k = j_{k,k+1}(\nu_k)$.

We say that a condition $p \in j_{0,n}(\mathcal{P}^1)$ is **compatible** with the structural iteration \vec{M} , if there exists a sequence $\vec{p} = \langle p^k \mid k \leq n \rangle$ with the following properties:

- (1) $p^0 \in \mathcal{P}^1$ in V^0 .
- (2) $p^1 = \begin{cases} j_{0,1}(p^0) = j^0_{(\beta, \alpha)}(p^0) & \text{if } \beta \leq \alpha, \\ (p \frown q)^{+(\kappa, j^0_{(\alpha, \beta)}(\kappa))} \text{ as in Definition 3.11} & \text{if } \alpha < \beta. \end{cases}$
- (3) For every $1 \leq k < n$,

$$p^{k+1} = (p \frown q \frown (j_{k,k+1}(p) \setminus \nu^1_k))^{+(\nu_k, \nu^1_k)},$$

where

- $q \geq^* j_{k,k+1}(p) \upharpoonright [\nu_k, \nu^1_k]$,
- $p \frown q \Vdash \nu_k \in j_{k,k+1}(p \nu_k) = j_{k,k+1}(p)_{\nu^1_k}$. Note that the existence of such q is guaranteed by Definition 3.8.
- (4) $p \geq^* p^n$.

Comparing the last definition with the definition of the iteration T^0 for $U^1_{(\alpha, \beta)}$, it clear that structural iterations are all finite subiterations of T^0 and that the T^0 -resulting limit, $\pi^0_{\alpha, \beta} : V^0 \rightarrow Z^0_{\alpha, \beta}$, is also the limit of the directed system of all structural iterations for $U^1_{(\alpha, \beta)}$.

Before proceeding, we point out the following simple facts:

⁶ Note that we must have $\alpha_k < \beta_k$ since $\nu \in j_{0,k}(\Delta')$.

Remarks 4.7: Let $\vec{M} = \langle M_m, j_{k,m} \mid k < m \leq n \rangle$ be a structural iteration for $U_{(\alpha,\beta)}^1$.

- (1) The embedding $j_{0,1}$ coincides with the ultrapower embedding used for the definition of $U_{(\alpha,\beta)}^1$ in Definitions 3.8 and 3.11 for $U_{(\alpha,\beta)}^1$.
- (2) For every $k < n - 1$, ν_k, ν_k^1 are not moved by $j_{k+1,n}$.
- (3) For every k , $1 \leq k < n$, $\nu_k \in j_{0,k}(\Delta')$ implies that $\nu_k^1 \in j_{0,k+1}(\Delta')$. Furthermore, since $U_{(\beta_k, \alpha_k)}^0$ (the measure generating $j_{k,k+1}$) does not include $j_{0,k}(\Delta') \cap \nu_k$ we get that $\nu_k \notin j_{0,k+1}(\Delta')$. Similarly, we have that $\kappa \notin k_{\alpha,\beta}^0(\Delta')$ when $\alpha < \beta$, and $\kappa \notin j_{(\beta,\alpha)}^0(\Delta')$ when $\alpha \geq \beta$. Therefore $\nu_k \notin j_{0,n}(\Delta')$ for every $k < n$.
- (4) If $p \in j_{0,n}(\mathcal{P}^1)$ is compatible with the iteration \vec{M} which is witnessed by a sequence $\langle p^i \mid i \leq n \rangle$, then $p^{k+1} \Vdash j_{0,k+1}(\dot{d})(\nu_k^1) = \nu_k$ for every $k < n$. Both ν_k, ν_k^1 are not moved by the rest of the iteration, hence

$$p \Vdash j_{0,n}(\dot{d})(\nu_k^1) = \nu_k$$

whenever ν_k^1 is defined.

- (5) Suppose that $p_0, p_1 \in j_{0,n}(\mathcal{P}^1)$ are compatible with \vec{M} . Let $\langle p_0^k \mid k \leq n \rangle, \langle p_1^k \mid k \leq n \rangle$ be witnessing sequences for p_0, p_1 respectively. It is easy to see by induction on $k \leq n$ that if $p_0^0, p_1^0 \in \mathcal{P}^1$ are compatible in \mathcal{P}^1 , then p_0^k, p_1^k are compatible in $j_{0,k}(\mathcal{P}^1)$.

LEMMA 4.8: *Let $\vec{M}_0 = \langle M_m, j_{k,m} \mid k < m \leq n_0 \rangle$ be a structural iteration of length n_0 and $p \in j_{0,n_0}(\mathcal{P}^1)$ compatible with \vec{M}_0 . We have that for every structural function $f = f^n$ of degree n which avoids $\text{supp}(p) \cup \kappa$, there exists a structural iteration $\vec{M} = \langle M_m, j_{k,m} \mid k < m \leq n_0 + n \rangle$ of length $n_0 + n$, extending \vec{M}_0 , and $q \in j_{0,n_0+n}(\mathcal{P}^1)$ so that:*

- (1) q is compatible with \vec{M} , and
- (2) q is a structural extension of $j_{n_0,n_0+n}(p)$ by $j_{n_0,n_0+n}(f)$.

Proof. Let us denote $n_0 + n$ by n^* . For every k , $n_0 \leq k < n^*$, we choose $j_{k,k+1} : M_k \rightarrow M_{k+1}$, $p^{k+1} \in j_{0,k+1}(\mathcal{P}^1)$, and g^{n^*-k} of degree $n^* - k$, so that p^{k+1} is compatible with $\vec{M} \upharpoonright (k + 1)$, and g^{n^*-k} avoids $\text{supp}(p^{k+1}) \cup \kappa$. Let M_{n_0} be the last model in \vec{M}_0 , $p^{n_0} = p$, and $g^{n^*-n_0} = g^n = f$. Suppose that $\vec{M} \upharpoonright k + 1, p^k, g^{n^*-k}$ have been defined. Note that $\{\nu_i^1 \mid i < k\} \subset \text{supp}(p^k)$ and $\nu_k \in j_{0,k}(\Delta') \setminus (\kappa \cup \text{supp}(p^k))$. Suppose that $\nu_k \in \Delta_{\alpha_k}(\beta_k)$, and let

$$j_{k,k+1} : M_k \rightarrow M_{k+1} \cong \text{Ult}(M_k, U_{\nu_k, (\beta_k, \alpha_k)}^0).$$

Let $X_{g^{n^*-k}}$ be the name associated with g^{n^*-k} , and let Y_k be a name for $X_{g^{n^*-k}} \cap p_{\nu_k}^k$. Since Y_k is a name of a set in $U_{(\alpha_k, \beta_k)}^1$, then by the definition of $U_{(\alpha_k, \beta_k)}^1$ there is some $q \geq^* j_{k,k+1}(p^k) \upharpoonright [\nu_k, \nu_k^1]$, so that $p^k \frown q \Vdash \check{\nu}_k \in j_{k,k+1}(Y_k)$. Define $p^{k+1} = (p^k \frown q \frown (j_{k,k+1}(p^k) \setminus \nu_k^1))^{+(\nu_k, \nu_k^1)}$.

Let ν_k be an ordinal in Y_k which is interpreted as $\check{\nu}_k$ by every condition which forces $\check{\nu}_k \in j_{k,k+1}(X_{g^{n^*-k}})$. Clearly $\nu_k \in \text{dom}(j_{k,k+1}(g^{n^*-k}))$ so we can define $g^{n^*-(k+1)} = j_{k,k+1}(g^{n^*-k})(\nu_k)$. The inductive hypothesis implies that p^{k+1} is compatible with $\vec{M} \upharpoonright k+2$ and that $g^{n^*-(k+1)}$ avoids $\kappa \cup \text{supp}(p^{k+1})$.

The construction terminates after $n = n^* - n_0$ steps. We obtain an iteration $\vec{M} = \langle M_m, j_{k,m} \mid k < m \leq n^* \rangle, \langle p^i \mid i \leq n^* \rangle$ and a structural function of degree 0, $g^0 = g^{n^*-n^*} = \emptyset$. For every $k \leq n^*$, ν_k, ν_k^1 are fixed by $j_{k+1,n}$, hence g^0 can also be described as follows:

- (1) $h^0 = j_{n_0, n^*}(f)$ is of degree n ,
- (2) $h^{i+1} = h^i(\nu_{n_0+i})$ is of degree $n - i - 1$ for all $i < n$,
- (3) $g^0 = h^n$.

We conclude that the sequence $\langle j_{k, n^*}(p^k) \in j_{n_0, n^*}(\mathcal{P}^1) \rangle$ is a witness for the fact that $q = p^{n^*} = j_{n^*, n^*}(p^{n^*})$ is a structural extension of $j_{n_0, n_0+n}(p)$ by $j_{n_0, n_0+n}(f)$. ■

The following concludes the findings of Lemmata 4.5 and 4.8.

COROLLARY 4.9: *Let \vec{M}_0 be a structural iteration of length n_0 , and D be a \mathcal{P}^1 -name of a dense open set in $j_{0, n_0}(\mathcal{P}^1) \setminus \kappa$. For every \vec{M}_0 compatible condition $p \in j_{0, n_0}(\mathcal{P}^1)$ there is a structural iteration \vec{M} extending \vec{M}_0 and a \vec{M} compatible condition $q \in j_{0, n^*}(\mathcal{P}^1)$ so that:*

- (1) $q \geq j_{n_0, n^*}(p)$ (here $n^* = |\vec{M}|$),
- (2) $q \upharpoonright \kappa = j_{n_0, n^*}(p) \upharpoonright \kappa = p \upharpoonright \kappa$, and
- (3) $q \upharpoonright \kappa \Vdash (q \setminus \kappa) \in D$.

4.2. A PROOF FOR PROPOSITION 4.3. Let $\alpha, \beta < \lambda$ and

$$T^0 = \langle Z_i^0, \sigma_{i,j}^0 \mid 0 \leq i < j < \theta \rangle$$

be the iteration associated with $U_{(\alpha, \beta)}^1$ (Definition 4.1), and let $\pi_{\alpha, \beta}^0 : V^0 \rightarrow Z_{\alpha, \beta}^0$ be the resulting elementary embedding; $\pi_{\alpha, \beta}^0$ is also the limit of the directed system which consists of all structural iterations of $U_{(\alpha, \beta)}^1$.

Definition 4.10:

- (1) For every structural iteration $\vec{M} = \langle M_k, j_{k,m} \mid k \leq m \leq n \rangle$ with respect to $U_{(\alpha,\beta)}^1$, let $j_{\vec{M}} : V^0 \rightarrow M_n$ denote $j_{0,n}$ and $k_{\vec{M}} : M_n \rightarrow Z_{\alpha,\beta}^0$ denote the direct limit embedding of M_n in $Z_{\alpha,\beta}^0$.
- (2) Let $G^1 \subset \mathcal{P}^1$ be a V^0 generic. We say that a condition $p \in j_{\vec{M}}(\mathcal{P}^1)$ is compatible with both \vec{M} and G^1 if there is a witnessing sequence $\langle p^k \mid k \leq n \rangle$ so that $p^0 \in G^1$.
- (3) Let $F_{\vec{M},G^1} \subset j_{\vec{M}}(\mathcal{P}^1)$ be the set of all the conditions $p \in j_{\vec{M}}(\mathcal{P}^1)$ which are compatible with \vec{M} and G^1 .

Proof of Proposition 4.3. Define $G_{\alpha,\beta}^1 \subset \pi_{\alpha,\beta}^0(\mathcal{P}^1)$,

$$G_{\alpha,\beta}^1 = \bigcup \{ k_{\vec{M}} \text{ “} F_{\vec{M},G^1} \mid \vec{M} \text{ is a structural iteration } \}.$$

It is clear that $\pi_{\alpha,\beta}^0 \text{ “} G^1 \subset G_{\alpha,\beta}^1$. Furthermore, the last remark in Remarks 4.7 implies that every two conditions in $G_{\alpha,\beta}^1$ are compatible.

We show that $G_{\alpha,\beta}^1$ is generic over $Z_{\alpha,\beta}^0$. Clearly, $G_{\alpha,\beta}^1 \upharpoonright \kappa = G^1$ is $\mathcal{P}^1 = \pi_{\alpha,\beta}^0(\mathcal{P}^1) \upharpoonright \kappa$ generic over $Z_{\alpha,\beta}^0$. Let D' be a \mathcal{P}^1 name for a dense open set in $\pi_{\alpha,\beta}^0(\mathcal{P}^1) \setminus \kappa$. Let \vec{M}_0 be a structural iteration for which there is $D \subset j_{\vec{M}_0}(\mathcal{P}^1)$ such that $k_{\vec{M}_0}(D) = D'$. Fix a condition $p \in F_{\vec{M}_0,G^1}$. By Corollary 4.9, there is a structural iteration \vec{M} extending \vec{M}_0 and a compatible condition $q \in j_{\vec{M}}(\mathcal{P}^1)$, so that $q \upharpoonright \kappa = p \upharpoonright \kappa$ and $q \upharpoonright \kappa \Vdash q \setminus \kappa \in D$. This implies that $q \in F_{\vec{M},G^1}$, which in turn implies that $q' = k_{\vec{M}}(q) \in D' \cap G_{\alpha,\beta}^1$. It follows that $G_{\alpha,\beta}^1 \cap D \neq \emptyset$.

We can therefore extend $\pi_{\alpha,\beta}^0 : V^0 \rightarrow Z_{\alpha,\beta}^0$ to an elementary embedding $\mathcal{P}_{\alpha,\beta}^1 : V^0[G^1] \rightarrow Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1]$ so that for every set $x = (\dot{x})_{G^1}$, $\pi_{\alpha,\beta}^1(x) = (\pi_{\alpha,\beta}^0(\dot{x}))_{G_{\alpha,\beta}^1}$.

Let U denote the normal measure on κ in $V^0[G^1]$ defined by $X \in U$ if $\kappa \in \pi_{\alpha,\beta}^1(X)$. We first show that $U = U_{(\alpha,\beta)}^1$, and then prove that $\pi_{\alpha,\beta}^1$ coincides with the ultrapower embedding of U . Let $X \in U_{(\alpha,\beta)}^1$. By the definition of $U_{(\alpha,\beta)}^1$ (Definitions 3.8 and 3.11) there is a G^1 name \dot{X} for X and a $j_{9,1}$ compatible condition $t \in j_{0,1} \text{ “} G^1$ so that $t \Vdash_{j_{0,1}(\mathcal{P}^1)} \check{\kappa} \in j_{0,1}(\dot{X})$. It follows that t is compatible with G^1 , i.e., $t \in F_{Z_1^0,G^1}$. Let $k : Z_1^0 \rightarrow Z_{\alpha,\beta}^0$ be the direct limit embedding of the iteration. We get that $k(t) \in G_{\alpha,\beta}^1$. As k does not move κ (the iteration T after Z_1^0 is above κ) it follows that $k(t) \Vdash \check{\kappa} \in \pi_{\alpha,\beta}^0(\dot{X})$, thus $X \in U$. We conclude that $U_{(\alpha,\beta)}^1 \subseteq U$. $U = U_{(\alpha,\beta)}^1$ as both are ultrafilters.

It follows that the ultrapower embedding of V^1 by U is

$$j_{(\alpha,\beta)}^1 : V^1 \rightarrow M_{(\alpha,\beta)}^1 \cong \text{Ult}(V^1, U_{(\alpha,\beta)}^1)$$

and that $\pi_{\alpha,\beta}^1$ can be factored into $e_{\alpha,\beta}^1 \circ j_{(\alpha,\beta)}^1$, where $e_{\alpha,\beta}^1 : M_{\alpha,\beta}^1 \rightarrow Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1]$ maps $[f]_{U_{(\alpha,\beta)}^1}$ to $\pi_{\alpha,\beta}^1(f)(\kappa)$. Therefore, in order to show

$$(j_{(\alpha,\beta)}^1, M_{\alpha,\beta}^1) = (\pi_{\alpha,\beta}^1, Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1])$$

it suffices to prove $e_{\alpha,\beta}^1$ is surjective. Suppose $x \in Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1]$ and let \dot{x} be a $\pi_{\alpha,\beta}^1(\mathcal{P}^1)$ name for x , i.e., $x = (\dot{x})_{G_{\alpha,\beta}^1}$ since $G_{\alpha,\beta}^1 = \pi_{\alpha,\beta}^1(G^1) \in \text{rng}(e_{\alpha,\beta}^1)$. Let us show that $\dot{x} \in \text{rng}(e_{\alpha,\beta}^1)$. To this end, $\dot{x} \in Z_{\alpha,\beta}^0$ implies there is a structural iteration \vec{M} and a $j_{\vec{M}}(\mathcal{P}^1)$ name \dot{y} so that $\dot{x} = k_{\vec{M}}(\dot{y})$. Let $\langle \nu_k \mid k < n \rangle$ be the list of critical points of $\vec{M} = \langle M_k, j_{k,m} \mid k < m \leq n \rangle$. Thus $\dot{y} = j_{0,n}(h)(\nu_0, \dots, \nu_{n-1})$ for some $h : \kappa^n \rightarrow V^0$ in V^0 . For every $k < n$ let $i_k < \theta$ such that $\nu_{i_k} = k_{\vec{M}}(\nu_k)$. By applying $k_{\vec{M}}$ we get $\dot{x} = \pi_{\alpha,\beta}^0(h)(\nu_{i_0}, \dots, \nu_{i_k})$. It remains to show that $\nu_{i_m} \in \text{rng}(e_{\alpha,\beta}^1)$ for each $m < n$. This is proved by induction. The case $m = 0$ is trivial as $\nu_0 = \kappa$ and $k_{\vec{M}}(\kappa) = \kappa$. Let $0 < m < n$ and suppose that the claim holds for every $m' < m$; ν_m is the critical point of the m -stage of the iteration \vec{M} . As a member of M_m (the m -th iterand in \vec{M}) we can write $\nu_m = j_{0,m}(h)(\nu_0, \dots, \nu_{m-1})$ in M_m , where $h : \kappa^m \rightarrow V^0$ belongs to V^0 . By applying $j_{m,m+1}$ we get $\nu_m^1 = j_{0,m+1}(h)(\nu_0, \dots, \nu_{m-1})$ in M_{m+1} . As $j_{m+1,n}$ does not move $\nu_0, \dots, \nu_{m-1}, \nu_m, \nu_m^1$, $\nu_m^1 = j_{0,n}(h)(\nu_0, \dots, \nu_{m-1})$ in M_n . Now for every $p \in F_{\vec{M},G^1}$ we have $p \Vdash \check{\nu}_m = j_{0,n}(\dot{d})(\check{\nu}_m^1) = j_{0,m}(\dot{h}')(\nu_0, \dots, \nu_{m-1})$, where \dot{d} is the name of the generic Prikry function and $h' = d \circ h$. It follows that if $q = k_{\vec{M}}(p)$, then $q \in G_{\alpha,\beta}^1$ and $q \Vdash \check{\nu}_{i_m} = \pi_{\alpha,\beta}^0(h')(\nu_{i_0}, \dots, \nu_{i_{m-1}})$. The result is therefore a consequence of the inductive assumption for $\nu_{i_0}, \dots, \nu_{i_{m-1}}$. ■

COROLLARY 4.11 ($j_{\alpha,\beta}^1 \upharpoonright V$): *The restriction $j_{\alpha,\beta}^1 \upharpoonright V$ results from the following iteration $T = \langle Z_i, \sigma_{i,j} \mid 0 \leq i < j < \theta \rangle$:*

(1) $Z_0 = V$. For $\sigma_{0,1} = \sigma_{0,1}^0 \upharpoonright V : Z_0 \rightarrow Z_1$, i.e.,

$$\sigma_{0,1} = \begin{cases} j_{(\beta,\alpha)}^0 \upharpoonright V = j_\beta & \text{if } \alpha \geq \beta, \\ k_{\alpha,\beta}^0 \upharpoonright V = j_{(\alpha,\beta),(\beta,\alpha)}^0 \upharpoonright V = j_{\alpha,\beta} & \text{if } \alpha > \beta. \end{cases}$$

(2) Given $T \upharpoonright i$ and $\sigma_{j,i} : Z_j \rightarrow Z_i$ for $j < i$, so that $Z_j = \mathcal{K}(Z_j^0)$ (i.e., the core of Z_j), and $\sigma_{j,i} = \sigma_{j,i}^0 \upharpoonright Z_j$, we have

$$\sigma_{i,i+1} = j_{\nu_i,(\beta_i,\alpha_i)}^0 \upharpoonright Z_i = j_{\nu_i,\beta_i} : Z_i \rightarrow Z_{i+1}.$$

5. The structure of $\triangleleft(\kappa)$ in V^1

We prove that in V^1 , the restriction of \triangleleft to $\{U^1_{(\alpha,\beta)} \mid \alpha \leq \beta < \lambda\}$ is isomorphic to R_λ .

PROPOSITION 5.1: *Suppose that $\alpha' \leq \beta'$, $\alpha \leq \beta$ are ordinals below $\lambda = o(\kappa)$. In $V^0[G^1]$, $U^1_{(\alpha',\beta')} \triangleleft U^1_{(\alpha,\beta)}$ if and only if $\beta' < \alpha$.*

Note that while V^1 contains normal measures $U^1_{(\alpha,\beta)}$ for $\alpha > \beta$, Proposition 5.1 refers only to $U^1_{(\alpha,\beta)}$ for $\alpha \leq \beta < \lambda$. These additional measures are omitted from Proposition 5.1 since they do not add any essential structure to $\triangleleft(\kappa)^{V^1}$. More precisely, the proof of Proposition 5.1 shows that for every $\alpha > \beta$, $U^1_{(\alpha,\beta)}$ is equivalent to $U^1_{(\alpha,\alpha)}$ in the order $\triangleleft(\kappa) \upharpoonright \vec{U}^1$, where

$$\vec{U}^1 = \{U^1_{(\alpha,\beta)} \mid \alpha, \beta < o(\kappa)\}.$$

As Section 6 proves that the measures in \vec{U}^1 are all the normal measures on κ in V^1 , we conclude that R_λ is isomorphic to the reduction of $\triangleleft(\kappa)^{V^1}$. We separate the proof of Proposition 5.1 to “if” and “only if” claims.

CLAIM 5.2: *If $\beta' < \alpha$ then $U^1_{(\alpha',\beta')} \triangleleft U^1_{(\alpha,\beta)}$.*

Proof. Suppose that $\alpha' \geq \beta'$. It is clear from Definitions 3.8 and 3.11 that $U^1_{(\alpha',\beta')}$ is defined in every inner model of V^1 which contains G^0 , G^1 , and $U^0_{(\beta',\alpha')}$. Similarly, if $\alpha' < \beta'$ then $U^1_{(\alpha',\beta')}$ is defined in every inner model containing G^0 , G^1 , $U^0_{(\beta',\alpha')}$, and $U^0_{(\alpha',\beta')}$.

For every α, β ,

$$\text{Ult}(V^1, U^1_{(\alpha,\beta)}) \cong Z^0_{\alpha,\beta}[G^1_{\alpha,\beta}] = Z_{\alpha,\beta}[G^0_{\alpha,\beta} * G^1_{\alpha,\beta}],$$

where $Z^0_{\alpha,\beta}$ results from the iterated ultrapower T^0 . Furthermore,

$$G^0 = G^0_{\alpha,\beta} \upharpoonright (\kappa + 1) \quad \text{and} \quad G^1 = G^1_{\alpha,\beta} \upharpoonright \kappa.$$

By Corollary 3.5 we have that $U^0_{(\alpha',\beta')} \triangleleft U^0_{(\alpha,\beta)}$ whenever $\alpha' < \alpha$. Also, by Definition 4.1 we know that the embedding $\pi^0_{\alpha,\beta}$ factors into $\sigma^0_{1,\theta} \circ \sigma^0_{0,1}$, where

$$\sigma^0_{0,1} = \begin{cases} j^0_{(\beta,\alpha)} & \text{if } \beta \leq \alpha, \\ k^0_{\alpha,\beta} & \text{if } \alpha \geq \beta, \end{cases}$$

and that $\text{cp}(\sigma^0_{1,\theta}) > \kappa$. Therefore, if $U^0_{(\alpha',\beta')} \in M^0_{(\alpha,\beta)}$ then $U^0_{(\alpha',\beta')} \in Z^0_{\alpha,\beta}$. We can now conclude the desired result by a simple case-by-case inspection:

(1) When $\alpha' = \beta'$ and $\alpha = \beta$, we get that

$$\beta' < \alpha \implies U_{(\beta',\beta')}^0 \in M_{(\alpha,\alpha)}^0 \implies U_{(\beta',\beta')}^0 \in Z_{\alpha,\alpha}^0.$$

(2) When $\alpha' = \beta'$ and $\alpha < \beta$, we have

$$\beta' < \alpha \implies U_{(\beta',\beta')}^0 \in M_{(\alpha,\beta)}^0 \implies U_{(\beta',\beta')}^0 \in Z_{\alpha,\alpha}^0.$$

(3) When $\alpha' < \beta'$ and $\alpha = \beta$, then

$$\beta' < \alpha \implies U_{(\alpha',\beta')}^0, U_{(\beta',\alpha')}^0 \in M_{(\alpha,\alpha)}^0 \implies U_{(\alpha',\beta')}^0, U_{(\beta',\alpha')}^0 \in Z_{\alpha,\alpha}^0.$$

(4) Finally, when $\alpha' < \beta'$ and $\alpha < \beta$, then

$$\beta' < \alpha \implies U_{(\alpha',\beta')}^0, U_{(\beta',\alpha')}^0 \in M_{(\alpha,\beta)}^0 \implies Uz\alpha'\beta', U_{(\beta',\alpha')}^0 \in Z_{\alpha,\alpha}^0. \quad \blacksquare$$

This concludes the “if” part of the proof. Before proceeding to the second part, let us first list several corollaries of inner model theory ([19]).

Suppose that $U_{(\alpha',\beta')}^1 \in M_{\alpha,\beta}^1$; let $j'_{\alpha',\beta'} : M_{\alpha,\beta}^1 \rightarrow M'_{\alpha',\beta'} \cong \text{Ult}(M_{\alpha,\beta}^1, U_{(\alpha',\beta')}^1)$ be the ultrapower embedding.

(A)

(1) By the uniqueness of the core model, $\mathcal{K}(M_{\alpha,\beta}^1) = Z_{\alpha,\beta}$ ($Z_{\alpha,\beta}$ is described in Corollary 4.11).

(2) The restriction $\pi'_{\alpha',\beta'} = j'_{\alpha',\beta'} \upharpoonright Z_{\alpha,\beta}$ can be realized as the limit of a normal iteration T' of $Z_{\alpha,\beta}$.

(3) Let $G = G_{\alpha,\beta}^0 * G_{\alpha,\beta}^1$; then:

(a) $G' = j'_{\alpha',\beta'}(G) \subset \pi'_{\alpha',\beta'}(\mathcal{P}^0 * \mathcal{P}^1)$ is $\pi'_{\alpha',\beta'}(\mathcal{P}^0 * \mathcal{P}^1)$ generic over $Z'_{\alpha',\beta'}$,

(b) $M'_{\alpha',\beta'} = Z'_{\alpha',\beta'}[G']$, and

(c) for every $x \in M_{(\alpha,\beta)}^1 = Z_{\alpha,\beta}[G]$, if $x = (\dot{x})_G$ then

$$j'_{\alpha',\beta'}(x) = \pi'_{\alpha',\beta'}(\dot{x})_{G'}.$$

(B) The models $M'_{\alpha',\beta'} \cong \text{Ult}(M_{\alpha,\beta}^1, U_{(\alpha',\beta')}^1)$ and $M_{(\alpha',\beta')}^1 \cong \text{Ult}(V^1, U_{(\alpha',\beta')}^1)$ have the same initial segment of the cumulative hierarchy, $V_{(j_{\alpha',\beta'}^1(\kappa))}$. Indeed, $M_{(\alpha,\beta)}^1 \cap (V^1)_{\kappa+1} = (V^1)_{\kappa+1}$ because $M_{(\alpha,\beta)}^1$ is an ultrapower of V^1 by a κ complete ultrafilter. Therefore, when applying a $U_{(\alpha',\beta')}^1$ ultrapower to V^1 and $M_{(\alpha,\beta)}^1$, we find that

$$j_{\alpha',\beta'}^1 \upharpoonright (\kappa^+ + 1) = j'_{\alpha',\beta'} \upharpoonright (\kappa^+ + 1)$$

and

$$M'_{\alpha',\beta'} \cap V_{j_{\alpha',\beta'}^1(\kappa)} = M_{(\alpha',\beta')}^1 \cap V_{j_{\alpha',\beta'}^1(\kappa)}.$$

It follows that:

- (1) $\pi'_{\alpha',\beta'} \upharpoonright \kappa^+ = \pi^1_{\alpha',\beta'} \upharpoonright \kappa^+$,
- (2) $Z'_{\alpha',\beta'} \upharpoonright j^1_{\alpha',\beta'}(\kappa) = Z_{\alpha',\beta'} \upharpoonright j^1_{\alpha',\beta'}(\kappa)$, and
- (3) the following normal iterations agree up to $j_{\alpha',\beta'}(\kappa^+) = \pi_{\alpha',\beta'}(\kappa^+)$:
 - T which generates $\pi_{\alpha',\beta'} : V \rightarrow M_{\alpha',\beta'}$, and
 - T' , generating $\pi'_{\alpha',\beta'} : M_{\alpha,\beta} \rightarrow M'_{\alpha',\beta'}$.

CLAIM 5.3: *If $U^1_{(\alpha',\beta')} \triangleleft U^1_{(\alpha,\beta)}$ then $\beta' < \alpha$.*

Proof. We use the notations and results listed above. Since $Z'_{\alpha',\beta'}$ and $Z_{\alpha',\beta'}$ agree up to $j^1_{(\alpha',\beta')}(\kappa)$, we get that $o^{Z_{\alpha',\beta'}}(\nu) = o^{Z'_{\alpha',\beta'}}(\nu)$ for every $\nu < j^1_{(\alpha',\beta')}(\kappa)$.

The first step of the iteration T' coincides with the first step of the iteration T . According to Corollary 4.11, the first step of T is an ultrapower by $U_{\alpha'}$; thus $U_{\alpha'} \in Z_{\alpha,\beta}$. Since $o^{Z_{\alpha,\beta}}(\kappa) = \alpha$ it follows that $\alpha' < \alpha$. Therefore, if $\alpha' = \beta'$ then $\beta' < \alpha$, as desired.

Suppose now that $\alpha' < \beta'$. Since $\pi'_{\alpha',\beta'}$ is the embedding generated by T' , it factors into $\pi'_{\alpha',\beta'} = k' \circ j_{\alpha'}$, where $j_{\alpha'} : Z_{\alpha,\beta} \rightarrow N'$ and $k' : N' \rightarrow Z'_{\alpha',\beta'}$, with $\text{cp}(k') > \kappa$. We have that $j_{\alpha',\beta'}(\kappa) > j_{\alpha'}(\kappa)$, therefore the iterations T and T' agree at $j_{\alpha'}(\kappa)$. We also know that $j_{\alpha'}(\kappa)$ is a critical point in T via the ultrapower by $U = j_{\alpha'}(U_{\beta'})$. Note that $o(U) = j_{\alpha'}(\beta')$. Therefore the same holds for T' , and we must have that $U \in N'$. It follows that

$$j_{\alpha'}(\beta') = o(U) < o^{N'}(j_{\alpha'}(\kappa)).$$

Finally, $o^{Z_{\alpha,\beta}}(\kappa) = \alpha$ and we get that $o^{N'}(j_{\alpha'}(\kappa)) = j_{\alpha'}(o^{Z_{\alpha,\beta}}(\kappa)) = j_{\alpha'}(\alpha)$. We conclude that $j_{\alpha'}(\beta') < j_{\alpha'}(\alpha)$, therefore $\beta' < \alpha$. ■

6. The normal measures on κ in V^1

PROPOSITION 6.1: *The measures $U^1_{(\alpha,\beta)}$, $\alpha, \beta < \lambda$ are the only measures on κ in V^1 .*

Proof. Let W be a normal measure on κ in V^1 , and

$$j_W : V^1 \rightarrow M_W \cong \text{Ult}(V^1, W).$$

There is a normal iteration T^W of V such that the resulting embedding $\pi : V \rightarrow M$ coincides with $j_W \upharpoonright V$. Moreover, if $V^1 = V[G]$ and

$$G = G^0 * G^1 \subset \mathcal{P}^0 * \mathcal{P}^1,$$

then $j_W(G) = j_W(G^0) * j_W(G^1)$ is $\pi(\mathcal{P}^0 * \mathcal{P}^1)$ generic over M_W . Denote $j_W(G^0)$, $j_W(G^1)$ by G_W^0 , G_W^1 respectively. For every M_W -inaccessible $\tau < j_W(\kappa)$, let $s_\tau^{G_W^0}$ be the G_W^0 -induced generic Sacks function at τ .

According to Friedman–Magidor ([7]), π “ G^0 determines the values of every G_W^0 Sacks function, $s_\tau^{G_W^0}$, with the exception of the values $s_{\gamma^1}^{G_W^0}(\gamma)$, where γ is a critical point in T_W and γ^1 is its image.⁷ In particular, $\kappa = \text{cp}(\pi)$ and π factors into $\pi = k \circ j_\beta$, where $\beta < o(\kappa)$ and $\text{cp}(k) > \kappa$. Let T_W^0 be the lift of the iteration T_W to $V^0 = V[G^0]$, and determined by G_W^0 , and let $\pi^0 : V[G^0] \rightarrow M_W[G_W^0]$ be its induced embedding.

Let $\gamma = s_{j_\beta(\kappa)}^{G_W^0}(\kappa)$, and $G_{U_{(\beta,\gamma)}^0}^0 = j_{(\beta,\gamma)}^0(G^0)$ be the $j_\beta(\mathcal{P}^1)$ -generic filter over M_β , associated with $U_{(\beta,\gamma)}^0$. It follows that $\pi^0 = k^0 \circ j_{(\beta,\gamma)}^0$, where

$$k^0 : M_\beta[G_{U_{(\beta,\gamma)}^0}^0] \rightarrow M_W[G_W^0]$$

is an extension of k .

SUBCLAIM 1: $\gamma \geq \beta$.

$o^{M_\beta}(\kappa) = \beta$ and $s_{j_W(\kappa)}^{G_W^0}(\kappa) = \gamma$, therefore $\kappa \in j_W(\Delta_\beta(\gamma))$. If γ was smaller than β , we would get that $\Delta_\beta(\gamma) \subset \Delta'$, i.e., W concentrates on the set of non-trivial iteration stages in \mathcal{P}^1 . Yet this contradicts the normality of W , as the generic Prikry function $d : \Delta' \rightarrow \kappa$ is regressive and injective outside a bounded set.

Recall that we have defined Γ to be the set of all generic Prikry points, i.e., $\Gamma = \text{rng}(d) = d \text{ “ } \Delta'$.

SUBCLAIM 2: If $\Gamma \not\subseteq W$ then $W = U_{(\gamma,\beta)}^1$.

It is sufficient to show that $U_{(\gamma,\beta)}^1 \subset W$. Suppose that $X \in U_{(\gamma,\beta)}^1$, and let \dot{X} be a G^1 name for X in V^0 . According to Remark 3.9 there is a condition $p \in G^1$ so that $j_{(\beta,\gamma)}^0(p)^{-\kappa} \Vdash \check{\kappa} \in j_{(\beta,\gamma)}^0(\dot{X})$. By applying k^0 we get that

$$\pi_W^0(p)^{-\kappa} \Vdash \check{\kappa} \in \pi^0(\dot{X})$$

Let $\Sigma \subset \kappa$ be the set of closure points of d^{-1} (namely, $\nu \in \Sigma$ if and only if $d^{-1}(\nu) \subset \nu$). Using the Magidor iteration support, it is not difficult to verify that Σ is closed unbounded in κ (also, see [3]). Since $\Gamma \not\subseteq W$, it follows that the set $\{\nu < \kappa \mid p^{-\nu} \in G^1\}$ belongs to W , thus $\pi^0(p)^{-\kappa} \in G_W^1$ and $X \in W$.

⁷ Namely, if $\gamma = \text{cp}(\pi_{i,i+1})$ is the critical point of the i -th stage of T_W then $\gamma^1 = \pi_{i,i+1}(\gamma)$.

SUBCLAIM 3: If $\Gamma \in W$ then $\gamma < \beta$ and $W = U^1_{(\beta,\gamma)}$.

Suppose now that $\Gamma \in W$. Let $\Gamma' = \{\alpha < \kappa : |d^{-1}(\alpha)| = 1\}$. It is not difficult to verify that $\Gamma \setminus \Gamma'$ is bounded in κ .⁸ Therefore, if $\Gamma \in W$ then there exists a unique $\mu < j_W(\kappa)$ such that $j_W(d)(\mu) = \kappa$.

According to the results in [3],⁹ there is a finite subiteration of T_W , by which $\pi = k \circ j_\beta$ factors into $\pi = e \circ j_{U'} \circ j_\beta$ so that:

- (1) $j_{U'}$ is an ultrapower embedding by a normal measure U' on $j_\beta(\kappa)$,
- (2) $U' = j_\beta(U_{\beta'})$ for some $\beta' < o(\kappa)$,
- (3) $\mu = e(j_\beta(\kappa))$,
- (4) $\text{cp}(e) > \kappa$.

Let $\pi^0 = e^0 \circ j_{U^0} \circ j^0_{(\beta,\gamma)}$ be the corresponding factorization of the extension π^0 of π . In particular, $U^0 \in M^0_{(\beta,\gamma)}$ extends $U' = j_\beta(U_{\beta'}) \in M_\beta$, and $\mu = e^0 \circ j^0_{(\beta,\gamma)}(\kappa)$.

We have $\kappa \in j_W(\Delta_\beta(\gamma))$. It follows from the definition of \mathcal{P}^1 that $\beta < \gamma$ and that $\mu = j_W(d^{-1})(\kappa) \in \pi^0(\Delta_\gamma(\beta))$, i.e.,

$$e^0 \circ j^0_{(\beta,\gamma)}(\kappa) \in e^0 \circ j_{U^0} \circ j^0_{(\beta,\gamma)}(\Delta_\gamma(\beta)).$$

Therefore, it must mean that $U^0 = j^0_{(\beta,\gamma)}(U_\gamma(\beta))$, so we can rewrite π^0 as $\pi^0 = e^0 \circ k^0_{\beta,\gamma}$ (i.e., $k^0_{\beta,\gamma}$ in Definition 3.11).

According to Remark 3.12, there is a condition $p \in G^1$ so that

$$(4) \quad k^0_{\beta,\gamma}(p)^{+(\kappa, j^0_{(\beta,\gamma)}(\kappa)) - \kappa - j^0_{(\beta,\gamma)}(\kappa)} \Vdash \check{\kappa} \in k^0_{\beta,\gamma}(\dot{X}).$$

Let

$$\Pi = \{\nu \in \Gamma' \mid \text{for every } \mu < \kappa \text{ if } \mu > d^{-1}(\nu) \text{ then } (d(\mu) \notin [\nu, d^{-1}(\nu)])\}.$$

It is not difficult to verify that $\Gamma \setminus \Pi$ is bounded in κ (see [3]), and that for every $p \in G^1$ and $\mu < \kappa$, $p^{+(\mu, d^{-1}(\mu)) - \mu - d^{-1}(\mu)} \in G^1$ whenever $\mu \in \Pi \cap \Sigma$.

Since $\Gamma \in W$ and $\Sigma \subset \kappa$ is a club, it follows that $\Pi \cap \Sigma \in W$. We conclude that $\pi^0(p)^{+(\kappa, \mu) - \kappa - \mu} \in G^1_W$. By Applying e^0 to equation (4) we conclude that

$$\pi^0(p)^{+(\kappa, \mu) - \kappa - \mu} \Vdash \check{\kappa} \in \pi^0(\dot{X}).$$

Therefore $X \in W$ ■

⁸ Also, see [3].

⁹ Namely, the proofs of Proposition 3.2 and Lemma 3.6 in [3]

7. A final cut

According to Friedman and Magidor ([7]), there is a sequence $\vec{X}^\kappa = \langle X_i^\kappa \mid i < \kappa^+ \rangle$ of pairwise disjoint stationary subsets of $\kappa^+ \cap \text{Cof}(\kappa)$ in V^1 and a function $f : \kappa \rightarrow V^1$, so that $j(f)(\kappa) = \vec{X}^\kappa$ for every elementary embedding j in V^1 with $\text{cp}(j) = \kappa$.¹⁰ We may assume that $f(\nu) = \langle X_i^\nu \mid i < \nu^+ \rangle$ is a ν^+ -sequence of disjoint stationary subsets of $\nu^+ \cap \text{Cof}(\nu)$ for each $\nu < \kappa$.

Definition 7.1 ($\text{Code}^*(\nu)$, $\nu < \kappa$): A condition in $\text{Code}^*(\nu)$ is a closed, bounded subset c of ν^+ which is disjoint from X_0^ν . For conditions $c, d \in \text{Code}^*(\nu)$, $d \geq c$, if and only if:

- (1) d end extends c ,
- (2) for $i \leq \max(c)$: if i belongs to c then $d \setminus c$ is disjoint from X_{1+2i}^ν ; if i does not belong to c then $d \setminus c$ is disjoint from X_{1+2i+1}^ν .

For a set $X \subset \kappa$ in V^1 , let \mathcal{P}^X be a variation of the Friedman–Magidor iteration, $\mathcal{P}^X = \mathcal{P}_\kappa^X = \langle \mathcal{P}_\nu^X, \mathcal{Q}_\nu^X \mid \nu < \kappa \rangle$, where

$$\mathcal{Q}_\nu^X = \begin{cases} \text{Code}^*(\nu) & \text{if } \nu \in X \text{ is inaccessible,} \\ \text{The trivial poset} & \text{otherwise.} \end{cases}$$

LEMMA 7.2: *Let $X \subset \kappa$ be a set in V^1 , and $G^X \subset \mathcal{P}^X$ be a generic filter over V^1 . For every normal measure U on κ in V^1 , if $X \notin U$ then U has a unique extension U^X in $V^1[G^X]$. Furthermore, these are the only normal measures on κ in $V^1[G^X]$.*

Proof. Let $U \in V^1$ be a normal measure on κ such that $X \notin U$, and

$$j : V^1 \rightarrow M^1 \cong \text{Ult}(V^1, U)$$

be its ultrapower embedding. We have $j(\mathcal{P}^X) \upharpoonright \kappa = \mathcal{P}^X$. Also, stage κ in $j^1(\mathcal{P}^X)$ is trivial as $\kappa \notin j(X)$. Like the Friedman–Magidor poset, \mathcal{P}^X satisfies that for every dense open set $D \subset j(\mathcal{P}^X)$, there is some $g \in G^X$ so that $j(g)$ reduces D to a dense open set in $j(\mathcal{P}^X) \upharpoonright (\kappa + 1) = \mathcal{P}^X$, which is intersected by G^X . Thus $j^1 G^X$ determines a unique generic filter $H^X \subset j(\mathcal{P}^X) \setminus \kappa$ over $M^1[G^X]$. Setting $G^* = G^X * H^X$, we get that $G^* \subset j(\mathcal{P}^X)$ is the unique generic filter over M^1 for which $j^1 G^X \subset G^*$. It follows that $j^* : V^1[G^X] \rightarrow M^1[G^*]$ is

¹⁰ Namely, we can use a \diamond_{κ^+} sequence in $V = \mathcal{K}(V^1)$ which is definable from $H(\kappa^+)^V$.

the only extension of $j : V^1 \rightarrow M^1$ to $V^1[G^X]$ and that

$$U^X = \{Y \subset \kappa \mid \kappa \in j^*(Y)\}$$

is the only extension of U in $V^1[G^X]$. For $\alpha, \beta < o(\kappa)$ such that $X \notin U^1_{(\alpha, \beta)}$, we denote $(U^1_{(\alpha, \beta)})^X$ by $U^X_{(\alpha, \beta)}$.

Suppose now that $W \in V^1[G^X]$ is a normal measure on κ and let

$$j_W : V^1[G^X] \rightarrow M_W$$

be the resulting ultrapower embedding. Then $j = j_W \upharpoonright V : V \rightarrow M$ is an iteration of V and $G_W = j_W(G) \subset j(\mathcal{P}^0 * \mathcal{P}^1 * \mathcal{P}^X)$ is generic over M . We first claim $X \notin W$. Otherwise, $\kappa \in j_W(X)$, so κ is a non-trivial forcing stage in $j_W(\mathcal{P}^X)$, and $\mathcal{Q}^X_\kappa = \text{Code}^*(\kappa)$. It follows that G_W introduces a club $D \subset \kappa^+$, disjoint from $j_W(f)(\kappa)_0 = X^*_0$. Note that D is a club in $V^1[G^X]$ since M_W is closed under κ -sequences. This is absurd as X^*_0 is stationary in V^1 and $|\mathcal{P}^X| = \kappa$.

To show that $W = U^X_{(\alpha, \beta)}$ for some $\alpha, \beta < o(\kappa) = \lambda$, it is sufficient to verify that $U^1_{(\alpha, \beta)} \subset W$. This is an immediate consequence of the proof of Proposition 6.1: Considering the restrictions

$$\pi : j \upharpoonright V : V \rightarrow M_W \quad \text{and} \quad \pi^0 : j_W \upharpoonright V^0 : V[G^0] \rightarrow M_W[G^0_W],$$

we get that α, β are determined from the values $o^{M_W}(\kappa)$, $s^{G^0_W}_{j_W(\kappa)}(\kappa)$, and whether $\Gamma \in W$. The proof of Proposition 6.1 relies solely on the analysis of the iterations of π, π^0 , and therefore applies here as well. ■

Suppose $U^X_{(\alpha, \beta)} \in V^1[G^X]$ is a normal measure on κ , and let

$$j^X_{(\alpha, \beta)} : V^1[G^X] \rightarrow M^X_{(\alpha, \beta)}$$

be its ultrapower embedding. We have that $j^X_{(\alpha, \beta)} \upharpoonright V^1 = j^1_{(\alpha, \beta)}$, thus

$$j^X_{(\alpha, \beta)} \upharpoonright V^0 = \pi^0_{\alpha, \beta} : V^0 \rightarrow Z^0_{\alpha, \beta}.$$

Note that $\pi^0_{\alpha, \beta}, Z^0_{\alpha, \beta}$ were used to determine the Mitchell order on $U^1_{(\alpha, \beta)}$, in Proposition 5.1. It follows that the proof of this Proposition applies to $U^X_{(\alpha, \beta)}$ as well.

COROLLARY 7.3: *Suppose that $U^X_{(\alpha', \beta')}, U^X_{(\alpha, \beta)} \in V^1$ where $\alpha' \leq \beta'$ and $\alpha \leq \beta$. We have that $U^X_{(\alpha', \beta')} \triangleleft U^X_{(\alpha, \beta)}$ if and only if $\beta' < \alpha$.*

LEMMA 7.4 (The final cut): *Let κ be a measurable cardinal in a transitive model of set theory V so that the normal measures on κ are separated by sets. Suppose that for every $X \subset \kappa$ there is a poset $\mathcal{P}^X \in V$ so that:*

- (1) *The normal measures on κ which extend in a \mathcal{P}^X generic extension are exactly the normal measures $U \in V$ which do not contain X . Furthermore, if $X \notin U$ then U has a unique extension $U^X \in V^{\mathcal{P}^X}$.*
- (2) *\mathcal{P}^X preserves the Mitchell order in V^1 . Namely, for every $U, W \in V$ which extend to U^X, W^X respectively, $U^X \triangleleft W^X$ if and only if $U \triangleleft W$.*

Then for every $\mathcal{W} \subset \triangleleft(\kappa)^V$ of cardinality $\leq \kappa$ there is a set $X \subset \kappa$ such that

$$\triangleleft(\kappa)^{V^{\mathcal{P}^X}} \cong \triangleleft(\kappa)^V \upharpoonright \mathcal{W}.$$

Proof. Let $\langle U_i \mid i < \rho \rangle$ be an enumeration of \mathcal{W} , where $\rho \leq \kappa$ is a cardinal. For every $i < \rho$ let $X_i \subset \kappa$ be a set which separates U_i from the rest of the normal measures on κ . Let $X_{\mathcal{W}} = \Delta_{i < \rho} X_i$, where $\Delta_{i < \rho}$ is the diagonal union if $\rho = \kappa$, and an ordinary union otherwise. It follows that the set $X = \kappa \setminus X_{\mathcal{W}}$ belongs to a normal measure $U \in V$, if and only if $U \notin \mathcal{W}$. Thus, it follows from the rest of the assumptions that $\triangleleft(\kappa)^{V^{\mathcal{P}^X}} \cong \triangleleft(\kappa)^V \upharpoonright \mathcal{W}$. ■

THEOREM 7.5: *Suppose that $V = L[\mathcal{U}]$ is a Mitchell model and $(S, <_S)$ is a tame order so that $|S| \leq \kappa$ and $\text{Trank}(S) \leq \mathcal{d}^{\mathcal{U}}(\kappa)$. Then there is a cofinality preserving generic extension V^* of V such that $\triangleleft(\kappa)^{V^*} \cong (S, <_S)$.*

Proof. If $\lambda \leq \kappa$ and $(S, <_S)$ is reduced, then $(S, <_S)$ embeds into R_λ for every $\lambda \geq \text{Trank}(S, <_S)$ (Proposition 2.10). We verify that the claim is an immediate consequence of the results established in Sections 5 and 6. We may assume that $S \subset R_\lambda$, and force with $\mathcal{P}^0 * \mathcal{P}^1$ over $V = L[\mathcal{U}]$ to obtain a generic extension V^1 of V , so that:

- the normal measures on κ are separated by sets (Proposition 6.1 and Corollary 3.14), and
- there are distinguished normal measures $U^1_{(\alpha, \beta)}$, $\alpha \leq \beta < \lambda$, so that $\triangleleft(\kappa)^{V^1} \upharpoonright \{U^1_{(\alpha, \beta)} \mid \alpha \leq \beta\} \cong R_\lambda$ (Proposition 5.1).

Let $\mathcal{W} = \{U^1_{(\alpha, \beta)} \mid (\alpha, \beta) \in S\}$; $|\mathcal{W}| \leq \kappa$ since $|S| \leq \kappa$, and by Lemma 7.4 there is a set $X \subset \kappa$ so that in a generic extension of V^1 by \mathcal{P}^X ,

$$\triangleleft(\kappa) \cong \triangleleft(\kappa)^{V^1} \upharpoonright S \cong (S, <_S).$$

Next, we describe how to modify \mathcal{P}^0 and \mathcal{P}^1 to deal with arbitrary tame orders $(S, <_S)$ of cardinality $\leq \kappa$. Let $\langle \rho_\tau \mid \tau < \kappa^+ \rangle$ be a sequence of canonical functions on κ , so that each ρ_τ has Galvin–Hajnal norm τ . If $j : V \rightarrow M$ with $\text{cp}(j) = \kappa$, then $j(\rho_\tau)(\kappa) = \tau$ for every $\tau < \kappa^+$. Also, for every $\alpha < \beta < \kappa^+$ the set $\{\nu < \kappa \mid \rho_\alpha(\nu) \not\leq \rho_\beta(\nu)\}$ is bounded in κ . Since $\lambda < \kappa^+$, we may choose the functions $\langle \rho_\tau \mid \tau < \lambda \rangle$ so that $\{\nu < \kappa \mid \rho_\alpha(\nu) \geq \rho_\beta(\nu)\} = \emptyset$ for every $\alpha < \beta \leq \lambda$. For each $\alpha < \lambda$ let $\Delta_\alpha = \{\nu < \kappa \mid o(\nu) = \rho_\alpha(\nu)\}$. It follows that the sets Δ_α , $\alpha < \lambda$, are pairwise disjoint. We proceed as follows:

1. It is not difficult to verify that there is a set $\Delta \in \bigcap_{\alpha < \lambda} U_{\kappa, \alpha}$, so that each $\nu \in \Delta$ is an inaccessible cardinal, a closure point of ρ_λ , and satisfies that $\rho_\lambda \upharpoonright \nu$ has a Galvin–Hajnal rank $\rho_\lambda(\nu) < \nu^+$. Also, $\mathcal{P}^0 = \langle \mathcal{P}_\nu^0, \mathcal{Q}_\nu^0 \mid \nu \leq \kappa \rangle$ is a Friedman–Magidor iteration where, for each $\nu < \kappa$, \mathcal{Q}_ν^0 is non-trivial if and only if $\nu \in \Delta \cup \{\kappa\}$, where $\mathcal{Q}_\nu^0 = \text{Sacks}_{\rho_\lambda \upharpoonright \nu}(\nu) * \text{Code}(\nu)$ is defined by:

- Conditions $T \in \text{Sacks}_{\rho_\lambda \upharpoonright \nu}(\nu)$ are the trees $T \subset <^\nu \nu \times \nu$ for which there is a club $C \subset \nu$, so that if $s \in T$ and $\text{len}(s) \in C$ then $s \frown \langle (\eta, \mu) \rangle \in T$ for every $\eta < \rho_\lambda(\text{len}(s))$ and $\mu < \text{len}(s)$.

The forcing $\text{Sacks}_{\rho_\lambda \upharpoonright \nu}(\nu)$ introduces a generalized Sacks function $s_\nu : \nu \rightarrow \rho_\lambda(\nu) \times \nu$.

- $\text{Code}(\nu)$ is a Friedman–Magidor coding poset, which introduces a club $C_\nu \subset \nu^+$ coding both s_ν and itself.

Let $V^0 = V[G^0]$ where $G^0 \subset \mathcal{P}^0$ is a V -generic filter. For each $(\eta, \mu) \in \lambda \times \kappa$ and $\alpha < o(\kappa)$, define

$$\Delta_\alpha(\eta, \mu) = \{\nu \in \Delta \cap \Delta_\alpha \mid s_\nu = s_\kappa \upharpoonright \nu \text{ and } s_\kappa(\nu) = (\rho_\eta(\nu), \mu)\}.$$

We get that

$$\{\Delta_\alpha(\eta, \mu) \mid \alpha < o(\kappa), \eta < \lambda, \mu < \kappa\}$$

are pairwise disjoint. The description of the normal measures on κ in Section 3 shows that each normal measure U_α in V extends in V^0 to

$$\{U_{(\alpha, \eta, \mu)}^0 \mid \eta < \lambda, \mu < \kappa\}$$

and that $\Delta(\eta, \mu) \in U_{(\alpha, \eta, \mu)}^0$.

The parameters $\alpha, \eta < \lambda$ in $U_{(\alpha, \eta, \mu)}^0$ will be associated with elements $(\alpha, \eta) \in R_\lambda$. The additional parameter $\mu < \kappa$ will guarantee that there are κ \leftarrow -equivalent copies of each $(\alpha, \eta) \in R_\lambda$, thus allowing us to realize non-reduced orders $(S, <_S)$ where each \sim_S equivalent class has cardinality $\leq \kappa$.

2. Next, we force over V^0 with a Magidor iteration of Prikry forcings, $\mathcal{P}^1 = \langle \mathcal{P}_\nu^1, \mathcal{Q}_\nu^1 \mid \nu < \kappa \rangle$. The recipe for choosing the normal measure on ν to be used at non-trivial iteration stages is similar to the recipe used in Section 3 (Definition 3.6), i.e., if $\nu \in \Delta_\alpha(\beta, \mu)$ for some $\beta < \lambda$ and $\mu < \kappa$, then

$$\mathcal{Q}_\nu^1 = \begin{cases} Q(U_{\nu,(\alpha,\beta,\mu)}^1) & \text{if } \beta < \alpha, \\ 0\text{---the trivial forcing} & \text{otherwise.} \end{cases}$$

Here, $U_{\nu,(\alpha,\beta,\mu)}^1$ is a normal measure on ν in $V^0[G^1 \upharpoonright \nu]$ which extends the measure $U_{\nu,(\beta,\alpha,\mu)}^0 \in V^0$ (thus extending $U_{\nu,\beta} \in V$). The definitions of $U_{(\alpha,\beta,\mu)}^1$ ($\alpha \geq \beta$ and $\alpha < \beta$) are similar to those of $U_{\alpha,\beta}^1$. Here, for $\alpha \geq \beta$, the $U_{\beta,\alpha}^0$ ultrapower in Definition 3.8 is replaced with an ultrapower by $U_{\beta,\alpha,\mu}^0$; for $\alpha < \beta$, the ultrapower by $U_{\alpha,\beta}^0 \times U_{\beta,\alpha}^0$ in Definition 3.10 is replaced with an ultrapower by $U_{\alpha,\beta,\mu}^0 \times U_{\beta,\alpha,\mu}^0$.

Therefore, a V^0 -generic filter $G^1 \subset \mathcal{P}^1$ introduces a Prikry (partial) function $d : \Delta \rightarrow \kappa$, where:

- $\nu \in \text{dom}(d)$ if and only if there are $\alpha < \beta < \lambda$ and $\mu < \nu$ so that $\nu \in \Delta_\beta(\alpha, \mu)$, and then
- $d(\nu) \in \Delta_\alpha(\beta, \mu) \cap \nu$ (for all but finitely many ν).

It follows that for every $\alpha < \beta < \lambda$ and $\mu < \kappa$, the function $\nu \mapsto (\nu, d^{-1}(\nu))$ introduces a projection of $U_{(\alpha,\beta,\mu)}^1 \in V^1$ to an extension of the product

$$U_{(\alpha,\beta,\mu)}^0 \times U_{(\beta,\alpha,\mu)}^0 \in V^0$$

(thus extending $U_\alpha \times U_\beta \in V$). When $\alpha = \beta$, $U_{(\alpha,\alpha,\mu)}^1$ extends $U_{(\alpha,\alpha,\mu)}^0 \in V^0$. The obvious modification of the proof of Proposition 5.1 implies that for every $U_{(\alpha',\beta',\mu')}^1, U_{(\alpha,\beta,\mu)}^1$ in V^1 , where $\alpha \leq \beta$ and $\alpha' \leq \beta'$, we have that

$$U_{(\alpha',\beta',\mu')}^1 \triangleleft U_{(\alpha,\beta,\mu)}^1 \iff \beta' < \alpha.$$

In particular, when restricting \triangleleft to these measures, we see that for every $\alpha \leq \beta < \lambda$ the normal measures in $\{U_{(\alpha,\beta,\mu)}^1 \mid \mu < \kappa\}$ are \triangleleft equivalent.

3. Let $([S], <_{[S]})$ be the reduction of $(S, <_S)$; $([S], <_{[S]})$ is reduced and $\text{Trank}([S], <_{[S]}) = \text{Trank}(S, <_S) = \lambda$. Proposition 2.10 implies that $([S], <_{[S]})$ embeds in $(R_\lambda, <_{R_\lambda})$. Since each equivalent class in $[S]$ has size at most κ , it follows that there is a subset $\mathcal{W} \in V^1$ of normal measures on κ such that $\triangleleft(\kappa)^{V^1} \upharpoonright \mathcal{W} \cong (S, <_S)$. By Lemma 7.4 there is a set $X \subset \kappa$ so that, in a generic extension of V^1 by \mathcal{P}^X , $\triangleleft(\kappa) \cong \triangleleft(\kappa)^{V^1} \upharpoonright \mathcal{W} \cong (S, <_S)$. ■

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