GEOMETRY AND TOPOLOGY OF RANDOM 2-COMPLEXES

BY

A. E. COSTA AND M. FARBER

Queen Mary, University of London, London E1 4NS, UK
e-mail: A.Costa@qmul.ac.uk, M.Farber@qmul.ac.uk

ABSTRACT

We study random 2-dimensional complexes in the Linial–Meshulam model and prove that the fundamental group of a random 2-complex Y has cohomological dimension ≤ 2 if the probability parameter satisfies $p \ll n^{-3/5}$. Besides, for $n^{-3/5} \ll p \ll n^{-1/2-\epsilon}$ the fundamental group $\pi_1(Y)$ has elements of order two and is of infinite cohomological dimension. We also prove that for $p \ll n^{-1/2-\epsilon}$ the fundamental group of a random 2-complex has no m-torsion, for any given odd prime $m \geq 3$. We find a simple algorithmically testable criterion for a subcomplex of a random 2-complex to be aspherical; this implies that (for $p \ll n^{-1/2-\epsilon}$) any aspherical subcomplex of a random 2-complex satisfies the Whitehead conjecture. We use inequalities for Cheeger constants and systoles of simplicial surfaces to analyse spheres and projective planes lying in random 2-complexes. Our proofs exploit the uniform hyperbolicity property of random 2-complexes (Theorem 3.4).

1. Introduction and statements of the main results

A model producing random simplicial complexes was recently suggested and studied by Linial and Meshulam [13] and further by Meshulam and Wallach [14]; it is a high-dimensional generalisation of one of the Erdős and Rényi models of random graphs [7]. In the Linial–Meshulam model one generates a random d-dimensional complex Y by considering the full d-dimensional skeleton of the simplex Δ_n on vertices $\{1, \ldots, n\}$ and retaining d-dimensional faces independently

with probability p. The work of Linial–Meshulam and Meshulam–Wallach provides threshold functions for the vanishing of the $(d-1)$-st homology groups of random complexes with coefficients in a finite abelian group. Threshold functions for the vanishing of the d-th homology groups were subsequently studied by Kozlov [12].

Significant progress in understanding the topology of random 2-complexes was made by Babson, Hoffman and Kahle [2] who investigated the fundamental groups of random 2-complexes. They showed that the fundamental group of a random 2-complex is either nontrivial and Gromov hyperbolic (for $p \ll n^{-1/2-\epsilon}$) or trivial (if $p^2 n - 3\log n \to \infty$), a.a.s.[1]

In the paper [5] it was proven that a random 2-complex Y collapses to a graph if $p \ll n^{-1}$ and therefore its fundamental group is free.

The paper [1] suggests an explicit constant γ such that for any $c < \gamma$ a random $Y \in Y(n, c/n)$ is either collapsible to a graph or contains a tetrahedron.

It is important to have a model producing random aspherical 2-dimensional complexes Y. A connected simplicial complex Y is said to be **aspherical** if $\pi_i(Y) = 0$ for all $i \geq 2$; this is equivalent to the requirement that the universal cover of Y is contractible. For 2-dimensional complexes Y the asphericity is equivalent to the vanishing of the second homotopy group $\pi_2(Y) = 0$, or equivalently, that any continuous map $S^2 \to Y$ is homotopic to a constant map. Random aspherical 2-complexes could be helpful for testing probabilistically the open problems of two-dimensional topology, such as the Whitehead conjecture. This conjecture stated by J. H. C. Whitehead in 1941 claims that a subcomplex of an aspherical 2-complex is also aspherical. Surveys of results related to the Whitehead conjecture can be found in [3], [17].

The Linial–Meshulam model of random 2-complexes produces an aspherical complex Y if $p \ll n^{-1}$, when Y is homotopy equivalent to a graph, a.a.s.; however, for $p \gg n^{-1}$ a random 2-complex is not aspherical since it contains a tetrahedron as a subcomplex, a.a.s. In [6] the authors studied **asphericable** 2-complexes, which can be made aspherical by deleting a few faces belonging to tetrahedra. The result of [6] states that random 2-complexes in the Linial–Meshulam model are asphericable in the range $p \ll n^{-46/47}$.

---

[1] The symbol a.a.s. is an abbreviation of "asymptotically almost surely", which means that the probability that the corresponding statement is valid tends to 1 as $n \to \infty$.

In this paper we revisit the hyperbolicity theorem of Babson, Hoffman and Kahle [2] having in mind two principal goals: firstly, we simplify and make more transparent the main ideas of the proof; secondly we observe that the arguments of [2] give in fact a slightly stronger statement: a uniform lower bound on the isoperimetric constants of all subcomplexes of random complexes (see Theorem 3.4 below and its full proof in §6).

The key role in this paper is played by Theorem 2.5 which gives a topological classification of minimal cycles Z with the property $\mu(Z) > 1/2$.

Now we state the major results obtained in this paper.

THEOREM A (see Corollary 5.9 and Proposition 5.11): *If $p \ll n^{-3/5}$, then the fundamental group $\pi_1(Y)$ of a random 2-complex $Y \in Y(n,p)$ has cohomological dimension at most 2, a.a.s. In particular, $\pi_1(Y)$ is torsion free, a.a.s. Moreover, if $\frac{c}{n} < p \ll n^{-3/5}$, where $c > 3$, then $\mathrm{cd}(\pi_1(Y)) = 2$, a.a.s.*

The following theorem states that 2-torsion appears in the fundamental group $\pi_1(Y)$ once the probability parameter p crosses the threshold $n^{-3/5}$.

THEOREM B (see Theorem 4.5): *If for some $0 < \epsilon < 0.1$ the probability parameter p satisfies*

$$n^{-3/5} \ll p \ll n^{-1/2-\epsilon},$$

then the fundamental group $\pi_1(Y)$ of a random 2-complex $Y \in Y(n,p)$ has nontrivial elements of order two and consequently $\mathrm{cd}(\pi_1(Y)) = \infty$, a.a.s.

The next result complements Theorem B.

THEOREM C (see Theorem 4.6): *Let $m \geq 3$ be an odd prime. If for some $\epsilon > 0$ the probability parameter p satisfies $p \ll n^{-1/2-\epsilon}$, then, with probability tending to one as $n \to \infty$, a random 2-complex $Y \in Y(n,p)$ has the following property: the fundamental group $\pi_1(Y')$ of any subcomplex $Y' \subset Y$ has no m-torsion.*

The next statement describes aspherical subcomplexes of random 2-complexes. It is a strengthening of the main result of [6].

THEOREM D (see Theorem 5.5): *Assume that $p \ll n^{-1/2-\epsilon}$ for a fixed $\epsilon > 0$. Then a random 2-complex $Y \in Y(n,p)$ has the following property with probability tending to one as $n \to \infty$: any subcomplex $Y' \subset Y$ is aspherical if and only if it contains no subcomplexes $S \subset Y'$ with at most $2\epsilon^{-1}$ faces which*

are homeomorphic either to the sphere S^2, the projective plane $\mathbf{RP}^2$ or the complexes Z_2, Z_3 shown on Figure 3.

Theorem D implies the following result which can be viewed as probabilistic confirmation of The Whitehead Conjecture.

COROLLARY E (The Whitehead Conjecture): *Assume that $p \ll n^{-1/2-\epsilon}$ for a fixed $\epsilon > 0$. Then a random 2-complex $Y \in Y(n, p)$ has the following property with probability tending to one as $n \to \infty$: any aspherical subcomplex $Y' \subset Y$ satisfies The Whitehead Conjecture, i.e., any subcomplex $Y'' \subset Y'$ is also aspherical.*

Proposition 5.11 describes the second Betti number of random aspherical 2-complexes obtained from a random 2-complex $Y \in Y(n, p)$ by deleting a sequence of faces. It follows that these complexes have non-free fundamental groups.

The following corollary states that a random 2-complex in the range $p \ll n^{-1/2-\epsilon}$ contains no tori and no large spheres and projective planes.

COROLLARY F (see Corollaries 3.5, 3.6, 3.7): *Suppose that $p \ll n^{-1/2-\epsilon}$, where $\epsilon > 0$ is fixed. Then:*

(a) *A random 2-complex $Y \in Y(n, p)$ contains no subcomplex homeomorphic to the torus T^2, a.a.s.*

(b) *A random 2-complex $Y \in Y(n, p)$ contains no subcomplex with more than $2\epsilon^{-1}$ faces which is homeomorphic to the sphere S^2, a.a.s.*

(c) *A random 2-complex $Y \in Y(n, p)$ contains no subcomplex with more than ϵ^{-1} faces which is homeomorphic to the real projective plane $\mathbf{RP}^2$, a.a.s.*

Note that under the assumptions of Corollary F (as follows from the results of [5]) any triangulated sphere with $\leq 2\epsilon^{-1}$ faces is contained in a random 2-complex as a subcomplex and, besides, any triangulated real projective plane with $\leq \epsilon^{-1}$ faces is contained in a random 2-complex, a.a.s. Thus Corollary F lists all spheres and projective planes which cannot be found as subcomplexes of random complexes with probability tending to one.

We know that for $p = n^\alpha$ with $\alpha > -1/2$ a random 2-complex $Y \in Y(n, p)$ is simply connected and hence it has homotopy type of a wedge of 2-spheres; see [2]. Thus, a significant change in geometry and topology of random 2-complexes

happens around $p = n^{-1/2}$, and an outstanding challenge for the research field
is to analyse this event in more detail.

Figure 1 illustrates the behaviour of the cohomological dimension of the fundamental group $\pi_1(Y)$ of a random 2-complex $Y \in Y(n, p)$. For simplicity we assume here that $p = n^\alpha$ where α is a real parameter, $\alpha \in (-\infty, 0)$.
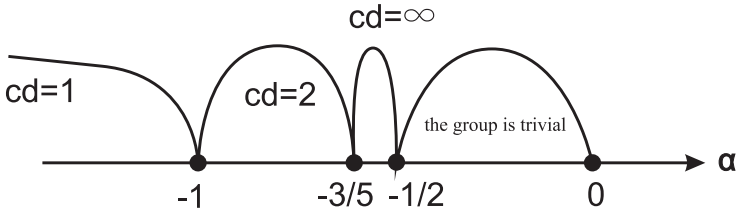

Figure 1. The cohomological dimension cd $=$ cd$(\pi_1(Y))$ of the fundamental group of a random 2-complex $Y \in Y(n, p)$ where $p = n^\alpha$.

The behaviour of the torsion in the fundamental group $\pi_1(Y)$ of a random 2-complex $Y \in Y(n, p)$ is illustrated by Figure 2. Again, we assume here that $p = n^\alpha$, where $\alpha \in (-\infty, 0)$. We also know that $\pi_1(Y)$ has no m-torsion, where m is an odd prime, for $p = n^\alpha$ with $\alpha \neq -1/2$.
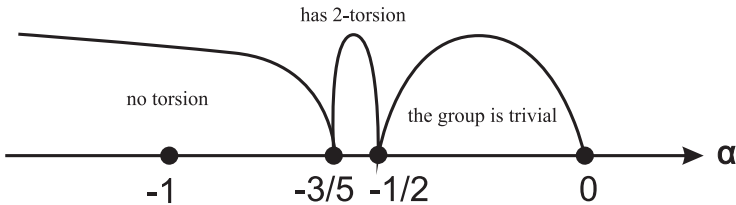

Figure 2. Torsion in the fundamental group of a random 2-complex $Y \in Y(n, p)$ where $p = n^\alpha$.

A few words about terminology we use in this paper. By a **2-complex** we understand a finite simplicial complex Z of dimension ≤ 2. The i-dimensional simplexes of a 2-complex are called **vertices** (for $i = 0$), **edges** (for $i = 1$) and **faces** (for $i = 2$). A 2-complex is said to be **pure** if any vertex and any edge are incident to a face. For an edge $e \subset Z$ we denote by deg $e = \deg_Z(e)$ the degree of e, i.e., the number of faces of Z containing e. The union of edges of degree

one is denoted ∂Z and is called the **boundary** of Z. We say that a 2-complex Z is **closed** if $\partial Z = \emptyset$.

We use the notations P^2 and $\mathbf{RP}^2$ for the real projective plane.

For a 2-complex X we denote by $\mu(X)$ the ratio v/f, where $v = v(X)$ is the number of vertices and $f = f(X)$ is the number of faces in X. The symbol $\tilde{\mu}(X)$ denotes

$$\tilde{\mu}(X) = \min \mu(S)$$

where $S \subset X$ runs over all subcomplexes. More information about the properties of the invariants $\mu(X)$ and $\tilde{\mu}(X)$ and their relevance to the containment problem for random complexes can be found in [2], [5]. For convenience of the reader we shall state here one of the results which will be used frequently in the paper; compare Theorem 15 in [5].

LEMMA 1.1: *Let $\mathcal{F}$ be a finite collection of finite simplicial complexes. Denote $\alpha = \min\{\tilde{\mu}(X), X \in \mathcal{F}\}$, and $\beta = \max\{\tilde{\mu}(X), X \in \mathcal{F}\}$. Let $A_n^{\mathcal{F}} \subset Y(n,p)$ denote the set of complexes $Y \in Y(n,p)$ which contain each of the complexes $X \in \mathcal{F}$ as a simplicial subcomplex. Let $B_n^{\mathcal{F}} \subset Y(n,p)$ denote the set of complexes $Y \in Y(n,p)$ which contain at least one of the complexes $X \in \mathcal{F}$ as a simplicial subcomplex. Clearly $A_n^{\mathcal{F}} \subset B_n^{\mathcal{F}}$. Then one has:*

(1) *If $p \ll n^{-\beta}$ then $\mathbb{P}(B_n^{\mathcal{F}}) \to 0$ as $n \to \infty$;*
(2) *If $p \gg n^{-\alpha}$ then $\mathbb{P}(A_n^{\mathcal{F}}) \to 1$ as $n \to \infty$.*

In other words, (1) states that for small p none of the complexes $X \in \mathcal{F}$ is embeddable into a random 2-complex, a.a.s.; statement (2) states that for large p any complex $X \in \mathcal{F}$ is embeddable into a random 2-complex, a.a.s.

2. Topology of minimal cycles

In this section we classify topologically the minimal cycles satisfying the condition $\mu(Z) > 1/2$; this result will be important for our study of aspherical subcomplexes and The Whitehead Conjecture in §5.

If $v \in Z$ is a vertex of a 2-complex Z, then the link of v is a graph; we denote by $\mathrm{Sing}(v) = \mathrm{Sing}_Z(v)$ the number of connected components of the link of v minus one.

Let Z be a finite pure 2-complex with the set of vertices $V(Z)$. The closure of a connected component of the complement $Z - V(Z)$ is called a **strongly connected component** of Z. A finite pure complex Z is called **strongly connected** if there is a unique strongly connected component. Note that if Z is path-connected and $\mathrm{Sing}(v) = 0$ for any vertex $v \in V(Z)$, then Z is strongly connected.

Given a pure 2-complex Z, there is a canonically defined 2-complex $\tilde{Z}$ with a natural surjective map $\tilde{Z} \to Z$, which is 1-1 on $\tilde{Z} - V(\tilde{Z})$, such that $\mathrm{Sing}_{\tilde{Z}}(w) = 0$ for any vertex w of $\tilde{Z}$. The complex $\tilde{Z}$ is obtained by multiplying every vertex of Z as many times as there are connected components in the link of this vertex (in other words, we cut open any vertex v with $\mathrm{Sing}(v) > 0$ along the cut point). Clearly, $\tilde{Z}$ is strongly connected if Z is strongly connected. In this case one observes that Z is homotopy equivalent to the wedge sum $\tilde{Z} \vee S^1 \vee S^1 \vee \cdots \vee S^1$, where the number of S^1 summands equals $\sum_{v \in V(Z)} \mathrm{Sing}(v)$. This gives the following corollary:

COROLLARY 2.1: *For any pure strongly connected 2-complex Z one has*

$$b_1(Z) = b_1(\tilde{Z}) + \sum_{v \in V(Z)} \mathrm{Sing}(v) \geq \sum_{v \in V(Z)} \mathrm{Sing}(v).$$

In particular, if Z has at least one vertex with $\mathrm{Sing}(v) \geq 1$ then $b_1(Z) \geq 1$.

Definition 2.2: A finite pure 2-complex Z is said to be a **minimal cycle** if $b_2(Z) = 1$ and for any proper subcomplex $Z' \subset Z$ one has $b_2(Z') = 0$.

Clearly a minimal cycle must be closed (i.e., $\partial Z = \emptyset$) and strongly connected. Indeed, if $Z_1, \ldots, Z_k$ are strongly connected components of Z, $k \geq 2$, then $b_2(Z) = \sum b_2(Z_i)$ and hence exactly one component Z_i has the second Betti number 1. Removing a face from another component Z_j, $j \neq i$, will not affect the second Betti number, which gives a contradiction.

The following easy statement will be useful later.

LEMMA 2.3: *Let $\phi : Z_1 \to Z_2$ be a simplicial map between finite pure 2-complexes such that $b_2(Z_1) \geq 1$ and Z_2 is a minimal cycle. Suppose that ϕ maps bijectively the set of faces of Z_1 onto the set of faces of Z_2 (and, in*

particular, faces of Z_1 do not degenerate under ϕ). Then Z_1 is also a minimal cycle; in particular, Z_1 is strongly connected.

Proof. Assuming that Z_1 is not a minimal cycle we may find a proper subcomplex $Z_1' \subset Z_1$ with $b_2(Z_1') = 1$. Then our assumptions imply that the subcomplex $\phi(Z_1') \subset Z_2$ is proper and carries a nontrivial 2-cycle, i.e., $b_2(\phi(Z_1')) \geq 1$ contradicting the minimality of Z_2. ∎

LEMMA 2.4: *Let Z be a minimal cycle and let $D \subset Z$ be a subcomplex homeomorphic to a 2-disc such that the interior Int(D) is open in Z. Denote by $Z' = Z - \text{Int}(D)$ the result of deleting the interior of D. Then the 2-complex Z' is strongly connected.*

Proof. Assume the contrary, i.e., let Z' have at least two strongly connected components $Z_1', Z_2', \ldots$. Then the quotient Z/D is homotopy equivalent to the wedge sum of the spaces $Z_i'/(Z_i' \cap D)$ (where $i = 1, 2, \ldots$) and some number of circles. It is obvious that in the case $Z_i' \cap D = \partial D$ for some i the complex Z' is strongly connected. Thus we may assume that each intersection $Z_i' \cap D$ is a proper subcomplex of the circle ∂D, i.e., $Z_i' \cap D$ is homeomorphic to a disjoint union of intervals. Hence the quotient $Z_i'/(Z_i' \cap D)$ is homotopy equivalent to the wedge sum of Z_i' with several copies of S^1 which gives $b_2(Z_i'/(Z_i' \cap D)) = b_2(Z_i')$. Thus we have

$$(1) \qquad b_2(Z) = b_2(Z/D) = \sum_i b_2(Z_i'/(Z_i' \cap D)) = \sum_i b_2(Z_i').$$

From (1) it follows that among the numbers $b_2(Z_i')$ exactly one equals 1 and the others vanish. This contradicts the assumption that Z is a minimal cycle since it contains a proper subcomplex $Z_i' \subset Z$ with $b_2(Z_i') = 1$. ∎

THEOREM 2.5 (Classification of minimal cycles): *Let Z be a minimal cycle satisfying*

$$\mu(Z) > 1/2.$$

Then Z is homeomorphic to one of the following 2-complexes Z_1, Z_2, Z_3, Z_4, where $Z_1 = S^2$; Z_2 is the quotient of S^2 with two distinct points identified; Z_3 is the quotient of S^2 with two adjacent arcs identified; and Z_4 is defined as $P^2 \cup \Delta^2$, where $\partial \Delta^2 = P^2 \cap \Delta^2$ equals the curve $P^1 \subset P^2$.
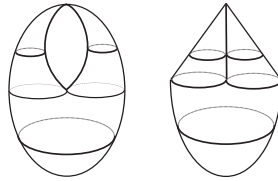
Figure 3. Minimal cycles Z_2 (left) and Z_3 (right).

Proof. Recall that for an edge $e \subset Z$ one denotes by $\deg e$ the degree of e, i.e., the number of faces containing e. Using the following formula

$$(2) \qquad \mu(Z) = \frac{1}{2} + \frac{2\chi(Z) + L(Z)}{2f(Z)},$$

where

$$L(X) = \sum_e (2 - \deg(e)),$$

the sum being taken over all edges e of X (see (7), (8) in [6]), we see that the condition $\mu(Z) > 1/2$ translates into

$$(3) \qquad\qquad\qquad 2\chi(Z) + L(Z) > 0.$$

Note that $L(Z) \leq 0$ since Z is pure and closed. We consider now a few special cases:

CASE A: $L(Z) = 0$. Then Z is a pseudo-surface[2], i.e., each edge has degree 2. The link of every vertex is a disjoint union of several circles. In other words, Z can be obtained from a connected surface $\tilde{Z}$ by identifying several vertices. By (3) one has $\chi(Z) > 0$ and hence Z is either the sphere Z_1 or the sphere with two points identified Z_2. Hence we see that in the case A the complex Z is homeomorphic either to Z_1 or Z_2.

CASE B: $L(Z) = -1$. Let us show that this is impossible. Indeed, in this case all edges have degree 2 except one edge which has degree 3. If e is this edge of degree 3, then for one of the incident vertices v one considers the link of v; it is a graph in which all vertices except one have degree 2 and one vertex has degree 3. This is impossible since the sum of degrees of all vertices in a graph is always even (as it equals twice the number of edges of the graph).

---

[2] A **pseudo-surface** is a 2-complex which is strongly connected and has the property that each edge is incident to exactly 2 faces.

If $b_1(Z) \geq 1$, then $\chi(Z) \leq 1$ and (3) gives $L(Z) \geq -1$, i.e., $L(Z) = 0$ or $L(Z) = -1$. Thus for minimal cycles satisfying $b_1(Z) \geq 1$ we are in one of the cases A or B considered above. We shall assume below that $b_1(Z) = 0$.

By Corollary 2.1 the condition $b_1(Z) = 0$ implies that the link of every vertex of Z is connected.

The assumptions $b_2(Z) = 1$ and $b_0(Z) = 1$ imply $\chi(Z) = 2$ and hence $\mu(Z) > 1/2$ is equivalent to $L(Z) \geq -3$. Thus, we have to consider the following four special cases $L(Z) = 0, -1, -2, -3$ with the two first analysed above.

CASE C: $L(Z) = -2$. There are two possibilities: either Z contains two edges of degree 3 or one edge of degree 4. Consider the first possibility: let e, e' be two edges of degree 3 and let v be a vertex incident to e but not to e'. Then the link of v is a graph with all vertices of degree 2 except one which has degree 3, which is impossible.
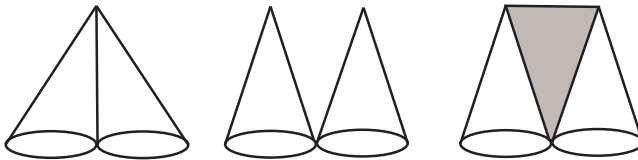

Figure 4. Cones.

Now suppose that Z contains an edge e of degree 4 and all other edges of Z have degree 2. Denote by v, w the endpoints of e. The links of v and w are connected graphs with one vertex of degree 4 and all other vertices of degree 2. Hence these links are homeomorphic to the figure eight Γ_8 (the wedge sum of two circles). The regular neighbourhood of v is the star of v which is homeomorphic to the cone over Γ_8. Replacing the cone $C(\Gamma_8)$ by the wedge of two cones over the base circles (see Figure 4, middle) gives a pseudo-surface Z' on which the original edge e of degree 4 is represented by two adjacent edges e_1, e_2 of degree 2. Clearly Z' is homotopy equivalent to Z: indeed, attaching a triangle to Z' as shown on Figure 4 (right) gives a homotopy equivalent space as the triangle can be collapsed along the free edge; on the other hand, collapsing the whole triangle gives a space homotopy equivalent to Z. By Lemma 2.3 the complex Z' is a minimal cycle and therefore it is strongly connected. Thus, Z' is a pseudo-surface with $b_1(Z') = 0$ and $b_2(Z') = 1$. Using Corollary 2.1 we see that Z' is nonsingular and hence is homeomorphic to S^2. The complex Z is obtained

from this sphere by folding two adjacent edges e_1, e_2. Thus, Z is homeomorphic to Z_3.

CASE D: $L(Z) = -3$. There are three possibilities: either (1) Z contains one edge e of degree 5 and all other edges have degree 2, or (2) there are two edges e_1, e_2 with $\deg(e_1) = 3$, and $\deg(e_2) = 4$ and all other edges have degree 2, or (3) there are 3 edges e_i with $\deg(e_i) = 3$ for $i = 1, 2, 3$ and all other edges of Z have degree 2. In the case (1), if v is a vertex incident to e, then the link of v is a graph with one vertex of degree 5 and all other vertices of degree 2 which is impossible. Similarly, the case (2) is impossible; one may apply these arguments to the vertex of e_1 which is not incident to e_2. The same reasoning shows that in the case (3) the only possibility is that the edges e_1, e_2, e_3 form a triangle.

Below we assume that the edges of degree three e_1, e_2, e_3 form a triangle with vertices v_1, v_2, v_3, i.e., the edge e_1 has vertices v_1, v_2, the edge e_2 has vertices v_2, v_3 and e_3 has vertices v_3, v_1.

Consider the link of v_i in complex Z. It is a graph with two vertices of degree three and with all other vertices of degree 2. Hence the link is homeomorphic to one of the graphs Γ_1, Γ_2 shown on Figure 5.
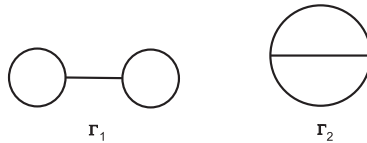

Figure 5. Possible links of v_i.

Let us show that the graph Γ_1 is in fact impossible. Indeed, suppose that the link of v_i is homeomorphic to Γ_1. The arc connecting two triple points of Γ_1 corresponds to a sequence of 2-faces $\sigma_1, \ldots, \sigma_r$ such that $\sigma_j \cap \sigma_{j+1}$ is an edge containing v_i. The union $\sigma = \sigma_1 \cup \cdots \cup \sigma_r$ is a subcomplex homeomorphic to a disc. Consider $Z' = Z - \mathrm{Int}(\sigma)$. By Lemma 2.4, the complex Z' is strongly connected. We have $\chi(Z') = 1$ and $b_2(Z') = 0$ and hence $b_1(Z') = 0$. This contradicts Corollary 2.1 since the link of v_i in Z' has two connected components.

Suppose that there is a 2-simplex σ in Z having the vertices v_1, v_2, v_3. Then removing the interior of this simplex we obtain a pseudo-surface Z' (i.e., $L(Z') = 0$) with $b_1(Z') = b_2(Z') = 0$. By Lemma 2.4, the complex Z' must

be strongly connected. Then $\text{Sing}(v) = 0$ for every vertex $v \in Z'$, since otherwise $b_1(Z') > 0$; see Corollary 2.1. Hence Z' is a genuine closed surface and since we know the Betti numbers of Z', clearly Z' must be homeomorphic to the real projective plane. Thus Z is homeomorphic to the union of P^2 and a triangle Δ^2. The intersection $P^2 \cap \Delta^2$ is a simple closed curve. If this curve is null-homologous on P^2, then this curve must separate P^2 into two connected components one of which must be homeomorphic to a 2-disc and together with Δ^2 this disc would form a proper 2-cycle in Z contradicting the minimality of Z (see Definition 2.2). Hence, the curve $P^2 \cap \Delta^2$ is homologous to $P^1 \subset P^2$. Therefore Z is homeomorphic to Z_4.

Next we consider the case that there is no 2-simplex with sides e_1, e_2, e_3. Let us label the three 2-faces incident to e_1 by the symbols $1, 2, 3$. In the link of v_2 the edge e_1 corresponds to one of the vertices of degree three and the symbols $1, 2, 3$ are now associated with the three arcs (see the graph Γ_2 on Figure 5) incident to this vertex. Viewing the other triple point of the link of v_2 we obtain a labelling by the symbols $1, 2, 3$ of the three triangles incident to the edge e_1. Thus we see that a labelling by the symbols $1, 2, 3$ of the three triangles containing an edge e_i determines a labelling of the following edge e_{i+1}. Performing this process in the following sequence $e_1 \to e_2 \to e_3 \to e_1$ we obtain a permutation τ of the symbols $1, 2, 3$. We denote by $k (= 1, 2, 3)$ the number of orbits of the action of τ on $\{1, 2, 3\}$.

CASE D1: Suppose that $k = 1$. Let $c = \sum_j n_j \sigma_j$ be a nontrivial 2-cycle of Z with integral coefficients n_j; here the symbols σ_i denote distinct oriented simplexes of Z. Due to our minimality assumption, $n_i \neq 0$ for any i. If two simplexes σ_i and $\sigma_{i'}$ have a common edge of degree two, then $n_i = \pm n_{i'}$. The complement $Z - E$ (where E is the union of the triple edges e_1, e_2, e_3) is strongly connected since Z is strongly connected and $k = 1$, i.e., one may pass from any face incident to E to any other face incident to E by jumping across edges not in E. Thus, any two 2-simplexes σ, σ' of Z can be connected by a chain $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_r = \sigma'$ such that each σ_i and σ_{i+1} has a common edge of degree two. Hence we obtain that $n_i = \pm n_j$ for any i, j. This leads to a contradiction with $\partial c = 0$, since near an edge e_r of degree three we will have three 2-simplexes, each contribution the same amount, up to a sign. Thus, Case D1 is impossible.

CASE D2: Suppose that $k = 2$ or $k = 3$. Then near the union of the triple edges E the complex Z is homeomorphic either to $Y \times S^1$ (where Y is the

graph representing the letter Y) or to the union of the Möbius band M and the cylinder $N = [0, 1] \times S^1$ such that $M \cap N = 0 \times S^1$ is the central circle of the Möbius band. In either case we may cut Z along the triple circle E, separating a cylinder from the rest, such that the result is a surface $\hat{Z}$ with boundary. Points of E will be represented on $\hat{Z}$ by two circles, one being the base of the cylindrical part and another the middle circle of a cylinder or a Möbius band. If $\hat{Z}$ is connected, then it is homotopy equivalent to a graph and hence $\chi(\hat{Z}) = \chi(Z) \le 1$. Since $L(Z) = -3$, we see that (3) is violated. Therefore the surface $\hat{Z}$ must have two connected components $\hat{Z}_1 \sqcup \hat{Z}_2$, one of which (say, $\hat{Z}_1$) is a closed surface, while the other $\hat{Z}_2$ has one boundary circle. The minimality condition implies that the closed surface $\hat{Z}_1$ must be non-orientable. Thus we have $2 = \chi(Z) = \chi(\hat{Z}) = \chi(\hat{Z}_1) + \chi(\hat{Z}_2)$ and $\chi(\hat{Z}_1) \le 1$, $\chi(\hat{Z}_2) \le 1$ which implies that $\chi(\hat{Z}_1) = 1 = \chi(\hat{Z}_2)$. Therefore, $\hat{Z}_1$ is the real projective plane and $\hat{Z}_2$ is a disk. We obtain that Z is the union of the projective plane and a disk intersecting along the triangle E. Clearly E is not null-homologous in $\hat{Z}_1$ (since otherwise it would cut the projective plane into two connected components one of which would be orientable and together with $\hat{Z}_2$ it would make a cycle, contradicting minimality of Z).

Clearly the pair $(\hat{Z}_1, E)$ is homeomorphic to (P^2, P^1), and Z is homeomorphic to Z_4.

This completes the proof.   ∎

COROLLARY 2.6: *In any minimal cycle Z with $\mu(Z) > 1/2$ there exists a 2-face σ such that the boundary curve $\partial\sigma$ is null-homotopic in the complement $Z - \mathrm{Int}(\sigma)$.*

LEMMA 2.7: *Let X be a finite closed strongly connected pure 2-complex with $b_2(X) = 0$. Then either $\mu(X) \le 1/2$ or X is homeomorphic to the projective plane $\mathbf{RP}^2$.*

Proof. Assume that $\mu(X) > 1/2$; this is equivalent (using formula (2)) to

$$(4) \qquad \mathcal{L}(X) \equiv 2\chi(X) + L(X) = 2 - 2b_1(X) + L(X) > 0.$$

Besides, we have $L(X) \le 0$, since X is closed. The inequality (4) implies that $L(X)$ can be either -1 or 0. In the proof of Theorem 2.5 (see Case B) we showed that $L(X) = -1$ is impossible. Hence, $L(X) = 0$, i.e., X is a pseudo-surface. Inequality (4) now implies that $b_1(X) = 0$. Using Corollary 2.1 we obtain that

X is a genuine surface without singularities. Thus X is a compact nonorientable surface with $b_1(X) = 0$, i.e., X is homeomorphic to the projective plane. ∎

We shall say that a 2-complex X is an **iterated wedge of projective planes** if it can be represented as a union of subcomplexes $X = \bigcup_{i=1}^{n} A_i$ where each A_j is homeomorphic to a projective plane and the intersection

$$A_j \cap \left(\bigcup_{i=1}^{j-1} A_i \right)$$

is a single point set for $j = 1, \dots, n$.

LEMMA 2.8: *Let X be a finite pure connected 2-complex with $b_1(X) = 0$. Then X is an iterated wedge of its strongly connected components.*

Proof. We shall use the following remark. Let $A \cup B = X$ be two connected subcomplexes which are unions of strongly connected components of X and such that every strongly connected component of X is contained either in A or in B but not in both A and B. Then the intersection $A \cap B$ is a finite set of vertices and using $b_1(X) = 0$ in the Mayer–Vietoris sequence implies that $\tilde{H}_0(A \cap B) = 0$, i.e., $A \cap B$ is a single point.

Let A_1 be a strongly connected component of X. Using the connectedness of X and the previous remark, we may find another strongly connected component A_2 of X such that $A_1 \cap A_2$ is a single point. Inductively, we may find a sequence $A_1, \dots, A_r$ of strongly connected components of X such that $X = A_1 \cup \cdots \cup A_r$ and for any j the intersection

$$A_j \cap \left(\bigcup_{i=1}^{j-1} A_i \right)$$

is a single point. Hence X is an iterated wedge of its strongly connected components. ∎

A version of Lemma 2.7 with the words "strongly connected" replaced by "connected" reads as follows.

LEMMA 2.9: *Let X be a finite closed connected pure 2-complex with $b_2(X) = 0$ and $\mu(X) > 1/2$. Then X is homeomorphic to an iterated wedge of projective planes.*

Proof. We will use induction on the number of strongly connected components k of X. Lemma 2.7 covers the case $k = 1$. Assume now that Lemma 2.9 has been proven for all complexes X with less than k strongly connected components.

If X has k strongly connected components and $b_2(X) = 0$, $\mu(X) > 1/2$, then $b_1(X) = 0$ (as follows from the inequality (3) combined with $L(X) \leq 0$). By Lemma 2.8, X can be represented as a wedge (one-point-union) $X = A \vee B$ with A strongly connected and B having $k-1$ strongly connected components. The complexes A and B are pure, closed, connected and $b_2(A) = 0 = b_2(B)$. One has $L(X) = L(A) + L(B)$ and $\chi(X) = \chi(A) + \chi(B) - 1$ which imply (using the notation introduced in the proof of Lemma 2.7)

$$\mathcal{L}(X) = \mathcal{L}(A) + \mathcal{L}(B) - 2.$$

We know by the induction hypothesis that either $\mathcal{L}(A) \leq 0$, or A is homeomorphic to the projective plane and then $\mathcal{L}(A) = 2$. Besides, either $\mathcal{L}(B) \leq 0$ or B is an iterated wedge of projective planes and then $\mathcal{L}(B) = 2$. Thus, we see that either $\mathcal{L}(X) \leq 0$ (which is equivalent to $\mu(X) \leq 1/2$ and contradicts our assumption $\mu(X) > 1/2$) or $\mathcal{L}(X) = 2$ and X is an iterated wedge of projective planes. ∎

COROLLARY 2.10 (compare with Lemma 4.1 from [2]): *Let Z be a connected 2-complex with $\tilde{\mu}(Z) > 1/2$. Then Z is homotopy equivalent to a wedge of circles, spheres and projective planes.*

Proof. Without loss of generality we may additionally assume that Z is strongly connected, since otherwise Z is homotopy equivalent to a wedge sum of its strongly connected components and a number of circles. Besides, we may assume that Z is closed, $\partial Z = \emptyset$ (otherwise, we may perform a sequence of collapses to obtain a closed subcomplex $Z' \subset Z$ having the same homotopy type). Similarly, without loss of generality we may assume that Z is pure, since otherwise we can apply the arguments given below to its pure part.

We shall act by induction on $b_2(Z)$.

If $b_2(Z) = 0$ and $\tilde{\mu}(Z) > 1/2$, then using Lemma 2.9 we see that the complex Z is homotopy equivalent to a wedge of circles and projective planes.

Assume that Corollary 2.10 was proven for all connected 2-complexes Z satisfying $\tilde{\mu}(Z) > 1/2$ and $b_2(Z) < k$.

Consider a 2-complex Z satisfying $b_2(Z) = k > 0$ and $\tilde{\mu}(Z) > 1/2$. Find a minimal cycle $Z' \subset Z$. Then the homomorphism $H_2(Z'; \mathbf{Z}) = \mathbf{Z} \to H_2(Z; \mathbf{Z})$

is an injection. Let $\sigma \subset Z'$ be a simplex given by Corollary 2.6. Then $Z'' = Z - \mathrm{Int}(\sigma)$ satisfies $b_2(Z'') = k - 1$. Indeed, in the exact sequence

$$0 \to H_2(Z'') \to H_2(Z) \to H_2(Z, Z'') (= \mathbf{Z}) \xrightarrow{\partial_*} H_1(Z'') \to \cdots$$

the homomorphism $\partial_* = 0$ is trivial since the curve $\partial\sigma$ is null-homotopic in Z''. Since $\tilde{\mu}(Z'') > 1/2$, by induction hypothesis Z'' is homotopy equivalent to a wedge of spheres, circles and projective planes. Then $Z \simeq Z'' \vee S^2$ also has this property. ∎

3. Isoperimetric constants of simplicial complexes

Let X be a finite simplicial 2-complex. For a simplicial loop $\gamma: S^1 \to X^{(1)} \subset X$ we denote by $|\gamma|$ the length of γ. If γ is null-homotopic, $\gamma \sim 1$, we denote by $A_X(\gamma)$ the **area** of γ, i.e., the minimal number of triangles in any simplicial filling V for γ. A simplicial filling (or a simplicial Van Kampen diagram) for a loop γ is defined as a pair of simplicial maps $S^1 \xrightarrow{i} V \xrightarrow{b} X$ such that $\gamma = b \circ i$ and the mapping cylinder of i is a disc with boundary $S^1 \times 0$; see [2].

Note that for $X \subset Y$ and $\gamma: S^1 \to X$, $\gamma \sim 1$, one has $A_X(\gamma) \geq A_Y(\gamma)$.

Define the following invariant of X:

$$I(X) = \inf\left\{\frac{|\gamma|}{A_X(\gamma)}; \gamma: S^1 \to X^{(1)}, \gamma \sim 1 \text{ in } X\right\} \in \mathbf{R}.$$

The inequality $I(X) \geq a$ means that for any null-homotopic loop γ in X one has the isoperimetric inequality $A_X(\gamma) \leq a^{-1} \cdot |\gamma|$. The inequality $I(X) < a$ means that there exists a null-homotopic loop γ in X with $A_X(\gamma) > a^{-1} \cdot |\gamma|$, i.e., γ is null-homotopic but does not bound a disk of area less than $a^{-1} \cdot |\gamma|$.

It is well known that $I(X) > 0$ if and only if $\pi_1(X)$ is **hyperbolic** in the sense of M. Gromov [9].

Example 3.1: For $X = T^2$ one has $I(X) = 0$.

Example 3.2: Let X be a finite 2-complex satisfying $\tilde{\mu}(X) > 1/2$. Then by Corollary 2.10 the fundamental group $\pi_1(X)$ is a free product of several copies of $\mathbf{Z}$ and $\mathbf{Z}_2$ and is hyperbolic. Hence $I(X) > 0$; compare Theorem 6.2 below.

Remark 3.3: Suppose that $\gamma = \gamma_1 \cdot \gamma_2$ is the concatenation of two null-homotopic loops γ_1, γ_2 in X. Then

$$|\gamma| = |\gamma_1| + |\gamma_2|, \quad \text{and} \quad A_X(\gamma) \leq A_X(\gamma_1) + A_X(\gamma_2).$$

This implies that

$$\frac{|\gamma|}{A_X(\gamma)} \geq \frac{|\gamma_1| + |\gamma_2|}{A_X(\gamma_1) + A_X(\gamma_2)} \geq \min\left\{\frac{|\gamma_1|}{A_X(\gamma_1)}, \frac{|\gamma_2|}{A_X(\gamma_2)}\right\}$$

and leads to the following observation:

  The number $I(X)$ coincides with the infimum of the ratios $|\gamma| \cdot A_X(\gamma)^{-1}$ where γ runs over all null-homotopic simplicial prime loops in X, i.e., such that their lifts to the universal cover $\tilde{X}$ of X are simple. Note that any simplicial filling $S^1 \xrightarrow{i} V \xrightarrow{b} X$ for a prime loop $\gamma : S^1 \to X$ has the property that V is a simplicial disc and i is a homeomorphism $i : S^1 \to \partial V$. Hence for prime loops γ the area $A_X(\gamma)$ coincides with the minimal number of 2-simplexes in any simplicial spanning disc for γ.

  The following Theorem 3.4 gives a uniform isoperimetric constant for random complexes $Y \in Y(n, p)$. It is a slightly stronger statement than simply hyperbolicity of the fundamental group of Y. Theorem 3.4 is implicitly contained in [2] although it was not stated there explicitly.

THEOREM 3.4: *Suppose that for some $\epsilon > 0$ the probability parameter p satisfies*

$$(5) \qquad\qquad p \ll n^{-1/2-\epsilon}.$$

Then there exists a constant $c_\epsilon > 0$ depending only on ϵ such that a random 2-complex $Y \in Y(n, p)$, with probability tending to 1 as $n \to \infty$, has the following property: any subcomplex $Y' \subset Y$ satisfies $I(Y') \geq c_\epsilon$ and, in particular, the fundamental group $\pi_1(Y')$ is hyperbolic.

  We give a detailed proof of Theorem 3.4 in the Appendix.
  We state now two immediate corollaries of Theorem 3.4.

COROLLARY 3.5: *Under the assumption (5), a random 2-complex $Y \in Y(n, p)$ contains no subcomplexes homeomorphic to the torus T^2, a.a.s.*

Proof. By Example 3.1 any torus S satisfies $I(S) = 0$ and the inclusion $S \subset Y$ would contradict Theorem 3.4. ∎

COROLLARY 3.6: *Under the assumption (5), a random 2-complex $Y \in Y(n, p)$ contains no subcomplex having more than $2\epsilon^{-1}$ faces which is homeomorphic to the sphere S^2, a.a.s.*

Proof. Suppose that a random 2-complex $Y \in Y(n, p)$ contains a sphere $S \subset Y$ as a subcomplex.

Firstly we show that the number of faces $f(S)$ is bounded above by a constant depending on ϵ. Indeed, removing a 2-simplex from S gives a disc S' and by Theorem 3.4 the isoperimetric ratio of the boundary curve γ of this disc must satisfy

$$\frac{|\gamma|}{A_{S'}(\gamma)} = \frac{3}{f(S) - 1} \geq c_\epsilon,$$

where $c_\epsilon > 0$ is the constant of Theorem 3.4. Thus we obtain

$$(6) \qquad\qquad f(S) \leq 3 \cdot c_\epsilon^{-1} + 1.$$

Secondly, we observe that there are finitely many isomorphism types of triangulations of the sphere S^2 satisfying (6) and we therefore may apply Theorem 15 from [5] or Lemma 1.1. By Theorem 27 and formula (8) from [5] we obtain $\tilde{\mu}(S) = \frac{1}{2} + \frac{2}{f(S)}$. If $\tilde{\mu}(S) < 1/2 + \epsilon$, then (by Lemma 1.1) S cannot be embedded into a random 2-complex $Y \in Y(n, p)$, a.a.s. Hence, for the triangulated spheres S embeddable into Y (a.a.s.) we must have $\tilde{\mu}(S) = \frac{1}{2} + \frac{2}{f(S)} \geq \frac{1}{2} + \epsilon$, implying $f(S) \leq 2 \cdot \epsilon^{-1}$. ∎

Let X be a simplicial 2-complex. By sys(X) we denote the **systole** of X, which is defined as the length of the shortest homotopically nontrivial simplicial loop in X. By Theorem 1.3 from [11] (see also Theorem 6.7.A from [8]) one has

$$(7) \qquad\qquad \mathrm{sys}(X) < 6 \cdot A(X)^{1/2},$$

assuming that the fundamental group of X is not a free group. Here $A(X)$ denotes the number of 2-simplexes in X. To deduce (7) from [11], Theorem 1.3 we apply this theorem to X equipped with a piecewise flat metric g in which each edge has length 1 and each 2-simplex has area $\sqrt{3}/4$. Let $\mathrm{sys}_g(X)$ and $A_g(X)$ denote the systole and the area of X with respect to this metric. Then $A_g(X) = A(X) \cdot \sqrt{3}/4$ and elementary estimates show that

$$\mathrm{sys}_g(X) \leq \mathrm{sys}(X) \leq 2 \cdot \mathrm{sys}_g(X).$$

Hence,

$$\frac{\mathrm{sys}^2(X)}{A(X)} \leq \frac{\sqrt{3}\left(\mathrm{sys}_g(X)\right)^2}{A_g(X)} \leq \sqrt{3} \cdot 12 < 36,$$

which implies (7).

COROLLARY 3.7: *Under the assumption* (5), *a random 2-complex* $Y \in Y(n, p)$ *contains no subcomplex having more than* ϵ^{-1} *faces which is homeomorphic to the projective plane* $\mathbf{RP}^2$, *a.a.s.*

Proof. Suppose that a random 2-complex $Y \in Y(n, p)$ contains a subcomplex $P \subset Y$ homeomorphic to the projective plane. By Theorem 3.4 the isoperimetric constant $I(P)$ satisfies $I(P) \geq c_\epsilon$. Consider the simple simplicial curve γ in P which is not null-homotopic and has length equal to $\mathrm{sys}(P)$. The twice passed curve γ is null-homotopic in P and the area of γ^2 equals $A(P)$. Thus we have

$$(8) \qquad \frac{2\mathrm{sys}(P)}{A(P)} \geq c_\epsilon.$$

Applying (7) we obtain $\mathrm{sys}(P)^2 \leq 36 \cdot A(P)$, and combining with (8) we get $A(P) \leq 144 \cdot c_\epsilon^{-2}$.

There are finitely many isomorphism types of triangulations of the real projective plane $\mathbf{RP}^2$ with at most $144c_\epsilon^{-2}$ faces. By Theorem 27 and formula (8) from [5] we obtain $\tilde{\mu}(S) = \frac{1}{2} + \frac{1}{f(S)}$. If $\tilde{\mu}(S) < 1/2 + \epsilon$, then (by Lemma 1.1) S cannot be embedded into a random 2-complex $Y \in Y(n, p)$, a.a.s. Hence, for the triangulated real projective plane S embeddable into Y (a.a.s.) we must have $\tilde{\mu}(S) = \frac{1}{2} + \frac{1}{f(S)} \geq \frac{1}{2} + \epsilon$, implying $f(S) \leq \epsilon^{-1}$. ∎

4. Torsion in fundamental groups of random 2-complexes

One of the main results of this section is Theorem 4.5, which states that the fundamental group of a random 2-complex has a nontrivial element of order 2 assuming that $n^{-3/5} \ll p \ll n^{-1/2-\epsilon}$. The proof of Theorem 4.5 uses Theorem 3.4 together with Theorem 4.3, which is stated and proven below.

The second main result presented in this section is Theorem 4.6, which claims that for any odd prime $m \geq 3$ (assuming that $p \ll n^{-1/2-\epsilon}$) the fundamental group of any subcomplex $Y' \subset Y$ of a random complex $Y \in Y(n, p)$ has no m-torsion, a.a.s. The proof uses Theorem 3.4 as well as the inequalities for systoles of Moore surfaces.

4.1. THE NUMBERS OF EMBEDDINGS. Consider two 2-complexes $S_1 \supset S_2$. Denote by v_i and f_i the numbers of vertices and faces of S_i. We have $v_1 \geq v_2$ and $f_1 \geq f_2$. We will assume that $f_1 > f_2$.

Let $\mu(S_1, S_2)$ denote the ratio

$$\mu(S_1, S_2) = \frac{v_1 - v_2}{f_1 - f_2}.$$

If $\mu(S_1) < \mu(S_2)$, then

(9) $$\mu(S_1, S_2) < \mu(S_1) < \mu(S_2).$$

If $\mu(S_1) > \mu(S_2)$, then

(10) $$\mu(S_1, S_2) > \mu(S_1) > \mu(S_2).$$

These two observations can be summarised by saying that $\mu(S_1)$ always lies in the interval connecting $\mu(S_2)$ and $\mu(S_1, S_2)$.

One has the following formula:

(11) $$\mu(S_1, S_2) = \frac{1}{2} + \frac{2(\chi(S_1) - \chi(S_2)) + L(S_1) - L(S_2)}{2(f_1 - f_2)},$$

which follows from the equation $2v_i = f_i + 2\chi(S_i) + L(S_i)$; the latter is equivalent to (2).

Remark 4.1: Let S_2 be a pseudo-surface and $2(\chi(S_1) - \chi(S_2)) + L(S_1) < 0$. Then $\mu(S_1, S_2) < 1/2$. Note that $L(S_2) = 0$ since S_2 is a pseudo-surface. This remark applies to the case when S_1 is a union of a pseudo-surface S_2 and a number of simplicial discs.

THEOREM 4.2: *Let $S_1 \supset S_2$ be two fixed 2-complexes and*[3]

(12) $$n^{-\tilde{\mu}(S_2)} \ll p \ll n^{-\mu(S_1, S_2)}.$$

Then the number of simplicial embeddings of S_1 into a random 2-complex $Y \in Y(n, p)$ is smaller than the number of simplicial embeddings of S_2 into Y, a.a.s. In particular, under the assumptions (12), with probability tending to one, there exists a simplicial embedding $S_2 \to Y$ which does not extend to an embedding $S_1 \to Y$.

Proof. Let $X_i : Y(n, p) \to \mathbf{Z}$ be the random variable counting the number of simplicial embeddings of S_i into $Y \in Y(n, p)$, $i = 1, 2$. We know that

$$\mathbb{E}(X_i) = \binom{n}{v_i} v_i! p^{f_i} \sim n^{v_i} p^{f_i}.$$

---

[3] The assumption (12) is meaningful iff $\mu(S_1, S_2) < \tilde{\mu}(S_2) \leq \mu(S_2)$ which, as follows from (9) and (10), implies that $\mu(S_1) < \mu(S_2)$. Thus, if Theorem 4.2 is applicable, then $\mu(S_1) < \mu(S_2)$.

Our goal is to show that $X_1 < X_2$, a.a.s. The left-hand side of inequality (12) implies (via Theorem 15 from [5]) that $X_2(Y) > 0$, a.a.s., i.e., S_2 admits an embedding into Y with probability tending to one. We have

$$\frac{\mathbb{E}(X_1)}{\mathbb{E}(X_2)} \sim n^{v_1 - v_2} p^{f_1 - f_2} = [n^{\mu(S_1, S_2)} p]^{f_1 - f_2} \to 0$$

under our assumption (12) (the right-hand side).

Below we shall find $t_1, t_2 > 0$ such that

$$t_1 + t_2 = \mathbb{E}(X_2) - \mathbb{E}(X_1)$$

and $\mathbb{E}(X_1)/t_1 \to 0$ while $\mathbb{E}(X_2)/t_2$ is bounded.

One of the following three statements holds: either $X_1 < X_2$ or $X_1 \geq \mathbb{E}(X_1) + t_1$ or $X_2 < \mathbb{E}(X_2) - t_2$, and therefore

$$(13) \qquad P(X_1 < X_2) \geq 1 - P(X_1 \geq \mathbb{E}(X_1) + t_1) - P(X_2 < \mathbb{E}(X_2) - t_2).$$

By Markov's inequality

$$P(X_1 \geq \mathbb{E}(X_1) + t_1) \leq \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1) + t_1} = \frac{\frac{\mathbb{E}(X_1)}{t_1}}{1 + \frac{\mathbb{E}(X_1)}{t_1}} \to 0$$

while by Chebyschev's inequality

$$P(X_2 < \mathbb{E}(X_2) - t_2) < \frac{\mathrm{Var}(X_2)}{t_2^2}.$$

It is known (see [5], proof of Theorem 15) that under our assumptions (12) the ratio $\frac{\mathrm{Var}(X_2)}{\mathbb{E}(X_2)^2}$ tends to zero. Therefore, combining the last two inequalities with the inequality (13) we see that $P(X_1 < X_2)$ tends to 1 as $n \to \infty$.

To make a specific choice of t_1 and t_2 one may take $t_1 = \sqrt{\mathbb{E}(X_1)\mathbb{E}(X_2)}$ and $t_2 = \mathbb{E}(X_2) - \mathbb{E}(X_1) - t_1$. Then

$$\frac{\mathbb{E}(X_1)}{t_1} = \sqrt{\frac{\mathbb{E}(X_1)}{\mathbb{E}(X_2)}} \to 0$$

and

$$\frac{\mathbb{E}(X_2)}{t_2} = \frac{1}{1 - \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_2)} - \sqrt{\frac{\mathbb{E}(X_1)}{\mathbb{E}(X_2)}}} \to 1$$

is bounded. This completes the proof. ∎

THEOREM 4.3: *Let*

$$T_j \supset S, \quad j = 1, \ldots, N,$$

be a finite family of 2-complexes containing a given 2-complex S and satisfying $\mu(T_j) < \mu(S)$. Assume that

(14) $n^{-\tilde{\mu}(S)} \ll p \ll n^{-\mu(T_j, S)}, \quad \text{for any } j = 1, \ldots, N.$

Then, with probability tending to one, for a random 2-complex $Y \in Y(n, p)$ there exists a simplicial embedding $S \to Y$ which does not extend to a simplicial embedding $T_j \to Y$, for any $j = 1, \ldots, N$.

Proof. Let $X_{1,j} : Y(n, p) \to \mathbf{Z}$ denote the random variable counting the number of embeddings of T_j into $Y \in Y(n, p)$. Denote $X_1 = \sum_{j=1}^{N} X_{1,j}$. Besides, let $X_2 : Y(n, p) \to \mathbf{Z}$ denote the number of embeddings of S into a random 2-complex $Y \in Y(n, p)$. As in the proof of the provious theorem one has

$$\frac{\mathbb{E}(X_1)}{\mathbb{E}(X_2)} = \sum_{j=1}^{N} \frac{\mathbb{E}(X_{1,j})}{\mathbb{E}(X_2)} \to 0$$

thanks to our assumption (14) (the right-hand-side inequality). Taking $t_1 = \sqrt{\mathbb{E}(X_1)\mathbb{E}(X_2)}$ and $t_2 = \mathbb{E}(X_2) - \mathbb{E}(X_1) - t_1$ (as in the proof of the previous theorem) one has $t_1 + t_2 = \mathbb{E}(X_2) - \mathbb{E}(X_1)$ and $\mathbb{E}(X_1)/t_1 \to 0$ while $\mathbb{E}(X_2)/t_2$ is bounded. Repeating the arguments used in the proof of the previous theorem we see that $X_1 < X_2$, a.a.s. Since every embedding $T_j \to Y$ determines (by restriction) an embedding $S \to Y$, the inequality $X_1(Y) < X_2(Y)$ implies that there there are embeddings $S \to Y$ which admit no extensions to an embedding $T_j \to Y$, for any $j = 1, \ldots, N$. ∎

4.2. PROJECTIVE PLANES IN RANDOM 2-COMPLEXES. In this section we prove the existence of 2-torsion in fundamental groups of random 2-complexes; see Theorem 4.5. The proof uses simplical embeddings of projective planes into random 2-complexes. Note that for a triangulation X of the real projective plane one has

$$\mu(X) = 1/2 + \frac{1}{f(X)},$$

where $f(X)$ is the number of faces of X, see (2), and therefore the maximal value of $\mu(X)$ happens when $f(X)$ is minimal.

It is well known that the complex S shown in Figure 6 is the minimal triangulation of P^2; it has 10 faces and therefore $\mu(S) = 3/5$. Moreover,

$\tilde{\mu}(S) = \mu(S) = 3/5$ by Theorem 27 from [5]. By Lemma 1.1 for $p \gg n^{-3/5}$ the complex S embeds into a random 2-complex, a.a.s. This explains the appearance of the exponent $3/5$ in Theorem 4.5 below.
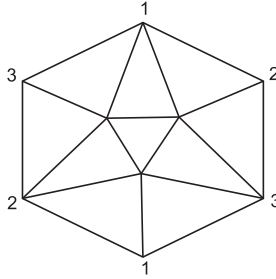

Figure 6. Minimal triangulation of the real projective plane; the antipodal points on the peripheral hexagon must be identified.

We may mention here that for $p \ll n^{-3/5}$ the fundamental group of a random 2-complex $Y \in Y(n, p)$ is torsion free; see Corollary 5.9.

Definition 4.4: A subcomplex $S \subset Y$ is said to be **essential** if the induced homomorphism $\pi_1(S) \to \pi_1(Y)$ is injective.

THEOREM 4.5: *Let S be a triangulation of the real projective plane $\mathbf{RP}^2$ having 6 vertices, 15 edges and 10 faces. Assume that $\epsilon > 0$ and*

$$n^{-3/5} \ll p \ll n^{-1/2-\epsilon}.$$

Then a random 2-complex $Y \in Y(n, p)$ contains S as an essential subcomplex, a.a.s. In particular, the fundamental group $\pi_1(Y)$ contains an element of order two and hence its cohomological dimension is infinite.

Proof. Consider the set $\mathcal{S}_\epsilon$ of isomorphism types of pure connected closed 2-complexes X satisfying the following conditions:

(a) X contains S as a subcomplex.
(b) The inclusion $S \to X$ induces a trivial homomorphism $\pi_1(S) \to \pi_1(X)$.
(c) For any subcomplex $S \subset X' \subset X$, $X' \neq X$, the homomorphism $\pi_1(S) \to \pi_1(X')$ is nontrivial.
(d) X has at most $10+3c_\epsilon^{-1}$ faces where c_ϵ is the constant given by Theorem 3.4.

(e) $\tilde{\mu}(X) > 1/2 + \epsilon$.

Let us show that any $X \in \mathcal{S}_\epsilon$ is homeomorphic to the complex Z_4, as defined in Theorem 2.5. In the exact sequence

$$0 \to H_2(X) \to H_2(X, S) \to H_1(S) = \mathbf{Z}_2 \to 0$$

the middle group has no torsion and hence $b_2(X) = b_2(X, S) \geq 1$. If $Z \subset X$ is a minimal cycle, then $\mu(Z) > 1/2 + \epsilon$ (by (e)) and Theorem 2.5 implies that Z is homeomorphic to one of the 2-complexes Z_1, Z_2, Z_3, Z_4.

If Z is homeomorphic to one of the complexes Z_1, Z_2, Z_3, then any face $\sigma \subset Z$ satisfies the conditions of Corollary 2.6. Therefore, removing from X any face $\sigma \subset Z - S$ would produce 2-complex X' violating (c).

This shows that any minimal cycle $Z \subset X$ is homeomorphic to $Z_4 = P^2 \cup \Delta^2$.

The union of the edges of degree 3 in Z is a closed curve and cutting along this curve disconnects Z onto a copy of P^2 and Δ^2. Thus, we may think of P^2 as being a subcomplex of Z. If $P^2 \not\subset S$, then there is a 2-simplex $\sigma \subset P^2 - S$. Then the curve $\partial\sigma \subset Z - \sigma \subset X - \sigma$ is null-homotopic and hence the inclusion $X' = Z - \sigma \to X$ induces an isomorphism $\pi_1(X') \to \pi_1(X)$ contradicting (c). This argument shows that $P^2 \subset S$ and hence $P^2 = S$. Since $\pi_1(S) \to \pi_1(Z)$ is trivial, it follows from the minimality of X (property (c)) that $Z = X$.

Since $\mu(X) > 1/2$, by formula (2) we obtain that $2\chi(X) + L(X) > 0$ or equivalently $L(X) \geq -3$ (as $\chi(X) = 2$ since X is homotopy equivalent to the 2-sphere). On the other hand, $L(X) \leq -3$ since X is closed and has at least 3 edge of degree 3. Thus $L(X) = -3$.

Using formula (11) we find that

$$\mu(X, S) = \frac{1}{2} - \frac{1}{2(f(X) - 10)} < \frac{1}{2}$$

and then applying Theorem 4.3 we find that for

$$n^{-3/5} \ll p \ll n^{-1/2-\epsilon},$$

with probability tending to 1, there exists embedding $S \to Y$ (where $Y \in Y(n, p)$ is random) which cannot be extended to an embedding of $X \to Y$ for any $X \in \mathcal{S}_\epsilon$.

Let $Y'_n \subset Y(n, p)$ denote the set of complexes $Y \in Y(n, p)$ such that there exists a simplicial embedding $S \to Y$ which cannot be extended to an embedding $X \to Y$, for any $X \in \mathcal{S}_\epsilon$. We have shown earlier that $\mathbb{P}(Y'_n) \to 1$ as $n \to \infty$.

Consider the finite family $\mathcal{F}$ of isomorphism types of simplicial complexes T having at most $10 + 3c_\epsilon^{-1}$ faces and satisfying $\tilde{\mu}(T) \leq 1/2 + \epsilon$. By Lemma 1.1, since we assume that $p \ll n^{-1/2-\epsilon}$, the set $Y_n'' \subset Y(n, p)$ of 2-complexes $Y \in Y(n, p)$ having the property that none of the complexes $T \in \mathcal{F}$ is embeddable into Y satisfies $\mathbb{P}(Y_n'') \to 1$, as $n \to \infty$.

Besides, let $Y_n''' \subset Y(n, p)$ denote the set of complexes satisfying Theorem 3.4. We know that $\mathbb{P}(Y_n''') \to 1$ as $n \to \infty$.

Now, let $Y \in Y_n' \cap Y_n'' \cap Y_n'''$ and let $S \subset Y$ be an embedding which cannot be extended to an embedding $X \to Y$ for $X \in \mathcal{S}_\epsilon$. Let us show that $S \subset Y$ is essential. Assuming the contrary, if the embedding $S \subset Y$ is not essential then the central circle γ of S (of length 3) extends to a simplicial map of a simplicial disc into Y. Under the assumption $p \ll n^{-1/2-\epsilon}$, using Theorem 3.4, we find that the circle γ extends to a simplicial map $b : \Delta^2 \to Y$, $b|\partial\Delta^2 = \gamma$, into Y of a simplicial disc of area $\leq 3c_\epsilon^{-1}$ where $c_\epsilon > 0$ depends only on the value of ϵ. Since $S \cup b(\Delta^2)$ is embedded into Y we obtain that $\tilde{\mu}(S \cup b(\Delta^2)) > 1/2 + \epsilon$. Hence, if $S \subset Y$ is not essential then there would exist a complex $X \in \mathcal{S}_\epsilon$, where $S \subset X \subset S \cup b(\Delta^2)$, such that the embedding $S \subset Y$ extends to an embedding $X \subset Y$. This gives a contradiction. ∎

4.3. ABSENCE OF HIGHER TORSION. In this subsection we prove the following statement complementing Theorem 4.5.

THEOREM 4.6: *Assume that the probability parameter p satisfies*

$$(15) \qquad\qquad p \ll n^{-1/2-\epsilon},$$

where $\epsilon > 0$. Let $m \geq 3$ be a prime number. Then a random 2-complex $Y \in Y(n, p)$, with probability tending to 1, has the following property: for any subcomplex $Y' \subset Y$ the fundamental group $\pi_1(Y')$ has no elements of order m.

Let Σ be a simplicial 2-complex homeomorphic to the **Moore surface**

$$M(\mathbf{Z}_m, 1) = S^1 \cup_{f_m} e^2, \quad \text{where } m \geq 3;$$

it is obtained from the circle S^1 by attaching a 2-cell via the degree m map $f_m : S^1 \to S^1$, $f_m(z) = z^m$, $z \in S^1$. The 2-complex Σ has a well defined circle $C \subset \Sigma$ (which we shall call the **singular circle**) which is the union of all edges of degree m; all other edges of Σ have degree 2. Clearly, the homotopy class of the singular circle generates the fundamental group $\pi_1(\Sigma) \simeq \mathbf{Z}_m$.

Next we define an integer $N_m(Y) \geq 0$ associated to any connected 2-complex Y. If $\pi_1(Y)$ has no m-torsion we set $N_m(Y) = 0$. If $\pi_1(Y)$ has elements of order m we shall consider homotopically nontrivial simplicial maps $\gamma : C_r \to Y$, where C_r is the simplicial circle with r edges, such that

(a) γ^m is null-homotopic (as a free loop in Y);
(b) r is minimal: for $r' < r$ any simplicial loop $\gamma : C_{r'} \to Y$ satisfying (a) is homotopically trivial.

Any such simplicial map $\gamma : C_r \to Y$ can be extended to a simplicial map $f : \Sigma \to Y$ of a triangulation Σ of the Moore surface, such that the singular circle C of Σ is isomorphic to C_r and $f|C = \gamma$. We shall say that a simplicial map $f : \Sigma \to Y$ is m-**minimal** if it satisfies (a), (b) and the number of 2-simplexes in Σ is the smallest possible. Now, we denote by

$$N_m(Y) \in \mathbf{Z}$$

the number of 2-simplexes in a triangulation of the Moore surface Σ admitting an m-minimal map $f : \Sigma \to Y$.

LEMMA 4.7: *Let Y be a 2-complex satisfying $I(Y) \geq c > 0$ (where the quantity $I(Y)$ is defined in §3). Let $m \geq 3$ be an odd prime. Then one has*

$$N_m(Y) \leq \left(\frac{6m}{c} \right)^2 .$$

Proof. We shall assume that the fundamental group of Y contains an element of order m; otherwise $N_m(Y) = 0$. Consider an m-minimal simplicial map $f : \Sigma \to Y$,

$$(16) \qquad\qquad\qquad A(\Sigma) = N_m(Y),$$

where $A(\Sigma)$ denotes the number of 2-simplexes in Σ. It is obvious from the m-minimality of f that the singular circle $C \subset \Sigma$ is the shortest (in terms of the number of edges) homotopically nontrivial simplicial loop in Σ, i.e.,

$$(17) \qquad\qquad\qquad |C| = \mathrm{sys}(\Sigma).$$

Consider the loop $\gamma = f|C$, $|\gamma| = |C| = \mathrm{sys}(\Sigma)$. We know that γ^m is homotopically trivial in Y and the inequality $I(Y) \geq c > 0$ implies that γ^m bounds in Y a disc of area at most $m \cdot |\gamma| \cdot c^{-1}$. The 2-complex Σ is obtained from a simplicial disc D by dividing its boundary into m intervals of equal length and identifying them to each other. If $A(D) > m \cdot |\gamma| \cdot c^{-1}$, then one could

replace the disc D by the minimal spanning disc for γ^m obtaining a simplicial map $\Sigma' \to Y$ which has smaller area, contradicting the m-minimality. Thus, we obtain

$$(18) \qquad\qquad A(\Sigma) \le c^{-1} \cdot m \cdot \text{sys}(\Sigma).$$

Inequality (7) gives

$$(19) \qquad\qquad \text{sys}(\Sigma) \le 6 \cdot A(\Sigma)^{1/2}.$$

Combining (16), (18) and (19) we obtain

$$N_m(Y) = A(\Sigma) \le \left(\frac{6m}{c}\right)^2. \qquad \blacksquare$$

THEOREM 4.8: *Assume that the probability parameter p satisfies $p \ll n^{-1/2-\epsilon}$ where $\epsilon > 0$ is fixed. Let $m \ge 3$ be an odd prime. Then there exists a constant $C_\epsilon > 0$ such that a random 2-complex $Y \in Y(n,p)$ with probability tending to 1 has the following property: for any subcomplex $Y' \subset Y$ one has*

$$(20) \qquad\qquad N_m(Y') \le C_\epsilon.$$

Proof. We know from Theorem 3.4 that, with probability tending to 1, a random 2-complex Y has the following property: for any subcomplex $Y' \subset Y$ one has $I(Y') \ge c_\epsilon > 0$, where $c_\epsilon > 0$ is the constant given by Theorem 3.4. Then, setting $C_\epsilon = (\frac{6m}{c_\epsilon})^2$, the inequality (20) follows from Lemma 4.7. $\qquad \blacksquare$

Proof of Theorem 4.6. Let $c_\epsilon > 0$ be the number given by Theorem 3.4. Consider the finite set of all isomorphism types of triangulations $\mathcal{S}_m = \{\Sigma\}$ of the Moore surface $M(\mathbf{Z}_m, 1)$ having at most $(\frac{6m}{c_\epsilon})^2$ two-dimensional simplexes. Let $\mathcal{X}_m$ denote the set of isomorphism types of images of all surjective simplicial maps $\Sigma \to X$ inducing injective homomorphisms $\pi_1(\Sigma) = \mathbf{Z}_m \to \pi_1(X)$, where $\Sigma \in \mathcal{S}_m$. The set $\mathcal{X}_m$ is also finite.

From Theorem 4.8 we obtain that, with probability tending to one, for any subcomplex $Y' \subset Y$, either $\pi_1(Y')$ has no m-torsion, or there exists an m-minimal map $f : \Sigma \to Y'$ with Σ having at most $(\frac{6m}{c_\epsilon})$ simplexes of dimension 2; in the second case the image $X = f(\Sigma)$ is a subcomplex of Y' and $f : \Sigma \to X$ induces a monomorphism $\pi_1(\Sigma) \to \pi_1(X)$, i.e., $X \in \mathcal{X}_m$.

From Corollary 2.10 we know that the fundamental group of any 2-complex satisfying $\tilde{\mu}(X) > 1/2$ is a free product of several copies of $\mathbf{Z}$ and $\mathbf{Z}_2$ and has no m-torsion, as we assume that $m \ge 3$. Since the fundamental group

of any $X \in \mathcal{X}_m$ has m-torsion, where $m \geq 3$, one has $\tilde{\mu}(X) \leq 1/2$ for any $X \in \mathcal{X}_m$. Hence, using the finiteness of $\mathcal{X}_m$ and the results on the containment problem (cf. Lemma 1.1) we see that for $p \ll n^{-1/2-\epsilon}$ the probability that a random complex $Y \in Y(n, p)$ contains a subcomplex isomorphic to one of the complexes $X \in \mathcal{X}_m$ tends to 0 as $n \to \infty$. Hence, we obtain that (a.a.s.) any subcomplex $Y' \subset Y$ does not contain $X \in \mathcal{X}_m$ as a subcomplex and therefore the fundamental group of Y' has no m-torsion. ∎

5. Minimal spheres and The Whitehead Conjecture

Definition 5.1: Let Y be a simplicial complex with $\pi_2(Y) \neq 0$. We define a numerical invariant $M(Y) \in \mathbf{Z}$, $M(Y) \geq 4$, as the minimal number of faces in a 2-complex Σ homeomorphic to the sphere S^2 such that there exists a homotopically nontrivial simplicial map $\Sigma \to Y$. We define $M(Y) = 0$, if $\pi_2(Y) = 0$.

Our first goal in this section is to prove the following theorem:

THEOREM 5.2: *Assume that the probability parameter p satisfies $p \ll n^{-1/2-\epsilon}$ where $\epsilon > 0$ is fixed. Then there exists a constant $C = C_\epsilon$ such that a random 2-complex $Y \in Y(n, p)$ with probability tending to 1 has the following property: for any subcomplex $Y' \subset Y$ one has*

$$(21) \qquad\qquad\qquad M(Y') \leq C.$$

In other words, in the specified range of p, the numbers $M(Y')$ have a uniform upper bound, a.a.s.

In the proof of Theorem 5.2 we will use the following version of the notion of Cheeger's constant. For a pure simplicial 2-complex Σ one defines the Cheeger's constant $h(\Sigma)$ as

$$(22) \qquad\qquad h(\Sigma) = \min_{S \subset \Sigma} \left\{ \frac{|\partial S|}{A(S)}; A(S) \leq A(\Sigma)/2 \right\},$$

where $S \subset \Sigma$ runs over all pure subcomplexes. Here $|\partial S|$ denotes the length of the boundary (the number of edges) and $A(S)$ and $A(\Sigma)$ denote the area of S and Σ, i.e., the number of 2-simplexes.

Note that in (22) one may always assume that $S \subset \Sigma$ is strongly connected. Indeed, if $S = \bigcup_j S_j$ are the strongly connected components of S, then

$$|\partial S| = \sum_j |\partial S_j| \quad \text{and} \quad A(S) = \sum_j A(S_j)$$

and therefore,

$$\frac{|\partial S|}{A(S)} \geq \min_j \frac{|\partial S_j|}{A(S_j)}.$$

In the case when Σ is homeomorphic to the sphere S^2, one can require the subcomplex $S \subset \Sigma$ which appears in the formula (22) to be homeomorphic to a disc. Indeed, if $S \subset \Sigma$ is strongly connected and $\frac{|\partial S|}{A(S)} = h(S)$, $A(S) \leq A(\Sigma)/2$, consider the strongly connected components of the closure of the complement $\overline{\Sigma - S} = \bigcup_j D_j$. Each D_j is a disc and

$$|\partial S| = \sum_j |\partial D_j| \quad \text{and} \quad A(S) \leq A(\Sigma - S) = \sum_j A(D_j).$$

Hence

$$h(\Sigma) = \frac{|\partial S|}{A(S)} \geq \frac{\sum_j |\partial D_j|}{\sum_j A(D_j)} \geq \min_j \frac{|\partial D_j|}{A(D_j)}$$

and, in particular, for some j_0 one has $\frac{|\partial D_{j_0}|}{A(D_{j_0})} \leq h(\Sigma)$. From now on we may assume that $A(D_{j_0}) > A(\Sigma)/2$ since otherwise the proof is complete. Consider the disc $C = \overline{\Sigma - D_{j_0}}$. Then $A(C) \leq A(\Sigma)/2$ and

$$A(C) \geq A(S), \quad |\partial C| \leq |\partial S|$$

and we obtain

$$h(\Sigma) = \frac{|\partial S|}{A(S)} \geq \frac{|\partial C|}{A(C)}.$$

Thus the value $h(\Sigma)$ in (22) is achieved on subdiscs.

Our next two results are deterministic. We show that the isoperimetric constant can be used to estimate above the value $M(Y)$.

LEMMA 5.3: *Let Y be a 2-complex with $I(Y) \geq c > 0$. Let $f : \Sigma \to Y$ be a homotopically nontrivial simplicial map where Σ is homeomorphic to S^2 and $A(\Sigma) = M(Y)$. Then $h(\Sigma) \geq c$.*

Proof. Assume that $h(\Sigma) < c$. Then (due to our discussion above) there exists a simplicial subdisc $S \subset \Sigma$ with $A(S) \leq A(\Sigma)/2$ and $\frac{|\partial S|}{A(S)} < c$. The curve $\gamma = f|\partial S$ is a null-homotopic loop in Y and $f|S : S \to Y$ is a bounding disk for γ. We claim that this disc has the smallest area among all bounding discs

for γ in Y. Indeed, if there existed a spanning disc $b : D \to Y$, $b|\partial D = \gamma$, with $A(D) < A(S)$, then we may define two maps of the sphere $S^2 \to Y$, both having smaller area than f. These maps are $\phi : S \cup D \to Y$ and $\phi' : S' \cup D \to Y$, where $S' = \overline{\Sigma - S}$. Here $\phi|S = f|S$, $\phi|D = b = \phi'|D$, $\phi'|S' = f|S'$. Clearly at least one of ϕ and ϕ' is not null-homotopic, since if both ϕ and ϕ' were null-homotopic then the original map $S^2 \to Y$ was null-homotopic as well. Thus we obtain $A(S) = A_Y(\gamma)$ and $\frac{|\gamma|}{A_Y(\gamma)} < c$, contradicting our assumption $I(Y) \geq c$.   ∎

COROLLARY 5.4: *Let Y be a 2-complex with $I(Y) \geq c > 0$. Then*

$$(23) \qquad\qquad M(Y) \leq \left(\frac{16}{c}\right)^2.$$

Proof. Papasoglu [15] proved the inequality

$$h(\Sigma) \leq \frac{16}{\sqrt{A(\Sigma)}}$$

valid for any simplicial sphere. By Lemma 5.3, one has $h(\Sigma) \geq c$ for the homotopically nontrivial map $\Sigma \to Y$ with the minimal $A(\Sigma)$. This implies that $A(\Sigma) \leq 16^2/c^2$, which is equivalent to our statement.   ∎

Proof of Theorem 5.2. By Theorem 3.4 one has $I(Y') > c_\epsilon$, a.a.s. where $c_\epsilon > 0$ depends only on ϵ. Thus, in view of Corollary 5.4, the inequality (21) is satisfied with $C = 256/c_\epsilon^2$.   ∎

Next we characterise aspherical subcomplexes of random 2-complexes:

THEOREM 5.5: *Assume that $p \ll n^{-1/2-\epsilon}$ for a fixed $\epsilon > 0$. Then a random 2-complex $Y \in Y(n,p)$ has the following property with probability tending to one as $n \to \infty$: for any subcomplex $Y' \subset Y$ the following properties are equivalent:*

  (A): *Y' is aspherical;*
  (B): *Y' contains no subcomplexes $S \subset Y'$ with at most $2\epsilon^{-1}$ faces which are homeomorphic to the sphere S^2, the projective plane $\mathbf{RP}^2$ or the complexes Z_2, Z_3 shown in Figure 3.*

In particular, a random 2-complex $Y \in Y(n,p)$ has the following property with probability tending to one as $n \to \infty$: any subcomplex $Y' \subset Y$ with $b_2(Y') = 0$ is aspherical if and only if

  (B'): *Y' contains no subcomplexes $S \subset Y'$ with at most $2 \cdot \epsilon^{-1}$ faces which are homeomorphic to the projective plane $\mathbf{RP}^2$.*

Proof. We first show that **(A)** $\Rightarrow$ **(B)**. It is obvious that an aspherical 2-complex cannot contain as a subcomplex none of S^2 or Z_2, Z_3; otherwise there would be a nontrivial spherical homology class. By a theorem of W. H. Cockcroft [4], an aspherical 2-complex cannot contain a projective plane.

Next we show that **(B)** $\Rightarrow$ **(A)** a.a.s. Let $Y' \subset Y$ be a subcomplex where $Y \in Y(n, p)$ is random. Assume that Y' contains no subcomplexes with at most $2/\epsilon$ faces which are homeomorphic to either S^2, $\mathbf{RP}^2$ or Z_2, Z_3; see above. If Y' is not aspherical then by Theorem 5.2, $M(Y') \leq C$ where C depends only on ϵ. There exist finitely many isomorphism types $\{S_j\}_{j \in J}$ of triangulations of S^2 with at most C faces. By Lemma 1.1 the probability that a random 2-complex Y contains an image $\phi_j(S_j)$ satisfying $\tilde{\mu}(\phi_j(S_j)) < 1/2 + \epsilon$ tends to zero as $n \to \infty$. Thus we only have to consider the images of homotopically nontrivial simplicial maps $\phi_j : S_j \to Y'$ satisfying $\tilde{\mu}(\phi_j(S_j)) \geq 1/2 + \epsilon$.

If the second Betti number of the image is nonzero, $b_2(\phi_j(S_j)) \neq 0$, then the image $\phi_j(S_j)$ contains a minimal cycle Z satisfying $\tilde{\mu}(Z) \geq 1/2 + \epsilon$. By Theorem 2.5, such Z must be homeomorphic to one of the complexes Z_1, Z_2, Z_3, Z_4. For $i = 1, 2, 3$ one has (using formula (2))

$$\epsilon \leq \tilde{\mu}(Z_i) - \frac{1}{2} \leq \frac{2}{f(Z_i)}$$

and hence $f(Z_i) \leq 2/\epsilon$. For $i = 4$, we have $Z_4 = P^2 \cup \Delta^2$ and

$$\mu(P^2) = \frac{1}{2} + \frac{1}{f(P^2)} \geq \tilde{\mu}(Z_4) \geq \frac{1}{2} + \epsilon$$

implying that $f(P^2) \leq 2\epsilon^{-1}$. Now we may use our assumptions concerning Y', implying that Y' contains no subcomplexes homeomorphic to $Z_1, \ldots, Z_4$ leading to a contradiction.

Consider now the remaining case $b_2(\phi_j(S_j)) = 0$. As above one has

$$\mu(\phi_j(S_j)) \geq \tilde{\mu}(\phi_j(S_j)) \geq \frac{1}{2} + \epsilon.$$

Now, applying Lemma 2.9, we see that any closed pure subcomplex of the image $\phi_j(S_j)$ is an iterated wedge of projective planes. The argument similar to the one used in the previous paragraph shows that each of these projective planes P^2 has at most $2\epsilon^{-1}$ faces. Since we assume that Y' contains no projective planes with at most $2\epsilon^{-1}$ faces, we obtain that $\phi_j(S_j)$ contains no closed pure subcomplexes and therefore the image $\phi_j(S_j)$ is homotopy equivalent to a wedge

of circles implying that $\pi_2(\phi_j(S_j)) = 0$. Hence we obtain a contradiction to our assumption that the map $\phi_j : S_j \to Y'$ is essential. ∎

The following results are corollaries of Theorem 5.5:

COROLLARY 5.6: *Assume that $p \ll n^{-1/2-\epsilon}$ where $\epsilon > 0$ is fixed. Then a random 2-complex $Y \in Y(n, p)$ has the following property with probability tending to one as $n \to \infty$: any aspherical subcomplex $Y' \subset Y$ satisfies The Whitehead Conjecture, i.e., if a subcomplex $Y' \subset Y$ is aspherical then all subcomplexes of Y' are also aspherical.*

Proof. Let $Y' \subset Y$ be aspherical. Then by the previous Theorem, Y' has property **(B)**. Hence any subcomplex $Y'' \subset Y'$ has property **(B)**. Applying Theorem 5.5 again we obtain that Y'' is aspherical. This completes the proof. ∎

COROLLARY 5.7: *Assume that $p \ll n^{-1/2-\epsilon}$ where $\epsilon > 0$ is fixed. Then a random 2-complex $Y \in Y(n, p)$ has the following property with probability tending to one as $n \to \infty$: a subcomplex $Y' \subset Y$ is aspherical if and only if any subcomplex $S \subset Y'$ with at most $2\epsilon^{-1}$ faces is aspherical.*

Proof. Indeed, in one direction the result follows from Corollary 5.6. In the other direction the result follows from the implication **(B)** $\Rightarrow$ **(A)** of Theorem 5.5. ∎

COROLLARY 5.8: *Assume that $p \ll n^{-1/2-\epsilon}$ where $\epsilon > 0$ is fixed. Then a random 2-complex $Y \in Y(n, p)$ has the following property with probability tending to one as $n \to \infty$: any subcomplex $Y' \subset Y$ with $H_2(Y'; \mathbf{Z}_2) = 0$ is aspherical.*

Proof. If $H_2(Y'; \mathbf{Z}_2) = 0$, then Y' cannot have subcomplexes homeomorphic to S^2, $\mathbf{RP}^2$, Z_2 and Z_3. Therefore Y' is aspherical by Theorem 5.5. ∎

COROLLARY 5.9: *Assume that $p \ll n^{-3/5}$. Then for a random 2-complex $Y \in Y(n, p)$ the fundamental group $\pi_1(Y)$ has cohomological dimension at most 2, a.a.s. In particular, under this assumption on p the fundamental group $\pi_1(Y)$ has no torsion, a.a.s. Moreover, for $p \ll n^{-3/5}$, any subcomplex $Y' \subset Y$ with $b_2(Y') = 0$ is aspherical a.a.s.*

Proof. We shall apply Theorem 5.5 with $\epsilon = 1/10$, so that $1/2 + \epsilon = 3/5$ and $2\epsilon^{-1} = 20$. Thus we shall consider embeddings of triangulated S^2, Z_2, Z_3 and $\mathbf{RP}^2$ into $Y \in Y(n, p)$ having at most 20 faces.

For any triangulation of $\mathbf{RP}^2$ one has $\tilde{\mu}(\mathbf{RP}^2) \leq 3/5$ (see [5], Corollary 22), and by Lemma 1.1 no projective plane with at most 20 faces is contained in $Y \in Y(n, p)$ under the assumption $p \ll n^{-3/5}$.

Thus Y may only contain copies of S^2, Z_2, Z_3 with at most 20 faces. We produce now a sequence of subcomplexes $Y = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_k$ with $k \leq b_2(Y)$ as follows. Each Y_{i+1} is obtained from Y_i by removing a face belonging to a subcomplex homeomorphic to S^2, Z_2 or Z_3. Then clearly $\pi_1(Y_{i+1}) = \pi_1(Y_i)$ and the last subcomplex $Y_k \subset Y$ is aspherical by Theorem 5.5.

Therefore, we obtain that there exists a 2-dimensional aspherical complex Y_k having fundamental group $\pi_1(Y_k) = \pi_1(Y)$. This implies that the cohomological dimension of $\pi_1(Y)$ is at most 2.

As for the second statement of the Corollary, we observe that any subcomplex $Y' \subset Y$ with $b_2(Y') = 0$ cannot contain S^2, Z_2, Z_3. It also cannot contain copies of the real projective plane with at most 20 faces, as explained above. Thus, by Theorem 5.5, Y' must be aspherical, a.a.s. ∎

Remark 5.10: The method of the proof of Corollary 5.9 gives in fact a stronger result: for $p \ll n^{-3/5}$ a random 2-complex $Y \in Y(n, p)$ has the following property with probability tending to one: the fundamental group of any subcomplex $Y' \subset Y$ has cohomological dimension at most 2, i.e., $cd(\pi_1(Y')) \leq 2$.

Using Theorem 5.5 one may generate random aspherical 2-complexes as follows. Start with a random 2-complex $Y \in Y(n, p)$ where $p \ll n^{-1/2-\epsilon}$. Let $\mathcal{A}_\epsilon = \{Z\}$ be the set of all isomorphism types of minimal cycles having at most $2\epsilon^{-1}$ faces with the property $\mu(Z) > 1/2$. The set $\mathcal{A}_\epsilon$ is finite; each $Z \in \mathcal{A}_\epsilon$ is homeomorphic to one of the 2-complexes Z_1, Z_2, Z_3, Z_4 described in Theorem 2.5. Now one finds all subcomplexes of the random complex Y which are isomorphic to the complexes $Z \in \mathcal{A}_\epsilon$ and removes randomly (or by applying certain rule) one of the faces[4]   which satisfies the condition of Corollary 2.6.

The obtained complex $Y' \subset Y$ is aspherical, a.a.s., by Theorem 5.5; clearly $\pi_1(Y') = \pi_1(Y)$.

---

[4] Note that in the case of $Z_4 = P^2 \cup \Delta^2$ we remove a simplex of P^2.

In the next statement we show that the aspherical 2-complex Y' has large second Betti number and hence the fundamental group $\pi_1(Y') = \pi_1(Y)$ has cohomological dimension 2. Combined with Corollary 5.9, Proposition 5.11 implies Theorem A from the Introduction.

PROPOSITION 5.11: *Assume that for some $c > 3$ and $\epsilon > 0$ one has*

$$(24) \qquad \frac{c}{n} < p < n^{-1/2-\epsilon}.$$

Then the second Betti number of the random aspherical 2-complex Y' described in the preceding paragraph satisfies

$$(25) \qquad n^2 \cdot \frac{c-3}{8} \le b_2(Y') \le n^{5/2-\epsilon},$$

a.a.s.

This result is similar to Theorem 3 from [6] and its proof uses the same sequence of arguments.

Proof. Let $f_2, b_2 : Y(n,p) \to \mathbf{Z}$ denote the number of 2-simplexes in a random complex and the second Betti number, viewed as random variables. Note that f_2 is binomially distributed with expectation $p\binom{n}{3}$. We have $f_2 - \binom{n-1}{2} \le b_2 \le f_2$. The inequality (2.6) on page 26 of [10] with $t = n^{7/4}$ gives

$$\mathbb{P}\left(f_2 \le p\binom{n}{3} - t\right) \le \exp\left(-\frac{t^2}{2p\binom{n}{3}}\right) \le \exp(-\sqrt{n}).$$

We see that with probability at least $1 - \exp(-\sqrt{n})$, one has

$$b_2 \ge f_2 - \binom{n-1}{2} \ge p\binom{n}{3} - \binom{n-1}{2} - t \ge \binom{n-1}{2} \cdot \left(\frac{c-3}{3}\right) - t \ge n^2 \cdot \frac{c-3}{7}$$

for large n; here $b_2 = b_2(Y)$.

For any $Z \in \mathcal{A}_\epsilon$ consider the random variable $k_Z : Y(n,p) \to \mathbf{Z}$, where for $Y \in Y(n,p)$ the value $k_Z(Y)$ is the number of subcomplexes of Y which are isomorphic (as simplicial complexes) to Z. The expectation of k_Z satisfies $\mathbb{E}(k_Z) \le n^{v(Z)}p^{f_2(Z)}$. For any $Z \in \mathcal{A}_\epsilon$ one has

$$L(Z) \le 0 \quad \text{and} \quad \chi(Z) \le 2$$

implying

$$\mu(Z) = \frac{v(Z)}{f_2(Z)} \le \frac{1}{2} + \frac{2}{f_2(Z)},$$

as follows from formula (2). Hence we have

$$v(Z) \leq f_2(Z)/2 + 2$$

and therefore

$$\begin{aligned}
\mathbb{E}(k_Z) &\leq n^{v(Z)} p^{f_2(Z)} \\
&\leq n^{f_2(Z)/2+2} p^{f_2(Z)} \\
&\leq n^{f_2(Z)/2+2} n^{(-1/2-\epsilon)f_2(Z)} \\
&= n^{2-\epsilon f_2(Z)} \leq n^{2-4\epsilon}.
\end{aligned}$$

Let $k = \sum_{Z \in \mathcal{A}_\epsilon} k_Z$ be the sum random variable. Then we obtain

$$\mathbb{E}(k) = \sum_Z \mathbb{E}(k_Z) \leq |\mathcal{A}_\epsilon| \cdot n^{2-4\epsilon}.$$

The Markov inequality $\mathbb{P}(k \geq t) \leq \frac{\mathbb{E}(k)}{t}$ with $t = n^{2-3\epsilon}$ gives

$$\mathbb{P}(k \geq n^{2-3\epsilon}) \leq |\mathcal{A}_\epsilon| \cdot n^{-\epsilon}.$$

Thus, with probability tending to one as $n \to \infty$, one has

$$b_2(Y') \geq b_2(Y) - k(Y) \geq n^2 \cdot \frac{c-3}{7} - n^{2-3\epsilon} \geq n^2 \cdot \frac{c-3}{8} > 0$$

for $n \to \infty$. This proves the left inequality in (25).

From inequality (2.5) on page 26 of [10] with $t = n^{7/4}$ one obtains that with probability at least $1 - \exp(-\sqrt{n})$ one has $f_2 \leq p\binom{n}{3} + t$ and hence $b_2(Y') \leq b_2(Y) \leq f_2 \leq p\binom{n}{3} + t \leq n^{5/2-\epsilon}$. This proves the right inequality in (25).   ∎

As above, for a positive integer m, let M_m denote the Moore surface, i.e., the quotient space of the disc $D^2 \subset \mathbf{C}$ (the unit disc on the complex plane) where points $z_1, z_2 \in \partial D^2$ are identified iff $z_1^m = z_2^m$.

COROLLARY 5.12: *If $p \ll n^{-1/2-\epsilon}$, then a random 2-complex $Y \in Y(n, p)$ with probability tending to one as $n \to \infty$ has the following property: no subcomplex $Y' \subset Y$ homeomorphic to a Moore surface M_m with $m \geq 3$.*

Proof. This is a corollary of Theorem 5.5, since for $m \geq 3$ the Moore surface M_m (a) contains no subcomplexes homeomorphic to the sphere, to the real projective space and to complexes Z_2, Z_3 (shown in Figure 3) and (b) the Moore surface M_m is not aspherical.   ∎

A similar argument provides an alternative proof of Corollary 3.7.

6. Appendix: Proof of Theorem 3.4

The proof of Theorem 3.4 given below is similar to the arguments of [2] and is based on two auxiliary results: (1) the local-to-global principle of Gromov [9] and (2) Theorem 6.2 giving uniform isoperimetric constants for complexes satisfying $\tilde{\mu}(X) \geq 1/2 + \epsilon$.

The local-to-global principle of Gromov can be stated as follows:

THEOREM 6.1: *Let X be a finite 2-complex and let $C > 0$ be a constant such that any pure subcomplex $S \subset X$ having at most $(44)^3 \cdot C^{-2}$ two-dimensional simplexes satisfies $I(S) \geq C$. Then $I(X) \geq C \cdot 44^{-1}$.*

Theorem 6.1 follows from Theorem 3.9 from [2]. Indeed, suppose that the assumptions of Theorem 6.1 are satisfied. Let $\gamma : S^1 \to X$ be a simplicial loop with $A_X(\gamma) < (44)^3 \cdot C^{-2} = 44\rho^2$, where $\rho = 44/C$. Then there is a pure subcomplex $S \subset X$ with at most $(44)^3 \cdot C^{-2}$ faces, which contains γ and the minimal spanning disc for γ in X, such that

$$\frac{|\gamma|}{A_S(\gamma)} \geq I(S) \geq C = 44/\rho,$$

by our assumption, i.e.,

$$A_X(\gamma) = A_S(\gamma) \leq \frac{\rho}{44} \cdot |\gamma|.$$

Applying Theorem 3.9 from [2] we obtain $I(X) \geq \rho^{-1} = C/44$.

Let X be a 2-complex satisfying $\tilde{\mu}(X) > 1/2$. Then by Corollary 2.10 the fundamental group of X is hyperbolic as it is a free product of several copies of cyclic groups $\mathbf{Z}$ and $\mathbf{Z}_2$. Hence, $I(X) > 0$. The following theorem gives a uniform lower bound for the numbers $I(X)$.

THEOREM 6.2: *Given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for any finite pure 2-complex X with $\tilde{\mu}(X) \geq 1/2 + \epsilon$ one has $I(X) \geq C_\epsilon$.*

This statement is equivalent to Lemma 3.5 from [2] and plays a crucial role in the proof of Theorem 3.4. A proof of Theorem 6.2 (which uses the method of [2] but is presented in a slightly different form) is given below in §6.1.

Proof of Theorem 3.4 using Theorem 6.1 and Theorem 6.2. Let C_ϵ be the constant given by Theorem 6.2. Consider the set $\mathcal{S}$ of isomorphism types of all pure 2-complexes having at most $44^3 \cdot C_\epsilon^{-2}$ faces. Clearly, the set $\mathcal{S}$ is finite. We may present it as the disjoint union $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$ where any $S \in \mathcal{S}_1$ satisfies $\tilde{\mu}(S) \geq 1/2 + \epsilon$ while for $S \in \mathcal{S}_2$ one has $\tilde{\mu}(S) < 1/2 + \epsilon$. By Lemma 1.1, a random 2-complex $Y \in Y(n,p)$ contains as subcomplexes complexes $S \in \mathcal{S}_2$ with probability tending to zero as $n \to \infty$. Hence, $Y \in Y(n,p)$ may contain as subcomplexes only complexes $S \in \mathcal{S}_1$, a.a.s. By Theorem 6.2, any $S \in \mathcal{S}_1$ satisfies $I(S) \geq C_\epsilon$. Hence we see that with probability tending to one, any subcomplex S of Y having at most $44^3 \cdot C_\epsilon^{-2}$ faces satisfies $I(S) \geq C_\epsilon$. Now applying Theorem 6.1 we obtain $I(Y') \geq C_\epsilon \cdot 44^{-1} = c_\epsilon$, for any subcomplex $Y' \subset Y$, a.a.s. ∎

6.1. PROOF OF THEOREM 6.2.

Definition 6.3: We will say that a finite 2-complex X is **tight** if for any proper subcomplex $X' \subset X$, $X' \neq X$, one has $I(X') > I(X)$.

Clearly, one has

$$(26) \qquad\qquad I(X) \geq \min\{I(Y)\}$$

where $Y \subset X$ is a tight subcomplex. Since $\tilde{\mu}(Y) \geq \tilde{\mu}(X)$ for $Y \subset X$, it is obvious from (26) that it is enough to prove Theorem 6.2 under the additional assumption that X is tight.

Remark 6.4: Suppose that X is pure and tight and suppose that $\gamma : S^1 \to X$ is a simplicial loop with $|\gamma| \cdot A_X(\gamma)^{-1}$ less than the minimum of the numbers $I(X')$, where $X' \subset X$ is a proper subcomplex. Let $b : D^2 \to X$ be a minimal spanning disc for γ; then $b(D^2) = X$, i.e., b is surjective. Indeed, if the image of b does not contain a 2-simplex σ, then removing it we obtain a subcomplex $X' \subset X$ with $A_{X'}(\gamma) = A_X(\gamma)$ and hence $I(X') \leq |\gamma| \cdot A_X(\gamma)^{-1}$, contradicting the assumption on γ.

LEMMA 6.5: *If X is a tight complex with $\tilde{\mu}(X) > 1/2$, then $b_2(X) = 0$.*

Proof. Assume that $b_2(X) \neq 0$. Then there exists a minimal cycle $Z \subset X$ satisfying $\mu(Z) > 1/2$. Hence, by Corollary 2.6, we may find a 2-simplex $\sigma \subset Z \subset X$ such that $\partial\sigma$ is null-homotopic in $Z - \sigma \subset X - \sigma = X'$. Note that $X'^{(1)} = X^{(1)}$ and a simplicial curve $\gamma : S^1 \to X'$ is null-homotopic in X' if and only if it is

null-homotopic in X. Besides, $A_X(\gamma) \leq A_{X'}(\gamma)$ and hence

$$\frac{|\gamma|}{A_X(\gamma)} \geq \frac{|\gamma|}{A_{X'}(\gamma)},$$

which implies that $I(X) \geq I(X') > I(X)$, a contradiction. ∎

LEMMA 6.6: *Given $\epsilon > 0$, there exists a constant $C'_\epsilon > 0$ such that for any finite pure tight connected 2-complex with $\tilde{\mu}(X) \geq 1/2 + \epsilon$ and $L(X) \leq 0$ one has $I(X) \geq C'_\epsilon$.*

This Lemma is similar to Theorem 6.2, but it has an additional assumption that $L(X) \leq 0$. It is clear from the proof that the assumption $L(X) \leq 0$ can be replaced by any assumption of the type $L(X) \leq 1000$, i.e., by any specific upper bound, without altering the proof.

Proof. We show that the number of isomorphism types of complexes X satisfying the conditions of the Lemma is finite; hence the statement of the Lemma follows by setting $C'_\epsilon = \min I(X)$ and using Corollary 2.10, which gives $I(X) > 0$ (since $\pi_1(X)$ is hyperbolic) and hence $C'_\epsilon > 0$. The inequality

$$\mu(X) = \frac{1}{2} + \frac{2\chi(X) + L(X)}{2f(X)} \geq \frac{1}{2} + \epsilon$$

is equivalent to

$$f(X) \leq \epsilon^{-1} \cdot (\chi(X) + L(X)/2),$$

where $f(X)$ denotes the number of 2-simplexes in X. By Lemma 6.5 we have $\chi(X) = 1 - b_1(X) \leq 1$ and, using the assumption $L(X) \leq 0$, we obtain $f(X) \leq \epsilon^{-1}$. This implies the finiteness of the set of possible isomorphism types of X and completes the proof. ∎

We will also use a relative isoperimetric constant $I(X, X') \in \mathbf{R}$ for a pair consisting of a finite 2-complex X and its subcomplex $X' \subset X$ which will be defined as the infimum of all ratios $|\gamma| \cdot A_X(\gamma)^{-1}$, where $\gamma : S^1 \to X'$ runs over simplicial loops in X' which are null-homotopic in X. Clearly, $I(X, X') \geq I(X)$ and $I(X, X') = I(X)$ if $X' = X$. Below is a useful strengthening of Lemma 6.6.

LEMMA 6.7: *Given $\epsilon > 0$, let $C'_\epsilon > 0$ be the constant given by Lemma 6.6. Then for any finite pure tight connected 2-complex with $\tilde{\mu}(X) \geq 1/2 + \epsilon$ and for a connected subcomplex $X' \subset X$ satisfying $L(X') \leq 0$, one has $I(X, X') \geq C'_\epsilon$.*

Proof. We show below that under the assumptions on X, X' one has

$$(27) \qquad\qquad I(X, X') \geq \min_Y I(Y),$$

where Y runs over all subcomplexes $X' \subset Y \subset X$ satisfying $L(Y) \leq 0$. Clearly, $\tilde{\mu}(Y) \geq 1/2 + \epsilon$ for any such Y. In the proof of Lemma 6.6 we showed that $b_2(X) = 0$, which implies that $b_2(Y) = 0$. Besides, without loss of generality we may assume that Y is connected. The arguments of the proof of Lemma 6.6 now apply (i.e., Y may have finitely many isomorphism types, each having a hyperbolic fundamental group) and the result follows: $I(X, X') \geq \min_Y I(Y) \geq C'_\epsilon$.

Suppose that inequality (27) is false, i.e., $I(X, X') < \min_Y I(Y)$, and consider a simplicial loop $\gamma : S^1 \to X'$ satisfying $\gamma \sim 1$ in X and $|\gamma| \cdot A_X(\gamma)^{-1} < \min_Y I(Y)$. Let $\psi : D^2 \to X$ be a simplicial spanning disc of minimal area. It follows from the arguments of Ronan [16] that ψ is non-degenerate in the following sense: for any 2-simplex σ of D^2 the image $\psi(\sigma)$ is a 2-simplex, and for two distinct 2-simplexes σ_1, σ_2 of D^2 with $\psi(\sigma_1) = \psi(\sigma_2)$ the intersection $\sigma_1 \cap \sigma_2$ is either $\emptyset$ or a vertex of D^2. In other words, we exclude **foldings**, i.e., situations such that $\psi(\sigma_1) = \psi(\sigma_2)$ and $\sigma_1 \cap \sigma_2$ is an edge. Consider $Z = X' \cup \psi(D^2)$. Note that $L(Z) \leq 0$. Indeed, since

$$L(Z) = \sum_e (2 - \deg_Z(e)),$$

where e runs over the edges of Z, we see that for $e \subset X'$, $\deg_{X'}(e) \leq \deg_Z(e)$, and for a newly created edge $e \subset \psi(D^2)$, clearly $\deg_Z(e) \geq 2$. Hence,

$$L(Z) \leq L(X') \leq 0.$$

On the other hand, $A_X(\gamma) = A_Z(\gamma)$ and hence

$$I(Z) \leq |\gamma| \cdot A_X(\gamma)^{-1} < \min_Y I(Y),$$

a contradiction. ■

The main idea of the proof of Theorem 6.2 in the general case is to find a planar complex (a "singular surface") Σ with one boundary component $\partial_+ \Sigma$ being the initial loop and such that "the rest of the boundary" $\partial_- \Sigma$ is a "product of negative loops" (i.e., loops satisfying Lemma 6.7). The essential part of the proof is in estimating the area (the number of 2-simplexes) of such Σ.

Proof of Theorem 6.2. Consider a connected tight pure 2-complex X satisfying

$$\tilde{\mu}(X) \geq \frac{1}{2} + \epsilon \tag{28}$$

and a simplicial prime loop $\gamma : S^1 \to X$ such that the ratio $|\gamma| \cdot A_X(\gamma)^{-1}$ is less than the minimum of the numbers $I(X')$ for all proper subcomplexes $X' \subset X$. Consider a minimal spanning disc $b : D^2 \to X$ for $\gamma = b|_{\partial D^2}$; here D^2 is a triangulated disc and b is a simplicial map. As we showed in Remark 6.4, the map b is surjective. As explained in the proof of Lemma 6.7, due to arguments of Ronan [16], we may assume that b has no foldings.

For any integer $i \geq 1$ we denote by $X_i \subset X$ the pure subcomplex generated by all 2-simplexes σ of X such that the preimage $b^{-1}(\sigma) \subset D^2$ contains $\geq i$ two-dimensional simplexes. One has $X = X_1 \supset X_2 \supset X_3 \supset \cdots$. Each X_i may have several connected components and we will denote by Λ the set labelling all the connected components of the disjoint union $\bigsqcup_{i \geq 1} X_i$. For $\lambda \in \Lambda$ the symbol X_λ will denote the corresponding connected component of $\bigsqcup_{i \geq 1} X_i$ and the symbol $i = i(\lambda) \in \{1, 2, \ldots\}$ will denote the index $i \geq 1$ such that X_λ is a connected component of X_i, viewed as a subset of $\bigsqcup_{i \geq 1} X_i$. We endow Λ with the following partial order:

$$\lambda_1 \leq \lambda_2 \quad \text{iff} \quad X_{\lambda_1} \supset X_{\lambda_2}$$

(where X_{λ_1} and X_{λ_2} are viewed as subsets of X) and $i(\lambda_1) \leq i(\lambda_2)$.

Next we define the sets

$$\Lambda^- = \{\lambda \in \Lambda; L(X_\lambda) \leq 0\}$$

and

$$\Lambda^+ = \{\lambda \in \Lambda; \text{ for any } \mu \in \Lambda \text{ with } \mu \leq \lambda, L(X_\mu) > 0\}.$$

Finally we consider the following subcomplex of the disk D^2:

$$\Sigma' = D^2 - \bigcup_{\lambda \in \Lambda^-} \text{Int}(b^{-1}(X_\lambda)) \tag{29}$$

and we shall denote by Σ the connected component of Σ' containing the boundary circle ∂D^2.

Recall that for a 2-complex X the symbol $f(X)$ denotes the number of 2-simplexes in X. We have

$$f(D^2) = \sum_{\lambda \in \Lambda} f(X_\lambda) \tag{30}$$

and

$$(31) \qquad f(\Sigma) \leq f(\Sigma') \leq \sum_{\lambda \in \Lambda^+} f(X_\lambda).$$

Formula (30) follows from the observation that any 2-simplex of $X = b(D^2)$ contributes to the RHS of (30) as many units as its multiplicity (the number of its preimages under b). Formula (31) follows from (30) and from the fact that for a 2-simplex σ of Σ' the image $b(\sigma)$ lies always in the complexes X_λ with $L(X_\lambda) > 0$.
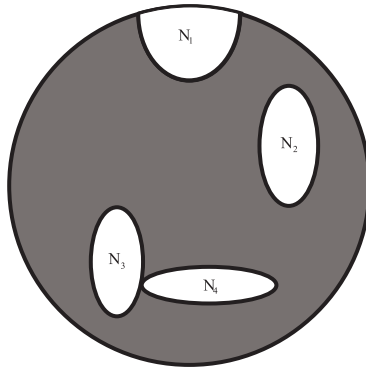

Figure 7. The complex $\Sigma \subset D^2$.

LEMMA 6.8: *One has the following inequality:*

$$(32) \qquad \sum_{\lambda \in \Lambda^+} L(X_\lambda) \leq |\partial D^2|.$$

Proof. For an edge e of X and for $\lambda \in \Lambda$ we denote

$$\widetilde{\deg}_\lambda(e) = \begin{cases} 2 - \deg_{X_\lambda}(e), & \text{if } e \subset X_\lambda, \\ 0, & \text{if } e \not\subset X_\lambda. \end{cases}$$

Note that $\widetilde{\deg}_\lambda(e) \le 1$ and $\widetilde{\deg}_\lambda(e) = 1$ iff e belongs to a unique 2-simplex of X_λ, i.e., $e \subset \partial X_\lambda$. One has

$$\sum_{\lambda \in \Lambda^+} L(X_\lambda) = \sum_{e \subset X} \sum_{\lambda \in \Lambda^+} \widetilde{\deg}_\lambda(e) \le \sum_{e \subset X} \max\left\{ \sum_{\Lambda^+} \widetilde{\deg}_\lambda(e), 0 \right\}$$

$$(33) \qquad\qquad\qquad\qquad\qquad\qquad \le \sum_{e \subset X} \max\left\{ \sum_{\Lambda} \widetilde{\deg}_\lambda(e), 0 \right\}$$

$$(34) \qquad\qquad\qquad\qquad\qquad\qquad \le \sum_{e \subset X} |b^{-1}(e) \cap \partial D^2| = |\partial D^2|.$$

Here $|b^{-1}(e) \cap \partial D^2|$ denotes the number of boundary edges which b maps onto the edge e. The inequality (33) follows from the fact that $\lambda \mapsto \widetilde{\deg}_\lambda(e)$ is a non-decreasing function on the poset $\Lambda_e = \{\lambda \in \Lambda; e \subset X_\lambda\}$. Note that Λ_e is in fact linearly ordered. Indeed, suppose that $\lambda_1, \lambda_2 \in \Lambda_e$ are such that $i_1 = i(\lambda_1) \ge i_2 = i(\lambda_2)$. Then X_{λ_2}, as a path-component of X_{i_2}, is contained in a path-component of X_{i_1}; the latter must coincide with X_{λ_1} since the intersection $X_{\lambda_1} \cap X_{\lambda_2} \ne \emptyset$ is nonempty (since it contains the edge e). Thus we see that the assumption $i(\lambda_1) \ge i(\lambda_2)$ implies that $X_{\lambda_2} \subset X_{\lambda_1}$, i.e., $\lambda_1 \ge \lambda_2$. Denoting $\Lambda_e \cap \Lambda^+ = \Lambda_e^+$ we observe that if the sum

$$(35) \qquad\qquad \sum_{\Lambda^+} \widetilde{\deg}_\lambda(e) = \sum_{\Lambda_e^+} \widetilde{\deg}_\lambda(e) > 0$$

is positive, then (due to the monotonicity) one has $\widetilde{\deg}_{\lambda_0}(e) > 0$ for the maximal element λ_0 of Λ_e^+ and hence $\widetilde{\deg}_\lambda(e) > 0$ for all $\lambda \ge \lambda_0$, $\lambda \in \Lambda_e$. Thus, the positivity (35) implies $\sum_{\Lambda^+} \widetilde{\deg}_\lambda(e) < \sum_{\Lambda} \widetilde{\deg}_\lambda(e)$, giving (33).

To prove the inequality (34) we observe that for an edge e of X one has

$$(36) \qquad \sum_{\lambda \in \Lambda} \deg_{X_\lambda}(e) = \sum_{e' \in b^{-1}(e)} \deg_{D^2}(e') = 2|b^{-1}(e)| - |b^{-1}(e) \cap \partial D^2|$$

and hence

$$\sum_{\lambda \in \Lambda} \widetilde{\deg}_\lambda(e) = 2 \cdot |\Lambda^e| - \sum_{\lambda \in \Lambda} \deg_\lambda(e)$$

$$(37) \qquad\qquad\qquad \le 2 \cdot |b^{-1}(e)| - \sum_{\lambda \in \Lambda} \deg_\lambda(e)$$

$$\qquad\qquad\qquad = |b^{-1}(e) \cap \partial D^2| \quad \text{(using (36))}.$$

Inequality (37) is based on

$$(38) \qquad |\Lambda_e| = \max_{\sigma \supset e} |b^{-1}(\sigma)| \leq |b^{-1}(e)|,$$

where σ runs over all 2-simplexes of X containing e. The equality which appears in (38) is a consequence of the set Λ_e being linearly ordered; see above. This completes the proof. ∎

Now we continue with the proof of Theorem 6.2. Consider a tight pure 2-complex X satisfying (28) and a simplicial loop $\gamma : S^1 \to X$ as above. We will use the notation introduced earlier. The complex Σ is a connected subcomplex of the disk D^2; it contains the boundary circle ∂D^2 which we will denote also by $\partial_+ \Sigma$. The closure of the complement of Σ,

$$N = \overline{D^2 - \Sigma} \subset D^2,$$

is a pure 2-complex. Let $N = \bigcup_{j \in J} N_j$ be the strongly connected components of N. Each N_j is PL-homeomorphic to a disc and we define

$$\partial_- \Sigma = \bigcup_{j \in J} \partial N_j,$$

the union of the circles ∂N_j which are the boundaries of the strongly connected components of N. It may happen that $\partial_+ \Sigma$ and $\partial_- \Sigma$ have nonempty intersection. Also, the circles forming $\partial_- \Sigma$ may not be disjoint.

We claim that for any $j \in J$ there exists $\lambda \in \Lambda^-$ such that $b(\partial N_j) \subset X_\lambda$. Indeed, let $\lambda_1, \ldots, \lambda_r \in \Lambda^-$ be the minimal elements of Λ^- with respect to the partial order introduced earlier. The complexes $X_{\lambda_1}, \ldots, X_{\lambda_r}$ are connected and pairwise disjoint and for any $\lambda \in \Lambda^-$ the complex X_λ is a subcomplex of one of the sets X_{λ_i}, where $i = 1, \ldots, r$. From our definition (29) it follows that the image of the circle $b(\partial N_j)$ is contained in the union $\bigcup_{i=1}^r X_{\lambda_i}$, but since $b(\partial N_j)$ is connected it must lie in one of the sets X_{λ_i}.

We may apply Lemma 6.7 to each of the circles ∂N_j. We obtain that each of the circles ∂N_j admits a spanning discs of area $\leq K_\epsilon |\partial N_j|$, where $K_\epsilon = C_\epsilon'^{-1}$ is the inverse of the constant given by Lemma 6.7. Using the minimality of the disc D^2 we obtain that the circles ∂N bound in D^2 several discs with the total area $A \leq K_\epsilon \cdot |\partial_- \Sigma|$ (here we use Lemma 6.6); otherwise one could reduce the area of the initial disc.

For $\lambda \in \Lambda^+$ one has $L(X_\lambda) \geq 1$ and $\chi(X_\lambda) \leq 1$ (since $b_2(X_\lambda) = 0$), hence we have

$$3L(X_\lambda) \geq 2\chi(X_\lambda) + L(X_\lambda) \geq 2\epsilon f(X_\lambda),$$

where on the last stage we used the inequality $\mu(X_\lambda) \geq 1/2 + \epsilon$. Summing up we get

$$f(\Sigma) \leq \sum_{\lambda \in \Lambda^+} f(X_\lambda) \leq \frac{3}{2\epsilon} \sum_{\lambda \in \Lambda^+} L(X_\lambda) \leq \frac{3}{2\epsilon} |\partial D^2|.$$

The rightmost inequality is given by Lemma 6.8.

Next, we observe that

$$(39) \qquad\qquad |\partial_- \Sigma| \leq 2f(\Sigma) + |\partial_+ \Sigma|.$$

To explain this inequality we note that each edge of $\partial_- \Sigma$ which does not belong to $\partial_+ \Sigma$ is incident to a face of Σ and a face of Σ can have at most two edges lying on $\partial_- \Sigma$. Therefore, we obtain

$$\begin{aligned} f(D^2) \leq &f(\Sigma) + A \leq \frac{3}{2\epsilon}|\gamma| + K_\epsilon \cdot 2 \cdot f(\Sigma) + K_\epsilon|\gamma| \\ \leq &\left(\frac{3}{2\epsilon}(1 + 2K_\epsilon) + K_\epsilon\right) \cdot |\gamma|, \end{aligned}$$

implying

$$(40) \qquad\qquad I(X) \geq \frac{2\epsilon}{3 + 6K_\epsilon + 2\epsilon K_\epsilon}.$$

This completes the proof of Theorem 6.2. ∎

References

[1] L. Aronshtam, N. Linial, T. Łuczak and R. Meshulam, *Collapsibility and vanishing of top homology in random simplicial complexes*, Discrete & Computational Geometry **49** (2013), 317–334.

[2] E. Babson, C. Hoffman and M. Kahle, *The fundamental group of random 2-complexes*, Journal of the American Mathematical Society **24** (2011), 1–28; see also the latest archive version arXiv:0711.2704 revised on 20.09.2012.

[3] W. A. Bogley, *J. H. C. Whitehead's asphericity question*, in *Two-dimensional Homotopy and Combinatorial Group Theory*, London Mathematical Society Lecture Note Series, Vol. 197, Cambridge University Press, Cambridge, 1993, pp. 309–334.

[4] W. H. Cockcroft, *On two-dimensional aspherical complexes*, Proceedings of the London Mathematical Society **4** (1954), 375–384.

[5] D. Cohen, A. E. Costa, M. Farber and T. Kappeler, *Topology of random 2-complexes,* Discrete & Computational Geometry **47** (2012), 117–149.

[6] A. E. Costa and M. Farber, *The asphericity of random 2-dimensional complexes*, Random Structures and Algorithms **46** (2015), 261–273.

[7] P. Erdős and A. Rényi, *On the evolution of random graphs*, Publications of the Mathematical Institute of the Hungarian Academy of Sciences **5** (1960), 17–61.

[8] M. Gromov, *Filling Riemannian manifolds*, Jornal Differential Geometry **18** (1983), 1–147.

[9] M. Gromov, *Hyperbolic groups*, in *Essays in Group Theory*, Mathematical Sciences Research Institute Publications, Vol. 8, Springer, New York, 1987, pp. 75–265.

[10] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley Interscience Series in Discrete Mathematics, and Optimization, Wiley-Interscience, New York, 2000.

[11] M. G. Katz, Y. B. Rudyak and S. Sabourau, *Systoles of 2-complexes, Reeb graphs, and Grushko decomposition*, International Mathematics Research Notices (2006), Art. ID 54936, 30 pp.

[12] D. Kozlov, *The threshold function for vanishing of the top homology group of random d-complexes*, Proceedings of the American Mathematical Society **138** (2010), 4517–4527.

[13] N. Linial and R. Meshulam, *Homological connectivity of random 2-complexes*, Combinatorica **26** (2006), 475–487.

[14] R. Meshulam and N. Wallach, *Homological connectivity of random k-complexes*, Random Structures & Algorithms **34** (2009), 408–417.

[15] P. Papasoglu, *Cheeger constants of surfaces and isoperimetric inequalities*, Transactions of the American Mathematical Society **361** (2009), 5139–5162.

[16] M. A. Ronan, *On the second homotopy group of certain simplicial complexes and some combinatorial applications*, Quarterly Journal of Mathematics **32** (1981), 225–233.

[17] S. Rosenbrock, *The Whitehead Conjecture—an overview*, Siberian Electronoc Mathematical Reports **4** (2007), 440–449.