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NORMAL ZETA FUNCTIONS OF THE HEISENBERG GROUPS OVER NUMBER RINGS II — THE NON-SPLIT CASE

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ABSTRACT

We compute explicitly the normal zeta functions of the Heisenberg groups $H(R)$, where R is a compact discrete valuation ring of characteristic zero. These zeta functions occur as Euler factors of normal zeta functions of Heisenberg groups of the form $H(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an arbitrary number field K, at the rational primes which are non-split in K. We show that these local zeta functions satisfy functional equations upon inversion of the prime.

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1. Introduction

Let G be a finitely generated abstract or profinite group. For $m \in \mathbb{N}$, let $a^{\triangleleft}_m(G)$ denote the number of (open) normal subgroups of G of index m in G. The *normal zeta function* of G is the Dirichlet generating series

$$\zeta^{\triangleleft}_G(s) = \sum_{m=1}^{\infty} a^{\triangleleft}_m(G) m^{-s},$$

where s is a complex variable. If G is a finitely generated nilpotent group, then its normal zeta function converges on a complex half-plane and satisfies the Euler product

$$\zeta^{\triangleleft}_G(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{G,p}(s).$$

Here, for a prime p, the Euler factor $\zeta^{\triangleleft}_{G,p}(s) = \sum_{k=0}^{\infty} a^{\triangleleft}_{p^k}(G) p^{-ks}$ enumerates the normal subgroups of G of p-power index in G. It may also be viewed as the normal zeta function of the pro-p completion $\widehat{G}^p$ of G. The Euler product reflects the facts that the normal zeta function of G coincides with the normal zeta function of its profinite completion $\widehat{G}$ and that $\widehat{G} \cong \prod_{p \text{ prime}} \widehat{G}^p$. The zeta functions $\zeta^{\triangleleft}_{G,p}(s)$ are known to be rational functions in p^{-s}; cf. [2, Theorem 1].

Given a ring $\mathcal{R}$, the *Heisenberg group* $H(\mathcal{R})$ over $\mathcal{R}$ is the group of upper unitriangular 3×3 matrices over $\mathcal{R}$:

$$H(\mathcal{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathcal{R} \right\}.$$

If $\mathcal{R}$ is a finitely generated torsion-free $\mathbb{Z}$-module of rank n, say, then $H(\mathcal{R})$ is a finitely generated torsion-free nilpotent group of nilpotency class 2 and Hirsch length $3n$. Given a prime p, the pro-p completion of $H(\mathcal{R})$ is isomorphic to the $3n$-dimensional nilpotent p-adic analytic pro-p group $H(\mathcal{R}_p)$, where $\mathcal{R}_p = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and we have

$$\zeta^{\triangleleft}_{H(\mathcal{R}),p} = \zeta^{\triangleleft}_{H(\mathcal{R}_p)}.$$

In this article we compute an explicit formula for the normal zeta function of the Heisenberg group over an arbitrary compact discrete valuation ring R of characteristic zero, i.e. a finite extension of the ring $\mathbb{Z}_p$ of p-adic integers. Let $\mathfrak{m}$ be the maximal ideal of the local ring R. The residue field $k_R = R/\mathfrak{m}$ is a finite extension of the prime field $\mathbb{F}_p$. Its degree $f = [k_R : \mathbb{F}_p]$ is called the **inertia degree** of R. The (**absolute**) **ramification index** e of R is given by $pR = \mathfrak{m}^e$.

The ring R is called **unramified** (over $\mathbb{Z}_p$) if $e = 1$ and **totally ramified** (over $\mathbb{Z}_p$) if $f = 1$. The **degree** of R as an extension of $\mathbb{Z}_p$ is $n = ef$. It coincides with the rank of R as a $\mathbb{Z}_p$-module.

Normal zeta functions of Heisenberg groups of the form $H(R)$ occur as Euler factors of normal zeta functions of Heisenberg groups over number rings. Indeed, let $\mathcal{O}_K$ be the ring of integers of a number field K. Then $(\mathcal{O}_K)_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a local ring precisely if p does not split in K, i.e. it decomposes in K as $p\mathcal{O}_K = \mathfrak{p}^e$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$. In this case, $f = [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ and $n = ef = [K : \mathbb{Q}]$ is the degree of K. We call such primes **non-split** (in K). Note that all finite extensions of $\mathbb{Z}_p$ arise in this way.

It follows from the general result [2, Theorem 1] that normal zeta functions of groups of the form $H(R)$ are rational in p^{-s}. The more specific result [2, Theorem 3] asserts that the Euler factors of (normal) zeta functions of $H(\mathcal{O}_K)$ are rational in the two parameters p^{-s} and p on sets of rational primes with fixed decomposition type in K; cf. also [7] for details. There are, in particular, rational functions $W^{\triangleleft}_{e,f}(X, Y) \in \mathbb{Q}(X, Y)$ such that for all rational primes p and rings R as above, the following holds:

$$\zeta^{\triangleleft}_{H(R)}(s) = W^{\triangleleft}_{e,f}(p, p^{-s}).$$

In Theorem 3.8, our main result, we compute the rational functions $W^{\triangleleft}_{e,f}(X, Y)$ explicitly. Moreover, we prove the following functional equation in Corollary 3.13.

THEOREM 1.1: *Let $e, f \in \mathbb{N}$ with $ef = n$. Then*

$$W^{\triangleleft}_{e,f}(X^{-1}, Y^{-1}) = (-1)^{3n} X^{\binom{3n}{2}} Y^{5n+2(e-1)f} W^{\triangleleft}_{e,f}(X, Y).$$

Note that $3n = \dim(H(R))$ and $5n = \dim(H(R)) + \dim(H(R)/H(R)')$, where $H(R)'$ is the derived subgroup of $H(R)$. Here "dim" refers to the dimensions as p-adic analytic pro-p groups. The term $2(e-1)f$ in the exponent of Y describes the deviation from the "generic" symmetry factor in the functional equations for the local factors of normal zeta functions of finitely generated nilpotent groups of nilpotency class 2; cf. [10, Theorem C].

Prior to our work, the normal zeta functions $\zeta^{\triangleleft}_{H(R)}$ had been calculated for all cases occurring for $n \leq 3$; see [1, Theorems 2.3, 2.7, and 2.9].

1.1. METHODOLOGY. The results of the current paper complement those of [7], where we carry out analogous computations of the normal zeta functions of the groups $H((\mathcal{O}_K)_p)$ for primes p which are unramified in the number field K. In [7, Theorem 1.2] we establish functional equations for these zeta functions that are comparable to those in Theorem 1.1. Our results agree, of course, in the common special case of primes p which are inert in K; see Theorem 3.2. In [7, Conjecture 1.4] we conjecture a functional equation for $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}(s)$ for arbitrary (not necessarily unramified or non-split) primes.

The methods used in the present paper are, however, quite different from those of [7]. There the problem of computing the relevant zeta functions reduces to that of effectively enumerating subgroups of finite abelian p-groups varying in infinite, combinatorially described families. The precise shape these families may take is determined by the decomposition type of the rational prime p in the number field K. The sum defining the local zeta function is organized as a finite sum, indexed by certain Dyck words.

The decomposition type that leads to the combinatorially simplest situation is that of inert primes, namely the case where $p\mathcal{O}_K$ is a prime ideal. We view the non-split case considered in this paper as a degeneration of the inert case and tackle it using geometric and Coxeter-group-theoretic ideas introduced in [9] and [4], as we now explain.

The paper [9] argues that the normal subgroup growth of a finitely generated nilpotent group G of nilpotency class 2 is, to a large extent, determined by the geometry of its **Pfaffian hypersurface**. This is a projective hypersurface, defined explicitly by the Pfaffian of an antisymmetric matrix of linear forms encoding the group's structure constants with respect to a chosen (Mal'cev) basis. If the Pfaffian hypersurface of G is smooth and contains no lines, and G satisfies some other mild hypotheses, then [9, Theorem 3] gives an explicit formula for the Euler factors $\zeta^{\triangleleft}_{G,p}$, at almost all primes p, in terms of the numbers of $\mathbb{F}_p$-rational points on the Pfaffian hypersurface. This formula presents the Euler factor as the sum of an **approximative term**, which coincides with the Euler factor if and only if the Pfaffian hypersurface has no $\mathbb{F}_p$-rational point, and a **correction term**, which corrects the approximation along the hypersurface's $\mathbb{F}_p$-points. In the special case $G = H(\mathcal{O}_K)$, the results of [9] are not directly applicable. The Pfaffian hypersurface is the union of n hyperplanes in general position in $(n-1)$-dimensional projective space. It has no $\mathbb{Q}$-rational points; over $\mathbb{Q}_p$ it splits as a union of (restrictions of scalars of) hyperplanes, in a way determined by the decomposition behaviour of p in K. The computation of the relevant local zeta function comes down to a detailed quantitative analysis of the interplay between these fixed hyperplanes and varying p-adic lattices.

In the case of inert primes p, the ideas of [9] do apply directly to the Euler factors $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}$, as in this case the Pfaffian hypersurface has no $\mathbb{F}_p$-rational points. Thus the Euler factor is equal to the approximative term mentioned above. In the setup of [9], this means that the set of solutions of a certain system of linear congruences has a particularly simple form. For non-split primes, ramification complicates this system only slightly. The main idea of the current article is to control this complication using parabolic length functions on symmetric groups. These functions generalize the usual Coxeter length and were used to solve related enumeration problems in [4]. Theorem 3.8 expresses $\zeta^{\triangleleft}_{H(R)}(s)$ in terms of parabolic length functions on the symmetric group S_n, whereas Corollary 3.18 gives a formula in the totally ramified case in terms of parabolic length functions on S_{n-1}. The functional equation expressed in Theorem 1.1 reflects the good behaviour of the relevant parabolic length functions under (left-)multiplication by the Coxeter group's longest element.

1.2. Outlook. In this section we briefly describe some directions for future research building on the methods of the present paper and of [7].

It would be of great interest to match the geometric setup of [9] precisely with the combinatorial approach taken in [7], for the Heisenberg groups $H(\mathcal{O}_K)$ and also more generally. It is plausible that the presence of lines and higher-dimensional linear spaces on the Pfaffian hypersurface necessitates further **correction terms**, accounting for the possible intersection types of flags with coordinate hyperplanes. We note that Dyck words and possible intersection behaviours of a flag with a fixed set of hyperplanes in general position are both enumerated by the Catalan numbers; moreover, there is a natural bijection between these two types of objects. For the case of $[K:\mathbb{Q}] = 3$, the correction terms arising from a generalization of the approach of [9] appear to coincide with the functions associated to Dyck words that were computed in [7]. This is likely to be a special case of a very general phenomenon.

Let $g \in \mathbb{N}$. Given g-tuples $\mathbf{e} = (e_1, \ldots, e_g) \in \mathbb{N}^g$ and $\mathbf{f} = (f_1, \ldots, f_g) \in \mathbb{N}^g$ satisfying $\sum_{i=1}^{g} e_i f_i = [K:\mathbb{Q}]$, we say that a (rational) prime p is of **decomposition type** $(\mathbf{e}, \mathbf{f})$ in the number field K if

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g},$$

where the $\mathfrak{p}_i$ are distinct prime ideals in $\mathcal{O}_K$ with ramification indices e_i and inertia degrees $f_i = [\mathcal{O}_K/\mathfrak{p}_i : \mathbb{F}_p]$ for $i = 1, \ldots, g$. We call the decomposition type $(\mathbf{e}, \mathbf{f})$ **unramified** if $\mathbf{e} = \mathbf{1} = (1, \ldots, 1)$.

Taken together, [7] and the present paper give explicit formulae for all but finitely many Euler factors of the global ideal zeta functions $\zeta^{\triangleleft}_{H(\mathcal{O}_K)}(s)$. Still outstanding is an analysis of the general ramified decomposition types. In view of the geometric picture sketched above, it is suggestive to view a general decomposition type $(\mathbf{e}, \mathbf{f})$ as a degeneration of an associated unramified decomposition type $(\mathbf{1}, \mathbf{f}')$, where $\mathbf{f}' = (e_1 f_1, \ldots, e_g f_g)$. The methods of this paper suggest trying to describe the effect of this degeneration on the zeta function, computed in [7], of the unramified type $(\mathbf{1}, \mathbf{f}')$ by means of suitable parabolic length functions or similar combinatorially described functions. The current paper carries out this idea for $g = 1$.

We see this paper and [7] as first steps in a systematic study of the behaviour of (normal) subgroup growth of general nilpotent groups under base extension. Specifically, one may ask the following: given a finitely generated nilpotent group of the form $G = \mathbf{G}(\mathbb{Z})$, arising as the group of $\mathbb{Z}$-rational points of a unipotent group scheme $\mathbf{G}$ defined over $\mathbb{Z}$, how does the normal subgroup growth sequence $(a^{\triangleleft}_m(\mathbf{G}(\mathcal{O})))_{m\in\mathbb{N}}$ vary as $\mathcal{O}$ ranges over the rings of integers of number fields? For instance, it seems reasonable to expect that the local factors of the associated normal zeta functions should admit some kind of uniform description on sets of (rational) primes of fixed decomposition type.

The same expectation holds for zeta functions encoding other data, such as the subgroup growth sequence $(a_m(\mathbf{G}(\mathcal{O})))_{m\in\mathbb{N}}$ counting *all* finite index subgroups of $\mathbf{G}(\mathcal{O})$. The associated Dirichlet series $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ are known to have Euler decompositions analogous to those of $\zeta^{\triangleleft}_{\mathbf{G}(\mathcal{O})}(s)$. It is very natural to try to extend the methodology developed in this paper and in [7] to the subgroup zeta factors $\zeta_{\mathbf{G}(\mathcal{O}),p}(s)$. For the Heisenberg group, it is conjectured in [2, p. 188] that for every decomposition type $(\mathbf{e}, \mathbf{f})$ there exists a rational function $W_{\mathbf{e},\mathbf{f}}(X, Y) \in \mathbb{Q}(X, Y)$ such that for all rational primes p of decomposition type $(\mathbf{e}, \mathbf{f})$ in K the following holds:

$$\zeta_{H(\mathcal{O}_K),p}(s) = W_{\mathbf{e},\mathbf{f}}(p, p^{-s}).$$

While the analogous statement for normal zeta functions was already proved in [2], to our knowledge this conjecture has not even been completely settled for $[K : \mathbb{Q}] = 2$ (but see [1, Theorem 2.4] for the case of split primes). That counting all finite index subgroups is a far more complex task than counting normal such subgroups is reflected in the fact that systems of **quadratic** Diophantine equations take the role played by the systems of linear such equations that we work with in this paper and in [7].

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2. Preliminaries

Let p be a rational prime. For an integer $m \geq 1$, we write $[m]$ for $\{1, 2, \ldots, m\}$ and $[m]_0$ for $\{0, 1, \ldots, m\}$. Given integers a, b with $a \leq b$, we write $[a, b]$ for $\{a, a+1, \ldots, b\}$. Given a finite set I of integers, we write $I = \{i_1, \ldots, i_\ell\}_<$ to indicate that $i_1 < \cdots < i_\ell$.

2.1. COXETER GROUPS. The symmetric group S_n of degree n is a Coxeter group with Coxeter generating set $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$, where, for each $i \in [n-1]$, we denote by $s_i = (i\ i{+}1)$ the transposition of the letters i and $i{+}1$ in the standard permutation representation of S_n. We will frequently identify elements of S_n with permutations of $[n]$ in this way.

We write $\mathrm{len} : S_n \to [\binom{n}{2}]_0$ for the usual Coxeter length function: for $w \in S_n$, $\mathrm{len}(w)$ denotes the length of a shortest word representing w as a product of elements of $\mathcal{S}$.

Given $I \subseteq [n-1]$, we write $W_I = \langle s_i \mid i \in I \rangle$ for the parabolic subgroup of S_n generated by the elements of $\mathcal{S}$ indexed by elements of I. The restriction of len to W_I coincides with the standard length function on the Coxeter group W_I. Every element $w \in S_n$ can be factorized uniquely as $w = w^I w_I$, where $w_I \in W_I$ and w^I is the unique element of shortest length in the coset wW_I. Moreover, $\mathrm{len}(w) = \mathrm{len}(w_I) + \mathrm{len}(w^I)$; cf. [3, Section 1.10]. We set $\mathrm{len}^I(w) := \mathrm{len}(w^I)$, and call len^I the (*right*) *parabolic length function* associated to I; cf. [4, Definition 2.2].

The group S_n has a unique longest element w_0 with respect to len, namely the inversion $w_0(i) = n + 1 - i$ for $i \in [n]$. Parabolic length functions are well-behaved with respect to (left) multiplication with w_0: for every $I \subseteq [n-1]$ and $w \in S_n$,

$$\mathrm{len}^I(w_0 w) = \mathrm{len}^I(w_0) - \mathrm{len}^I(w); \tag{2.1}$$

cf. [4, Lemma 2.3]. Clearly $\mathrm{len} = \mathrm{len}^{\varnothing}$. The other parabolic length function relevant for us is $\mathrm{len}^{[n-2]}$. It is easy to check that $\mathrm{len}^{[n-2]}(w) = n - w(n)$ for all $w \in S_n$, and in particular that $\mathrm{len}^{[n-2]}(w_0) = n - 1$.

The (**right**) **descent set** $\mathrm{Des}(w)$ of an element $w \in S_n$ is defined as

$$\mathrm{Des}(w) = \{i \in [n-1] \mid \mathrm{len}(ws_i) < \mathrm{len}(w)\}.$$

It is easily seen that $\mathrm{Des}(w) = \{i \in [n-1] \mid w(i+1) < w(i)\}$ and

$$\mathrm{Des}(w_0 w) = [n-1] \setminus \mathrm{Des}(w). \tag{2.2}$$

Example 2.1: Consider the element $w \in S_6$ corresponding to the permutation matrix

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here $\mathrm{Des}(w) = \{3\}$, $\mathrm{len}(w) = 7$ and $\mathrm{len}^{[4]}(w) = 6 - w(6) = 2$.

For a variable Y and integers $a, b \in \mathbb{N}_0$ with $a \geq b$, the **Gaussian binomial coefficient** is defined to be

$$\binom{a}{b}_Y = \frac{\prod_{i=a-b+1}^{a}(1 - Y^i)}{\prod_{i=1}^{b}(1 - Y^i)} \in \mathbb{Z}[Y].$$

Given an integer $n \in \mathbb{N}$ and a subset $I = \{i_1, \ldots, i_\ell\}_< \subseteq [n-1]$, the associated **Gaussian multinomial** is defined as

$$\binom{n}{I}_Y = \binom{n}{i_\ell}_Y \binom{i_\ell}{i_{\ell-1}}_Y \cdots \binom{i_2}{i_1}_Y \in \mathbb{Z}[Y].$$

Then (cf. [8, Section 1.7]) for $I \subseteq [n-1]$ we have

$$\sum_{w \in S_n,\ \mathrm{Des}(w) \subseteq I} Y^{\mathrm{len}(w)} = \binom{n}{I}_Y. \tag{2.3}$$

2.2. Grassmannians. Given an integer $i \in [n]_0$, we denote by $\mathrm{Gr}(n, n-i)$ the Grassmannian of $(n-i)$-dimensional subspaces of affine n-dimensional space. This $i(n-i)$-dimensional projective variety has a decomposition

$$\mathrm{Gr}(n, n-i) = \bigcup_{w \in S_n,\ \mathrm{Des}(w) \subseteq \{i\}} \Omega_w$$

into (*Schubert*) *cells* Ω_w, indexed by $\binom{n}{n-i}$ elements of S_n. These cells have an elementary realization as follows. Fix a vector space basis for affine n-dimensional space. Subspaces of dimension $n-i$ may then be represented by GL_{n-i}-left cosets of matrices of size $n \times (n-i)$ of full rank $n-i$. A set of such matrices of the form

$$\begin{pmatrix}
* & * & \cdots & & * \\
\vdots & & & & \vdots \\
* & * & \cdots & & * \\
1 & 0 & \cdots & \cdots & 0 \\
0 & * & \cdots & \cdots & * \\
\vdots & \vdots & & & \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * \\
 & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & & & & 0 \\
\vdots & & & & \vdots \\
0 & 0 & \cdots & & 0
\end{pmatrix}_{n \times (n-i)}$$

where $*$ stands for arbitrary field elements, is a set of unique coset representatives. More precisely, for any such matrix there is a subset $J \subseteq [n]$ of cardinality $n-i$ such that the submatrix comprising rows labeled by elements of J is the $(n-i)$-identity matrix. The matrix above has zeroes in all entries below or to the right of a 1 in this submatrix, and arbitrary entries in the remaining positions. The set of cosets corresponding to such matrices for a fixed subset $J = \{j_1, \ldots, j_{n-i}\}_< \subseteq [n]$ may be identified with the cell Ω_w, where $w \in S_n$ is the unique element in S_n whose descent set is contained in $\{i\}$ and which satisfies $w(i+m) = j_m$ for all $m \in [n-i]$. This illustrates that each cell Ω_w is an affine space of dimension $i(n-i) - \mathrm{len}(w)$, which is the number of symbols $*$ in the above matrix. Hence, given a prime p, the number $\#\,\mathrm{Gr}(n, n-i; \mathbb{F}_p)$

of $\mathbb{F}_p$-rational points of $\mathrm{Gr}(n, n-i)$ is given by the formula

$$\begin{aligned}\#\,\mathrm{Gr}(n,n-i;\mathbb{F}_p) &= \sum_{w\in S_n,\ \mathrm{Des}(w)\subseteq\{i\}} p^{i(n-i)-\mathrm{len}(w)} \\ &= \binom{n}{n-i}_{p^{-1}} p^{i(n-i)} = \binom{n}{n-i}_{p};\end{aligned} \tag{2.4}$$

cf. (2.3). We refer to [5, Section 3.2] for further information about Schubert cells.

2.3. Lattices. For the reader's convenience, we recall some notation used in [9] to parameterize sublattices $\Lambda \leq \mathbb{Z}_p^n$. A sublattice $\Lambda \leq \mathbb{Z}_p^n$ of finite index in $\mathbb{Z}_p^n$ is **maximal in** $\mathbb{Z}_p^n$ if $p^{-1}\Lambda \not\leq \mathbb{Z}_p^n$. Such a lattice is called **of type** $\nu(\Lambda) = (I, \mathbf{r}_I)$, where $I = \{i_1, \ldots, i_\ell\}_< \subseteq [n-1]$ and $\mathbf{r}_I = (r_{i_1}, \ldots, r_{i_\ell}) \in \mathbb{N}^\ell$, if Λ has elementary divisors

$$p^\nu := (\underbrace{1,\ldots,1}_{i_1}, \underbrace{p^{r_{i_1}},\ldots,p^{r_{i_1}}}_{i_2-i_1}, \ldots, \underbrace{p^{\sum_{\iota\in I} r_\iota},\ldots,p^{\sum_{\iota\in I} r_\iota}}_{n-i_\ell})$$

with respect to $\mathbb{Z}_p^n$. (Note that this ordering differs from the one used in [10, Section 3.1].)

Fix a $\mathbb{Z}_p$-basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbb{Z}_p^n$. The group $\Gamma = \mathrm{SL}_n(\mathbb{Z}_p)$ acts transitively on the finite set of maximal sublattices of $\mathbb{Z}_p^n$ of given type $\nu = (I, \mathbf{r}_I)$. Denote by $\Gamma_{(I,\mathbf{r}_I)}$ the stabilizer in Γ of the diagonal lattice $\bigoplus_{j=1}^n (p^\nu)_j \mathbb{Z}_p \varepsilon_j$. This allows us to identify a given maximal lattice with a coset $\boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)}$, where $\boldsymbol{\alpha} \in \Gamma$. The number of maximal lattices of type $(I, \mathbf{r}_I)$ inside $\mathbb{Z}_p^n$ is given by

$$|\Gamma : \Gamma_{(I,\mathbf{r}_I)}| = \binom{n}{I}_{p^{-1}} p^{\sum_{\iota\in I} r_\iota \iota(n-\iota)}; \tag{2.5}$$

see, for instance, [10, Eq. (26)].

2.4. Linearization. The problem of counting finite-index normal subgroups of $H(R)$ turns out to be equivalent to the problem of counting finite-index ideals in a certain Lie ring, which we now introduce. Given a ring $\mathcal{R}$, the **Heisenberg Lie ring** $L(\mathcal{R})$ over $\mathcal{R}$ is defined as

$$L(\mathcal{R}) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathcal{R} \right\},$$

equipped with the Lie bracket induced from $\mathfrak{gl}_3(\mathcal{R})$. The derived subring $L(\mathcal{R})'$ of $L(\mathcal{R})$ is equal to the center of $L(\mathcal{R})$ and consists of those matrices for which $a = b = 0$. Let $\overline{L(\mathcal{R})} = L(\mathcal{R})/L(\mathcal{R})'$ be the abelianization.

If $\mathcal{R}$ is an A-module of finite rank, for some commutative ring A, then so is $L(\mathcal{R})$. In this case, $L(\mathcal{R})$ has only finitely many A-ideals of each finite index. The (A-)**ideal zeta function** of $L(\mathcal{R})$ is then defined as the Dirichlet generating function

$$\zeta^{\triangleleft}_{L(\mathcal{R})}(s) = \sum_{n=1}^{\infty} a^{\triangleleft}_n(L(\mathcal{R}))n^{-s}, \tag{2.6}$$

where $a^{\triangleleft}_n(L(\mathcal{R}))$ denotes the number of A-ideals of index n in $L(\mathcal{R})$. In the cases considered in this paper, we have $A = \mathbb{Z}_p$.

3. Computation of the functions $W^{\triangleleft}_{e,f}(X,Y)$

3.1. The set-up. Let R be a compact discrete valuation ring of characteristic zero, with maximal ideal $\mathfrak{m}$ and finite residue field $k_R = R/\mathfrak{m}$. Fix a uniformizer $\pi \in \mathfrak{m}$, and let val be the discrete valuation on R, normalized so that $\mathrm{val}(\pi) = 1$. Let p be the characteristic of k_R and $f = [k_R : \mathbb{F}_p]$ the inertia degree. Denote by e the ramification index of R, which satisfies $pR = \mathfrak{m}^e$. Note that there is a natural ring embedding of $\mathbb{Z}_p$ into R, endowing R with a $\mathbb{Z}_p$-module structure.

Let $(\overline{\beta}_1, \dots, \overline{\beta}_f)$ be an ordered $\mathbb{F}_p$-basis of k_R. For each $i \in [f]$, we fix a lift $\beta_i \in R$ of $\overline{\beta}_i$. Then R is a free $\mathbb{Z}_p$-module of rank $n = ef$ and the set

$$\mathcal{B} = \left\{\beta_i \pi^j \mid i \in [f], j \in [e-1]_0\right\}$$

is a $\mathbb{Z}_p$-basis; cf. [6, Proposition II.6.8]. We order it as follows: $\mathcal{B} = (d_1, \dots, d_n)$, where $d_{i+fj} = \beta_i \pi^j$. Setting

$$a_i = \begin{pmatrix} 0 & d_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{n+i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_i \\ 0 & 0 & 0 \end{pmatrix}, \quad c_i = \begin{pmatrix} 0 & 0 & d_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for each $i \in [n]$, we obtain the following presentation of the Heisenberg Lie ring $L(R)$ defined in Section 2.4:

$$L(R) = \langle a_1, \dots, a_{2n}, c_1, \dots, c_n \mid [a_i, a_j] = M(\mathbf{c})_{ij},\ i,j \in [2n] \rangle. \tag{3.1}$$

Here $M(\mathbf{Y}) \in \mathrm{Mat}_{2n}(\mathbb{Z}_p[Y_1, \dots, Y_n])$ is a matrix whose entries are $\mathbb{Z}_p$-linear forms in the variables $Y_1, \dots, Y_n$, and $M(\mathbf{c})$ is the matrix obtained after making the substitution $Y_i = c_i$ for all $i \in [n]$. More precisely, $M(\mathbf{Y})$ has the form

$$M(\mathbf{Y}) = \begin{pmatrix} 0 & B(\mathbf{Y}) \\ -B(\mathbf{Y}) & 0 \end{pmatrix}, \tag{3.2}$$

where $B(\mathbf{Y})$ is an $n \times n$ matrix whose entries are given by $B(\mathbf{Y})_{ij} = \sum_{k=1}^{n} \gamma_k^{ij} Y_k$, and the "structure constants" $\gamma_k^{ij} \in \mathbb{Z}_p$ are defined by the relations $d_i d_j = \sum_{k=1}^{n} \gamma_k^{ij} d_k$. Note that (3.1) differs from the presentation appearing in [7, Section 2.1] by a reordering of the generators a_i.

Remark 3.1: Given integers $i, j \in [n]$, write $i = i_1 f + i_0$ and $j = j_1 f + j_0$, where $i_0, j_0 \in [f]$ and $i_1, j_1 \in [e-1]_0$. Define $\eta = \pi^e / p \in \mathbb{Z}_p^*$. It is immediate from the definition of the basis elements d_i that $d_i d_j = \pi^{i_1 + j_1} d_{i_0} d_{j_0}$. We write $i_1 + j_1 = \ell_1 e + \ell_0$ for $\ell_0 \in [e-1]_0$ and $\ell_1 \in \{0, 1\}$. The structure constants γ_k^{ij} satisfy the following:

(1) $\gamma_k^{ij} = \gamma_k^{ji}$ for all $i, j, k \in [n]$.
(2) $\gamma_{\ell_0 f + k}^{ij} = p^{\ell_1} \eta^{\ell_1} \gamma_k^{i_0 j_0}$ for $k \in [(e - \ell_0) f]$.
(3) $\gamma_k^{ij} \in p^{\ell_1 + 1} \mathbb{Z}_p$ for all $k \in [\ell_0 f]$.

In particular, $B(\mathbf{Y})$ is symmetric and has the following block decomposition:

$$B(\mathbf{Y}) = \begin{pmatrix} B^{(0)}(\mathbf{Y}) & B^{(1)}(\mathbf{Y}) & \dots & B^{(e-1)}(\mathbf{Y}) \\ B^{(1)}(\mathbf{Y}) & B^{(2)}(\mathbf{Y}) & \dots & pB^{(e)}(\mathbf{Y}) \\ \vdots & \vdots & & \vdots \\ B^{(e-1)}(\mathbf{Y}) & pB^{(e)}(\mathbf{Y}) & \dots & pB^{(2e-2)}(\mathbf{Y}) \end{pmatrix}, \tag{3.3}$$

for suitable square matrices $B^{(\mu)}(\mathbf{Y}) \in \mathrm{Mat}_f(\mathbb{Z}_p[\mathbf{Y}])$ of $\mathbb{Z}_p$-linear forms, for $\mu \in [2e-2]_0$.

By the remark after [2, Lemma 4.9], we have that

$$\zeta^{\triangleleft}_{H(R)} = \zeta^{\triangleleft}_{L(R)}, \tag{3.4}$$

where the ideal zeta function on the right hand side was defined in (2.6); see the discussion in [7, Section 1.3] for more details.

It is well known that, for all $d \in \mathbb{N}$, the (normal) zeta function of the free abelian pro-p group $\mathbb{Z}_p^d$ of rank d is given by

$$\zeta_{\mathbb{Z}_p^d}(s) = \prod_{i=0}^{d-1} \zeta_p(s-i), \tag{3.5}$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ is the Euler factor of the Riemann zeta function $\zeta(s)$ at the prime p; see, for instance, [2, Proposition 1.1].

3.2. The unramified case. First suppose that $e = 1$, covering the case of finite unramified extensions R of $\mathbb{Z}_p$.

Theorem 3.2: *Let p be a prime and R a finite unramified extension of $\mathbb{Z}_p$. Then*

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s)\frac{1}{1-x_0}\sum_{I\subseteq[n-1]}\binom{n}{I}_{p^{-1}}\prod_{i\in I}\frac{x_i}{1-x_i}, \tag{3.6}$$

with numerical data $x_i = p^{(2n+i)(n-i)-(3n-i)s}$ for $i \in [n-1]_0$.

Proof. In order to keep the notation of this paper compatible with [9], we have labeled the numerical data in reverse order to that of [7]. By [7, Corollary 3.7], we have

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s)\frac{1}{1-x_0}\sum_{I\subseteq[n-1]}\binom{n}{I}_{p^{-1}}\prod_{i\in I}\frac{x_{n-i}}{1-x_{n-i}}.$$

Define $n-I \subseteq [n-1]$ to be the set $\{n-i \mid i \in I\}$. Our claim follows by the identity

$$\binom{n}{n-I}_{p^{-1}} = \binom{n}{I}_{p^{-1}};$$

cf. [7, Remark 2.13]. ∎

The object of this section is to give a second proof of Theorem 3.2, based on the ideas of [9]. This will prepare the way for arguments in the general case in the remainder of the article.

Note that, since $e = 1$, we have $\mathcal{B} = (\beta_1, \ldots, \beta_n)$. Hence $\mathcal{B}$ reduces modulo $\mathfrak{m} = pR$ to an $\mathbb{F}_p$-basis of the residue field k_R. As in [9], we consider the Pfaffian hypersurface $\mathfrak{P}_{H(R)} \subseteq \mathbb{P}^{n-1}$ defined by the equation $\det(B(\mathbf{Y})) = 0$.

Lemma 3.3: *Let $q = p^n$, and let $T : \mathbb{F}_q \to \mathbb{F}_p$ be a non-zero $\mathbb{F}_p$-linear map. Let $\{x_1, \ldots, x_n\}$ be an $\mathbb{F}_p$-basis of $\mathbb{F}_q$. Then the matrix $A_T = (T(x_ix_j))_{ij} \in \mathrm{Mat}_n(\mathbb{F}_p)$ is nonsingular.*

Proof. Given $x \in \mathbb{F}_q$, consider the $\mathbb{F}_p$-linear map $U_{T,x} : \mathbb{F}_q \to \mathbb{F}_p$ given by $U_{T,x}(y) = T(xy)$. Let $\mathbb{F}_q^\vee = \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_q, \mathbb{F}_p)$ be the dual space of $\mathbb{F}_q$. Observe that the map

$$\mathbb{F}_q \to \mathbb{F}_q^\vee, \quad x \mapsto U_{T,x}$$

is $\mathbb{F}_p$-linear and injective and therefore is an isomorphism of $\mathbb{F}_p$-vector spaces. The matrix A_T is just the matrix of this map with respect to the $\mathbb{F}_p$-basis $\{x_1, \dots, x_n\}$ and its dual basis $\{x_1^\vee, \dots, x_n^\vee\}$, so the claim follows. ∎

Lemma 3.4: *The Pfaffian hypersurface $\mathfrak{P}_{H(R)}$ has no $\mathbb{F}_p$-rational points.*

Proof. Let $\mathbf{v} = (v_1, \dots, v_n)^t \in \mathbb{Z}^n$ be a column vector, and set $q = p^n$. Let $\overline{v_i} \in \mathbb{F}_p$ be the reduction modulo p of $v_i \in \mathbb{Z}$. Choose an isomorphism $k_R \simeq \mathbb{F}_q$ and use it to identify these two fields. Now let $\overline{\beta_i} \in \mathbb{F}_q$ be the reduction modulo p of $\beta_i \in R$, for $i \in [n]$. Recall that $\{\overline{\beta_1}, \dots, \overline{\beta_n}\}$ is an $\mathbb{F}_p$-basis of $\mathbb{F}_q$ and consider the $\mathbb{F}_p$-linear map $T_{\mathbf{v}} : \mathbb{F}_q \to \mathbb{F}_p$ given by $T_{\mathbf{v}}(\overline{\beta_i}) = \overline{v_i}$. Observe that the reduction modulo p of the matrix $B(\mathbf{v})$ is just the matrix $A_{T_{\mathbf{v}}}$ defined in the statement of Lemma 3.3. The conclusion of that lemma then implies that $\det(B(\overline{\mathbf{v}})) = 0$ only if $\overline{\mathbf{v}} = 0$. ∎

In the notation of [9], Lemma 3.4 states that $n_{\mathfrak{P}_{H(R)}}(p) = 0$ if R is unramified. Furthermore, it implies that the Pfaffian hypersurface has no points defined over $\mathbb{Q}$. Therefore it is vacuously smooth and has no lines. Hence, in the notation of [9, Theorem 3], $\zeta^{\triangleleft}_{H(R),p}(s) = W_0(p, p^{-s})$, where $W_0(p, p^{-s})$ is implicitly computed in [9, Section 4.2.1]. It is easily seen to match the formula given in (3.6). This concludes the second proof of Theorem 3.2.

3.3. The general case. We start off by describing the elementary divisors of matrices of the form $B(\alpha) \in \mathrm{Mat}_n(\mathbb{Z}_p)$, where $B(\mathbf{Y}) \in \mathrm{Mat}_n(\mathbb{Z}_p[\mathbf{Y}])$ is defined following (3.2) and $\alpha \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p$. Recall the block decomposition of $B(\mathbf{Y})$ given in (3.3).

Given a real number x, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x, and by $\lfloor x \rfloor$ the largest integer less than or equal to x.

Definition 3.5: Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$, put

$$\mu(\alpha) = \max\{i \in [n] \mid \mathrm{val}(\alpha_i) = 0\} \in [e]$$

and define $\lceil \alpha \rceil = \lceil \frac{\mu(\alpha)}{f} \rceil$.

Lemma 3.6: *Let $\alpha \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$ with $\lceil \alpha \rceil = m$. Then:*

1. $B^{(m-1)}(\alpha) \in \mathrm{GL}_f(\mathbb{Z}_p)$ *and* $B^{(\mu)}(\alpha) \in p\,\mathrm{Mat}_f(\mathbb{Z}_p)$ *for all* $\mu \in [m, e-1]$.
2. $B^{(m+e-1)}(\alpha) \in \mathrm{GL}_f(\mathbb{Z}_p)$ *and* $B^{(\mu)}(\alpha) \in p\,\mathrm{Mat}_f(\mathbb{Z}_p)$ *for all* $\mu \in [m+e, 2e-2]$.

Proof. Let $\mu \in [e-1]$ and let $\bar{A}^{(\mu)} \in \mathrm{Mat}_f(\mathbb{F}_p)$ be the reduction modulo p of $B^{(\mu)}(\alpha)$. From Remark 3.1(3) it follows that $\bar{A}^{(\mu)}_{i,j} = \sum_{\ell=1}^{(e-\mu)f} \overline{\gamma}^{ij}_\ell \overline{\alpha}_{\mu f+\ell}$ for all $i,j \in [f]$, where the overline denotes reduction modulo p and the γ^{ij}_ℓ are as defined immediately following (3.2). Our assumption on α immediately implies the second part of (1), whereas if $\mu = m-1$ then we obtain $\bar{A}^{(m-1)}_{i,j} = \sum_{\ell=1}^{f} \overline{\gamma}^{ij}_\ell \overline{\alpha}_{(m-1)f+\ell}$. We would like to prove that $\bar{A}^{(m-1)}$ is invertible.

Since $\overline{\beta}_i \overline{\beta}_j = \sum_{\ell=1}^{f} \overline{\gamma}^{ij}_\ell \overline{\beta}_\ell$ for all $i,j \in [f]$, we find that $\bar{A}^{(m-1)} = A_T$, in the notation of Lemma 3.3, where $T : k_R \to \mathbb{F}_p$ is the non-zero $\mathbb{F}_p$-linear operator given by $T(\overline{\beta}_i) = \overline{\alpha}_{(m-1)f+i}$ for all $i \in [f]$. Lemma 3.3 thus implies that $\bar{A}^{(m-1)}$ is non-singular. This establishes the first part of claim (1).

The second part of (2) follows similarly from Remark 3.1 and the hypothesis on α. To establish the first part of (2), we let $\psi : k_R \to k_R$ denote the $\mathbb{F}_p$-linear isomorphism corresponding to multiplication by $\overline{\pi^e/p} \in k_R^\times$ and set $(\overline{\alpha}'_1, \ldots, \overline{\alpha}'_f) = \psi(\overline{\alpha}_{(m-1)f+1}, \ldots, \overline{\alpha}_{mf})$. It follows similarly to the previous case that $\bar{A}^{(m+e-1)} = A_{T'}$, where $T'(\overline{\beta}_i) = \overline{\alpha}'_i$ for all $i \in [f]$. Thus we again have $\bar{A}^{(m+e-1)} \in \mathrm{GL}_f(\mathbb{F}_p)$ by Lemma 3.3. ∎

Write I_ℓ for the $\ell \times \ell$ identity matrix. For $m \in [e]$ we define

$$J_m = \begin{pmatrix} 0 & I_{mf} \\ p^{-1} I_{(e-m)f} & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{Q}_p).$$

Corollary 3.7: *Let $m \in [e]$, and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$ be a vector such that $\lceil \alpha \rceil = m$. Then $B(\alpha) J_m \in \mathrm{GL}_n(\mathbb{Z}_p)$.*

Proof. This is immediate from Lemma 3.6 and (3.3). ∎

We now state the main result of this article. Recall the statistics len, $\mathrm{len}^{[n-2]}$, and Des on the Coxeter group S_n that were defined in Section 2.1.

Theorem 3.8: *Let R be a finite extension of $\mathbb{Z}_p$ with inertia degree f and ramification index e. Set $n = ef$. Then*

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s) \frac{\sum_{w \in S_n} p^{-\mathrm{len}(w) + 2f \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor s} \prod_{j \in \mathrm{Des}(w)} x_j}{\prod_{i=0}^{n-1} (1 - x_i)}, \tag{3.7}$$

with numerical data $x_i = p^{(2n+i)(n-i)-(3n-i)s}$ for $i \in [n-1]_0$.

Proof. We saw in (3.4) that $\zeta^{\triangleleft}_{H(R)}(s) = \zeta^{\triangleleft}_{L(R)}(s)$, where $L(R)$ is the Heisenberg Lie ring over R; cf. Section 2.4. The abelianization $\overline{L(R)}$ and the derived subring $L(R)'$ are free $\mathbb{Z}_p$-modules of rank $2n$ and n, respectively. By [9, Lemma 1], which is essentially [2, Lemma 6.1], we have

(3.8)
$$\zeta^{\triangleleft}_{L(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s)\zeta_p(3ns - 2n^2) \sum_{\substack{\Lambda' \leq L(R)' \\ \Lambda' \text{ maximal}}} |L(R)' : \Lambda'|^{2n-s} |L(R) : X(\Lambda')|^{-s},$$

where, for every finite-index sublattice $\Lambda' \leq L(R)'$, we define $X(\Lambda')$ to be the sublattice of $L(R)$ such that $X(\Lambda')/\Lambda'$ is the center of $L(R)/\Lambda'$. Note that $\zeta_p(3ns - 2n^2) = \frac{1}{1-x_0}$.

Let $\Lambda' \leq L(R)' \simeq \mathbb{Z}_p^n$ be a maximal sublattice of finite index, of type $\nu(\Lambda') = (I, \mathbf{r}_I)$, where $I = \{i_1, \ldots, i_\ell\}_< \subseteq [n-1]$ and $\mathbf{r}_I = (r_{i_1}, \ldots, r_{i_\ell}) \in \mathbb{N}^\ell$; cf. Section 2.3. We write i for i_ℓ. As in Section 2.3, we identify Λ' with a coset $\boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)}$, where $\boldsymbol{\alpha} \in \mathrm{SL}_n(\mathbb{Z}_p)$. For $j \in [n]$, let α^j denote the j-th column vector of $\boldsymbol{\alpha}$. Recalling Definition 3.5 we set

$$\kappa(\Lambda') := e - \max\{\lceil \alpha^j \rceil \mid n - i < j \leq n\} \in [e-1]_0. \tag{3.9}$$

An informal description of $\kappa(\Lambda')$ is as follows. Consider the reduction modulo p of the $n \times (n-i)$ matrix composed of the last $n-i$ columns of $\boldsymbol{\alpha}$. Then $\kappa(\Lambda') = \kappa$ if and only if the last κf rows of this matrix are zero, but the $(\kappa+1)$-st block of f rows from the bottom contains a nonzero element.

The most mysterious ingredient of (3.8) is the quantity $|L(R) : X(\Lambda')|$, which we will now compute.

Lemma 3.9: *Let $\Lambda' \leq L(R)'$ be a maximal sublattice of type $\nu(\Lambda') = (I, \mathbf{r}_I)$. Then*
$$|L(R) : X(\Lambda')| = p^{2\left(n \sum_{\iota \in I} r_\iota - \kappa(\Lambda') f\right)}.$$

Proof. By [9, Theorem 6], $|L(R) : X(\Lambda')|$ is equal to the index in $\overline{L(R)} \cong \mathbb{Z}_p^{2n}$ of the sublattice of simultaneous solutions to the following system of linear congruences:

$$\overline{\mathbf{g}} M(\alpha^j) \equiv 0 \bmod (p^\nu)_j \quad \text{for } j \in [n] \tag{3.10}$$

in variables $\overline{\mathbf{g}} = (g_1, \ldots, g_{2n})$. Here $M(\alpha^j)$ is the commutator matrix $M(\mathbf{Y})$ of (3.2) evaluated at α^j. It is clear from (3.2) that the index of the solution sublattice of (3.10) in $\overline{L(R)}$ is the square of the index in $\mathbb{Z}_p^n$ of the solution sublattice of the system

$$\mathbf{h}B(\alpha^j) \equiv 0 \bmod (p^\nu)_j \qquad \text{for } j \in [n], \tag{3.11}$$

where $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_p^n$. By Lemma 3.7, each matrix $B(\alpha^j) \in \mathrm{Mat}_n(\mathbb{Z}_p)$ becomes invertible after all the entries in its last $(e - \lceil \alpha^j \rceil)f$ columns have been divided by p. Therefore, $\mathbf{h} = (h_t) \in \mathbb{Z}_p^n$ is a solution to (3.11) if and only if, for all $j \in [n]$,

$$\begin{aligned} h_t &\equiv 0 \bmod (p^\nu)_j \quad && \text{if } t \le \lceil \alpha^j \rceil f, \\ ph_t &\equiv 0 \bmod (p^\nu)_j \quad && \text{if } t > \lceil \alpha^j \rceil f. \end{aligned}$$

It follows that the congruences where $(p^\nu)_j$ is maximal, namely those with $j \in [i+1, n]$, dominate all the others. Hence, $\mathbf{h} \in \mathbb{Z}_p^n$ is a solution to (3.11) if and only if

$$\begin{aligned} h_t &\equiv 0 \bmod p^{\sum_{\iota \in I} r_\iota} \quad && \text{if } t \le (e - \kappa(\Lambda'))f, \\ h_t &\equiv 0 \bmod p^{(\sum_{\iota \in I} r_\iota)-1} \quad && \text{if } t > (e - \kappa(\Lambda'))f. \end{aligned}$$

Recalling that $n = ef$, it follows that the index in $\overline{L(R)}$ of the sublattice of simultaneous solutions to the congruences (3.10) is the quantity in the statement of the lemma. ∎

It is obvious from the definition of the type of a lattice that, if $\Lambda' \le L(R)'$ is a sublattice of type $(I, \mathbf{r}_I)$, then

$$|L(R)' : \Lambda'| = p^{\sum_{\iota \in I}(n-\iota) r_\iota}. \tag{3.12}$$

Given $\kappa \in [e-1]_0$ and a type $(I, \mathbf{r}_I)$, define

$$\mathcal{N}^\kappa_{(I, \mathbf{r}_I)} = \#\{\Lambda' \le L(R)' \mid \nu(\Lambda') = (I, \mathbf{r}_I), \kappa(\Lambda') = \kappa\}.$$

It follows from (3.8), (3.12), and Lemma 3.9 that

$$\begin{aligned} &\zeta^{\triangleleft}_{L(R)}(s) \\ &= \frac{\zeta_{\mathbb{Z}_p^{2n}}(s)}{1 - x_0} \sum_{I \subseteq [n-1]} \sum_{\mathbf{r}_I \in \mathbb{N}^{|I|}} \sum_{\kappa=0}^{e-1} \mathcal{N}^\kappa_{(I,\mathbf{r}_I)} p^{\left(\sum_{\iota \in I}(n-\iota) r_\iota\right)(2n-s) - 2s\left(n \sum_{\iota \in I} r_\iota - \kappa f\right)}. \end{aligned} \tag{3.13}$$

As preparation for computing the numbers $\mathcal{N}^{\kappa}_{(I,\mathbf{r}_I)}$, we fix a type $(I,\mathbf{r}_I)$ as above and consider the surjective map

$$\begin{aligned}\varphi_{(I,\mathbf{r}_I)} : \{\Lambda' \mid \nu(\Lambda') = (I,\mathbf{r}_I)\} &\to \mathrm{Gr}(n, n-i; \mathbb{F}_p)\\ \boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)} &\mapsto \langle \overline{\alpha^j} \mid i < j \leq n\rangle_{\mathbb{F}_p}.\end{aligned}$$

As before, we identify lattices of type $(I,\mathbf{r}_I)$ with cosets $\boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)}$ for $\boldsymbol{\alpha} \in \Gamma$. Informally, $\varphi(\boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)})$ is the subspace of $\mathbb{F}_p^n$ spanned by the reduction modulo p of the last $n-i$ columns of the matrix $\boldsymbol{\alpha} \in \Gamma = \mathrm{SL}_n(\mathbb{Z}_p)$.

Lemma 3.10: *The fibres of $\varphi_{(I,\mathbf{r}_I)}$ all have the same cardinality*

$$p^{-i(n-i)} \binom{i}{I \setminus \{i\}}_{p^{-1}} p^{\sum_{\iota \in I} r_\iota \iota(n-\iota)}. \tag{3.14}$$

Proof. The map $\varphi_{(I,\mathbf{r}_I)}$ is just the natural surjection

$$\Gamma/\Gamma_{(I,\mathbf{r}_I)} \to \Gamma/\Gamma_{(\{i\},1)}, \quad \boldsymbol{\alpha}\Gamma_{(I,\mathbf{r}_I)} \mapsto \boldsymbol{\alpha}\Gamma_{(\{i\},1)}.$$

Of course $|\Gamma : \Gamma_{(I,\mathbf{r}_I)}| = |\Gamma : \Gamma_{(\{i\},1)}||\Gamma_{(\{i\},1)} : \Gamma_{(I,\mathbf{r}_I)}|$. By (2.5) we have

$$\begin{aligned}|\Gamma : \Gamma_{(I,\mathbf{r}_I)}| &= \binom{n}{I}_{p^{-1}} p^{\sum_{\iota\in I} r_\iota \iota(n-\iota)}\\ &= \binom{n}{n-i}_{p^{-1}} p^{r_i i(n-i)} \cdot \binom{i}{I \setminus \{i\}}_{p^{-1}} p^{\sum_{\iota \in I\setminus\{i\}} r_\iota \iota(n-\iota)}.\end{aligned}$$

Formula (3.14) for the index $|\Gamma_{(\{i\},1)} : \Gamma_{(I,\mathbf{r}_I)}|$ follows now from (2.4), as

$$|\Gamma : \Gamma_{(\{i\},1)}| = \#\,\mathrm{Gr}(n, n-i; \mathbb{F}_p) = \binom{n}{n-i}_{p^{-1}} p^{i(n-i)}. \qquad \blacksquare$$

Consider the following filtration on $\mathrm{Gr}(n, n-i; \mathbb{F}_p)$. Let $(\varepsilon_1, \ldots, \varepsilon_n)$ denote the standard $\mathbb{F}_p$-basis of $V = \mathbb{F}_p^n$, and consider the flag $(V_d)_{d=0}^e = \left(\langle \varepsilon_1, \ldots, \varepsilon_{fd}\rangle_{\mathbb{F}_p}\right)_{d=0}^e$. Define

$$\psi : \mathrm{Gr}(n, n-i; \mathbb{F}_p) \to [e], \quad W \mapsto \min\{d \mid W \subseteq V_d\}.$$

One verifies easily that the fibres of ψ are unions of Schubert cells. Indeed, if $\lambda \in [e]$ and $W \in \mathrm{Gr}(n, n-i; \mathbb{F}_p)$, then $\psi(W) = \lambda$ if and only if the bottom $(e-\lambda)f$ rows of the matrix of W (cf. Section 2.2) consist of zeroes, whereas the previous block of f rows does contain a non-zero matrix element. It is clear from the discussion in Section 2.2 that the lowest-positioned non-zero element in the matrix of W is a 1 in the $(w(n), n-i)$ position, where $w \in S_n$ is such that $W \in \Omega_w(\mathbb{F}_p)$. In other words, $\psi(W) = \lambda$ if and only if $w(n) \in [(\lambda-1)f+1, \lambda f]$.

Recalling that $\mathrm{len}^{[n-2]}(w) = n - w(n)$, this condition is clearly equivalent to $\left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = e - \lambda$. Therefore, for every $\lambda \in [e]$,

$$\psi^{-1}(\lambda) = \bigcup_{\substack{w \in S_n,\ \mathrm{Des}(w) \subseteq \{i\} \\ \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = e-\lambda}} \Omega_w(\mathbb{F}_p). \tag{3.15}$$

Now consider the composition

$$\psi \circ \varphi_{(I,\mathbf{r}_I)} : \{\Lambda' \leq L(R)' \mid \nu(\Lambda') = (I, \mathbf{r}_I)\} \to [e],$$
$$\boldsymbol{\alpha}\Gamma_\nu \mapsto \psi\left(\langle \overline{\alpha^j} \mid i < j \leq n \rangle_{\mathbb{F}_p}\right).$$

From the definition of $\kappa(\Lambda')$ in (3.9) it is evident for all $\kappa \in [e-1]_0$ that $\kappa(\Lambda') = \kappa$ if and only if $\psi \circ \varphi_{(I,\mathbf{r}_I)}(\Lambda') = e - \kappa$. Thus, by (3.14), (3.15), and the fact that $|\Omega_w(\mathbb{F}_p)| = p^{i(n-i)-\mathrm{len}(w)}$ for all $w \in S_n$ (cf. Section 2.2), we obtain

(3.16)

$$\begin{aligned}
\mathcal{N}^\kappa_{(I,\mathbf{r}_I)} &= \#\{\Lambda' \leq L(R)' \mid \nu(\Lambda') = (I, \mathrm{r}_I), \kappa(\Lambda') = \kappa\} \\
&= \#\{\Lambda' \leq L(R)' \mid \nu(\Lambda') = (I, \mathbf{r}_I), \psi(\varphi_{(I,\mathbf{r}_I)}(\Lambda')) = e - \kappa\} \\
&= \left(\sum_{w \in S_n,\ \mathrm{Des}(w) \subseteq \{i\},\ \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = \kappa} p^{-\mathrm{len}(w)} \right) \binom{i}{I \setminus \{i\}}_{p^{-1}} p^{\sum_{\iota \in I} r_\iota \iota(n-\iota)} \\
&= \alpha_I^\kappa \prod_{\iota \in I} p^{r_\iota \iota(n-\iota)},
\end{aligned}$$

where we set

$$\alpha_I^\kappa = \left(\sum_{w \in S_n,\ \mathrm{Des}(w) \subseteq \{i\},\ \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = \kappa} p^{-\mathrm{len}(w)} \right) \binom{i}{I \setminus \{i\}}_{p^{-1}}. \tag{3.17}$$

Lemma 3.11: *Let α_I^κ be the quantity defined in* (3.17). *Then*

$$\alpha_I^\kappa = \sum_{\substack{w \in S_n, \mathrm{Des}(w) \subseteq I \\ \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = \kappa}} p^{-\mathrm{len}(w)}.$$

Proof. It is easy to see that an element $w \in S_n$ is the unique element of shortest length in its coset $wW_{[i-1]}$ if and only if $\mathrm{Des}(w) \cap [i-1] = \varnothing$. Writing an arbitrary element $w \in S_n$ in the form $w = w^{[i-1]} w_{[i-1]}$ as in Section 2.1, we find that $w^{[i-1]}$ is the unique element such that $\mathrm{Des}(w^{[i-1]}) \cap [i-1] = \varnothing$ and $w^{[i-1]}(j) = w(j)$ for all $j > i$; i.e., $\mathrm{Des}(w^{[i-1]}) \cap [i+1, n] = \mathrm{Des}(w) \cap [i+1, n]$, whereas $\mathrm{Des}(w_{[i-1]}) \cap [i-1] = \mathrm{Des}(w) \cap [i-1]$. It follows that the elements $w \in S_n$ satisfying $\mathrm{Des}(w) \subseteq I$ are precisely those for which $\mathrm{Des}(w^{[i-1]}) \subseteq \{i\}$ and $\mathrm{Des}(w_{[i-1]}) \subseteq I \setminus \{i\}$. Finally, it is clear that $w(n) = w^{[i-1]}(n)$ and hence $\mathrm{len}^{[n-2]}(w) = \mathrm{len}^{[n-2]}(w^{[i-1]})$. Since $\mathrm{len}(w) = \mathrm{len}(w^{[i-1]}) + \mathrm{len}(w_{[i-1]})$ and

$$\binom{i}{I \setminus \{i\}}_{p^{-1}} = \sum_{\substack{w \in W_{[i-1]} \\ \mathrm{Des}(w) \subseteq I \setminus \{i\}}} p^{-\mathrm{len}(w)},$$

the desired equality follows. ∎

Finally, we have all the ingredients necessary to compute $\zeta^{\triangleleft}_{H(R)}(s)$ and finish the proof of Theorem 3.8. Indeed, a simple calculation using (3.16) shows that

$$\sum_{\mathbf{r}_I \in \mathbb{N}^{|I|}} \mathcal{N}^{\kappa}_{(I,\mathbf{r}_I)} p^{\left(\sum_{\iota \in I}(n-\iota) r_\iota\right)(2n-s) - 2s\left(n \sum_{\iota \in I} r_\iota - \kappa f\right)} =$$

$$\alpha^{\kappa}_I p^{2s\kappa f} \sum_{\mathbf{r}_I \in \mathbb{N}^{|I|}} \prod_{\iota \in I} \left(p^{(2n+\iota)(n-\iota)-(3n-\iota)s}\right)^{r_\iota} = \alpha^{\kappa}_I p^{2s\kappa f} \prod_{\iota \in I} \frac{x_\iota}{1-x_\iota},$$

where $x_\iota = p^{(2n+\iota)(n-\iota)-(3n-\iota)s}$ for $\iota \in I \subseteq [n-1]$ as in the statement of Theorem 3.8. By (3.13) this implies

$$\zeta^{\triangleleft}_{H(R)}(s) \frac{1-x_0}{\zeta_{\mathbb{Z}_p^{2n}}(s)} = \sum_{I \subseteq [n-1]} \sum_{\kappa=0}^{e-1} \alpha^{\kappa}_I p^{2\kappa f s} \prod_{\iota \in I} \frac{x_\iota}{1-x_\iota}.$$

Bringing the right hand side to a common denominator, we get

$$\frac{\sum_{\kappa=0}^{e-1} \sum_{I \subseteq [n-1]} \beta^{\kappa}_I p^{2\kappa f s} \prod_{\iota \in I} x_\iota}{\prod_{i=1}^{n-1}(1-x_i)},$$

where

$$\beta^{\kappa}_I = \sum_{J \subseteq I} (-1)^{|I|-|J|} \alpha^{\kappa}_J = \sum_{\substack{w \in S_n,\, \mathrm{Des}(w) = I \\ \left\lfloor \frac{\mathrm{len}^{[n-2]}(w)}{f} \right\rfloor = \kappa}} p^{-\mathrm{len}(w)},$$

the last equality following from a simple inclusion-exclusion argument; cf. [8, (1.34)]. Hence,

$$\begin{aligned}\zeta^{\triangleleft}_{H(R)}(s)\frac{\prod_{i=0}^{n-1}(1-x_i)}{\zeta_{\mathbb{Z}_p^{2n}}(s)} &= \sum_{I\subseteq[n-1]}\sum_{\kappa=0}^{e-1} p^{2\kappa f s}\left(\sum_{\substack{w\in S_n,\,\mathrm{Des}(w)=I\\ \left\lfloor\frac{\mathrm{len}^{[n-2]}(w)}{f}\right\rfloor=\kappa}} p^{-\mathrm{len}(w)}\right)\prod_{\iota\in I}x_\iota\\ &= \sum_{w\in S_n} p^{-\mathrm{len}(w)+2f\left\lfloor\frac{\mathrm{len}^{[n-2]}(w)}{f}\right\rfloor s}\prod_{j\in\mathrm{Des}(w)}x_j,\end{aligned}$$

as claimed. ∎

Remark 3.12: Theorem 3.2 is indeed a special case of Theorem 3.8. If $e=1$ and $f=n$, then $\mathrm{len}^{[n-2]}(w)\le n-1<f$ for all $w\in S_n$, and hence $\left\lfloor\frac{\mathrm{len}^{[n-2]}(w)}{f}\right\rfloor=0$ for all $w\in S_n$. One verifies easily that

$$\frac{\sum_{w\in S_n}p^{-\mathrm{len}(w)}\prod_{j\in\mathrm{Des}(w)}x_j}{\prod_{i=0}^{n-1}(1-x_i)}=\frac{1}{1-x_0}\sum_{I\subseteq[n-1]}\binom{n}{I}_{p^{-1}}\prod_{i\in I}\frac{x_i}{1-x_i},$$

by bringing the right hand side to a common denominator as in [9, Section 4.1] and using (2.3).

Corollary 3.13: *Let R be a finite extension of $\mathbb{Z}_p$ with inertia degree f and ramification index e. Set $n=ef$. Then $\zeta^{\triangleleft}_{H(R)}(s)$ satisfies the following functional equation:*

$$\zeta^{\triangleleft}_{H(R)}(s)|_{p\to p^{-1}}=(-1)^{3n}p^{\binom{3n}{2}-(5n+2(e-1)f)s}\zeta^{\triangleleft}_{H(R)}(s).$$

Proof. Recall from (2.1) that, for all $w\in S_n$, we have $\mathrm{len}^{[n-2]}(w_0w)=\mathrm{len}^{[n-2]}(w_0)-\mathrm{len}^{[n-2]}(w)$, where $w_0\in S_n$ is longest element. The key observation is that the fractional part of $\frac{\mathrm{len}^{[n-2]}(w_0)}{f}=\frac{n-1}{f}=e-\frac{1}{f}$ is the largest possible. Hence, for all $w\in S_n$,

$$\left\lfloor\frac{\mathrm{len}^{[n-2]}(w_0w)}{f}\right\rfloor=\left\lfloor\frac{\mathrm{len}^{[n-2]}(w_0)}{f}\right\rfloor-\left\lfloor\frac{\mathrm{len}^{[n-2]}(w)}{f}\right\rfloor=(e-1)-\left\lfloor\frac{\mathrm{len}^{[n-2]}(w)}{f}\right\rfloor.$$

Using (3.5), (2.2), and (2.3) it then follows that

$$\begin{aligned}
&\zeta^{\triangleleft}_{H(R)}(s)|_{p\to p^{-1}}\\
=&\zeta_{\mathbb{Z}_p^{2n}}(s)|_{p\to p^{-1}}\frac{\sum_{w\in S_n} p^{\operatorname{len}(w)-2f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w)}{f}\right\rfloor s}\prod_{j\in\operatorname{Des}(w)} x_j^{-1}}{\prod_{i=0}^{n-1}(1-x_i^{-1})}\\
=&(-1)^{3n}p^{\binom{2n}{2}-2ns}\zeta_{\mathbb{Z}_p^{2n}}(s)x_0\frac{\sum_{w\in S_n} p^{\operatorname{len}(w)-2f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w)}{f}\right\rfloor s}\prod_{j\in[n-1]\setminus\operatorname{Des}(w)} x_j}{\prod_{i=0}^{n-1}(1-x_i)}\\
=&(-1)^{3n}p^{\binom{3n}{2}-(5n+2(e-1)f)s}\\
&\times\zeta_{\mathbb{Z}_p^{2n}}(s)\frac{\sum_{w_0w\in S_n} p^{-\operatorname{len}(w_0w)+2f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w_0w)}{f}\right\rfloor s}\prod_{j\in\operatorname{Des}(w_0w)} x_j}{\prod_{i=0}^{n-1}(1-x_i)}\\
=&(-1)^{3n}p^{\binom{3n}{2}-(5n+2(e-1)f)s}\zeta^{\triangleleft}_{H(R)}(s),
\end{aligned}$$

as claimed. ∎

3.4. An alternative formulation of the main result. We now prove an alternative formula for $\zeta^{\triangleleft}_{H(R)}(s)$ to that of Theorem 3.8 by showing that, in general, the fraction on the right hand side of (3.7) admits some cancellation.

Consider the n-cycle $c = (1\ 2\cdots n)\in S_n$. For $i\in[n-1]_0$, let x_i be as in Theorem 3.8.

Lemma 3.14: *Let $w\in S_n$, and let $m\in[n-1]_0$ be such that $w(1)\le n-m$. Then*

$$p^{-\operatorname{len}(c^m w)+2\operatorname{len}^{[n-2]}(c^m w)s}\prod_{j\in\operatorname{Des}(c^m w)} x_j = (p^{2n-3s})^m p^{-\operatorname{len}(w)+2\operatorname{len}^{[n-2]}(w)s}\prod_{j\in\operatorname{Des}(w)} x_j.$$

Proof. It suffices to prove the statement for $m=1$; the general case clearly follows from iterated application of this result. So let $w\in S_n$ and set $j = w^{-1}(n)$. Recall that $\operatorname{len}^{[n-2]}(w) = n-w(n)$ and observe that $\operatorname{len}(cw)-\operatorname{len}(w) = 2j-n-1$. Moreover, we observe that $\operatorname{Des}(cw) = (\operatorname{Des}(w)\cup\{j-1\})\setminus\{j\}$.

If $j<n$, then $w(n)<n$ and so $w(n)-cw(n) = -1$. If $j=n$, then $w(n)-cw(n) = n-1$. In either case we obtain, by setting $x_n := 1$, for $j\in[n]$,

that

$$\frac{p^{-\operatorname{len}(cw)+2\operatorname{len}^{[n-2]}(cw)s}\prod_{j\in\operatorname{Des}(cw)}x_j}{p^{-\operatorname{len}(w)+2\operatorname{len}^{[n-2]}(w)s}\prod_{j\in\operatorname{Des}(w)}x_j} = p^{n-2j+1+2(w(n)-cw(n))s}\frac{x_{j-1}}{x_j} = p^{2n-3s}. \qquad \blacksquare$$

Lemma 3.15: *Suppose that $w \in S_n$ and $f \in \mathbb{N}$ satisfies $w(1) \leq f$. Then for any $m \leq \lfloor \frac{n-f}{f} \rfloor$ the following holds:*

$$p^{-\operatorname{len}(c^{mf}w)+2f\left\lfloor \frac{\operatorname{len}^{[n-2]}(c^{mf}w)}{f}\right\rfloor s}\prod_{j\in\operatorname{Des}(c^{mf}w)}x_j = (p^{2n-3s})^{mf}p^{-\operatorname{len}(w)+2f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w)}{f}\right\rfloor s}\prod_{j\in\operatorname{Des}(w)}x_j.$$

Proof. Let w and m be as in the lemma. Then $\operatorname{len}^{[n-2]}(c^{mf}w) = \operatorname{len}^{[n-2]}(w) - mf$. Thus

$$f\left\lfloor\frac{\operatorname{len}^{[n-2]}(c^{mf}w)}{f}\right\rfloor - \operatorname{len}^{[n-2]}(c^{mf}w) = f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w)}{f}\right\rfloor - \operatorname{len}^{[n-2]}(w).$$

The lemma follows immediately from this and Lemma 3.14. ∎

Theorem 3.16: *Let R be a finite extension of $\mathbb{Z}_p$ with inertia degree f and ramification index e. Set $n = ef$ and $S_n^{(f)} = \{w \in S_n \mid w(1) \leq f\}$. Then*

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s)\frac{\sum_{w\in S_n^{(f)}} p^{-\operatorname{len}(w)+2f\left\lfloor\frac{\operatorname{len}^{[n-2]}(w)}{f}\right\rfloor s}\prod_{j\in\operatorname{Des}(w)}x_j}{(1-p^{f(2n-3s)})\prod_{i=1}^{n-1}(1-x_i)}.$$

Proof. This follows from Theorem 3.8, Lemma 3.15, and the observations that

$$1 - x_0 = 1-(p^{2n-3s})^n = (1-(p^{f(2n-3s)}))\sum_{m=0}^{e-1}p^{mf(2n-3s)}$$

and that every element of S_n can be written uniquely in the form $c^{mf}w$, where $m \in [e-1]_0$ and $w \in S_n^{(f)}$. ∎

Remark 3.17: An interesting question is whether the fraction in Theorem 3.16 is always in lowest terms and admits no more cancellation. T. Bauer has verified this for all pairs (e, f) with $n = ef \leq 10$.

In the case that $e = 1$, Theorem 3.16 is exactly Theorem 3.2. In the other extreme, the case $f = 1$, we obtain an interesting corollary. We identify $S_n^{(1)}$, the stabilizer in S_n of the letter 1, viz. the parabolic subgroup $(S_n)_{\{s_i \mid 2 \leq i \leq n-1\}}$, with S_{n-1}.

Corollary 3.18: *Let R be a totally ramified extension of $\mathbb{Z}_p$ of degree n. Then*

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{2n}}(s) \frac{\sum_{w \in S_{n-1}} p^{-\operatorname{len}(w) + 2 \operatorname{len}^{[n-3]}(w)s} \prod_{j \in \operatorname{Des}(w)} x_{j+1}}{(1 - p^{2n-3s}) \prod_{i=1}^{n-1}(1 - x_i)}, \tag{3.18}$$

with numerical data $x_i = p^{(2n+i)(n-i)-(3n-i)s}$ for $i \in [n-1]$.

Example 3.19: We illustrate our results in the case $e = 3$, $f = 1$. Thus let R be a totally ramified cubic extension of $\mathbb{Z}_p$. It is shown in [2, Proposition 8.15] that

$$\zeta^{\triangleleft}_{H(R)}(s) = \frac{1 + p^{7-5s}}{\left(\prod_{i=0}^{5}(1 - p^{i-s})\right)(1 - p^{6-3s})(1 - p^{8-7s})(1 - p^{14-8s})}. \tag{3.19}$$

Theorem 3.8 presents this zeta function as

$$\zeta^{\triangleleft}_{H(R)}(s) = \zeta_{\mathbb{Z}_p^{6}}(s) \frac{\sum_{w \in S_3} p^{-\operatorname{len}(w) + 2 \operatorname{len}^{\{1\}}(w)s} \prod_{j \in \operatorname{Des}(w)} x_j}{\prod_{i=0}^{2}(1 - x_i)}, \tag{3.20}$$

with numerical data

$$x_0 = p^{18-9s}, \quad x_1 = p^{14-8s}, \quad x_2 = p^{8-7s}.$$

The Coxeter group S_3 is generated by the involutions s_1 and s_2. We tabulate the values of the functions Des, len, and $\operatorname{len}^{\{1\}}$ on S_3.

$w \in S_3$	$\operatorname{Des}(w)$	$\operatorname{len}(w)$	$\operatorname{len}^{\{1\}}(w)$
1	$\varnothing$	0	0
s_1	$\{1\}$	1	0
s_2	$\{2\}$	1	1
$s_2 s_1$	$\{1\}$	2	1
$s_1 s_2$	$\{2\}$	2	2
$(s_2 s_1 s_2 =) s_1 s_2 s_1$	$\{1, 2\}$	3	2

We deduce that

$$\sum_{w\in S_3} p^{-\operatorname{len}(w)+2\operatorname{len}^{\{1\}}(w)s} \prod_{j\in\operatorname{Des}(w)} x_j$$
$$= 1+p^{13-8s}+p^{7-5s}+p^{12-6s}+p^{6-3s}+p^{19-11s} = \frac{(1-p^{18-9s})(1+p^{7-5s})}{1-p^{6-3s}},$$

showing that (3.19) accords with (3.20). Formula (3.19) also illustrates (3.18), as $\langle s_2\rangle \cong S_2$ and

$$1+p^{7-5s} = 1+p^{-1+2s}x_2 = \sum_{w\in S_2} p^{-\operatorname{len}(w)+2\operatorname{len}(w)s} \prod_{j\in\operatorname{Des}(w)} x_{j+1}.$$

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