

FULL RESIDUAL FINITENESS GROWTHS OF NILPOTENT GROUPS

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ABSTRACT

Full residual finiteness growth of a finitely generated group G measures how efficiently word metric n -balls of G inject into finite quotients of G . We initiate a study of this growth over the class of nilpotent groups. When the last term of the lower central series of G has finite index in the center of G we show that the growth is precisely n^b , where b is the product of the nilpotency class and dimension of G . In the general case, we give a method for finding an upper bound of the form n^b where b is a natural number determined by what we call a terraced filtration of G . Finally, we characterize nilpotent groups for which the word growth and full residual finiteness growth coincide.

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Introduction

Let G be a residually finite group endowed with a word metric given by a finite generating set X . A subset $S \subseteq G$ is **fully detected** by a group Q if there exists a homomorphism $\varphi : G \rightarrow Q$ such that $\varphi|_S$ is injective. For a natural number n , set $\Phi_G^X(n)$ to be the minimal order of a group Q that fully detects the ball of radius n in G (first studied in [4]). The **full residual finiteness growth of G with respect to X** is the growth of the function Φ_G^X , that is, its equivalence class under the equivalence relation defined by $f \approx g$ if and only if there is a constant C so that $f(n) \leq Cg(Cn)$ and $g(n) \leq Cf(Cn)$ for all natural numbers n . The growth of Φ_G^X is independent of choice of generating set X (see Lemma 1.1). Therefore full residual finiteness growth is an invariant of a finitely generated group, and can be denoted simply Φ_G .

This article focuses on finitely generated nilpotent groups. While it is known that word growth (defined below after Theorem 2) has precisely polynomial growth over this class of groups [1], computing other growth functions for this class has proved to be a serious task. Indeed, even computing the answer for subgroup growth [13] in the two-generated free nilpotent case takes work; see [16]. The main difficulty lies in that the structure of p -group quotients of a fixed finitely generated nilpotent group can depend heavily on the choice of the prime p . That is, it is difficult to draw global behavior (behavior over all finite quotients) from local behavior (behavior over all finite quotients that are p -groups). Moreover, comparisons between full residual finiteness growth and word growth, which is to our knowledge the only nontrivial growth function known to have precisely polynomial growth over the class of nilpotent groups, do not allow one to immediately draw much information on Φ_G^X . In fact, the growth of Φ_G^X is often, but not always, strictly larger than the word growth of G (see Theorem 3). Obtaining sharp control of full residual finiteness growth over this class requires new understanding of the structure theory of nilpotent groups.

To present our findings, we begin with some basic examples. In §2, we show that $\Phi_G(n) = n^k$ for $G = \mathbb{Z}^k$ and $\Phi_G(n) = n^6$ for G equal to the discrete Heisenberg group. The key property shared by these examples is that the

center of G is equal to the last term of the lower central series of G . In fact, we explicitly compute full residual finiteness growths for all groups satisfying a slightly weaker condition. To make this precise we introduce notation: for a nilpotent group G of class c we denote by $\gamma_c(G)$ the last nontrivial term of its lower central series, by $Z(G)$ its center, and by $\dim(G)$ its dimension. (See §1.2 for more explicit definitions.)

THEOREM 1: *Let G be a finitely generated nilpotent group of class c with $[Z(G) : \gamma_c(G)] < \infty$. Then*

$$\Phi_G(n) \approx n^{c \dim(G)}.$$

The conclusion of Theorem 1 does not generally hold when $[Z(G) : \gamma_c(G)] = \infty$. This is seen by taking G to be the direct product of the discrete Heisenberg group with \mathbb{Z} , which satisfies $\Phi_G(n) \approx n^7$ while $c = 2$ and $\dim(G) = 4$. Groups not satisfying the hypothesis of Theorem 1 are generally more complicated than this example. For instance, in Proposition 3.1 we provide an example of a nilpotent group Γ of class $c = 3$ with $\dim(\Gamma) = 8$ and $\Phi_\Gamma(n) \approx n^{22}$ that does not split as a direct product.

For general nilpotent group G of class c , we introduce methods to find an upper bound on the polynomial degree of Φ_G . Define a **terraced filtration** of G to be a filtration $1 = H_0 \leq H_1 \leq \dots \leq H_{c-1} \leq G$ where each H_i is a maximal normal subgroup of G satisfying $H_i \cap \gamma_{i+1}(G) = 1$. Every terraced filtration of G gives an explicit polynomial upper bound on growth of Φ_G .

THEOREM 2: *Let G be a finitely generated nilpotent group of class c . Suppose $1 = H_0 \leq H_1 \leq \dots \leq H_{c-1} \leq G$ is a terraced filtration of G . Then*

$$\Phi_G(n) \preceq n^{c \dim(G) - \sum_{i=1}^{c-1} \dim(H_i)}.$$

This upper bound generally depends on the choice of terraced filtration. See the comments following the proof of Theorem 2 in §3 for an explicit example demonstrating this dependence. It would be interesting to determine whether the lowest upper bound obtained from a terraced filtration by Theorem 2 is optimal.

Results on distortion in nilpotent groups from Osin [15] and Pittet [17] play an important role in all of our proofs.

We also compare full residual finiteness growth to word growth. Recall that the **word growth**, w_G , of a finitely generated group G is the growth of the

function $w_G^X(n) = |B_G^X(n)|$, which is independent of X . Gromov [9] has characterized nilpotent groups in the class of finitely generated groups as those for which w_G is polynomial. By applying this theorem, it is shown in [4] (see Theorem 1.3 there) that full residual finiteness growth enjoys the same conclusion. In spite of this similarity, these two growths rarely coincide. Our final result characterizes nilpotent groups for which full residual finiteness growth equals word growth.

THEOREM 3: *Let G be a finitely generated nilpotent group. Then $\Phi_G \approx w_G$ if and only if G is virtually abelian.*

In §1.1 we provide a geometric interpretation of Φ_G^X . From this point of view, Theorem 3 implies that virtually abelian groups are characterized in the class of finitely generated nilpotent groups solely in terms of the asymptotic data of the Cayley graph. A non-normal version of full residual finiteness growth, the systolic growth, is studied in [8]. There it is shown that systolic growth matches word growth if and only if the group is Carnot.

This paper is organized as follows: In §1 we present basic results on nilpotent groups and full residual finiteness growth, including important lemmas on word metric distortion of central subgroups of nilpotent groups following from work of Osin [15] and Pittet [17]. In §2.1 we prove Theorem 3. In §2.2 we compute the full residual finiteness growth of the Heisenberg group and prove Theorem 1. In §3 we give an illustrative example showing that the conclusion of Theorem 1 does not hold in general, and prove Theorem 2.

We finish the introduction with a bit of history. The concept of full residual finiteness growth was first studied by Ben McReynolds and K.B. in [4]. The full residual finiteness growth of the discrete Heisenberg group is presented in [8]. Compare full residual finiteness growth to the concept of **residual finiteness growth**, which measures how well individual elements are detected by finite quotients, appearing in [5], [7], [3], [11], [18], [6], [12]. Also compare this with Sarah Black's growth function defined and studied in [2]. Full residual finiteness growth measures how efficiently the word growth function can be recovered from Black's growth function. See remarks in [2] on p. 406 before §2 for further discussion.

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1. Some background and preliminary results

1.1. FULL RESIDUAL FINITENESS GROWTH. In this subsection we give a geometric interpretation of full residual finiteness growth for finitely presented groups.

Write $f \preceq g$ to mean there exists C such that $f(n) \leq Cg(Cn)$. We write $f \approx g$ if $f \preceq g$ and $g \preceq f$. Recall that the **growth** of a function f is the equivalence class of f with respect to \approx .

We first prove a lemma that implies that the growth of the function Φ_G^X defined in the introduction is independent of generating set X :

LEMMA 1.1: *Let G be finitely generated with finitely generated subgroup $H \leq G$. Fix finite generating sets X and Y for G and H . Then $\Phi_H^Y \preceq \Phi_G^X$.*

Proof. Since $H \leq G$, there exists $C > 0$ such that any element in Y can be written in terms of at most C elements in X . Thus, $B_H(n) \subseteq B_G(Cn)$ for any $n > 1$. Because any homomorphism from G restricts to a homomorphism from H , this gives

$$\Phi_G^X(Cn) \geq \Phi_H^Y(n),$$

as desired. ■

Lemma 1.1 in particular implies that if X and Y are two finite generating sets of a group G , then $\Phi_G^X \approx \Phi_G^Y$. Let Φ_G denote the equivalence class of Φ_G^X with respect to \approx for any finite generating set X of G .

We now provide a geometric interpretation of Φ_G in the case that G is a finitely presented group. Let G be a residually finite group with Cayley graph Γ with respect to a finite generating set S . Each edge of Γ is labeled by the corresponding generator. For a subset $X \subseteq \Gamma$, we set ∂X to be the collection of edges and vertices of X each of which has closure not contained in the interior of X . Let $\{A_k\}$ be an increasing sequence of finite connected subsets of Γ with

$$A_{k+1} = \partial A_{k+1} \sqcup A_k.$$

Then the sequence of subsets, $\{A_k\}$, is called a **growing sequence**. Let $B_G^S(n)$ denote the closed ball of radius n in the Cayley graph of G with respect to the word metric induced by S . We will omit the S from the notation when the generating set is understood and there is no chance for confusion. The prototypical example of a growing sequence is the sequence that assigns to each positive integer k the metric ball $B_G^S(k)$ in the Cayley graph of G with respect to S .

The **geometric full residual finiteness growth** of Γ with respect to $\{A_k\}$ is the growth of the function, $\Phi_\Gamma^{\{A_k\}} : \mathbb{N} \rightarrow \mathbb{N}$, given by

$$n \mapsto \min\{|Q| : Q \text{ is a group with } A_n \text{ isometrically embedding in one of its Cayley graphs}\}.$$

Our first lemma demonstrates that the growth of $\Phi_G^{\{A_k\}}$ does not depend on the growing sequence.

LEMMA 1.2: *Let $\{X_k\}$ and $\{Y_k\}$ be two growing sequences for a finitely generated group G . Then $\Phi_G^{\{X_k\}} \approx \Phi_G^{\{Y_k\}}$.*

Proof. We first assume that $\{X_k\}$ and $\{Y_k\}$ are growing sequences from the same Cayley graph realization of G . Then there exists $K \in \mathbb{N}$ such that

$$Y_1 \subseteq X_K \quad \text{and} \quad X_1 \subseteq Y_K.$$

Hence, $C_G^{\{Y_k\}}(n) \leq C_G^{\{X_k\}}(K + i)$ and $C_G^{\{X_k\}}(n) \leq C_G^{\{Y_k\}}(K + i)$. Thus, we can assume that $\{X_k\}$ and $\{Y_k\}$ are the word metric k -balls of G with respect to two different generating sets. It is straightforward to see that there exists $C > 0$ such that $Y_n \subseteq X_{Cn} \subseteq Y_{C^2n}$ for every natural number n . Hence

$$\Phi_G^{\{Y_k\}}(n) \leq \Phi_G^{\{X_k\}}(Cn) \leq \Phi_G^{\{Y_k\}}(C^2n),$$

as desired. ■

Next we show that the notions of full residual finiteness growth, given in the introduction, and geometric full residual finiteness growth, given in this section, agree in the case that the group G is finitely presented. It would be interesting to determine if this equivalence holds for all finitely generated groups.

LEMMA 1.3: *Let G be a finitely presented group. For any generating set X and growing sequence $\{A_k\}$ we have*

$$\Phi_G^{\{A_k\}} \approx \Phi_G^X.$$

Proof. Let X be a finite generating set for G and let R be the set of finite relations. It is clear that $\Phi_G^{\{A_k\}} \preceq \Phi_G^X$. We show the reverse inequality. We can, by Lemma 1.2, suppose that the growing set $\{A_k\}$ is simply the sequence $\{B_G^X(k)\}$. It suffices, then, to show that there exists $N \in \mathbb{N}$ such that for any $n > N$ and any finite group, Q , with $B_G(n)$ isometrically embedding in a Cayley graph realization of Q , there exists a homomorphism $\phi : G \rightarrow Q$ with $\phi|_{B_G(n)}$ being injective. Select N to be the maximal word length of any element in R . Then since B_n isometrically embeds in a Cayley graph of Q , we see that there exists a generating set for Q such that each relator R is satisfied by this generating set. This finishes the proof. ■

The next lemma controls some of the full residual finiteness growth of a direct product of groups.

LEMMA 1.4: *Let G and H be finitely generated groups. Then*

$$\Phi_{G \times H} \preceq \Phi_G \cdot \Phi_H.$$

Proof. Fix generating sets X and Y for G and H . Then $(X \times \{1\}) \cup (\{1\} \times Y)$ is a finite generating set for $G \times H$. Note that

$$B_{G \times H}(n) \subseteq (B_G(n) \times \{1\})(\{1\} \times B_H(n)).$$

Thus, if Q_1 is a quotient that fully detects $B_G(n)$ and Q_2 a quotient that fully detects $B_H(n)$, then $Q_1 \times Q_2$ fully detects $B_{G \times H}(n)$. We see then that $\Phi_{G \times H} \preceq \Phi_G \Phi_H$, as desired. ■

Can the conclusion of Lemma 1.4 be improved to $\Phi_{G \times H} \approx \Phi_G \Phi_H$? This can possibly be false: it is not even true that if $\varphi : G \rightarrow H$ is a surjective homomorphism, then $\Phi_G(n) \succeq \Phi_H(n)$. Consider a free group mapping onto one of Kharlampovich–Sapir’s solvable and finitely presented groups of arbitrarily large residual finiteness growth [12].

Full residual finiteness growth is well-behaved under taking the quotient by a finite normal subgroup:

PROPOSITION 1.5: *Let G be a finitely generated residually finite group. Let T be a finite normal subgroup of G . Then $\Phi_G \approx \Phi_{G/T}$.*

Proof. Fix a generating set X for G , and let Y be the image of X under the quotient map $G \rightarrow G/T$. Let K be the largest length, with respect to X , of an

element in T . We first claim

$$\Phi_G^X(K + n) \geq \Phi_{G/T}^Y(n).$$

Let $\phi : G \rightarrow Q$ be a finite quotient of minimal cardinality that fully detects $B_G^X(K + n)$. That is $|Q| = \Phi_G^X(K + n)$. Define $\psi : G/T \rightarrow Q/\phi(T)$ by $gT \mapsto \phi(g)\phi(T)$. Let $g \in B_{G/T}^Y(n) \cap \ker \psi$. By construction, we may lift g to an element $\tilde{g} \in G$ such that $\tilde{g} \in B_G^X(n)$ and $\phi(\tilde{g}) \in \phi(T)$. That is, there exists $t \in T$ such that $\phi(g) = \phi(t)$, which gives

$$\phi(\tilde{g}t^{-1}) = 1.$$

If $\tilde{g}t^{-1} \neq 1$, then this contradicts that ϕ fully detects $B_G(K + n)$. Hence $\tilde{g} = t$, and so $\ker \psi \cap B_{G/T}^Y(K + n)$ is trivial. It follows that ψ fully detects $B_{G/T}^Y(n)$, and so $\Phi_G^X(K + n) \geq \Phi_{G/T}^Y(n)$, as claimed.

Since G is residually finite and T is finite, there exists a normal subgroup, H , such that $T \cap H = 1$. To finish, we claim that $\Phi_G^X(n) \leq [G : H]\Phi_{G/T}^Y(n)$. Let $\psi : G/T \rightarrow Q$ be a quotient that fully detects $B_{G/T}^Y(n)$, with $|Q| = \Phi_{G/T}^Y(n)$. Let $\phi : G \rightarrow Q$ be the natural map $G \rightarrow G/T \rightarrow Q$. Set $N = \ker \phi \cap H$. Clearly, $[G : N] \leq [G : \ker \phi][G : H] = |Q|[G : H]$. Moreover, if $g \in B_G^X(n) \cap N$, then $g \notin T$. Hence, by the construction of Y , we have that $\phi(g) \neq 1$. It follows that G/N fully detects $B_G^X(n)$, and so $\Phi_G^X(n) \leq [G : H]\Phi_{G/T}^Y(n)$, as desired. ■

We finish the section with a lemma that, in some restrictive cases, allows us to pass to finite-index subgroups.

LEMMA 1.6: *Let G and H be finitely generated nilpotent groups with H normal subgroup in G of finite index. If every normal subgroup of H is normal in G , then $\Phi_G \approx \Phi_H$.*

Proof. By Lemma 1.1, it suffices to show that $\Phi_G \preceq \Phi_H$. Fix generating sets for G and H so that $B_H(n) \subseteq B_G(n)$ for all $n > 0$. Because H is of finite index in G and thus quasi-isometric to G , there exists $C > 0$ such that

$$H \cap B_G(2n) \subseteq B_H(Cn).$$

Let H/K be a quotient of H that fully detects $B_H(Cn)$. By our assumption, K is normal in G so G/K is well-defined. Then any element in $B_G(2n)$ not in H is mapped nontrivially onto G/K . And since $H \cap B_G(2n) \subseteq B_H(Cn)$, it follows that $B_G(2n)$ is mapped nontrivially onto G/K . Thus, $B_G(n)$ is fully detected by G/K , and so we are done. ■

1.2. NILPOTENT GROUPS. In this subsection we fix basic notation and present several lemmas that play important roles in our proofs. Let G be a group. The **lower central series** $\gamma_k(G)$ of G is the sequence of subgroups defined by $\gamma_1(G) = G$ and

$$\gamma_k(G) = [\gamma_{k-1}(G), G].$$

For any group H , let $Z(H)$ denote the center of H . The **upper central series** $\zeta_k(G)$ of G is given by $\zeta_0(G) = \{e\}$ and the formula

$$\zeta_k(G)/\zeta_{k-1}(G) = Z(G/\zeta_{k-1}(G)).$$

The group G is said to be **nilpotent** if $\gamma_k(G) = 1$ for some natural number k . Equivalently, G is nilpotent if and only if it is an element of its upper central series. Moreover, G is said to be **nilpotent of class c** if $\gamma_c(G) \neq 1$ and $\gamma_{c+1}(G) = 1$.

If G is a finitely generated nilpotent group, then the successive quotients of the upper central series of G are abelian groups of finite-rank. Thus, the upper central series has a refinement

$$G = G_1 \geq G_2 \geq \dots \geq G_{n+1} = 1,$$

such that G_i/G_{i+1} is cyclic for all $i = 1, \dots, n$. The number of infinite cyclic factors in this series does not depend on the series and is called the **dimension** of G , denoted by $\dim(G)$ [19, p. 16, Exercise 8]. Let this series be chosen so that n is minimal. An n -tuple of elements $(g_1, g_2, \dots, g_n) \in G^n$ is a **basis** for G if $g_i \in G_i$ and $G_i/G_{i-1} = \langle g_i G_{i-1} \rangle$ for each $i = 1, \dots, n$. In the case when G_i/G_{i+1} is infinite for all $i = 1, \dots, n$ we call the n -tuple a **Malcev basis** for G .

The set of torsion elements T in a finitely generated nilpotent group G is a finite normal subgroup, and the quotient G/T is a torsion-free nilpotent group [19, p. 13, Corollary 10]. A corollary of Proposition 1.5 is that G/T has the same full residual finiteness growth as G .

COROLLARY 1.7: *If G is a finitely generated nilpotent group and T is the subgroup of torsion elements, then $\Phi_G \approx \Phi_{G/T}$.*

We recall a folklore result, used in the proof of the following lemmas.

LEMMA 1.8: *Suppose G is a finitely generated nilpotent group of class c . The assignment $(x, y) \mapsto [x, y]$ defines a homomorphism*

$$(\zeta_k(G)/\zeta_{k-1}(G)) \times (\zeta_\ell(G)/\zeta_{\ell-1}(G)) \rightarrow \zeta_{k+\ell-c-1}(G)/\zeta_{k+\ell-c-2}(G).$$

Proof. This follows immediately from [14, Theorem 2.1], noting that the upper central series is a central filtration of G when indexed so that the i^{th} term of the filtration is $\zeta_{c+1-i}(G)$. ■

If G is a group generated by a finite set X , for $g \in G$ we use $\|g\|_X$ to denote the word length of g with respect to X . Let G be a finitely generated nilpotent group. The following lemma is a consequence of well-known distortion estimates.

LEMMA 1.9: *Let G be a nilpotent group of class c generated by a finite set X . Fix a positive integer i and a generating set X_i for $Z(G) \cap \gamma_i(G)$. Then there exists $C > 1$ such that for all $g \in Z(G) \cap \gamma_i(G)$,*

$$(1) \qquad \|g\|_X \leq C \|g\|_{X_i}^{1/i}.$$

Proof. We first assume G is torsion-free. First consider the case that $g = x^m$ for some $x \in X_i$ and $m \in \mathbb{Z} \setminus \{0\}$. Assume without loss of generality that X_i is a basis for the free abelian group $Z(G) \cap \gamma_i(G)$, so that $\|g\|_{X_i} = |m|$. Embed G as a cocompact lattice in a simply-connected nilpotent Lie group N , which identifies $Z(G) \cap \gamma_i(G)$ with a lattice in a simply-connected central subgroup $Z \leq N$. Fix any left-invariant Riemannian metric on N , which gives a norm $\|\cdot\|_{\mathfrak{n}}$ on \mathfrak{n} , the Lie algebra of N . Consider the path $\gamma : [0, |m|] \rightarrow Z$ defined so that $\gamma(|m|) = g$ and $\gamma(t) = \exp(tz)$ for some $z \in \mathfrak{n}$. Note that $\exp(z) = x$ if $m > 0$ and $\exp(z) = x^{-1}$ if $m < 0$. In particular, z does not depend on m . By [17, Prop. 4.1(1)], the length of γ is $\|z\|_{\mathfrak{n}} \|g\|_{X_i}$. Then applying [17, Prop. 4.1(2)] to the curve γ , there is a constant $C > 0$ depending on z so that

$$(2) \qquad d_N(e, g) \leq C \|g\|_{X_i}^{1/i}.$$

The quantity $d_N(e, g)$ is uniformly comparable to $\|g\|_X$, so this proves the desired inequality for g of the form x^m .

Now for any $g \in Z(G) \cap \gamma_i(G)$, write $g = \prod_{j=1}^k x_j^{m_j}$ where $X_i = \{x_1, \dots, x_k\}$. Let C be the largest constant appearing in equation (2) as x ranges over

x_1, \dots, x_k . Then there is some $D > 0$ so that

$$\begin{aligned} \|g\|_X &\leq \sum_{j=1}^k \|x_j^{m_j}\|_X \\ &\leq C \sum_{j=1}^k \|x_j^{m_j}\|_{X_i}^{1/i} \\ &\leq C \sum_{j=1}^k |m_j|^{1/i} \\ &\leq Ck \left(\sum_{j=1}^k |m_j| \right)^{1/i} \\ &\leq CkD \|g\|_{X_i}^{1/i}. \end{aligned}$$

The last step follows because $Z(G) \cap \gamma_i(G)$ is abelian. The penultimate step follows from the general fact that $(m_1^{1/i} + \dots + m_k^{1/i})^i \leq k^i(m_1 + \dots + m_k)$ when $m_j \geq 1$ for all j . This completes the proof in the case that G is torsion-free.

Now suppose G is an arbitrary finitely generated nilpotent group. There is a torsion-free normal subgroup H of finite index in G . Fix a generating set Y for H . The map $i : H \rightarrow G$ is a quasi-isometry because H is finite index in G . In fact, because distinct points in each of G and H are distance at least 1 and i is injective, it is easy to check that i is bi-Lipschitz. This means that there is some $C \geq 1$ so that:

- (1) For $g, h \in H$,

$$\frac{1}{C} \|gh^{-1}\|_Y \leq \|gh^{-1}\|_X \leq C \|gh^{-1}\|_Y.$$

- (2) For every element $g \in G$, there exists $h \in H$ such that

$$\|hg^{-1}\|_X \leq C.$$

Fix generating sets X_i for $Z(G) \cap \gamma_i(G)$ and Y_i for $Z(H) \cap \gamma_i(H)$. We claim that $Z(H) \leq Z(G)$. Indeed, if not then there exists $h \in Z(H)$, an integer $r \geq 1$, and elements $x_1, \dots, x_r \in G$ such that $h \in \zeta_{r+1}(G) \setminus \zeta_r(G)$ and

$$[h, x_1, \dots, x_r] \in Z(G) \setminus \{1\}.$$

Since H has finite index in G there exists $n \in \mathbb{N}$ such that $x_1^n \in H$. By Lemma 1.8 we have

$$[h, x_1^n, \dots, x_r] = [h, x_1, \dots, x_r]^n.$$

Since H is normal we have $[h, x_1, \dots, x_r] \in H$. This implies $[h, x_1^n, \dots, x_r^n] \neq 1$ because H is torsion-free. Therefore $[h, x_1^n]$ cannot be trivial, which contradicts the fact that $h \in Z(H)$. By the aforementioned claim, $Z(G) \cap \gamma_i(G)$ contains $Z(H) \cap \gamma_i(H)$ as a subgroup. In fact, it is not hard to show that $Z(H) \leq Z(G)$ and $\gamma_i(H) \leq \gamma_i(G)$ are, in both cases, subgroups of finite index. Hence, the inclusion $i_2 : Z(H) \cap \gamma_i(H) \rightarrow Z(G) \cap \gamma_i(G)$ is a bi-Lipschitz quasi-isometry with constant $D \geq 1$.

Now select $C' > 1$ such that inequality (1) holds for all $g \in G$ that are finite order (again, there are only finitely many of them). Next, let g be an infinite order element in G with

$$g \in Z(G) \cap \gamma_i(G).$$

We can suppose, without loss of generality, that X contains X_i . Then since i_2 is a D -quasi-isometry, there exists $h \in Z(H) \cap \gamma_i(H)$ such that

$$\|hg^{-1}\|_X \leq \|hg^{-1}\|_{X_i} \leq D.$$

Since H is torsion-free, by enlarging C if necessary we have

$$\|h\|_Y \leq C\|h\|_{Y_i}^{1/i}.$$

Thus

$$(3) \quad \|g\|_X = \|gh^{-1}h\|_X \leq \|gh^{-1}\|_X + \|h\|_X \leq C + \|h\|_X.$$

And, further,

$$(4) \quad \|h\|_X \leq C\|h\|_Y \leq C^2\|h\|_{Y_i}^{1/i}.$$

To finish,

$$(5) \quad \|h\|_{Y_i} \leq D\|h\|_{X_i} = D\|gg^{-1}h\|_{X_i} \leq D(\|g\|_{X_i} + D).$$

The desired inequality follows from equations (3)–(5), as all additive constants can be absorbed into the multiplicative constants. ■

Next, we show a technical lemma that will be important in our main proofs:

LEMMA 1.10: *Let G be a nilpotent group of class c generated by a finite set X . Fix a number $0 < i \leq c$, and fix generating sets Y_0 for $\zeta_i(G)$ and Y for $Z(G)$. There exists $C_i > 0$ such that for any $g \in \zeta_i(G) \setminus \zeta_{i-1}(G)$, there exists $x_1, \dots, x_{i-1} \in X$ such that for any $\gamma \in \zeta_{i-1}(G)$,*

$$0 < \|[g\gamma, x_1, \dots, x_{i-1}]\|_Y \leq C_i\|g\|_{Y_0}.$$

In fact, there is some $F_i > 0$ so that

$$0 < \|[g\gamma, x_1, \dots, x_{i-1}]\|_X \leq F_i \|g\|_{Y_0}^{1/t},$$

where t is the minimal k satisfying $[g\gamma, x_1, \dots, x_{i-1}] \notin \gamma_{k+1}(G)$.

Proof. Let $g \in \zeta_i(G) \setminus \zeta_{i-1}(G)$ be given. Since G is nilpotent, there exists $x_1, \dots, x_{i-1} \in X$ so that

$$[g, x_1, \dots, x_{i-1}] \in Z(G) \setminus \{1\}.$$

Note that for any $x \in \zeta_i(G)$ we have that

$$[x, x_1, \dots, x_{i-1}] \in Z(G).$$

Write $g = \prod_{i=1}^n g_i$ where $g_i \in Y_0$ and n is the word length of g with respect to Y_0 . Applying Lemma 1.8 repeatedly gives

$$\begin{aligned} [g, x_1, \dots, x_{i-1}] &= [g_1 g_2 \cdots g_n, x_1, \dots, x_{i-1}] \\ &= [g_1, x_1, \dots, x_{i-1}] [g_2, x_1, \dots, x_{i-1}] \cdots [g_n, x_1, \dots, x_{i-1}]. \end{aligned}$$

Set Y' to be Y union the set of all elements of the form $[\beta, \alpha_1, \alpha_2, \dots, \alpha_{i-1}]$ where $\beta \in Y_0$ and $\alpha_i \in X$. Notice that Y' does not depend on g . Further, by our above computation, we have

$$\|[g, x_1, \dots, x_{i-1}]\|_{Y'} \leq n.$$

Because Y' is finite, $(Z(G), d_Y)$ is bi-Lipschitz equivalent to $(Z(G), d_{Y'})$. This gives $C_i > 0$, depending only on Y' , such that

$$(6) \quad 0 < \|[g, x_1, \dots, x_{i-1}]\|_Y \leq C_i \|[g, x_1, \dots, x_{i-1}]\|_{Y'} \leq C_i n = C_i \|g\|_{Y_0}.$$

Let $\gamma \in \zeta_{i-1}(G)$ be arbitrary. Then as $[g, x_1, \dots, x_{i-1}]$ and $[\gamma, x_1, \dots, x_{i-1}]$ are central,

$$[g\gamma, x_1, \dots, x_{i-1}] = [g\gamma, x_1, \dots, x_{i-1}],$$

so the proof of the first assertion is complete.

Fix generating sets X_j for $\gamma_j(G) \cap Z(G)$ for each $1 \leq j \leq c$. These sets can be chosen independently of g and i . By Lemma 1.9, for each j we have that there exists $D_j > 1$ such that for all $w \in \gamma_j(G) \cap Z(G)$,

$$\|w\|_X \leq D_j \|w\|_{X_j}^{1/j}.$$

Set D to be the maximal such D_j . Notice that D only depends on X and G . Since $\gamma_t(G) \cap Z(G)$ is a subset of the abelian group, $Z(G)$, we have that there exists $E > 1$, depending only on Y and the selection of X_j , such that

$$\|[g\gamma, x_1, \dots, x_{i-1}]\|_{X_t} \leq E\|[g\gamma, x_1, \dots, x_{i-1}]\|_Y \leq E^2\|[g\gamma, x_1, \dots, x_{i-1}]\|_{X_t}.$$

Combining these inequalities with inequality (6) gives

$$\begin{aligned} 0 < \|[g\gamma, x_1, \dots, x_{i-1}]\|_X &\leq D\|[g\gamma, x_1, \dots, x_{i-1}]\|_{X_t}^{1/t} \\ &\leq ED\|[g\gamma, x_1, \dots, x_{i-1}]\|_Y^{1/t} \\ &\leq EDC_i\|g\|_{Y_0}^{1/t} = F_i\|g\|_{Y_0}^{1/t}, \end{aligned}$$

for some constant F_i that depends only on i and our choice of generating sets, as desired. ■

For any $g \in G$ of infinite order, the **weight** $\nu_G(g)$ of g in the group G is the maximal k such that $\langle g \rangle \cap \gamma_k(G) \neq \{1\}$. If G is a group and m a natural number, let G^m denote the normal subgroup of G generated by all m^{th} powers of elements of G . When G is nilpotent we have $[G : G^m] < \infty$ for any m (see, for instance, [10, p. 20, Lemma 4.2]). We need the following technical result for Lemma 1.12.

LEMMA 1.11: *Let G be a nilpotent group generated by a finite set X . Fix a positive integer i . Then there exists a constant $C > 1$ such that, for all $m \in \mathbb{N}$ and all $g \in (Z(G) \cap \gamma_i(G))^m$ with $\nu_G(g) = i$, we have*

$$m \leq C\|g\|_X^i.$$

Proof. Select $Y = \{x_1, \dots, x_r\}$ so that the image of Y under

$$\pi : (Z(G) \cap \gamma_i(G)) \rightarrow (Z(G) \cap \gamma_i(G))/\gamma_{i+1}(G)$$

generates a free abelian group of rank r , where r is the rank of

$$(Z(G) \cap \gamma_i(G))/\gamma_{i+1}(G).$$

Select N to be the order of $\pi(Z(G) \cap \gamma_i(G))/\pi(\langle Y \rangle)$.

Let f be the projection $G \rightarrow G/\gamma_{i+1}(G)$. We apply [15, Theorem 2.2] to the torsion-free subgroup $\Pi = f(\langle x_1, \dots, x_r \rangle) \leq G/\gamma_{i+1}(G)$ to get $D > 1$ such that

$$\sup_{h \in \Pi \cap B_{f(X)}(n)} \|h\|_{f(Y)} \leq Dn^i.$$

Thus, if $\|h\|_{f(X)} = n$, then $\|h\|_{f(Y)} \leq Dn^i = D(\|h\|_{f(X)})^i$. That is,

$$(7) \quad \|h\|_{f(Y)} \leq D\|h\|_{f(X)}^i.$$

Now suppose g is an element of $(Z(G) \cap \gamma_i(G))^m$ with $\nu_G(g) = i$. Since $\nu_G(g) = i$, we have that $\pi(g)$ is infinite order. Thus, we can write $g^N = h\gamma$ where $h \in \langle Y \rangle$ and $\pi(\gamma)$ is trivial. Thus $\gamma \in \gamma_{i+1}(G)$. The map $f|_{\langle Y \rangle}$ is an injection, thus

$$\|f(h)\|_{f(Y)} = \|h\|_Y.$$

Finally, by the fact that $g^N \equiv h \pmod{\gamma_{i+1}(G)}$ and inequality (7), we have

$$\begin{aligned} N\|g\|_X &\geq \|g^N\|_X \geq \|f(g^N)\|_{f(X)} \\ &= \|f(h)\|_{f(X)} \geq D^{1/i}\|f(h)\|_{f(Y)}^{1/i}. \end{aligned}$$

Notice that for any abelian group, A , and $\ell \in \mathbb{N}$ we have

$$A^\ell = \langle \{x^\ell : x \in A\} \rangle.$$

Using additive notation, this becomes

$$A^\ell := \langle \{\ell x : x \in A\} \rangle = \ell\{x : x \in A\} = \ell A.$$

So we have

$$mNA = m(NA) = m\{nx : x \in A\}.$$

To apply this, note that $A = (Z(G) \cap \gamma_i(G))$ is an abelian group, as it is contained in the center. For any element $y \in NA$, by the definition of N , we have $f(y)$ is an element of Π . Thus $f(my) = mf(y)$ is an element of Π^m , and so it follows that

$$f(A^{mN}) \leq \Pi^m.$$

In particular, $g^N \in (Z(G) \cap \gamma_i(G))^{Nm}$, so we have

$$f(g^N) \in \Pi^m.$$

Since $mf(Y)$ is a free basis for Π^m , $f(Y)$ is a free basis for Π , and $f(h) = f(g^N)$, we conclude that

$$\|f(h)\|_{f(Y)} = \|f(g^N)\|_{f(Y)} \geq m,$$

so we are done. ■

With the previous lemmas in hand, we finish with a proof that gives some control on the word lengths of elements in G^m .

LEMMA 1.12: *Let \tilde{G} be a finitely generated nilpotent group of nilpotence c . There exists $f \in \mathbb{N}$ such that $G = \tilde{G}^f$ is a torsion-free characteristic subgroup of \tilde{G} of finite index. Let $g \in G$, X , and $t \in \mathbb{N}$ be as in Lemma 1.10. Then there exists $C > 1$, $M \in \mathbb{N}$, depending only on G , such that if $g \in G^{M^m}$, we have that*

$$\|g\|_X \geq Cm^{1/t}.$$

Proof. Set $\tau(\tilde{G})$ to be the set of all elements of finite order in \tilde{G} . By [19, p. 13, Chapter 1, Corollary 10], this is a finite characteristic subgroup of \tilde{G} . Since G is residually finite and $\tau(H)$ is finite, there exists a finite Q that fully detects $\tau(\tilde{G})$. Set f to be the exponent of Q and set G to be the characteristic finite-index subgroup \tilde{G}^f [10, p. 20, Lemma 4.2]. Then the map $\tilde{G} \rightarrow Q$ factors through \tilde{G}/G , and thus $\tau(\tilde{G})$ is fully detected by \tilde{G}/G . Since $\tau(\tilde{G})$ contains all the torsion elements in \tilde{G} , it follows that G is torsion-free.

We will show by induction on d that for all $n > c$,

$$(8) \quad (\zeta_d(G))^{(d)! \cdots 2!n} \cap Z(G) \leq Z(G)^n.$$

The base case $\zeta_1(G) = Z(G)$ is immediate. For the inductive step, set

$$M = (d)!(d-1)! \cdots 2!$$

and let $H = \zeta_d(G) \leq G$. Let $h \in H^{Mn} \cap Z(G)$. Since h is in H^{Mn} we can write

$$h = g_1^{Mn} g_2^{Mn} \cdots g_k^{Mn} \in Z(G),$$

where g_1, \dots, g_k are elements in H .

To proceed, let $\tau_n(x_1, x_2, \dots, x_k) = \tau_n(\bar{x})$ be the n th **Petresco word** [10, p. 40], which is defined by the recursive formula

$$x_1^n x_2^n \cdots x_k^n = \tau_1(\bar{x})^n \tau_2(\bar{x})^{\binom{n}{2}} \cdots \tau_n(\bar{x})^{\binom{n}{n-1}}.$$

By the Hall–Petresco Theorem [10, p. 41, Theorem 6.3], we have that

$$\tau_n(H) \subset \gamma_n(H)$$

for all $n \in \mathbb{N}$. Thus, replacing n with Mn and using the Hall–Petresco Theorem, we get

$$g_1^{Mn} g_2^{Mn} \cdots g_k^{Mn} = \tau_1(\bar{g})^{Mn} \tau_2(\bar{g})^{\binom{Mn}{2}} \cdots \tau_d(\bar{g})^{\binom{Mn}{d}}.$$

By the Hall–Petresco Theorem, $\tau_k(\bar{g}) \in \zeta_{d-1}(G)$ for all $k > 1$, and by definition $h = g_1^{Mn} g_2^{Mn} \cdots g_k^{Mn} \in \zeta_{d-1}(G)$. Therefore, because $G/\zeta_{d-1}(G)$ is torsion-free, $\tau_1(\bar{g})$ is in $\zeta_{d-1}(G)$. We conclude that, for each $1 \leq k \leq d$, there exists

$z_k \in \zeta_{d-1}(G)$ such that

$$\tau_k(\bar{g})^{\binom{Mn}{k}} = (z_k)^{\frac{M}{(d)!}n} \in \zeta_{d-1}(G)^{\frac{M}{(d)!}}.$$

Further,

$$\frac{M}{d!} = (d - 1)! \cdots 2!.$$

Hence $h \in (\zeta_{d-1}(G))^{(d-1)!(d-2)! \cdots 2!}$, so by the inductive hypothesis we must have $h \in Z(G)^n$, which completes the proof of equation (8).

Let D be the product of all finite order elements in $G/\gamma_n(G)$ for all $n = 1, \dots, c$. Selecting $d = c$ in equation (8) we get, for $M = (c)!(c-1)! \cdots 2!D$, and $n > c$,

$$(9) \quad G^{Mn} \cap Z(G) \leq Z(G)^{Dn}.$$

Now suppose $g \in G^{Mm}$. By Lemma 1.10, there exists $x_1, \dots, x_{i-1} \in X$ such that $[g, x_1, \dots, x_{i-1}] \in \gamma_t(G) \cap Z(G)$. Thus, as G^{Mm} is normal, we have, by equation (9), $[g, x_1, \dots, x_{i-1}] \in Z(G)^{Dm}$. Hence, by our choice of D and the fact that $Z(G)$ is a free abelian group, we have $\nu_G([g, x_1, \dots, x_{i-1}]) = t$. Thus, applying Lemma 1.11 gives $C_1 > 0$, depending only on G , such that

$$\|[g, x_1, \dots, x_{i-1}]\|_X > C_1 m^{1/t}.$$

A simple counting argument gives a $C_2 > 0$, depending only on G , such that

$$\|g\|_X \geq C_2 \|[g, x_1, \dots, x_{i-1}]\|_X.$$

Thus, we have $C > 0$, depending only on G , such that

$$\|g\|_X > C m^{1/t},$$

as desired. ■

2. Some examples and basic results

2.1. ABELIAN GROUPS. In this section we discuss some facts concerning abelian groups and present a proof of Theorem 3. This begins with the simplest torsion-free group. Fix $\{1\}$ as the generating set \mathbb{Z} . Then

$$B_{\mathbb{Z}}(n) = \{-n, -n + 1, \dots, n - 1, n\}.$$

Clearly, $B_{\mathbb{Z}}(n)$ is fully detected by $\mathbb{Z}/(2n + 1)\mathbb{Z}$. Further, any quotient fully detecting $B_{\mathbb{Z}}(n)$ has cardinality greater than $2n$. So we get $\Phi_{\mathbb{Z}}(n) \approx n$. This result generalizes immediately to all torsion-free finitely generated abelian groups, and more generally to all finitely generated abelian groups.

COROLLARY 2.1: *Let A be a finitely generated abelian group. Then*

$$\Phi_A(n) \approx n^{\dim(A)}.$$

Proof. By Corollary 1.7, we may assume A is torsion-free. The computation in this case is straightforward. ■

One salient consequence of Corollary 2.1 is that an abelian group’s full residual finiteness growth Φ_A matches its word growth w_A . We now prove Theorem 3, which shows that this property characterizes abelian groups in the class of nilpotent groups. It also demonstrates that although Φ_G and w_G share properties, they are seldom the same.

Proof of Theorem 3. By Corollary 1.7, we may assume that G is torsion-free. Let’s further assume G is not abelian. Fix a Malcev basis x_1, \dots, x_k for G . For every n , let Q_n be a quotient fully detecting $B_G(n)$. Let c be the nilpotent class of G . Fix a tuple (x_1, \dots, x_m) consisting of all the basis elements not in $\zeta_{c-1}(G)$. We claim that there exists $C > 0$ such that, for any $\gamma \in \zeta_{c-1}(G)$, the image of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \gamma$ in Q_{Cn} is nontrivial in Q_{Cn} for any $|k_i| \leq n^2$ with $\sum_{i=1}^m |k_i| > 0$. Indeed, by the second assertion of Lemma 1.10 there exists $C > 0$ with $[x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}, y_1, y_2, \dots, y_{c-1}]$ being nontrivial and having word-length at most $Cn^{2/c} \leq Cn$ in G . Thus, as nontrivial elements of $B_G(Cn)$ are nontrivial in Q_{Cn} , the image of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \gamma$ in Q_{Cn} is nontrivial, so the claim is shown.

Consider the set

$$B^+(n) := \{x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \gamma : 1 \leq k_i \leq n^2, \gamma \in B_G(n) \cap \zeta_{c-1}(G)\}.$$

Given any $x, y \in B^+(n)$, the above claim implies that $y^{-1}x$ has nontrivial image in Q_{Cn} . It follows that $B^+(n)$ is fully detected by Q_{Cn} . On the other hand, by comparing with the explicit calculations for word growth in [1] and the appendix of [9] we see that the set $B^+(n)$ has cardinality at least $n^m w_G(n)$. Thus we have

$$\Phi_G(n) \succeq n^m w_G(n),$$

as desired. ■

2.2. SOME NON-ABELIAN GROUPS. We begin this section with the simplest non-abelian example. Recall that the **discrete Heisenberg group** is given by

$$H_3 = \langle x, y, z : [x, y] = z, z \text{ is central} \rangle.$$

PROPOSITION 2.2: *We have $\Phi_{H_3}(n) \approx n^6$.*

Proof. Let $B_{H_3}(n)$ be the ball of radius n in H_3 with respect to the generating set $\{x, y, z\}$. An exercise in the geometry of H_3 shows that there is some $D > 0$ so that if $x^{\alpha_1}y^{\alpha_2}z^{\alpha_3} \in B_{H_3}(Dn)$ then $|\alpha_i| \leq n$ for $i = 1, 2$ and $|\alpha_3| \leq n^2$. Therefore there is some $C > 0$ so that $B_{H_3}(n)$ injects into the quotient $H_3/H_3^{Cn^2}$, and so $\Phi_{H_3}(n) \preceq n^6$.

Now note that $B_{H_3}(5n)$ contains z^i for $-n^2 \leq i \leq n^2$, as

$$[x^n, y^j]z^k = z^{nj+k}$$

has word length at most $5n$ for each $1 \leq j, k \leq n$. Let Q_n be a quotient detecting $B_{H_3}(5n)$. Consider $w = x^a y^b x^c$. Then $[w, y] = z^a$ and $[w, x] = z^{-b}$. If w is trivial in Q_n then both $[w, y]$ and $[w, x]$ are also trivial. It follows that w has nontrivial image in Q_n for any values $0 < a, b, c \leq n^2$. Thus $|Q_n| \geq n^6$, as desired. ■

We now prove Theorem 1 from the introduction, which generalizes the conclusion of Proposition 2.2 to a large class of nilpotent groups.

For a finite k -tuple of elements $X = (x_1, \dots, x_k)$ from a group, we will use $B_X^+(n)$ to denote the set

$$B_X^+(n) = \{x_1^{\alpha_1} \cdots x_k^{\alpha_k} : 0 \leq \alpha_i \leq n\} \subseteq G.$$

Note that this is not generally the same as the semigroup ball of radius n .

Proof of Theorem 1. Lemma 1.12 demonstrates that $B_G(n)$ is fully detected by a quotient of the form G/G^{Mn^c} for some $M > 0$. We therefore have

$$\Phi_G(n) \preceq \left(\prod_{i=1}^c n^{\dim(\zeta_i(G)/\zeta_{i-1}(G))} \right)^c = n^{\dim(G)c}.$$

To show the reverse inequality, we will show that for any positive integer n , there exists a set of cardinality approximately $n^{\dim(G)c}$ that is fully detected by any finite quotient of G that realizes $\Phi_G(n)$. To this end, for each i , equip $\gamma_i(G)$ with a fixed generating set X_i . Let Q be a quotient of G that realizes $\Phi_G(n)$.

By [15, Theorem 2.2], for any generating set of $\gamma_c(G)$ there is a constant $C > 0$ such that for every h, h' in $\gamma_c(G)$, we have

$$d_{\gamma_c(G)}(h, h') \leq C[d_G(h, h')]^c.$$

Thus, the set $B_{\gamma_c(G)}(n^c/C)$ must inject into Q as it is contained in $B_G(n)$.

To continue, fix a basis $B = (g_1, \dots, g_k)$ obtained from the upper central series. For any i , let

$$B_i = \{g_j \in B \mid g_j \in \zeta_i(G) \setminus \zeta_{i+1}(G), g_j \text{ nontorsion in } G/\zeta_i(G)\}.$$

Set B^t to be the tuple consisting of elements from B_i respecting the ordering of the basis. That is,

$$B^t = (g_{a_1}, \dots, g_{a_k}),$$

where each entry is in some B_i and $a_i < a_{i+1}$. We claim that $B_{B^t}^+(Dn^c)$ is fully detected by Q for some $D > 0$. To prove this claim, we will use the fact that if any element in the normal closure of some $g \in G$ has nontrivial image in Q , then g has nontrivial image in Q . Let $x, y \in B_{B^t}^+(n^c)$ be elements with $x \neq y$. There is some $i \leq c$ so that $y^{-1}x \in \zeta_i(G) \setminus \zeta_{i-1}(G)$. Set

$$E = \max\{|B_j|\} \cdot \max\{\|\gamma\|_{X_j} : \gamma \in B_j\}.$$

There is some $\gamma \in \zeta_{i-1}(G)$ so that $\|y^{-1}x\gamma\|_{X_i} \leq En^c$. This statement follows by reducing the word $y^{-1}x$ to normal form with respect to the basis. Let E_0 be the largest constant C_i output by Lemma 1.10 for $i = 1, \dots, c$. By Lemma 1.10,

$$\|[y^{-1}x, x_1, \dots, x_r]\|_{X_c} \leq E_0\|xy^{-1}\gamma\|_{X_i} \leq E_0En^c.$$

It follows then that the set $B_{B^t}^+(n^c/(CE_0E))$ is fully detected by Q . Set $D = 1/(CE_0E)$. By the definition of a basis we have $|B_{B^t}^+(Dn^c)| \geq (Dn)^{c \dim(G)}$, so we get the desired inequality. ■

3. A general upper bound

The example $H_3 \times \mathbb{Z}$, which has full residual finiteness growth n^7 , demonstrates that the conclusion of Theorem 1 does not hold for any finitely generated nilpotent group. In this section we prove Theorem 2, providing a technique that provides for any finitely generated nilpotent group an explicit upper bound of $\Phi_G(n)$ of the form n^d . We first illustrate the technique in an example in

Proposition 3.1, where we show moreover that the upper bound is sharp in this example.

Let U_n denote the group of upper triangular unipotent matrices in $SL_n(\mathbb{Z})$. For $i \neq j$, let $e_{i,j}$ denote the elementary matrix differing from the identity matrix only in that its ij -entry is 1. We define the **coordinates** of the tuple (x_1, \dots, x_k) to be the set $\{x_1, \dots, x_k\}$. Recall that a **terraced** filtration of G is a filtration $1 = H_0 \leq H_1 \leq \dots \leq H_{c-1} \leq G$ where each H_i is a maximal normal subgroup of G satisfying $H_i \cap \gamma_{i+1}(G) = 1$.

PROPOSITION 3.1: Consider elementary matrices $x = e_{1,4}$ and $y = e_{1,5}$ in U_5 . Define a normal subgroup $N = \langle x, y \rangle \leq U_5$ and set $\Gamma = U_5/N$. Then

$$\Phi_\Gamma(n) \approx n^{22}.$$

Proof. Set $H_3 = \Gamma$ and $H_2 = \langle e_{1,2}, e_{1,3} \rangle$, and let $H_0 = H_1 = 1$. Note that $1 = H_0 \leq H_1 \leq H_2 \leq \Gamma$ forms a terraced filtration of Γ . Define two tuples of elements of Γ by $X_3 = (e_{1,3}, e_{1,2})$ and $X_2 = (e_{2,5}, e_{2,4}, e_{3,5}, e_{2,3}, e_{3,4}, e_{4,5})$. For each $i = 2, 3$, let Y_i be the set of coordinates of X_i . Clearly $Y = Y_2 \cup Y_3$ generates Γ .

To establish the upper bound, let Q be a quotient of Γ detecting $B_\Gamma(n)$. Each of $H_3^{n^3}$ and $H_2^{n^2}$ is normal in Γ , so we can define a normal subgroup

$$N = H_3^{n^3} H_2^{n^2} \leq \Gamma.$$

A simple induction shows that if $g \in B_\Gamma(n)$ then $|g_{ij}| \leq n^{j-i}$. In particular, this implies that there is some $C > 0$ so that $B_\Gamma(Cn)$ is fully detected by G/N . Since $|G/N| \approx n^{22}$, this establishes the desired upper bound on $\Phi_\Gamma(n)$.

To establish the lower bound, define the **depth** of an element $\gamma \in \Gamma$ to be the maximal i with $\gamma \notin \zeta_i(\Gamma)$. Order the elements Y in a tuple (y_1, y_2, \dots, y_8) of non-increasing depth. Set $B^+(n)$ to be

$$\left\{ \prod_{i=1}^8 y_i^{\alpha_i} : 0 \leq \alpha_i \leq n^2 \text{ if } y_i \in Y_2 \text{ and } 0 \leq \alpha_i \leq n^3 \text{ otherwise} \right\}.$$

We claim that there exists $C > 0$ such that any quotient Q in which $B_\Gamma(Cn)$ embeds restricts to $B^+(n)$ as an injection. This gives the desired lower bound, as $|B^+(n)| \geq n^{22}$. To see this claim, let x, y be distinct elements in $B^+(n)$. Set i to be the depth of $y^{-1}x$. We break up the rest of the proof of this claim into cases depending on i .

If $i = 0$, then $y^{-1}x$ is in the center of Γ and we have

$$y^{-1}x = e_{1,2}^{a_1}e_{2,5}^{a_2},$$

where $|a_1| \leq n^2$ and $|a_2| \leq n^3$. Note that $e_{1,2} \in \gamma_2(\Gamma)$ and $e_{2,5} \in \gamma_3(\Gamma)$. Applying Lemma 1.9 twice, we have that

$$\|y^{-1}x\|_\Gamma \leq \|e_{1,2}^{a_1}\|_\Gamma + \|e_{2,5}^{a_2}\|_\Gamma \leq Cn,$$

for some $C > 0$, independent of n . Thus $y^{-1}x$ cannot vanish in any quotient that fully detects $B^+(Cn)$.

If $i = 1$, then by definition we may write

$$y^{-1}x = e_{1,2}^{a_1}e_{2,4}^{a_2}e_{3,5}^{a_3}\gamma,$$

where $\gamma \in \zeta_i(\Gamma)$, $|a_1| \leq n^2$, $|a_2| \leq n^3$, and $|a_3| \leq n^3$. Since this $y^{-1}x$ is not in the center, there exists $z \in Y$ such that

$$[e_{1,2}^{a_1}e_{2,4}^{a_2}e_{3,5}^{a_3}\gamma, z] \neq 1.$$

This element is now in the center. Thus, by Lemma 1.8, we have

$$[e_{1,2}^{a_1}e_{2,4}^{a_2}e_{3,5}^{a_3}\gamma, z] = [e_{1,2}, z]^{a_1}[e_{2,4}, z]^{a_2}[e_{3,5}, z]^{a_3}.$$

Now by Lemma 1.9 applied three times, we see that the word length of

$$[e_{1,2}^{a_1}e_{2,4}^{a_2}e_{3,5}^{a_3}\gamma, z]$$

is less than a constant multiple of n , where the constant does not depend on n . Thus $y^{-1}x$ cannot vanish in any quotient that fully detects $B^+(Cn)$ for some $C > 0$ independent of n .

If $i = 2$, then by definition we may write

$$y^{-1}x = e_{2,3}^{a_1}e_{3,4}^{a_2}\gamma,$$

where $\gamma \in \zeta_i(\Gamma)$, $|a_1|, |a_2| \leq n^3$. Suppose, without loss of generality, that $a_1 \neq 0$. Then, using Lemma 1.8, we have that there exists $\gamma' \in \gamma_1(\Gamma)$ such that

$$[y^{-1}x, e_{3,4}] = [e_{2,3}, e_{3,4}]^{a_1}[e_{3,4}, e_{3,4}]^{a_2}\gamma' = e_{2,4}^{a_1}\gamma'.$$

Now it is clear that there exists $z \in \Gamma$ such that

$$[[y^{-1}x, e_{3,r}], z] = [e_{2,4}^{a_1}\gamma', z] \neq 1.$$

We can proceed as in case $i = 1$ to achieve the desired conclusion. Indeed, Lemma 1.9 applies, giving that $y^{-1}x$ is detected if $B(Cn)$ is fully detected for

some constant $C > 0$ independent of n . That is, we cannot have $y^{-1}x = 1$ in Q , if Q detects $B_\Gamma(Cn)$. The claim then follows, ending the proof. \blacksquare

We now prove Theorem 2.

Proof of Theorem 2. Let G be a finitely generated nilpotent group and suppose $1 = H_0 \leq H_1 \leq \dots \leq H_{c-1} \leq G$ is a terraced filtration. Set $H_c = G$.

Choose a basis X_1 of H_1 . Inductively construct tuples X_2, \dots, X_c by setting X_i to be a pull-back of a basis for H_i/H_{i-1} . Set Y_i to be the set of all coordinates of X_i and $Y = \bigcup_i Y_i$. It is clear from the construction that Y is a generating set for G . Note also that for any $n \in \mathbb{N}$, the subgroup

$$N(n) = \prod_{i=1}^c \langle y^{n^k} : y \in \langle Y_1 \cup Y_2 \cup \dots \cup Y_k \rangle \rangle$$

is normal in G .

We now claim that there exists a constant $D \in \mathbb{N}$ so that for any $n \in \mathbb{N}$, the ball $B_Y(n)$ is detected by $G/N(Dn)$. To prove the claim, let $f, M \in \mathbb{N}$ be as in Lemma 1.12. Then G^{fM} is torsion-free; let $K = G^{fM}$. Fix a finite generating set T for K . For each i and any $n \in \mathbb{N}$, Lemma 1.12 gives that any element $g \in K^n \cap H_i$ has word length at least $C_i n^{1/t_i}$ with respect to T . Thus we have that there exists $D_0 > 0$ such that $B_T(D_0 n)$ is fully detected by $K/N(fMn)$. Further, since K is of finite index in G , we have $D_1 > 1$ such that for any $g \in K$,

$$\|g\|_T \leq D_1 \|g\|_Y \leq D_1^2 \|g\|_T.$$

Therefore, as $N(fMn)$ is contained in K , any singleton contained in $B_Y(n/D_1)$ is fully detected by $G/N(fMn)$ and so $B_Y(n/(2D_1))$ is fully detected by $G/N(fMn)$. This proves the claim, as we can select $D = 2D_1 fM$.

We will now demonstrate that the order of $G/N(Dn)$ is dictated by a single polynomial of the form n^b for

$$b = \sum_{k=1}^c k \cdot \dim(H_k/H_{k-1}).$$

Set $G_k = H_k/H_{k-1}$. It is apparent from the definition of $N(Dn)$ that the index of $N(Dn)$ in G is bounded above by

$$\prod_{k=1}^c |G_k/G_k^{D n^k}|.$$

By the construction of D , the subgroup G_k^D is torsion-free in G_k . Thus it is clear that $|G_k^D/G_k^{D^k n^k}|$ has order $D^{\dim(G_k)(k-1)} n^{k \dim(G_k)}$. This gives an upper bound for the index of $N(Dn)$ in G of the form $C_0 n^{\sum_{k=1}^c k \dim(G_k)}$, where $C_0 > 0$ does not depend on n .

One can check that $b = c \dim(G) - \sum_{i=1}^{c-1} \dim(H_i)$ using the general fact that $\dim(G/H) = \dim(G) - \dim(H)$ for any finitely generated nilpotent group G with normal subgroup H . This completes the proof since $G/N(Dn)$ detects $B_Y(Cn)$. ■

We conclude with an example that shows that the upper bound to Φ_G given by Theorem 2 generally may depend on the choice of terraced filtration. Consider the group $\tilde{G} = U_3 \times U_4 \times U_5$, which is nilpotent of class $c = 4$. There is an isomorphism

$$Z(\tilde{G}) \cong Z(U_3) \times Z(U_4) \times Z(U_5).$$

Under identifications

$$Z(U_3) \cong Z(U_4) \cong Z(U_5) \cong \mathbb{Z},$$

define an infinite cyclic subgroup

$$Z = \{(x, y, z) \in Z(U_3) \times Z(U_4) \times Z(U_5) \mid x = y = z\} \leq Z(\tilde{G}).$$

Let $G = \tilde{G}/Z$ and let $\pi : \tilde{G} \rightarrow G$ be the quotient map. Then π restricts to an isomorphism $Z(U_3) \times Z(U_4) \cong Z(G)$. Under this identification, the last term of the lower central series of G is

$$\gamma_4(G) = \{(x, y) \in Z(U_3) \times Z(U_4) \mid x = y\}.$$

Since $\gamma_3(G)$ contains the image of $Z(U_4)$, we see that $Z(G) \leq \gamma_3(G)$. Since $H \cap Z(G)$ is nontrivial for any nontrivial normal subgroup $H \leq G$, it follows that H_2 is trivial for any terraced filtration of G .

Now define $H_0 = H_1 = H_2 = 1$ and $H_3 = \pi(U_3)$, and $H'_0 = H'_1 = H'_2 = 1$ and $H'_3 = \pi(U_4)$. It is easy to see that both $\pi(U_3)$ and $\pi(U_4)$ are maximal normal subgroups of G whose intersection with $\gamma_4(G)$ is trivial. It follows from the above comments that

$$H_0 \leq H_1 \leq H_2 \leq H_3 \leq G \quad \text{and} \quad H'_0 \leq H'_1 \leq H'_2 \leq H'_3 \leq G$$

are terraced filtrations of G . However, these filtrations give different upper bounds for Φ_G because $\dim(\pi(U_3)) = 3$ while $\dim(\pi(U_4)) = 6$.

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