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OPTIMAL HARDY–LITTLEWOOD TYPE INEQUALITIES FOR POLYNOMIALS AND MULTILINEAR OPERATORS

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ABSTRACT

In this paper we obtain quite general and definitive forms for Hardy–Littlewood type inequalities. Moreover, when restricted to the original particular cases, our approach provides much simpler and straightforward proofs and we are able to show that in most cases the exponents involved are optimal. The technique we used is a combination of probabilistic tools and of an interpolative approach; this former technique is also employed in this paper to improve the constants for vector-valued Bohnenblust–Hille type inequalities.

1. Introduction

In 1930 Littlewood [22] has shown the following result on bilinear forms on $c_0 \times c_0$, now called Littlewood's 4/3 inequality: for any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$,

$$\left(\sum_{i,j=1}^{+\infty} |A(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2}\|A\|$$

and, moreover, the exponent 4/3 is optimal. From now on, $m \geq 1$ is a positive integer, $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$ and

$$\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

For $1 \leq p < +\infty$, let us set $X_p := \ell_p$ and let us define $X_\infty = c_0$. As soon as Littlewood's 4/3 inequality appeared, it was rapidly extended to more general frameworks. For instance:

- (Bohnenblust and Hille, [6, Theorem I], 1931 (see also [12])) There exists a constant $C = C(m) \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} |A(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C\|A\| \tag{1.1}$$

  for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{C}$ and the exponent $\frac{2m}{m+1}$ is optimal.

- (Hardy and Littlewood, [18], 1934 (see also [19, page 224])/Praciano-Pereira, [27, Theorems A and B], 1981) Let $\mathbf{p} \in [1, +\infty]^m$ with
$$\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2},$$
then there exists a constant $C > 0$ such that, for every continuous m-linear form $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{C}$,
$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} |A(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|}}\right)^{\frac{m+1-2|\frac{1}{\mathbf{p}}|}{2m}} \leq C\|A\|. \tag{1.2}$$
- (Defant and Sevilla-Peris, [11, Theorem 1], 2009) If $1 \leq s \leq q \leq 2$, there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : c_0 \times \cdots \times c_0 \to \ell_s$, then
$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2(\frac{1}{s}-\frac{1}{q})}}\right)^{\frac{m+2(\frac{1}{s}-\frac{1}{q})}{2m}} \leq C\|A\|.$$

Very recently the previous results were generalized by the authors and by Dimant and Sevilla-Peris:

- ([1, Corollary 1.3], 2013) Let $1 \leq s \leq q \leq 2$ and $\mathbf{p} \in [1, +\infty]^m$ such that
$$\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right| \geq 0. \tag{1.3}$$
Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have
$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2(\frac{1}{s}-\frac{1}{q}-|\frac{1}{\mathbf{p}}|)}}\right)^{\frac{m+2(\frac{1}{s}-\frac{1}{q}-|\frac{1}{\mathbf{p}}|)}{2m}} \leq C\|A\|$$
and the exponent is optimal.
- (Dimant and Sevilla-Peris, [14, Proposition 4.4], 2013) Let $\mathbf{p} \in [1, +\infty]^m$ and $s, q \in 1, +\infty]$ be such that $s \leq q$. Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have
$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{\rho}\right)^{\frac{1}{\rho}} \leq C\|A\|,$$
where ρ is given by:

(i) If $s \leq q \leq 2$, and
(a) if $0 \leq |\frac{1}{\mathbf{p}}| < \frac{1}{s} - \frac{1}{q}$, then

$$\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\Big(\frac{1}{s} - \frac{1}{q} - \Big|\frac{1}{\mathbf{p}}\Big|\Big);$$

(b) if $\frac{1}{s} - \frac{1}{q} \leq |\frac{1}{\mathbf{p}}| < \frac{1}{2} + \frac{1}{s} - \frac{1}{q}$, then

$$\frac{1}{\rho} = \frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \Big|\frac{1}{\mathbf{p}}\Big|.$$

(ii) If $s \leq 2 \leq q$, and
(a) if $0 \leq |\frac{1}{\mathbf{p}}| < \frac{1}{s} - \frac{1}{2}$, then

$$\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\Big(\frac{1}{s} - \frac{1}{2} - \Big|\frac{1}{\mathbf{p}}\Big|\Big);$$

(b) if $\frac{1}{s} - \frac{1}{2} \leq |\frac{1}{\mathbf{p}}| < \frac{1}{s}$, then

$$\frac{1}{\rho} = \frac{1}{s} - \Big|\frac{1}{\mathbf{p}}\Big|.$$

(iii) If $2 \leq s \leq q$ and $0 \leq |\frac{1}{\mathbf{p}}| < \frac{1}{s}$, then

$$\frac{1}{\rho} = \frac{1}{s} - \Big|\frac{1}{\mathbf{p}}\Big|.$$

Moreover, the exponents in the cases (ia), (iib) and (iii) are optimal. Also, the exponent in (ib) is optimal for $\frac{1}{s} - \frac{1}{q} \leq |\frac{1}{\mathbf{p}}| < \frac{1}{2}$.

Our main intention, in this paper, is to improve the previous theorems in three directions.

(1) We study in depth the remaining cases of the Dimant and Sevilla-Peris result. Surprisingly, we show that in case (iia), the exponent given above is optimal whereas it is not optimal in case (ib) when $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$. We give a better exponent in that case and show a necessary condition on it. These two bounds coincide when $s = 1$. We can summarize this into the two following statements.

Theorem 1.1: *Let $\mathbf{p} \in [1, +\infty]^m$ and let $\rho > 0$. Assume moreover that either $q \geq 2$ or $q < 2$ and $|\frac{1}{\mathbf{p}}| < \frac{1}{2}$. Let*

$$\frac{1}{\lambda} := \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}} - \Big|\frac{1}{\mathbf{p}}\Big| > 0.$$

Then there exists $C > 0$ such that, for every continuous m-linear operator $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\rho} \right)^{\frac{1}{\rho}} \leq C\|A\|$$

if and only if

$$\frac{m}{\rho} \leq \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, s, 2\}}.$$

The following table summarizes the optimal value of $\frac{1}{\rho}$ following the respective values of $s, q, p_1, \ldots, p_m$:

$1 \leq s \leq q \leq 2,\ \lambda < 2$	$\frac{1}{2} + \frac{1}{ms} - \frac{1}{mq} - \frac{1}{m} \times \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq q \leq 2,\ \lambda \geq 2,\ \|\frac{1}{\mathbf{p}}\| < \frac{1}{2}$	$\frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq 2 \leq q,\ \lambda < 2$	$\frac{m-1}{2m} + \frac{1}{ms} - \frac{1}{m} \times \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq 2 \leq q,\ \lambda \geq 2$	$\frac{1}{s} - \left\|\frac{1}{\mathbf{p}}\right\|$
$2 \leq s \leq q$	$\frac{1}{s} - \left\|\frac{1}{\mathbf{p}}\right\|$

We note that (1.1) and (1.2) are recovered by Theorem 1.1 just by choosing $s = 1$ and $q = 2$.

When $q < 2$ and $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$ (observe that this automatically implies $\lambda \geq 2$), the situation is more difficult and we get the following statement.

THEOREM 1.2: *Let $\mathbf{p} \in [1, +\infty]^m$, $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$, $1 \leq s \leq q \leq 2$ and let $\rho > 0$. Let us consider the following property.*

There exists $C > 0$ such that, for every continuous m-linear operator $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\rho} \right)^{\frac{1}{\rho}} \leq C\|A\|.$$

(A) *The property is satisfied as soon as*

$$\frac{1}{\rho} \leq \frac{(\frac{1}{s}-\frac{1}{q})(\frac{1}{s}-|\frac{1}{\mathbf{p}}|)}{\frac{1}{2}-\frac{1}{s}}.$$

(B) *If the property is satisfied, then*

$$\frac{1}{\rho} \leq 2\Big(1-\Big|\frac{1}{\mathbf{p}}\Big|\Big)\Big(\frac{1}{s}-\frac{1}{q}\Big).$$

In particular, if $s = 1$, then the property is satisfied if and only if

$$\frac{1}{\rho} \leq 2\Big(1-|\frac{1}{\mathbf{p}}|\Big)\Big(1-\frac{1}{q}\Big).$$

(2) We give a simpler proof of the sufficient part of the Dimant and Sevilla-Peris theorem. It turns out that it is easier to prove a more general result.

Theorem 1.3: *Let $\mathbf{p} \in [1,+\infty]^m$ and $1 \leq s \leq q \leq \infty$ be such that*

$$\Big|\frac{1}{\mathbf{p}}\Big| < \frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}}.$$

Let

$$\frac{1}{\lambda} := \frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}}-\Big|\frac{1}{\mathbf{p}}\Big|.$$

If $\lambda > 0$ and $t_1,\ldots,t_m \in [\lambda, \max\{\lambda,s,2\}]$ are such that

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m} \leq \frac{1}{\lambda}+\frac{m-1}{\max\{\lambda,s,2\}}, \tag{1.4}$$

then there exists $C > 0$ satisfying, for every continuous m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$,

$$\left(\sum_{i_1=1}^{+\infty} \left(\cdots \left(\sum_{i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{t_m} \right)^{\frac{t_{m-1}}{t_m}} \cdots \right)^{\frac{t_1}{t_2}} \right)^{\frac{1}{t_1}} \leq C\|A\|. \tag{1.5}$$

Moreover, the exponents are optimal except eventually if $q \leq 2$ and $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$.

Remark 1.4: The optimality in the above theorem shall be understood in a strong sense: when $\lambda < 2$, we prove that if $t_1, \ldots, t_m \in [1, +\infty)$ are so that (1.5) holds, then (1.4) is valid. When $\lambda \geq 2$, note that $\lambda = \max\{\lambda, s, 2\}$ and we prove that if $t = t_1 = \cdots = t_m$ are in $[1, +\infty)$ and (1.5) is valid, then we have (1.4) and, as a direct consequence, $t \geq \lambda$.

(3) We prove similar results for m-linear mappings with arbitrary codomains which assume their cotype. For a Banach space X, let $q_X = \inf\{q \geq 2;\ X$ has cotype $q\}$.

The proof that (B) implies (A) in the theorem below appears in [14, Proposition 4.3].

Theorem 1.5: *Let $\mathbf{p} \in [2, +\infty]^m$, let X be an infinite-dimensional Banach space with cotype q_X, $|\frac{1}{\mathbf{p}}| < \frac{1}{q_X}$, and let $\rho > 0$. The following assertions are equivalent:*

(A) *Every bounded m-linear operator $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ is such that*

$$\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|^\rho < +\infty.$$

(B) $\frac{1}{\rho} \leq \frac{1}{q_X} - |\frac{1}{\mathbf{p}}|$.

Finally, in the last section of the paper we obtain better estimates for the constants of vector-valued Bohnenblust–Hille inequalities.

We conclude this introduction by noting that our theorems can be naturally stated in the context of homogeneous polynomials. Given an m-homogeneous polynomial $P : X \to Y$, we denote its coefficients $(c_\alpha(P))$. In [11, Lemma 5], it is shown that an inequality

$$\left(\sum_\alpha \|c_\alpha(P)\|^\rho \right)^{\frac{1}{\rho}} \leq C\|P\|$$

holds for every m-homogeneous polynomial $P : X \to Y$ if and only if a similar inequality

$$\left(\sum_{i_1,\ldots,i_m} \|T(e_{i_1}, \ldots, e_{i_m})\|^\rho \right)^{\frac{1}{\rho}} \leq C'\|T\|$$

is satisfied for every m-linear mapping $A : X \times \cdots \times X \to Y$, where X is a Banach sequence space.

Notations: For two positive integers n, k, we set

$$\mathcal{M}(k,n) := \{\mathbf{i} = (i_1, \dots, i_k); i_1, \dots, i_k \in \{1, \dots, n\}\}.$$

For $q \in [1, +\infty]$, q^* will denote its conjugate exponent.

2. Proof of Theorem 1.3 (sufficiency)

Let $1 \le q \le +\infty$. We recall that a Banach space X has **cotype** q if there is a constant $\kappa > 0$ such that, no matter how we select finitely many vectors $x_1, \dots, x_n \in X$,

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{\frac{1}{q}} \le \kappa \left(\int_I \left\|\sum_{k=1}^{n} r_k(t) x_k\right\|^2 dt\right)^{\frac{1}{2}}$$

where $I = [0,1]$ and r_k denotes the k-th Rademacher function. To cover the case $q = +\infty$, the left-hand side should be replaced by $\max_{1 \le k \le n} \|x_k\|$. The smallest of all these constants is denoted by $C_q(X)$ and named the cotype q constant of X.

An operator between Banach spaces $v : X \to Y$ is (r,s)-summing (with $s \le r \le +\infty$) if there exists $C > 0$ such that, for all $n \ge 1$ and for all vectors $x_1, \dots, x_n \in X$,

$$\left(\sum_{k=1}^{n} \|vx_k\|^r\right)^{\frac{1}{r}} \le C \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |x^*(x_k)|^s\right)^{\frac{1}{s}}.$$

The smallest constant in this inequality is denoted by $\pi_{r,s}(v)$.

We need a cotype q version of [1, Proposition 4.1], whose proof can be found in [14, Proposition 3.1]:

Lemma 2.1: *Let X be a Banach space, let Y be a cotype q space, let $r \in [1,q]$ and let $\mathbf{p} \in [1,+\infty]^m$ with*

$$\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{r} - \frac{1}{q}.$$

Define

$$\frac{1}{\lambda} := \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|.$$

Then, for every continuous m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ and every $(r,1)$-summing operator $v : X \to Y$, we have

$$\left(\sum_{i_k}\left(\sum_{\widehat{i_k}} \|vA(e_{i_1}, \dots, e_{i_m})\|_Y^q\right)^{\lambda/q}\right)^{1/\lambda} \le (\sqrt{2}C_q(Y))^{m-1}\pi_{r,1}(v)\|A\| \tag{2.1}$$

for all $k = 1, \dots, m$.

The symbol $\sum_{\widehat{i_k}}$ means that we are fixing the k-th index and that we are summing over all the remaining indices.

We shall deduce from this lemma the following theorem, which extends results of [1] and [14]:

THEOREM 2.2: *Let* $\mathbf{p} \in [1, +\infty]^m$, X *be a Banach space,* Y *be a cotype* q *space and* $1 \le r \le q$, *with* $|\frac{1}{\mathbf{p}}| < \frac{1}{r}$. *Define*

$$\frac{1}{\lambda} := \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|.$$

If $t_1, \dots, t_m \in [\lambda, \max\{\lambda, q\}]$ *are such that*

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} \le \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, q\}},$$

then, for every continuous m*-linear map* $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ *and every* $(r, 1)$*-summing operator* $v : X \to Y$, *we have*

$$\left(\sum_{i_1=1}^{+\infty} \left(\cdots \left(\sum_{i_m=1}^{+\infty} \|vA(e_{i_1}, \dots, e_{i_m})\|_Y^{t_m} \right)^{\frac{t_{m-1}}{t_m}} \cdots \right)^{\frac{t_1}{t_2}} \right)^{\frac{1}{t_1}} \le (\sqrt{2} C_{\max\{\lambda,q\}}(Y))^{m-1} \pi_{r,1}(v) \|A\|. \tag{2.2}$$

Proof. If $\lambda < q$, from Lemma 2.1, we have (2.2) for

$$(t_1, \dots, t_m) = (\lambda, q, \dots, q).$$

Since $\lambda < q$, the mixed (ℓ_λ, ℓ_q)-norm inequality (see [1, Proposition 3.1]), we also have (2.2) for the exponents

$$(t_1, \dots, t_m) = (q, \dots, q, \lambda, q, \dots, q)$$

with λ in the k-th position, for all $k = 1, \dots, m$. Now, using a general version of Hölder's inequality (see [15, Theorem 2.1]), we get (2.2) for all $(t_1, \dots, t_m) \in [\lambda, q]^m$ such that

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} = \frac{1}{\lambda} + \frac{m-1}{q} = \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, q\}}.$$

If $\lambda \ge q$, for any $\varepsilon > 0$, let $q_\varepsilon = \lambda + \varepsilon$. So $\lambda < q_\varepsilon$ and this automatically implies that

$$\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{r} - \frac{1}{q_\varepsilon}.$$

Since Y has cotype $q_\varepsilon > q$, we may apply Lemma 2.1 to get

$$\left(\sum_{i_1=1}^{N}\left(\sum_{i_2,\ldots,i_m=1}^{N}\|vA(e_{i_1},\ldots,e_{i_m})\|^{\lambda+\varepsilon}\right)^{\frac{\lambda}{\lambda+\varepsilon}}\right)^{\frac{1}{\lambda}}\leq(\sqrt{2}C_{\lambda+\varepsilon}(Y))^{m-1}\pi_{r,1}(v)\|A\|$$

for all positive integers N. Making $\varepsilon\to 0$, we get

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\|vA(e_{i_1},\ldots,e_{i_m})\|^{\lambda}\right)^{\frac{1}{\lambda}}\leq(\sqrt{2}C_{\lambda}(Y))^{m-1}\pi_{r,1}(v)\|A\|$$

for all N and the proof is done. ∎

Remark 2.3: If we take $t_1=\cdots=t_m$, then, upon polarization, we recover exactly [14, Theorem 1.2] with a much simpler proof due to the fact that the inequality is simpler to prove for the extremal values of $(t_1,\ldots,t_m)$.

We are now ready for the proof of the sufficient part of Theorem 1.3. We split the proof into three cases, and we combine Theorem 2.2 with the Bennett–Carl inequalities ([3, 9]): for $1\leq s\leq q\leq+\infty$, the inclusion map $\ell_s\hookrightarrow\ell_q$ is $(r,1)$-summing, where the optimal r is given by

$$\frac{1}{r}:=\frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{2,q\}}.$$

(i) $s\leq q\leq 2$: The Bennet–Carl inequalities ensure that the inclusion map $\ell_s\hookrightarrow\ell_q$ is $(r,1)$-summing with $\frac{1}{r}=\frac{1}{2}+\frac{1}{s}-\frac{1}{q}$, so the results follow from Theorem 2.2, with $t_1,\ldots,t_m$ satisfying

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m}=\frac{1}{2}+\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|+\frac{m-1}{\max\{\lambda,2\}}.$$

(ii) $s\leq 2\leq q$: Also by using Bennet–Carl inequalities, $\ell_s\hookrightarrow\ell_2$ is $(s,1)$-summing, thus we get (1.5) applying Theorem 2.2, with $t_1,\ldots,t_m$ satisfying

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m}=\frac{1}{s}-\left|\frac{1}{\mathbf{p}}\right|+\frac{m-1}{\max\{\lambda,2\}}.$$

(iii) $2\leq s\leq q$: Since $\ell_s\hookrightarrow\ell_s$ is $(s,1)$-summing, the result follows from Theorem 2.2, with $t_1=\cdots=t_m=\lambda$ and $\lambda\geq s$, since $r=s$ and

$$\frac{1}{\lambda}:=\frac{1}{s}-\left|\frac{1}{\mathbf{p}}\right|\leq\frac{1}{s}.$$

Remark 2.4: Let us set

$$c_{qs} := \begin{cases} q, & \text{if } s \le q \le 2, \\ 2, & \text{if } s \le 2 \le q, \\ s, & \text{if } 2 \le s \le q. \end{cases}$$

With the above notations, a careful look at the proof shows that the constant C which appears in Theorem 1.3 is dominated by

$$(\sqrt{2}C_{\max\{\lambda,s,2\}}(\ell_{c_{qs}}))^{m-1}\pi_{r,1}(\ell_s \hookrightarrow \ell_{c_{qs}}).$$

3. Proof of Theorem 1.3 (optimality)

In this section we show that the exponents in Theorem 1.3 are optimal except when $q \le 2$ and $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$. More precisely, if $(t_1, \dots, t_m) \in [1, +\infty)^m$ are such that there exists $C > 1$ satisfying, for any continuous multilinear map $X_{p_1} \times \cdots \times X_{p_m} \to X_s$,

$$\left(\sum_{i_1=1}^{+\infty}\left(\cdots\left(\sum_{i_m=1}^{+\infty}\|A(e_{i_1},\dots,e_{i_m})\|_{\ell_q}^{t_m}\right)^{\frac{t_{m-1}}{t_m}}\cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} \le C\|A\|, \tag{3.1}$$

then we prove that (1.4) holds. When $\lambda \ge 2$, we will always assume that $t_1 = \cdots = t_m = t$, since $\lambda = \max\{\lambda, s, 2\}$ and our inequality holds true when all the exponents are equal. We split the proof into several cases. Most of the cases are a consequence of a random construction. The main tool is the following lemma, from [1, Lemma 6.2].

Lemma 3.1: *Let $d, n \ge 1$, $q_1, \dots, q_{d+1} \in [1, +\infty]^{d+1}$ and let, for $q \ge 1$,*

$$\alpha(q) = \begin{cases} \frac{1}{2} - \frac{1}{q} & \textit{if } q \ge 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Then there exists a d-linear mapping $A : \ell_{p_1}^n \times \cdots \times \ell_{p_d}^n \to \ell_{p_{d+1}}^n$ which may be written

$$A(x^{(1)}, \dots, x^{(d)}) = \sum_{i_1,\dots,i_{d+1}=1}^{n} \pm x_{i_1}^{(1)} \cdots x_{i_d}^{(d)} e_{i_{d+1}}$$

such that

$$\|A\| \le C_d n^{\frac{1}{2}+\alpha(p_1)+\cdots+\alpha(p_d)+\alpha(p_{d+1}^*)}.$$

3.1. Case 1: $1 \le s \le q \le 2$ and $\lambda < 2$. This case has already been solved in [1, Section 6.2], using Lemma 3.1 with $d = m$ and $(q_1, \dots, q_{m+1}) = (p_1, \dots, p_m, s)$.

3.2. Case 2: $1 \le s \le q \le 2$, $\lambda \ge 2$ and $|\frac{1}{\mathbf{p}}| \le \frac{1}{2}$. This case has already been solved in [14, Proposition 4.4(ib)] using a Fourier matrix. We shall give an alternative probabilistic proof. Let $p \in [2, +\infty]$ be such that $\frac{1}{p} = |\frac{1}{\mathbf{p}}|$. By Lemma 3.1, there exists a linear map $T : \ell_p^n \to \ell_s^n$ which may be written $T(x) = \sum_{i,j} \varepsilon_{i,j} x_i e_j$ with $\varepsilon_{i,j} = \pm 1$ and such that

$$\|T\| \le C n^{\frac{1}{2} + \frac{1}{2} - \frac{1}{p} + \frac{1}{2} - \frac{1}{s^*}} = C n^{\frac{1}{2} + \frac{1}{s} - |\frac{1}{\mathbf{p}}|}.$$

Let $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \ell_s^n$ defined by

$$A(x^{(1)}, \dots, x^{(m)}) := \sum_{i,j} \varepsilon_{i,j} x_i^{(1)} \cdots x_i^{(m)} e_j.$$

By Hölder's inequality, it is plain that $\|A\| \le \|T\| \le C n^{\frac{1}{2} + \frac{1}{s} - |\frac{1}{\mathbf{p}}|}$. On the other hand, since $A(e_{i_1}, \dots, e_{i_m}) \ne 0$ if and only if $i_1 = \cdots = i_m$, and

$$\|A(e_i, \dots, e_i)\|_{\ell_q} = n^{1/q},$$

we have

$$\left(\sum_{\mathbf{i} \in \mathcal{M}(m,n)} \|A(e_{i_1}, \dots, e_{i_m})\|_{\ell_q}^t \right)^{\frac{1}{t}} = n^{\frac{1}{q} + \frac{1}{t}}.$$

This clearly implies

$$\frac{1}{t} \le \frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|.$$

3.3. Case 3: $1 \le s \le 2 \le q$ and $\lambda < 2$. Let $p \in [0, +\infty]$ be defined by

$$\frac{1}{p} = \frac{1}{p_m} + \frac{1}{s^*}.$$

Since $\lambda < 2$, it is easy to check that $p \ge 2$ and that $p_i \ge 2$ for any $i = 1, \dots, m$. We then apply Lemma 3.1 with $d = m-1$ and $(q_1, \dots, q_m) = (p_1, \dots, p_{m-1}, p^*)$. We get an $(m-1)$-linear form $T : \ell_{p_1}^n \times \cdots \times \ell_{p_{m-1}}^n \to \ell_{p^*}^n$ which can be written

$$T(x^{(1)}, \dots, x^{(m-1)}) = \sum_{i_1, \dots, i_m} \varepsilon_{i_1, \dots, i_m} x_{i_1}^{(1)} \cdots x_{i_{m-1}}^{(m-1)} e_{i_m}$$

and such that

$$\|T\| \le C n^{\frac{1}{2} + \frac{m}{2} - |\frac{1}{\mathbf{p}}| - \frac{1}{s^*}} = C n^{\frac{m-1}{2} - |\frac{1}{\mathbf{p}}| + \frac{1}{s}}.$$

We then define $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \ell_s^n$ by

$$A(x^{(1)}, \ldots, x^{(m)}) = \sum_{i_1,\ldots,i_m} \varepsilon_{i_1,\ldots,i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} e_{i_m}.$$

Then, for any $x^{(1)}, \ldots, x^{(m)} \in B_{\ell_{p_1}^n} \times \cdots \times B_{\ell_{p_m}^n}$,

$$\begin{aligned}\|A(x^{(1)}, \ldots, x^{(m)})\| &= \sup_{y \in B_{\ell_{s^*}^n}} \Big| \sum_{i_1,\ldots,i_m} \varepsilon_{i_1,\ldots,i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} y_{i_m} \Big| \\ &\leq \sup_{z \in B_{\ell_p^n}} \Big| \sum_{i_1,\ldots,i_m} \varepsilon_{i_1,\ldots,i_m} x_{i_1}^{(1)} \cdots x_{i_{m-1}}^{(m-1)} z_{i_m} \Big| \\ &\leq \|T\|.\end{aligned}$$

Moreover, given any $\mathbf{i} \in \mathcal{M}(m,n)$, $\|A(e_{i_1}, \ldots, e_{i_m})\|_q = \|e_{i_m}\|_q = 1$, so that

$$\left(\sum_{i_1=1}^{+\infty} \left(\cdots \left(\sum_{i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{t_m} \right)^{\frac{t_{m-1}}{t_m}} \cdots \right)^{\frac{t_1}{t_2}} \right)^{\frac{1}{t_1}} = n^{\frac{1}{t_1} + \cdots + \frac{1}{t_m}}.$$

Hence, provided (3.1) is satisfied, $(t_1, \ldots, t_m)$ has to satisfy

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} \leq \frac{m-1}{2} + \frac{1}{s} - \Big| \frac{1}{\mathbf{p}} \Big|.$$

3.4. Case 4 and Case 5: $1 \leq s \leq 2 \leq q$ and $\lambda \geq 2$, $2 \leq s \leq q$. These cases have a deterministic proof, as noted in [14, Proposition 4.4 (iib), (iii)], considering $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \ell_s^n$ given by

$$A(x^{(1)}, \ldots, x^{(m)}) := \sum_{i=1}^{n} x_i^{(1)} \cdots x_i^{(m)} e_i.$$

3.5. The proof of Theorem 1.1. From Theorem 1.3, by choosing $t_1 = \cdots = t_m$ we conclude that provided

$$\Big| \frac{1}{\mathbf{p}} \Big| < \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}},$$

the best exponent ρ in Theorem 1.1 satisfies

$$\frac{m}{\rho} = \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, s, 2\}}.$$

To conclude the proof, it remains to prove that, whenever

$$\Big| \frac{1}{\mathbf{p}} \Big| \geq \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}},$$

we cannot find an exponent $\rho > 0$ such that (1.1) is satisfied for all m-linear operators $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$. In fact, everything has already been done before: if $q \leq 2$, then we have just to follow the lines of Case 2, and if $q \geq 2$, then we may consider the m-linear mapping of Cases 4 and 5.

4. The case $1 \leq s \leq q \leq 2$, $\lambda \geq 2$ and $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$

4.1. A reformulation of the Hardy–Littlewood type inequalities. We shall improve in this section the bound given by Theorem 1.1. We shall proceed by interpolation. To do this, we need a reformulation of the result of this theorem, as Villanueva and Pérez-García reformulated the Bohnenblust–Hille inequality in [26]. The forthcoming result is a variant of [7, Proposition 2.2]; its proof will be omitted.

Theorem 4.1: *Let $1 \leq p_1, \ldots, p_m \leq +\infty$, $1 \leq s \leq q \leq \infty$ and let $\rho > 0$. The following assertions are equivalent.*

(A) *There exists $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have*

$$\left(\sum_{i_1,\ldots,i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{\rho} \right)^{1/\rho} \leq C\|A\|.$$

(B) *There exists $C > 0$ such that, for any $n \geq 1$, for any Banach spaces $Y_1, \ldots, Y_m$, for any continuous m-linear mapping $S : Y_1 \times \cdots \times Y_m \to X_s$, the induced operator*

$$\begin{aligned} T : \ell^n_{p_1^*,w}(Y_1) \times \cdots \times \ell^n_{p_m^*,w}(Y_m) \to & \ell_\rho^{n^m}(X_q) \\ (x^{(1)}, \ldots, x^{(m)}) \mapsto & (S(x^{(1)}_{i_1}, \ldots, x^{(m)}_{i_m}))_{\mathbf{i} \in \mathcal{M}(m,n)} \end{aligned}$$

satisfies $\|T\| \leq C\|S\|$.

We recall that, for any $p \in [1, +\infty]$ and any Banach space Y,

$$\ell^n_{p,w}(Y) = \left\{ (x_j)_{j=1}^n \subset Y;\ \|(x_j)\|_{w,p} := \sup_{\varphi \in B_{Y^*}} \left(\sum_{j=1}^n |\varphi(x_j)|^p \right)^{1/p} < +\infty \right\}$$

with the appropriate modifications for $p = \infty$.

4.2. Proof of the sufficient condition. We now prove our better upper bound in the case $1 \leq s \leq q \leq 2$, $|\frac{1}{\mathbf{p}}| > \frac{1}{2}$ (namely we prove the first part of Theorem 1.2). Let $n \geq 1$, let $Y_1, \dots, Y_m$ be Banach spaces and let $S : Y_1 \times \cdots \times Y_m \to X_s$ be bounded. Let $n \geq 1$ and let T be the operator induced by S on $\mathcal{Y} = \ell^n_{p_1^*,w}(Y_1) \times \cdots \times \ell_{p_m^*,w}(Y_m)$, defined by

$$T(x^{(1)}, \dots, x^{(m)}) = (S(x^{(1)}_{i_1}, \dots, x^{(m)}_{i_m})).$$

Then T is bounded as an operator from $\mathcal{Y}$ into $\ell^{n^m}_\infty(X_s)$ (this is trivial); T is also bounded as an operator from $\mathcal{Y}$ into $\ell^{n^m}_\rho(X_s)$ with $\frac{1}{\rho} = \frac{1}{s} - |\frac{1}{\mathbf{p}}|$ (this is Theorem 1.1 for $1 \leq s \leq 2$ and $q \geq 2$). We can interpolate between these two extreme situations. Hence, let $q \in [s, 2]$ and let $\theta \in [0, 1]$ be such that

$$\frac{1}{q} = \frac{1-\theta}{s} + \frac{\theta}{2} \iff \theta = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} - \frac{1}{2}}.$$

By [4, Theorem 4.4.1], T is bounded as an operator from $\mathcal{Y}$ into $\ell^{n^m}_t(X_q)$ where

$$\frac{1}{t} = \frac{1-\theta}{\infty} + \frac{\theta}{\rho} = \frac{(\frac{1}{s} - \frac{1}{q})(\frac{1}{s} - |\frac{1}{\mathbf{p}}|)}{\frac{1}{s} - \frac{1}{2}}.$$

Remark 4.2: It is easy to check that, for $1 \leq s \leq q \leq 2$ and $|\frac{1}{\mathbf{p}}| \geq \frac{1}{2}$, then the bound

$$\frac{(\frac{1}{s} - \frac{1}{q})(\frac{1}{s} - |\frac{1}{\mathbf{p}}|)}{\frac{1}{s} - \frac{1}{2}}$$

is always better (namely larger) than the bound $\frac{1}{2} + \frac{1}{s} - \frac{1}{q} - |\frac{1}{\mathbf{p}}|$ obtained in Theorem 1.1.

4.3. The necessary condition. We now prove the second part of Theorem 1.2. It also uses a probabilistic device for linear maps when the two spaces do not need to have the same dimension. The forthcoming lemma can be found in [3, Proposition 3.2].

Lemma 4.3: *Let $n, d \geq 1$, $1 \leq p, s \leq 2$. There exists $T : \ell^d_p \to \ell^n_s$, $T(x) = \sum_{i,j} \pm x_j e_i$ such that*

$$\|T\| \leq C_{p,s} \max(d^{1/s}, n^{1-\frac{1}{p}} d^{\frac{1}{s}-\frac{1}{2}}).$$

Coming back to the proof of Theorem 1.2, we first observe that we may always assume that $|\frac{1}{\mathbf{p}}| < 1$. Otherwise, we can consider the m-linear map

$A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$ defined by

$$A(x^{(1)}, \ldots, x^{(m)}) = \sum_{i \geq 1} x_i^{(1)} \ldots x_i^{(m)} e_0$$

and observe that it is bounded whereas it has infinitely many coefficients equal to 1. We then define $p \in [1,2]$ by $\frac{1}{p} = |\frac{1}{\mathbf{p}}|$ and we consider $T : \ell_p^d \to \ell_s^n$, $T(x) = \sum_{i,j} \varepsilon_{i,j} x_j e_i$ the map given by Lemma 4.3. We then define

$$\begin{aligned} A : \ell_{p_1}^d \times \cdots \times \ell_{p_m}^d &\to \ell_s^n \\ (x^{(1)}, \ldots, x^{(m)}) &\mapsto \sum_{i,j} \varepsilon_{i,j} x_j^{(1)} \cdots x_j^{(m)} e_i \end{aligned}$$

and we observe that, by Hölder's inequality, $\|A\| \leq \|T\|$. Furthermore,

$$\left(\sum_{i_1, \ldots, i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^t \right)^{1/t} = n^{1/t} d^{1/q}.$$

Taking $d^{1/2} = n^{1-\frac{1}{p}}$ (this is the optimal relation between d and n), we get that if

$$\left(\sum_{i_1, \ldots, i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^t \right)^{1/t} \leq C\|A\|,$$

then it is necessary that

$$\frac{1}{t} \leq 2\Big(1 - \frac{1}{p}\Big)\Big(\frac{1}{s} - \frac{1}{q}\Big).$$

Remark 4.4: This last condition is optimal when $s = 1$ or when $|\frac{1}{\mathbf{p}}| = \frac{1}{2}$ (with, in fact, the same proof as in Case 2 above). When $1 < s < 2$, another necessary condition is

$$\frac{1}{t} \leq \frac{1}{s} - \Big|\frac{1}{\mathbf{p}}\Big|$$

(see Case 4 or Case 5 above).

5. Optimal estimates under cotype assumptions

For a Banach space X, let $q_X := \inf\{q \geq 2;\ X \text{ has cotype } q\}$. For scalar-valued multilinear operators it is easy to observe that summability in multiple indexes behaves in a quite different way than summability in just one index.

For instance, for any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$,

$$\left(\sum_{i,j=1}^{+\infty} |A(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\|A\|$$

and the exponent $4/3$ is optimal. But, if we sum diagonally ($i = j$), the exponent $4/3$ can be reduced to 1 since

$$\sum_{i=1}^{+\infty} |A(e_i, e_i)| \leq \|A\|$$

for any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$. Now we prove Theorem 1.5 which shows that when replacing the scalar field by infinite-dimensional spaces the situation is quite different.

Proof. (A)⇒(B). From a deep result of Maurey and Pisier ([24] and [13, Section 14]), ℓ_{q_X} is finitely representable in X, which means that, for any $n \geq 1$, one may find unit vectors $z_1, \ldots, z_n \in X$ such that, for any $a_1, \ldots, a_n \in \mathbb{C}$,

$$\sum_{i=1}^{n} \|a_i z_i\|_X \leq 2\left(\sum_{i=1}^{n} |a_i|^{q_X}\right)^{1/q_X}.$$

We then consider the m-linear map $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to X$ defined by

$$A(x^{(1)}, \cdots, x^{(m)}) := \sum_{i=1}^{n} x_i^{(1)} \cdots x_i^{(m)} z_i.$$

Then, for any $(x^{(1)}, \ldots, x^{(m)})$ belonging to $B_{\ell_{p_1}^n} \times \cdots \times B_{\ell_{p_m}^n}$,

$$\begin{aligned}\|A(x^{(1)}, \cdots, x^{(m)})\| &\leq 2\left(\sum_{i=1}^{n} |x_i^{(1)}|^{q_X} \cdots |x_i^{(m)}|^{q_X}\right)^{1/q_X} \\ &\leq 2n^{\frac{1}{q_X} - \left|\frac{1}{\mathbf{p}}\right|},\end{aligned}$$

where the last inequality follows from Hölder's inequality applied to the exponents

$$\frac{p_1}{q_X}, \ldots, \frac{p_m}{q_X}, \left(1 - q_X\left|\frac{1}{\mathbf{p}}\right|\right)^{-1}.$$

On the other hand,

$$\left(\sum_{i=1}^{n} \|A(e_i, \ldots, e_i)\|^{\rho}\right)^{1/\rho} = n^{\frac{1}{\rho}}$$

and we obtain (B).

(B)⇒(A). This implication is proved in [14, Proposition 4.3]. ∎

If X does not have cotype q_X, the condition remains necessary. But now we just have the following sufficient condition:

$$\frac{m}{\rho} < \frac{1}{q_X} - \left|\frac{1}{\mathbf{p}}\right|.$$

Of course, it would be nice to determine what happens in this case. A look at [13, page 304] shows that the situation does not look simple.

As a consequence of the previous result we conclude that under certain circumstances the concepts of absolutely summing multilinear operator and multiple summing multilinear operator (see [8, 23, 25]) are precisely the same.

Corollary 5.1: *Let $p \in [2, +\infty]$, let X be an infinite dimensional Banach space with cotype $q_X < \frac{p}{m}$ and let $\rho > 0$. The following assertions are equivalent:*

(A) *Every bounded m-linear operator $A : X_p \times \cdots \times X_p \to X$ is absolutely $(\rho; p^*)$-summing.*
(B) *Every bounded m-linear operator $A : X_p \times \cdots \times X_p \to X$ is multiple $(\rho; p^*)$-summing.*
(C) $\frac{1}{\rho} \leq \frac{1}{q_X} - \frac{m}{p}$.

We stress the equivalence between (A) and (B) is not true, in general. For instance, every bounded bilinear operator $A : \ell_2 \times \ell_2 \to \ell_2$ is absolutely $(1; 1)$-summing but this is no longer true for multiple summability.

6. Constants of vector-valued Bohnenblust–Hille inequalities

A particular case of our main result is the following vector-valued Bohnenblust–Hille inequality (see [11, Lemma 3] and also [28, Section 2.2]):

Theorem 6.1: *Let X be a Banach space, Y a cotype q space and $v : X \to Y$ an $(r, 1)$-summing operator with $1 \leq r \leq q$. Then, for all m-linear operators $T : c_0 \times \cdots \times c_0 \to X$,*

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|vT(e_{i_1}, \ldots, e_{i_m})\|_Y^{\frac{qrm}{q+(m-1)r}} \right)^{\frac{q+(m-1)r}{qrm}} \leq C_{Y,m}\pi_{r,1}(v)\|T\|$$

with $C_{Y,m} = (\sqrt{2}C_q(Y))^{m-1}$.

In this section, in Theorem 6.2, we improve the above estimate for $C_{Y,m}$. The proof of Theorem 6.2 follows almost word by word the proof of [2, Proposition 3.1] using [10, Lemma 2.2] and Kahane's inequality instead of the Khinchine inequality. We present the proof for the sake of completeness. We need the following inequality due to Kahane:

KAHANE'S INEQUALITY: *Let $0 < p, q < +\infty$. Then there is a constant $K_{p,q} > 0$ for which*

$$\left(\int_I \| \sum_{k=1}^{n} r_k(t) x_k \, dt \|^q \right)^{\frac{1}{q}} \leq K_{p,q} \left(\int_I \left\| \sum_{k=1}^{n} r_k(t) x_k \, dt \right\|^p \right)^{\frac{1}{p}},$$

regardless of the choice of a Banach space X and of finitely many vectors $x_1, \ldots, x_n \in X$.

THEOREM 6.2: *For all m and all $1 \leq k < m$,*

$$C_{Y,m} \leq (C_q(Y) K_{\frac{qrk}{q+(k-1)r},2})^{m-k} C_{Y,k}.$$

Proof. Let $\rho := \frac{qrm}{q+(m-1)r}$ and to simplify notation let us write

$$vTe_{\mathbf{i}} = vT(e_{i_1}, \ldots, e_{i_m}).$$

Let us make use of [2, Remark 2.2] with $m \geq 2$, $1 \leq k \leq m-1$ and $s = \frac{qrk}{q+(k-1)r}$. So we have

$$\left(\sum_{\mathbf{i}} \|vTe_{\mathbf{i}}\|_Y^{\rho} \right)^{\frac{1}{\rho}} \leq \prod_{S \in P_k(m)} \left(\sum_{\mathbf{i}_S} \left(\sum_{\mathbf{i}_{\hat{S}}} \|vT(e_{\mathbf{i}_S}, e_{\mathbf{i}_{\hat{S}}})\|_Y^q \right)^{\frac{s}{q}} \right)^{\frac{1}{s\binom{m}{k}}}, \tag{6.1}$$

where $P_k(m)$ denotes the set of all subsets of $\{1, \ldots, m\}$ with cardinality k. For the sake of clarity, we shall assume that $S = \{1, \ldots, k\}$. By the multilinear cotype inequality (see [10, Lemma 2.2]) and the Kahane inequality, we have

$$\begin{aligned}
&\left(\sum_{\mathbf{i}_{\hat{S}}} \|vT(e_{\mathbf{i}_S}, e_{\mathbf{i}_{\hat{S}}})\|_Y^q \right)^{\frac{s}{q}} \\
\leq &(C_q(Y) K_{s,2})^{s(m-k)} \int_{I^{m-k}} \left\| \sum_{\mathbf{i}_{\hat{S}}} r_{\mathbf{i}_{\hat{S}}}(t_{\hat{S}}) vT(e_{\mathbf{i}_S}, e_{\mathbf{i}_{\hat{S}}}) \right\|_Y^s dt_{\hat{S}} \\
= &(C_q(Y) K_{s,2})^{s(m-k)} \int_{I^{m-k}} \left\| vT\left(e_{\mathbf{i}_S}, \sum_{\mathbf{i}_{\hat{S}}} r_{\mathbf{i}_{\hat{S}}}(t_{\hat{S}}) e_{\mathbf{i}_{\hat{S}}} \right) \right\|_Y^s dt_{\hat{S}} \\
= &(C_q(Y) K_{s,2})^{s(m-k)} \\
&\times \int_{I^{m-k}} \left\| v\left(T\left(e_{i_1}, \ldots, e_{i_k}, \sum_{i_{k+1}} r_{k+1}(t_{k+1}) e_{k+1}, \ldots, \sum_{i_m} r_m(t_m) e_m \right)\right) \right\|_Y^s dt_{k+1} \cdots dt_m.
\end{aligned}$$

But for a fixed choice of $(t_{k+1},\dots,t_m)\in I^{m-k}=[0,1]^{m-k}$, we know, by Theorem 6.1, that

$$\sum_{i_1,\dots,i_k}\left\|v\left(T\left(e_{i_1},\dots,e_{i_k},\sum_{i_{k+1}}r_{k+1}(t_{k+1})e_{k+1},\dots,\sum_{i_m}r_m(t_m)e_m\right)\right)\right\|_Y^s \leq (C_{Y,k}\pi_{r,1}(v)\|T\|)^s.$$

Thus

$$\begin{aligned}&\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat S}}\|vT(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat S}})\|_Y^q\right)^{\frac{s}{q}}\\ &\leq (C_q(Y)K_{s,2})^{s(m-k)}\\ &\quad\times\sum_{i_1,\dots,i_k}\left\|v\left(T\left(e_{i_1},\dots,e_{i_k},\sum_{i_{k+1}}r_{k+1}(t_{k+1})e_{k+1},\dots,\sum_{i_m}r_m(t_m)e_m\right)\right)\right\|_Y^s\\ &\leq ((C_q(Y)K_{s,2})^{m-k}\pi_{r,1}(v)C_{Y,k}\|T\|)^s,\end{aligned}\tag{6.2}$$

namely

$$\left(\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat S}}\|vT(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat S}})\|_Y^q\right)^{\frac{s}{q}}\right)^{\frac{1}{s}}\leq (C_q(Y)K_{s,2})^{m-k}\pi_{r,1}(v)C_{Y,k}\|T\|.$$

From (6.1) we conclude that

$$\left(\sum_{\mathbf{i}}\|vTe_{\mathbf{i}}\|_Y^{\rho}\right)^{\frac{1}{\rho}}\leq (C_q(Y)K_{s,2})^{m-k}C_{Y,k}\pi_{r,1}(v)\|T\|. \qquad \blacksquare$$

When m is even, the case $k=\frac{m}{2}$ recovers the constants from [28].

Corollary 6.3: *For all m,*

$$C_{Y,m}\leq C_q(Y)^{m-1}\prod_{k=1}^{m-1}K_{\frac{qrk}{q+(k-1)r},2}.$$

7. Other exponents

From now on $1\leq r\leq q$ and $(q_1,\dots,q_m)\in[r,q]^m$, so that

$$\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{q+(m-1)r}{qr}=\frac{1}{r}+\frac{m-1}{q}$$

are called vector-valued Bohnenblust–Hille exponents. From Theorem 2.2 we have:

THEOREM 7.1 (Multiple exponent vector-valued Bohnenblust–Hille inequality): *Let X be a Banach space and Y a cotype q space with $1 \leq r \leq q$. If $(q_1, \dots, q_m) \in [r, q]^m$ are vector-valued Bohnenblust–Hille exponents, then there exists $C_{Y,q_1,\dots,q_m} \geq 1$ such that, for all m-linear operators $T : c_0 \times \cdots \times c_0 \to X$ and every $(r, 1)$-summing operator $v : X \to Y$, we have*

$$\left(\sum_{i_1=1}^{+\infty} \cdots \left(\sum_{i_m=1}^{+\infty} \|vTe_{\mathbf{i}}\|_Y^{q_m}\right)^{\frac{q_{m-1}}{q_m}} \cdots\right)^{\frac{1}{q_1}} \leq C_{Y,q_1,\dots,q_m} \pi_{r,1}(v)\|T\|, \tag{7.1}$$

with $C_{Y,q_1,\dots,q_m} = (\sqrt{2}C_q(Y))^{m-1}$.

Our final result gives better estimates for the constants $C_{Y,q_1,\dots,q_m}$:

THEOREM 7.2: *If $(q_1, \dots, q_m)$ is a vector-valued Bohnenblust–Hille exponent and σ is a permutation of the indexes $\{1, \dots, m\}$ such that $q_{\sigma(1)} \leq \cdots \leq q_{\sigma(m)}$, then*

$$C_{Y,q_1\dots,q_m} \leq \prod_{k=1}^{m} \left((C_q(Y)K_{\frac{kqr}{q+(k-1)r},2})^{m-k} C_{Y,k}\right)^{\theta_k}$$

with

$$\theta_m = m\Big(\frac{1}{r} - \frac{1}{q}\Big)^{-1}\Big(\frac{1}{q_{\sigma(m)}} - \frac{1}{q}\Big) \tag{7.2}$$

and

$$\theta_k = k\Big(\frac{1}{r} - \frac{1}{q}\Big)^{-1}\Big(\frac{1}{q_{\sigma(k)}} - \frac{1}{q_{\sigma(k+1)}}\Big), \quad \text{for } k = 1, \dots, m-1. \tag{7.3}$$

Proof. In view of a consequence of Minkowski's inequality, which can be see in, e.g., [16, Corollary 5.4.2], we have that $C_{Y,q_1,\dots,q_m} \leq C_{Y,q_{\sigma(1)},\dots,q_{\sigma(m)}}$ Therefore, it suffices to prove the result for the exponent $(q_{\sigma(1)}, \dots, q_{\sigma(m)})$.

For each $k = 1, \dots, m$, define

$$s_k = \frac{kqr}{q + (k-1)r}.$$

From the proof of Theorem 6.2 we have (7.1) for each exponent

$$(s_k, \overset{k \text{ times}}{\dots}, s_k, q \dots, q).$$

More precisely, from (6.2) we have

$$\left(\sum_{i_1,\dots,i_k} \left(\sum_{i_{k+1},\dots,i_m} \|vTe_{\mathbf{i}}\|_Y^{q}\right)^{\frac{s_k}{q}}\right)^{\frac{1}{s_k}} \leq (C_q(Y)K_{s_k,2})^{m-k} C_{Y,k}\pi_{r,1}(v)\|T\|.$$

Consequently, for each $k = 1, \dots, m$ we have

$$C_{Y,s_k,\overset{k \text{ times}}{\dots},s_k,q\dots,q} \le (C_q(Y)K_{s_k,2})^{m-k} C_{Y,k}.$$

Since every vector-valued Bohnenblust–Hille exponent $(q_1, \dots, q_m)$ with $q_1 \le \dots \le q_m$ is obtained by interpolation of $\alpha_1, \dots, \alpha_m$ with

$$\alpha_k = (s_k, \overset{k \text{ times}}{\dots}, s_k, q \dots, q),$$

and $\theta_1, \dots, \theta_m$ as in (7.2) and (7.3), we conclude that

$$C_{Y,q_1,\dots,q_m} \le \prod_{k=1}^{m} (C_{Y,s_k,\overset{k \text{ times}}{\dots},s_k,q,\dots,q})^{\theta_k} \le \prod_{k=1}^{m} ((C_q(Y)K_{s_k,2})^{m-k} C_{Y,k})^{\theta_k}. \quad \blacksquare$$

A particular case of Kahane's inequality is Khintchine's inequality: if (ε_i) is a sequence of independent Rademacher variables, then, for any $p \in [1, 2]$, there exists a constant $\mathrm{A}_{\mathbb{R},p}$ such that, for any $n \ge 1$ and any $a_1 \dots, a_n \in \mathbb{R}$,

$$\left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} \le \mathrm{A}_{\mathbb{R},p}\left(\int_\Omega \left|\sum_{i=1}^{n} a_i \varepsilon(\omega)\right|^p d\omega\right)^{\frac{1}{p}}.$$

It has a complex counterpart: for any $p \in [1, 2]$, there exists a constant $\mathrm{A}_{\mathbb{C},p}$ such that, for any $n \ge 1$ and any $a_1 \dots, a_n \in \mathbb{C}$,

$$\left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} \le \mathrm{A}_{\mathbb{C},p}\left(\int_{\mathbb{T}^n} \left|\sum_{i=1}^{n} a_i z_i\right|^p dz\right)^{\frac{1}{p}}.$$

The best constants $\mathrm{A}_{\mathbb{R},p}$ and $\mathrm{A}_{\mathbb{C},p}$ are known (see [17] and [20]):

- $\mathrm{A}_{\mathbb{R},p} = \begin{cases} 2^{\frac{1}{p}-\frac{1}{2}}, & \text{if } 0 < p \le p_0 \approx 1.847, \\ \frac{1}{\sqrt{2}}\left(\frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}}\right)^{-\frac{1}{p}}, & \text{if } p > p_0; \end{cases}$
- $\mathrm{A}_{\mathbb{C},p} = \Gamma\left(\frac{1+p}{2}\right)^{-\frac{1}{p}}$, if $1 < p \le 2$.

Taking $X = Y = \mathbb{K}$ and $r = 1$ we obtain estimates for the constants of the scalar-valued Bohnenblust–Hille inequality with multiple exponents:

Corollary 7.3: *If* $(q_1, \dots, q_m) \in [1, 2]^m$ *fulfils*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{m+1}{2},$$

and σ is a permutation of the indexes $\{1,\dots,m\}$ such that $q_{\sigma(1)} \leq \cdots \leq q_{\sigma(m)}$, then

$$\left(\sum_{i_1=1}^{+\infty}\cdots\left(\sum_{i_m=1}^{+\infty}|T(e_{i_1},\dots,e_{i_m})|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{1}{q_1}}$$
$$\leq C_{\mathbb{K},m}^{2m\left(\frac{1}{q_{\sigma(m)}}-\frac{1}{2}\right)}\left(\prod_{k=1}^{m-1}\left(\mathrm{A}_{\mathbb{K},\frac{2k}{k+1}}^{m-k}C_{\mathbb{K},k}\right)^{2k\left(\frac{1}{q_{\sigma(k)}}-\frac{1}{q_{\sigma(k+1)}}\right)}\right)\|T\|$$

for all m-linear operators $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. In particular, for complex scalars, the left-hand side of the above inequality can be replaced by

$$\left(\prod_{j=1}^{m}\Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{2m\left(\frac{1}{q_{\sigma(m)}}-\frac{1}{2}\right)}$$
$$\times\left(\prod_{k=1}^{m-1}\left(\Gamma\left(\frac{3k+1}{2k+2}\right)^{\left(\frac{-k-1}{2k}\right)(m-k)}\prod_{j=1}^{k}\Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{2k\left(\frac{1}{q_{\sigma(k)}}-\frac{1}{q_{\sigma(k+1)}}\right)}\right)\|T\|.$$

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