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GAUSSIAN ANALYTIC FUNCTIONS IN THE UNIT BALL

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ABSTRACT

We study some properties of hyperbolic Gaussian analytic functions of intensity L in the unit ball of $\mathbb{C}^n$. First we deal with the asymptotics of fluctuations of linear statistics as $L \to \infty$. Then we estimate the probability of large deviations (with respect to the expected value) of such linear statistics and use this estimate to prove a hole theorem.

Introduction

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$ and let ν denote the Lebesgue measure in $\mathbb{C}^n$ normalised so that $\nu(\mathbb{B}_n) = 1$. Explicitly $d\nu = \frac{n!}{\pi^n} dm = \beta^n$, where dm is the Lebesgue measure and $\beta = \frac{i}{2\pi}\partial\bar{\partial}|z|^2$ is the fundamental form of the Euclidean metric.

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For $L > n$ consider the weighted Bergman space

$$B_L(\mathbb{B}_n) = \Big\{ f \in H(\mathbb{B}_n) : \|f\|^2_{n,L} := c_{n,L} \int_{\mathbb{B}_n} |f(z)|^2 (1-|z|^2)^L d\mu(z) < +\infty \Big\},$$

where

$$d\mu(z) = \frac{d\nu(z)}{(1-|z|^2)^{n+1}}, \tag{1}$$

and $c_{n,L} = \frac{\Gamma(L)}{n!\Gamma(L-n)}$ is chosen so that $\|1\|_{n,L} = 1$.

Let

$$e_\alpha(z) = \Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)} \Big)^{1/2} z^\alpha$$

denote the normalisation of the monomial z^α in the norm $\|\cdot\|_{n,L}$, so that $\{e_\alpha\}_\alpha$ is an orthonormal basis of $B_L(\mathbb{B}_n)$. As usual, here we denote $z = (z_1, \ldots, z_n)$ and use the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

The **hyperbolic Gaussian analytic function** (GAF) of **intensity** L is defined as

$$f_L(z) = \sum_\alpha a_\alpha \Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)} \Big)^{1/2} z^\alpha, \quad z \in \mathbb{B}_n,$$

where a_α are i.i.d. complex Gaussians of mean 0 and variance 1 ($a_\alpha \sim N_{\mathbb{C}}(0,1)$).

We choose the orthonormal basis $\{e_\alpha\}_\alpha$ for convenience, but any other basis would produce the same covariance kernel (see below) and therefore the same results.

The sum defining f_L can be analytically continued to $L > 0$, hence the discussion below is also valid for all such L. However, since we only study asymptotic properties of f_L as L tends to infinity, we shall not really use this extension.

The characteristics of the hyperbolic GAF are determined by its covariance kernel, which is given by (see [ST04, Section 1], [Sto94, pp. 17–18])

$$\begin{aligned} K_L(z,w) =& \mathbb{E}[f_L(z)\overline{f_L(w)}] = \sum_\alpha \frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)} z^\alpha \bar{w}^\alpha \\ &= \sum_{m=0}^\infty \frac{\Gamma(L+m)}{\Gamma(L)} \sum_{\alpha:|\alpha|=m} \frac{1}{\alpha!} z^\alpha \bar{w}^\alpha \\ &= \sum_{m=0}^\infty \frac{\Gamma(L+m)}{m!\Gamma(L)} (z\cdot\bar{w})^m = \frac{1}{(1-z\cdot\bar{w})^L}. \end{aligned}$$

A main feature of the hyperbolic GAF is that the distribution of its zero set

$$Z_{f_L} = \{z \in \mathbb{B}_n;\ f_L(z) = 0\}$$

is invariant under the group $\mathrm{Aut}(\mathbb{B}_n)$ of holomorphic automorphisms of the ball. Given $w \in \mathbb{B}_n$ there exists $\phi_w \in \mathrm{Aut}(\mathbb{B}_n)$ such that $\phi_w(w) = 0$ and $\phi_w(0) = w$, and all automorphisms are essentially of this form: for all $\psi \in \mathrm{Aut}(B_n)$ there exist $w \in \mathbb{B}_n$ and $\mathcal{U}$ in the unitary group such that $\psi = \mathcal{U}\phi_w$ (see [Rud08, 2.2.5]). Then the **pseudo-hyperbolic distance** ϱ in $\mathbb{B}_n$ is defined as

$$\varrho(z, w) = |\phi_w(z)|, \quad z, w \in \mathbb{B}_n,$$

and the corresponding pseudo-hyperbolic balls as

$$E(w, r) = \{z \in \mathbb{B}_n : \varrho(z, w) < r\}, \quad r < 1.$$

There is an immediate relation between the normalised covariance kernel

$$\theta_L(z, w) = \frac{K_L(z, w)}{\sqrt{K_L(z, z)}\sqrt{K_L(w, w)}} = \frac{(1 - |z|^2)^{L/2}(1 - |w|^2)^{L/2}}{(1 - \bar{z} \cdot w)^L}$$

and the pseudo-hyperbolic distance, given by the identity

$$1 - |\phi_w(z)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z} \cdot w|^2}. \tag{2}$$

The transformations

$$T_w(f)(z) = \Big(\frac{1 - |w|^2}{(1 - \bar{w} \cdot z)^2}\Big)^{L/2} f(\phi_w(z))$$

are isometries of $B_L(\mathbb{B}_n)$, hence the random zero sets Z_{f_L} and $Z_{f_L \circ \phi_w}$ have the same distribution. More specifically, the distribution of the (random) integration current

$$[Z_{f_L}] = \frac{i}{2\pi} \partial\bar{\partial} \log |f_L|^2$$

is invariant under automorphisms of the unit ball.

The Edelman–Kostlan formula (see [HKPV09, Section 2.4] and [Sod00, Theorem 1]) gives the so-called **first intensity** of the GAF:

$$\mathbb{E}[Z_{f_L}] = \frac{i}{2\pi} \partial\overline{\partial} \log K_L(z, z) = L\, \omega(z),$$

where ω is the invariant form

$$\omega(z) = \frac{i}{2\pi} \partial\overline{\partial} \log\Big(\frac{1}{1 - |z|^2}\Big) = \frac{1}{(1 - |z|^2)^2} \frac{i}{2\pi} \sum_{j,k=1}^{n} [(1 - |z|^2)\delta_{j,k} + z_k \overline{z_j}] dz_j \wedge d\overline{z_k}.$$

Notice that $\mu = \omega^n$ is also invariant by $\mathrm{Aut}(\mathbb{B}_n)$ [Sto94, p. 19].

In this paper we study some statistical properties of the zero variety Z_{f_L} for large values of the intensity L. The outline of the paper is as follows.

In Section 1 we study the fluctuations of linear statistics as the intensity L tends to ∞. Let $\mathcal{D}_{(n-1,n-1)}$ denote the space of compactly supported smooth forms of bidegree $(n-1,n-1)$. For $\varphi \in \mathcal{D}_{(n-1,n-1)}$, consider the integral of φ over Z_{f_L}:

$$I_L(\varphi) = \int_{Z_{f_L}} \varphi = \int_{\mathbb{B}_n} \varphi \wedge [Z_{f_L}].$$

By the Edelman–Kostlan formula,

$$\mathbb{E}[I_L(\varphi)] = L \int_{\mathbb{B}_n} \varphi \wedge \omega. \tag{3}$$

We compute the leading term of $\operatorname{Var}[I_L(\varphi)]$ in the limit as $L \to \infty$ and see that the rate of self-averaging of the integral of $I_L(\varphi)$ increases with the dimension. A quantitative statement is the following.

Theorem 1: *Let φ be a compactly supported, real-valued $(n-1,n-1)$-form with $\mathcal{C}^2$ coefficients and let $D\varphi$ be the function defined by $\frac{i}{2\pi}\partial\bar{\partial}\varphi = D\varphi d\mu$. Then*

$$\operatorname{Var}[I_L(\varphi)] = n!\zeta(n+2)\Big(\int_{\mathbb{B}_n} (D\varphi)^2 d\mu\Big)\frac{1}{L^n} + O\Big(\frac{\log L}{L^{n+1}}\Big).$$

Notice that this shows a strong form of self-averaging of the volume $I_L(\varphi)$, in the sense that

$$\frac{\operatorname{Var} I_L(\varphi)}{(\mathbb{E}[I_L(\varphi)])^2} = \mathrm{O}\Big(\frac{1}{L^{n+2}}\Big).$$

Notice also that the self-averaging increases with the dimension.

The same computations involved in the proof of this theorem show the asymptotic normality of $I_L(\varphi)$, i.e., that the distribution of

$$\frac{I_L(\varphi) - \mathbb{E}[I_L(\varphi)]}{\sqrt{\operatorname{Var}[I_L(\varphi)]}}$$

converges weakly to the (real) standard gaussian (Corollary 5), for each φ.

The proofs are rather straight-forward generalisations of the proof for the one-dimensional case given by Sodin and Tsirelson [ST04], or the analogous result in the context of compact manifolds given by Shiffman and Zelditch, which we now outline.

Let p_N be a Gaussian holomorphic polynomial in $\mathbb{CP}^n$ or, more generally, a section of a power $\mathscr{L}^N$ of a positive Hermitian line bundle $\mathscr{L}$ over an n-dimensional Kähler manifold M. Given a test form φ of bidegree $(n-1,n-1)$, define

$$I_N(\varphi) = \int_{Z_{p_N}} \varphi = \int_M \varphi \wedge [Z_{p_N}].$$

According to [SZ10, Theorem 1.1], for a real-valued $(n-1,n-1)$-form with $\mathcal{C}^3$ coefficients, as $N \to \infty$,

$$\operatorname{Var}[I_N(\varphi)] = \frac{\pi^{n-2}}{4}\zeta(n+2)\|\partial\bar{\partial}\varphi\|_2^2 \, \frac{1}{N^n} + \mathrm{O}\Big(\frac{1}{N^{n+1/2-\epsilon}}\Big).$$

The proof of this result is based on a bi-potential expression of $\operatorname{Var}[I_N(\varphi)]$ (see (4)) together with good estimates of the covariance kernel, something we certainly have for the GAF in the ball.

In Section 2, we deal with large deviations. We study the probability that the deviation of $I_L(\varphi)$ from its expected value is at least a fixed proportion of $\mathbb{E}[I_L(\varphi)]$.

Theorem 2: *For all $\varphi \in \mathcal{D}_{(n-1,n-1)}$ and $\delta > 0$, there exist $c > 0$ and $L_0(\varphi, \delta, n)$ such that for all $L \geq L_0$,*

$$\mathbb{P}[|I_L(\varphi) - \mathbb{E}(I_L(\varphi))| > \delta\mathbb{E}(I_L(\varphi))] \leq e^{-cL^{n+1}}.$$

Replacing $\delta \int_{\mathbb{B}_n} \varphi \wedge \omega$ by δ we get the equivalent formulation:

$$\mathbb{P}\Big[\Big|\frac{1}{L} I_L(\varphi) - \int_{\mathbb{B}_n} \varphi \wedge \omega\Big| > \delta\Big] \leq e^{-cL^{n+1}}.$$

Following the scheme of [SZZ08, p. 1994] we deduce a corollary that implies the upper bound in the hole theorem (Theorem 4 below). For a compactly supported function ψ in $\mathbb{B}_n$ denote

$$I_L(\psi) = \int_{Z_{f_L}} \psi\omega^{n-1} = \int_{\mathbb{B}_n} \psi \wedge \omega^{n-1} \wedge [Z_{f_L}].$$

Notice that (3) gives here

$$\mathbb{E}[I_L(\psi)] = L \int_{\mathbb{B}_n} \psi \, d\mu.$$

In particular, and for an open set U in the ball, let χ_U denote its characteristic function and let $I_L(U) = I_L(\chi_U)$. Then $\mathbb{E}[I_L(U)] = L\mu(U)$.

Corollary 3: *Suppose that U is an open set contained in a compact subset of $\mathbb{B}_n$. For all $\delta > 0$ there exist $c > 0$ and L_0 such that for all $L \geq L_0$,*

$$\mathbb{P}\Big[\Big|\frac{1}{L}I_L(U) - \mu(U)\Big| > \delta\Big] \leq e^{-cL^{n+1}}.$$

The case $n = 1$ of Theorem 2 is given in [Buc13, Theorem 5.7]. Our proof is inspired by the methods of B. Shiffman, S. Zelditch and S. Zrebiec for the study of the analogous problem for compact Kähler manifolds. According to [SZZ08, Theorem 1.5], given $\delta > 0$, and letting ω denote the Kähler form of the manifold,

$$\mathbb{P}\Big[\Big|\frac{1}{N}\int_{Z_{p_N}} \varphi - \frac{1}{\pi}\int_M \omega \wedge \varphi\Big| > \delta\Big] \leq e^{-cN^{n+1}},$$

where here N indicates the power of the positive Hermitian bundle over M.

In the last Section we study the probability that Z_{f_L} has a pseudohyperbolic hole of radius r. By the invariance under automorphisms of the distribution of the zero variety, this is the same as studying the probability that $Z_{f_L} \cap B(0, r) = \emptyset$.

Theorem 4: *Let $r \in (0, 1)$ be fixed. There exist $C_1 = C_1(n, r) > 0$, $C_2 = C_2(n, r) > 0$ and L_0 such that for all $L \geq L_0$,*

$$e^{-C_1 L^{n+1}} \leq \mathbb{P}[Z_{f_L} \cap B(0, r) = \emptyset] \leq e^{-C_2 L^{n+1}}.$$

This result is also inspired by an analogue for entire functions in the plane given by Sodin and Tsirelson [ST05]. Let

$$\mathcal{F}_L = \Big\{ f \in H(\mathbb{C}) : \int_{\mathbb{C}} |f(z)|^2 e^{-L|z|^2} dm(z) < +\infty \Big\}$$

and consider the Gaussian entire function

$$f_L(z) = \sum_{k=0}^{\infty} a_k e_k(z),$$

where a_k are i.i.d. complex standard Gaussians and $\{e_k(z)\}_{k=0}^{\infty}$ is an orthonormal basis of $\mathcal{F}_L$.

The Edelman–Kostlan formula gives

$$\mathbb{E}[Z_{f_L}] = \frac{L}{\pi}\, dm(z),$$

and for a test function φ,

$$I_L(\varphi) = L\int_{\mathbb{C}} \varphi(z)\frac{dm(z)}{\pi} = \int_{\mathbb{C}} \varphi(w/\sqrt{L})\frac{dm(z)}{\pi}.$$

In particular

$$\mathbb{E}[\#(Z_{f_L} \cap D(0,r))] = \mathbb{E}[\#(Z_{f_1} \cap D(0, r\sqrt{L}))],$$

and therefore studying the asymptotics as $L \to \infty$ is equivalent to replacing L by r^2 and letting $r \to \infty$.

Sodin and Tsirelson proved [ST05, Theorem 1] that, as $r \to \infty$,

$$e^{-Cr^4} \leq \mathbb{P}[Z_{f_1} \cap D(0,r) = \emptyset] \leq e^{-cr^4}.$$

Zrebiec extended this result to $\mathbb{C}^n$ [Zre07, Theorem 1.2], showing that the decay rate is then $e^{-Cr^{2n+2}}$, which matches with our Theorem 4.

Shiffman, Zelditch and Zrebiec proved also a hole theorem for sections of powers of a positive Hermitian line bundle over a compact Kähler manifold [SZZ08, Theorem 1.4]. In that case the decay rate of the hole probability is again $e^{-CN^{n+1}}$.

A final word about notation. By $A \lesssim B$ we mean that there exists $C > 0$ independent of the relevant variables of A and B for which $A \leq CB$. Then $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

1. Linear statistics

Proof of Theorem 1. The proof is as in [HKPV09, Section 3.5], so we keep it short. By Stokes and Fubini's theorems

$$\begin{aligned}
&\mathrm{Var}[I_L(\varphi)]\\
=&\mathbb{E}[|I_L(\varphi) - \mathbb{E}(I_L(\varphi))|^2] = \mathbb{E}\Big[\Big|\int_{\mathbb{B}_n} \varphi \wedge \frac{i}{2\pi}\partial\bar\partial \log\Big(\frac{|f_L|^2}{K_L(z,z)}\Big)\Big|^2\Big]\\
=&4\mathbb{E}\Big[\Big|\int_{\mathbb{B}_n} \log\Big(\frac{|f_L|}{\sqrt{K_L(z,z)}}\Big)\frac{i}{2\pi}\partial\bar\partial\varphi\Big|^2\Big]\\
=&4\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \mathbb{E}\Big[\log\Big(\frac{|f_L(z)|}{\sqrt{K_L(z,z)}}\Big)\log\Big(\frac{|f_L(w)|}{\sqrt{K_L(w,w)}}\Big)\Big]\frac{i}{2\pi}\partial\bar\partial\varphi(z)\frac{i}{2\pi}\partial\bar\partial\varphi(w).
\end{aligned}$$

Consider the normalised GAF

$$\hat f(z) = \frac{f_L(z)}{\sqrt{K_L(z,z)}}.$$

Then $(\hat f(z), \hat f(w))$ has joint gaussian distribution with mean 0 and marginal variances 1. Since $\hat f(z) \sim N_{\mathbb{C}}(0,1)$ the expectation $\mathbb{E}(\log|\hat f(z)|)$ is constant, and integrated against $\partial\bar{\partial}\varphi$ gives 0. Therefore, in the integral above, the expectation can be replaced by

$$\operatorname{Cov}(\log|\hat{f}(z)|, \log|\hat{f}(w)|) = \mathbb{E}[\log|\hat{f}(z)|\log|\hat{f}(w)|] - \mathbb{E}[\log|\hat{f}(z)|]\mathbb{E}[\log|\hat{f}(w)|].$$

This yields the following bi-potential expression of the variance, which is our starting point:

$$\begin{aligned} \operatorname{Var}[I_L(\varphi)] &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \rho_L(z,w)\frac{i}{2\pi}\partial\bar{\partial}\varphi(z)\frac{i}{2\pi}\partial\bar{\partial}\varphi(w) \\ &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \rho_L(z,w)D\varphi(z)D\varphi(w)d\mu(z)d\mu(w), \end{aligned} \tag{4}$$

where $\rho_L(z,w) = 4\operatorname{Cov}(\log|\hat{f}(z)|, \log|\hat{f}(w)|)$. By [HKPV09, Lemma 3.5.2]

$$\rho_L(z,w) = \sum_{m=1}^{\infty} \frac{|\theta_L(z,w)|^{2m}}{m^2},$$

where

$$\theta_L(z,w) = \frac{K_L(z,w)}{\sqrt{K_L(z,z)}\sqrt{K_L(w,w)}} = \frac{(1-|z|^2)^{L/2}(1-|w|^2)^{L/2}}{(1-\bar{z}\cdot w)^L} \tag{5}$$

is the normalised covariance kernel of f_L.

We see next that only the near diagonal part of the double integral (4) is relevant. Let $\varepsilon_L = 2/L^{n+1}$, and split the integral in three parts:

$$\operatorname{Var}[I_L(\varphi)] = \int_{\rho_L(z,w)\le\varepsilon_L} \rho_L(z,w)D\varphi(z)D\varphi(w)d\mu(z)d\mu(w) \tag{I1}$$

$$+ \int_{\rho_L(z,w)>\varepsilon_L} \rho_L(z,w)(D\varphi(z)-D\varphi(w))D\varphi(w)d\mu(z)d\mu(w) \tag{I2}$$

$$+ \int_{\rho_L(z,w)>\varepsilon_L} \rho_L(z,w)(D\varphi(w))^2 d\mu(z)d\mu(w). \tag{I3}$$

The bound for the first integral is straight-forward,

$$|\mathrm{I1}| \le \varepsilon_L \int_{\rho_L(z,w)\le\varepsilon_L} |D\varphi(z)D\varphi(w)|d\mu(z)d\mu(w) \le \varepsilon_L\left(\int_{\mathbb{B}_n} |D\varphi(z)|\,d\mu(z)\right)^2.$$

In order to bound (I2) let ϕ_z denote the automorphism of $\mathbb{B}_n$ exchanging z and 0, so that $|\theta_L(z,w)|^2 = (1-|\phi_z(w)|^2)^L$ (see (2)). By the uniform continuity of $i\partial\bar{\partial}\varphi$ there exists $\eta(t)$ with $\lim_{t\to 1}\eta(t) = 0$ such that for all $z, w \in \mathbb{B}_n$,

$$|D\varphi(z) - D\varphi(w)| \le \eta(1-|\phi_z(w)|^2).$$

An immediate estimate shows that

$$x \le \sum_{m=1}^{\infty} \frac{x^m}{m^2} \le 2x, \quad x \in [0,1],$$

and therefore

$$(1-|\phi_z(w)|^2)^L \le \rho_L(z,w) \le 2(1-|\phi_z(w)|^2)^L. \tag{6}$$

By the invariance by automorphisms of the measure $d\mu$, we get (after changing appropriately the value of C_φ at each step)

$$\begin{aligned}
|\mathrm{I2}| &\le 2C_\varphi \int_{\{\rho_L(z,w)>\varepsilon_L\}\cap(\operatorname{supp}\varphi\times\operatorname{supp}\varphi)} (1-|\phi_z(w)|^2)^L \eta(1-|\phi_z(w)|^2) d\mu(z)d\mu(w)\\
&\le C_\varphi \eta((\varepsilon_L/2)^{1/L}) \int_{\{\rho_L(z,w)>\varepsilon_L\}\cap(\operatorname{supp}\varphi\times\operatorname{supp}\varphi)} (1-|\phi_z(w)|^2)^L d\mu(z)d\mu(w)\\
&\le C_\varphi \eta((\varepsilon_L/2)^{1/L}) \int_{\operatorname{supp}\varphi} \left(\int_{z:\rho_L(z,0)>\varepsilon_L} (1-|z|^2)^L d\mu(z)\right) d\mu(w)\\
&\le C_\varphi \eta((\varepsilon_L/2)^{1/L}) \int_{z:\rho_L(z,0)>\varepsilon_L} (1-|z|^2)^L d\mu(z).
\end{aligned}$$

Since $\eta(t) \lesssim |1-t|$ for t near 1, we see that

$$\eta((\varepsilon_L/2)^{1/L}) \lesssim 1-(\varepsilon_L/2)^{1/L} \simeq \frac{\log L}{L}$$

and therefore

$$|\mathrm{I2}| \lesssim \frac{\log L}{L} \int_{z:\rho_L(z,0)>\varepsilon_L} (1-|z|^2)^L d\mu(z).$$

On the other hand, using again the invariance, we see that

$$\mathrm{I3} = \left(\int_{\mathbb{B}_n} (D\varphi(w))^2 d\mu(w) \right) \int_{z:\rho_L(z,0)>\varepsilon_L} (1-|z|^2)^L d\mu(z).$$

Since $\lim_{L\to\infty} \varepsilon_L^{1/L} = 1$ we have thus $\mathrm{I2} = \mathrm{o}(\mathrm{I3})$ and therefore

$$\operatorname{Var}[I_L(\varphi)] = \mathrm{I3}\Big(1+\mathrm{O}\Big(\frac{\log L}{L}\Big)\Big) + \mathrm{O}(\varepsilon_L). \tag{7}$$

It remains to compute the second factor in I3:

$$J := \int_{z:\rho_L(z,0)>\varepsilon_L} \rho_L(z,0) d\mu(z) = \sum_{m=1}^{\infty} \frac{1}{m^2} \int_{z:\rho_L(z,0)>\varepsilon_L} (1-|z|^2)^{mL} d\mu(z).$$

By (6),

$$\{|z|^2 < 1-\varepsilon_L^{1/L}\} \subset \{\rho_L(z,0) > \varepsilon_L\} \subset \{|z|^2 < 1-(\varepsilon_L/2)^{1/L}\}$$

and therefore

$$J=\sum_{m=1}^{\infty}\frac{1}{m^2}\int_{|z|^2<1-(\frac{\varepsilon_L}{2})^{1/L}}(1-|z|^2)^{mL}d\mu(z)-\int_{\substack{|z|^2<1-(\frac{\varepsilon_L}{2})^{1/L}\\ \rho_L(z,0)\le\varepsilon_L}}(1-|z|^2)^{mL}d\mu(z).$$

Claim 1: The sum of the negative terms is negligible. More precisely,

$$\sum_{m=1}^{\infty}\frac{1}{m^2}\int_{\substack{|z|^2<1-(\frac{\varepsilon_L}{2})^{1/L}\\ \rho_L(z,0)\le\varepsilon_L}}(1-|z|^2)^{mL}d\mu(z)=\mathrm{O}\Big(\frac{\log^{n-1}L}{L^{2n+1}}\Big).$$

Assuming this we have

$$J=\sum_{m=1}^{\infty}\frac{1}{m^2}I_m+\mathrm{o}(L^{-n}) \tag{8}$$

where, denoting $r_L=1-(\frac{\varepsilon_L}{2})^{1/L}$,

$$I_m=\int_{|z|^2<r_L}(1-|z|^2)^{mL}d\mu(z)=\int_{|z|^2<r_L}(1-|z|^2)^{mL-n-1}d\nu(z).$$

Integration in polar coordinates ([Rud08, 1.4.3]) shows that I_m is a truncated beta function:

$$I_m=n\int_0^{\sqrt{r_L}}(1-r^2)^{mL-n-1}r^{2(n-1)}2r\,dr=n\int_0^{r_L}(1-t)^{mL-n-1}t^{n-1}dt.$$

A repeated integration by parts yields, for $n,k>0$,

$$\begin{aligned}&n\int_0^r(1-t)^{k-1}t^{n-1}dt\\ &\quad=\frac{n!\Gamma(k)}{\Gamma(n+k)}(1-(1-r)^{k+n-1})-\sum_{j=1}^{n-1}\frac{n!\Gamma(k)}{\Gamma(n-j)\Gamma(k+j)}(1-r)^{k+j-1}r^{n-j},\end{aligned}$$

thus taking $k=mL-n$ we deduce from (8) that

$$\begin{aligned}J=n!\sum_{m=1}^{\infty}\frac{1}{m^2}\Big[&\frac{\Gamma(mL-n)}{\Gamma(mL)}[1-(1-r_L)^{mL-1}]\\ &-\sum_{j=1}^{n-1}\frac{\Gamma(mL-n)}{\Gamma(n-j)\Gamma(mL-n+j)}(1-r_L)^{mL-n+j-1}r_L^{n-j}\Big].\end{aligned}$$

Claim 2: The negative terms in this sum are again negligible. Specifically,

$$\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{j=1}^{n-1}\frac{\Gamma(mL-n)}{\Gamma(n-j)\Gamma(mL-n+j)}(1-r_L)^{mL-n+j-1}r_L^{n-j}=\mathrm{O}\Big(\frac{\log^{n+j}L}{L^{2n+3}}\Big).$$

The asymptotics of the Γ-function

$$\lim_{m\to\infty}\frac{\Gamma(m+n)}{\Gamma(m)m^n}=1 \tag{9}$$

and the fact that $(1-r_L)^{mL}=(\varepsilon_L/2)^m$ tends to 0 as $L\to\infty$ yield

$$\begin{aligned}J&=n!\sum_{m=1}^{\infty}\frac{1}{m^2}\frac{\Gamma(mL-n)}{\Gamma(mL)}+\mathrm{o}(L^{-n})=n!\sum_{m=1}^{\infty}\frac{1}{m^2}\frac{1}{(mL)^n}+\mathrm{o}(L^{-n})\\&=n!\frac{1}{L^n}\zeta(n+2)+\mathrm{o}(L^{-n}).\end{aligned}$$

Plugging this in (7) we finally obtain the stated result. ∎

Proof of Claim 1. Denote by N the sum we need to estimate. Using the fact that $\varepsilon_L=2L^{-(n+1)}$ and unwinding the condition $\rho_L(z,0)\le\varepsilon_L$, a rough estimate yields

$$\begin{aligned}N&=\sum_{m=1}^{\infty}\frac{1}{m^2}\int_{(\frac{\varepsilon_L}{2})^{1/L}\le 1-|z|^2\le\varepsilon_L^{1/L}}(1-|z|^2)^{mL-n-1}d\nu(z)\\&\lesssim\sum_{m=1}^{\infty}\frac{1}{m^2}(\varepsilon_L^{1/L})^{L-n-1}\nu\Big(\Big\{1-\varepsilon_L^{1/L}\le|z|^2\le 1-\Big(\frac{\varepsilon_L}{2}\Big)^{1/L}\Big\}\Big)\\&\lesssim\varepsilon_L^{1-\frac{n}{L}}\Big(1-\frac{1}{2^{1/L}}\Big)\Big(1-(\frac{\varepsilon_L}{2})^{1/L}\Big)^{n-1}\\&\le\frac{2}{L^{n+1}}\Big(\frac{\log 2}{L}+\mathrm{o}(L^{-1})\Big)\Big(\frac{n+1}{L}\log L+\mathrm{o}\Big(\frac{\log L}{L}\Big)\Big)^{n-1}\\&=\mathrm{O}\Big(\frac{\log^{n-1}L}{L^{2n+1}}\Big).\qquad ∎\end{aligned}$$

Proof of Claim 2. We have

$$\begin{aligned}(1-r_L)^{mL-n+j-1}r_L^{n-j}&=L^{-\frac{n+1}{L}(mL-n+j-1)}\Big(\frac{n+1}{L}\log L+\mathrm{o}(L^{-n})\Big)^{n-j}\\&=\mathrm{O}\Big(\frac{\log^{n+j}L}{L^{(n+1)m+n+j}}\Big).\end{aligned}$$

On the other hand, the number of terms in the sum in j is independent of L, so by (9), for L big enough and for all j

$$\lim_{L\to\infty}\frac{\Gamma(mL-n)}{\Gamma(mL-n+j)}=\frac{1}{(mL)^j}.$$

Thus, denoting by M the double sum in m and j we see that

$$M \simeq \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{j=1}^{n-1} \frac{1}{(mL)^j} \frac{\log^{n+j} L}{L^{(n+1)m+n+j}} = \mathrm{O}\Big(\frac{\log^{n+j} L}{L^{2n+3}}\Big). \qquad \blacksquare$$

As an immediate consequence of the results of M. Sodin and B. Tsirelson and the previous computations we obtain the asymptotic normality of $I_L(\varphi)$.

Corollary 5: *As $L \to \infty$ the distribution of the normalised random variable*

$$\frac{I_L(\varphi) - \mathbb{E}[I_L(\varphi)]}{\sqrt{\mathrm{Var}(I_L(\varphi))}}$$

tends weakly to the standard (real) gaussian, for each φ.

Proof. Consider the normalised GAF $\hat{f}_L(z)$, whose covariance kernel is $\theta_L(z,w)$. Notice that

$$J_L(\varphi) := \int_{\mathbb{B}_n} \log|\hat{f}_L(z)|^2 D\varphi(z) d\mu(z) = I_L(\varphi) - \int_{\mathbb{B}_n} \log K_L(z,z)\, D\varphi(z) d\mu(z),$$

and that the second term has no random part. Hence

$$(J_L(\varphi) - \mathbb{E}[J_L(\varphi)])/\sqrt{\mathrm{Var}[J_L(\varphi)]} \quad \text{and} \quad (I_L(\varphi) - \mathbb{E}[I_L(\varphi)])/\sqrt{\mathrm{Var}[I_L(\varphi)]}$$

have the same distribution, and according to [ST04, Theorem 2.2], to prove the asymptotic normality of $J_L(\varphi)$ it is enough to see that

(a) $$\liminf_{L\to\infty} \frac{\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} |\theta_L(z,w)|^2 |D\varphi(z)||D\varphi(w)| d\mu(z) d\mu(w)}{\sup_{w\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\theta_L(z,w)| d\mu(z)} > 0$$

(b) $$\lim_{L\to\infty} \sup_{w\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\theta_L(z,w)| d\mu(z) = 0.$$

By the invariance under automorphisms of the measure μ

$$\int_{\mathbb{B}_n} |\theta_L(z,w)|\, d\mu(z) = \int_{\mathbb{B}_n} (1-|z|^2)^{L/2} d\mu(z),$$

and (b) follows.

On the other hand, the double integral in the numerator of (a) is essentially the same as the integral we have estimated in the proof of the previous theorem (see (4)), and the same computations show that (a) holds. $\blacksquare$

2. Large deviations

We begin with the proof of Corollary 3 (assuming Theorem 2).

Proof of Corollary 3. Since $\omega^{n-1} \wedge [Z_{f_L}]$ is a positive current, the functional $I_L(\psi)$ is monotone, i.e., if $\psi_1 \leq \psi_2$ then $I_L(\psi_1) \leq I_L(\psi_2)$.

Let ψ_1, ψ_2 be smooth compactly supported functions in $\mathbb{B}_n$ such that $0 \leq \psi_1 \leq \chi_U \leq \psi_2 \leq 1$ and

$$\int_{\mathbb{B}_n} \psi_1 \, d\mu \geq \mu(U)(1-\delta), \quad \int_{\mathbb{B}_n} \psi_2 \, d\mu \leq \mu(U)(1+\delta).$$

Outside an exceptional set of probability $e^{-cL^{n+1}}$ we have, by Theorem 2,

$$I_L(U) \leq I_L(\psi_2) \leq (1+\delta)\mathbb{E}[I_L(\psi_2)] = (1+\delta)L \int_{\mathbb{B}_n} \psi_2 d\mu \leq (1+\delta)^2 L\mu(U).$$

Similarly, using ψ_1, we see that

$$I_L(U) \geq (1-\delta)^2 L\mu(U)$$

outside another set of probability $e^{-cL^{n+1}}$, which after appropiately changing the value of δ completes the proof. ∎

A different proof of Corollary 3 can be obtained by following the scheme of [HKPV09, Theorem 7.2.5], using the Poisson–Szegö representation of the averages $\int_{|\xi|=1} \log |f_L(\xi)| d\sigma(\xi)$ instead of Jensen's formula.

Proof of Theorem 2. Applying Stokes' theorem, we have

$$I_L(\varphi) - \mathbb{E}[I_L(\varphi)] = \int_{\mathbb{B}_n} \varphi \wedge \frac{i}{2\pi} \partial\overline{\partial} \log \frac{|f_L|^2}{K_L(z,z)} = \int_{\mathbb{B}_n} \log \frac{|\hat{f}_L|^2}{K_L(z,z)} \frac{i}{2\pi} \partial\overline{\partial}\varphi.$$

Thus,

$$|I_L(\varphi) - \mathbb{E}[I_L(\varphi)]| \leq \|D\varphi\|_\infty \int_{\operatorname{supp}\varphi} |\log |\hat{f}_L(z)|^2| d\mu(z).$$

By (3), the proof of Theorem 2 will be completed as soon as we prove the following Lemma.

Lemma 6: *For any regular compact set K and any $\delta > 0$ there exists $c = c(\delta, K)$ such that*

$$\mathbb{P}\left[\int_K |\log |\hat{f}_L(z)|^2| d\mu(z) > \delta L \right] \leq e^{-cL^{n+1}}.$$

The key ingredient in the proof of this lemma is given by the following control on the average of $\left|\log |\hat{f}_L|^2\right|$ over pseudo-hyperbolic balls.

Lemma 7: *There exists a constant $c > 0$ such that for a hyperbolic ball $E = E(z_0, s)$, $z_0 \in \mathbb{B}_n$, $s \in (0,1)$,*

$$\mathbb{P}\left[\frac{1}{\mu(E)}\int_E |\log|\hat{f}_L(\xi)|^2|d\mu(\xi) > 5L\mu(E)^{1/n}\right] \leq e^{-cL^{n+1}}.$$

Let us see first how this allows to complete the proof of Lemma 6, and therefore of Theorem 2.

Proof of Lemma 6. Cover K with pseudo-hyperbolic balls $E_j = E(\lambda_j, \epsilon)$, $j = 1, \dots, N$ of fixed invariant volume $\mu(E_j) = \eta$ (to be determined later on). A direct estimate shows that $N \simeq \mu(K)/\eta$.

By Lemma 7, outside an exceptional event of probability $Ne^{-cL^{n+1}} \leq e^{-c'L^{n+1}}$,

$$\begin{aligned}\int_K \log|\hat{f}_L(\xi)|^2|d\mu(\xi) &\leq \sum_{j=1}^N \int_{E_j} |\log|\hat{f}_L(\xi)|^2|d\mu(\xi)\\ &\leq \sum_{j=1}^N 5L\eta^{1+1/n} \simeq L\mu(K)\eta^{1/n}.\end{aligned}$$

Choosing η such that $\mu(K)\eta^{1/n} = \delta$ we are done. ∎

Now we proceed to prove Lemma 7. A first step is the following lemma.

Lemma 8: *Fix $r < 1$ and $\delta > 0$. There exists $c > 0$ and $L_0 = L_0(r, \delta)$ such that for all $L \geq L_0$ and all $z_0 \in \mathbb{B}_n$*

(a) $P[\max_{E(z_0,r)} \log|\hat{f}_L(z)|^2 < -\delta L] \leq e^{-cL^{n+1}}$,

(b) $P[\max_{E(z_0,r)} \log|\hat{f}_L(z)|^2 > \delta L] \leq e^{-ce^{L\delta/2}}$.

Combining both estimates $\mathbb{P}[\max_{E(z_0,r)} |\log|\hat{f}_L(z)|^2| > \delta L] \leq e^{-cL^{n+1}}$.

Proof. By the invariance of the distribution of $\hat{f}$, it is enough to consider the case $z_0 = 0$.

(a) Consider the event

$$\mathcal{E}_1 = \{\max_{|z|\leq r} \log|\hat{f}_L(z)|^2 < -\delta L\}.$$

Note that

$$\log|\hat{f}_L(z)|^2 = \log\frac{|f_L(z)|^2}{K_L(z,z)} = \log|f_L(z)|^2 - \log\frac{1}{(1-|z|^2)^L},$$

hence, by subharmonicity,

$$\begin{aligned}\mathcal{E}_1 &\subset \{\max_{|z|\le r} \log|f_L(z)|^2 \le L\log\frac{1}{1-r^2} - L\delta\}\\ &= \{\max_{|z|=r} \log|f_L(z)|^2 \le L(\log\frac{1}{1-r^2} - \delta)\}.\end{aligned}$$

Therefore, letting $\tilde{\delta} = \frac{\delta}{2}[\log(\frac{1}{1-r^2})]^{-1}$,

$$\mathbb{P}[\mathcal{E}_1] \le \mathbb{P}\Big[\max_{|z|=r} \frac{\log|f_L(z)|}{L} \le \Big(\frac{1}{2} - \tilde{\delta}\Big)\log\frac{1}{1-r^2}\Big].$$

The estimate of $\mathbb{P}[\mathcal{E}_1]$ will be done as soon as we prove the following lemma, which is the analogue of the upper bound in [HKPV09, Lemma 7.2.7].

Lemma 9: *For $0<\delta<1/2$ and $r\in(0,1)$ there exist $c=c(\delta,r)$ and $L_0=L_0(\delta,r)$ such that for all $L\ge L_0$*

$$\mathbb{P}\Big[\max_{|z|=r} \frac{\log|f_L(z)|}{L} \le \Big(\frac{1}{2} - \delta\Big)\log\frac{1}{1-r^2}\Big] \le e^{-cL^{n+1}}.$$

Proof of Lemma 9. Under the event we want to estimate

$$\max_{|z|=r} |f_L(z)| \le (1-r^2)^{-L(\frac{1}{2}-\delta)}.$$

We shall see that this implies that some coefficients of the series of f_L are necessarily "small", something that only happens with a probability less than $e^{-cL^{n+1}}$. Since

$$f_L(z) = \sum_\alpha \frac{\partial^\alpha f_L(0)}{\alpha!} z^\alpha = \sum_\alpha a_\alpha \Big(\frac{\Gamma(|\alpha|+L)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha,$$

we have

$$a_\alpha = \Big(\frac{\alpha!\Gamma(L)}{\Gamma(L+|\alpha|)}\Big)^{1/2} \frac{\partial^\alpha f_L(0)}{\alpha!},$$

and by Cauchy's formula [Rud08, p. 37]

$$\frac{\partial^\alpha f_L(0)}{\alpha!} = \frac{\Gamma(n+|\alpha|)}{\Gamma(n)\alpha! r^{|\alpha|}} \int_S f_L(r\xi)\overline{\xi}^\alpha d\sigma(\xi).$$

Hence

$$|a_\alpha| \le \Big(\frac{\alpha!\Gamma(L)}{\Gamma(L+|\alpha|)}\Big)^{1/2} \frac{\Gamma(n+|\alpha|)}{\Gamma(n)\alpha! r^{|\alpha|}} (\max_{\xi\in S}|\xi^\alpha|)(\max_{|z|=r}|f_L|).$$

Since for $m\in\mathbb{N}$,

$$\sum_{|\alpha|=m} \frac{\alpha^\alpha}{\alpha!|\alpha|^{|\alpha|}} = \frac{1}{m!}, \tag{10}$$

we have

$$|a_\alpha| \le \Big(\frac{\Gamma(L)}{\Gamma(L+|\alpha|)}\Big)^{1/2} \frac{\Gamma(n+|\alpha|)}{\Gamma(n)} \Big(\frac{\alpha^\alpha}{\alpha!|\alpha|^{|\alpha|}}\Big)^{1/2} (1-r^2)^{-L(\frac{1}{2}-\delta)} r^{-|\alpha|}.$$

Using

$$\sum_{|\alpha|=m} \frac{\alpha^\alpha}{\alpha!|\alpha|^{|\alpha|}} = \frac{1}{m!}, \tag{11}$$

Stirling's formula and the asymptotics for the Gamma function (9), we get (for $m \gg n$)

$$\begin{aligned}\sum_{|\alpha|=m} |a_\alpha|^2 &\le \frac{\Gamma(L)}{\Gamma(L+m)} \frac{\Gamma^2(n+m)}{\Gamma^2(n)m!} r^{-2m}(1-r^2)^{-L(1-2\delta)} \\ &\lesssim \frac{\Gamma(L)\Gamma(n+m)}{\Gamma(L+m)} m^{n-1} r^{-2m}(1-r^2)^{-L(1-2\delta)} \\ &\lesssim \frac{L^L(m+n)^{m+n}}{(L+m)^{L+m}} m^{n-1} r^{-2m}(1-r^2)^{-L(1-2\delta)} \\ &\lesssim \frac{L^L(m+n)^{m}}{(L+m)^{L+m}} m^{2n} r^{-2m}(1-r^2)^{-L(1-2\delta)}.\end{aligned}$$

(We use this lemma (and Lemma 8) in the proof of Lemma 7, which is in turn used in Lemma 6 with a radius $r = \epsilon$ such that $\mu(E(\lambda_j, \epsilon)) = (\delta/\mu(K))^n$. Since in Lemma 6 it is enough to consider δ small, here it is enough to consider r close to 0. We assume thus that r is close to 0, although the proof seems to work for all $r \in (0,1)$.)

For the indices m such that

$$m \le \frac{r^2L-n}{1-r^2} \tag{12}$$

we have $(1-r^2)m \le r^2L - n$ and therefore $\frac{m+n}{L+m} r^{-2} \le 1$. Hence

$$\sum_{|\alpha|=m} |a_\alpha|^2 \le \frac{L^L}{(L+m)^L} \frac{m^{2n}}{(1-r^2)^{L(1-2\delta)}} = \Big[\frac{Lm^{\frac{2n}{L}}}{(L+m)(1-r^2)^{1-2\delta}}\Big]^L.$$

Fix ϵ (possibly very small) and let us find conditions on m so that the term in the brackets is smaller than $(1+\epsilon)^{-1}$. Assume that m satisfies (12) and

$$m \ge (1-\delta)\frac{r^2L}{1-r^2}, \tag{13}$$

Then $\lim_{L\to\infty} m^{\frac{2n}{L}} = 1$ and we can take L_0 such that $m^{\frac{2n}{L}} \le 1+\epsilon$ for $L \ge L_0$. Then, for the term in the brackets to be smaller than $(1+\epsilon)^{-1}$ it is enough to have

$$\frac{L(1+\epsilon)}{(L+m)(1-r^2)^{1-2\delta}} \le \frac{1}{1+\epsilon},$$

that is

$$(1+\epsilon)^2 L \le (L+m)(1-r^2)^{1-\delta}.$$

This will occur for the m's in our range if

$$(1+\epsilon)^2 < \Big(1+\frac{(1-\delta)r^2}{1-r^2}\Big)(1-r^2)^{1-\delta}.$$

Thus for the existence of an $\epsilon > 0$ with this property it is enough to have

$$1 < \Big(1+\frac{(1-\delta)r^2}{1-r^2}\Big)(1-r^2)^{1-\delta} = (1-r^2)^{1-\delta} + \frac{(1-\delta)r^2}{(1-r^2)^{\delta}}.$$

The function $f(x) = (1-x)^{1-\delta} + \frac{(1-\delta)x}{(1-x)^{\delta}}$ has $f(0) = 1$ and $f'(x) = \frac{\delta(1-\delta)x}{(1-x)^{1+\delta}} > 0$, thus $f(x) > 1$ for $x > 0$.

All combined, for the indices m satisfying (12) and (13), i.e., in the set

$$A := \Big\{m:\ (1-\delta)\frac{r^2L}{1-r^2} \le m \le \frac{r^2L-n}{1-r^2}\Big\},$$

the following estimate holds

$$\sum_{|\alpha|=m} |a_\alpha|^2 \lesssim (1+\epsilon)^{-m}.$$

Let us see next that this happens with very small probability. Note that

$$\mathbb{P}\Big[\sum_{|\alpha|=m} |a_\alpha|^2 \le (1+\epsilon)^{-m},\ \forall m\in A\Big] = \prod_{m\in A} \mathbb{P}\Big[\sum_{j=1}^{N(n,m)} |\xi_j|^2 \le (1+\epsilon)^{-m}\Big],$$

where $\xi_j \sim \mathbb{N}_{\mathbb{C}}(0,1)$ are independent and $N(n,m) = \Gamma(n+m)/(m!\Gamma(n))$ is the number of indices α with $|\alpha| = m$. The variable $\sum_{j=1}^{N(n,m)} |\xi_j|^2$ follows a Gamma distribution of parameter $N(n,m)$, therefore,

$$\begin{aligned}\mathbb{P}\Big[\sum_{j=1}^{N(n,m)} |\xi_j|^2 \le (1+\epsilon)^{-m}\Big] &= \frac{1}{\Gamma(N(n,m))}\int_0^{(1+\epsilon)^{-m}} x^{N(n,m)-1}e^{-x}dx \\ &\le \frac{1}{\Gamma(N(n,m))}\frac{1}{N(n,m)}(1+\epsilon)^{-mN(n,m)}.\end{aligned}$$

Observe that for $m \in A$, $m \simeq L$ and, by (9), $N(n,m) \simeq m^{n-1} \simeq L^{n-1}$. With this and Stirling's formula we get

$$\begin{aligned}\log \mathbb{P}\Big[\sum_{j=1}^{N(n,m)} |\xi_j|^2 \leq (1+\epsilon)^{-(m+n)}\Big] \\ &\lesssim -\log\Gamma(L^{n-1}) - \log L^{n-1} - L\cdot L^{n-1}\log(1+\epsilon) \\ &\simeq -L^n\log(1+\epsilon)[1+o(1)] \leq -CL^n.\end{aligned}$$

Therefore, changing appropiately the value C at each step, we finally see that

$$\mathbb{P}\Big[\sum_{|\alpha|=m} |a_\alpha|^2 \leq (1+\epsilon)^{-m},\ \forall m\in A\Big] \leq (e^{-CL^n})^{\#A} = (e^{-CL^n})^{L+o(1)} \leq e^{-CL^{n+1}}.$$

This finishes the proof of (a) in Lemma 8. ∎

(b) Let now

$$\mathcal{E}_2 := \{\max_{|z|\leq r} \log|\hat{f}_L(z)|^2 > \delta L\} = \Big\{\max_{|z|\leq r}\Big[\log|f_L(z)| - \frac{L}{2}\log\frac{1}{1-|z|^2}\Big] > \delta L\Big\}.$$

We estimate the probability of this event by controlling the coefficients of the series of f_L. Let C be a constant to be determined later on. Split the sum defining $|f_L|$ as

$$\begin{aligned}|f_L(z)| &\leq \sum_{|\alpha|\leq C\delta L} |a_\alpha|\Big(\frac{\Gamma(|\alpha|+L)}{\alpha!\Gamma(L)}\Big)^{1/2}|z^\alpha| + \sum_{|\alpha|>C\delta L} |a_\alpha|\Big(\frac{\Gamma(|\alpha|+L)}{\alpha!\Gamma(L)}\Big)^{1/2}|z^\alpha| \\ &=: (I) + (II).\end{aligned} \tag{14}$$

We shall estimate each part separately.

Let us begin with the first sum. Using Cauchy–Schwarz inequality, (10) and (11), we obtain

$$\begin{aligned}(I) &\leq \Big(\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2\Big)^{1/2}\Big(\sum_{|\alpha|\leq C\delta L} \frac{\Gamma(|\alpha|+L)}{\alpha!\Gamma(L)}\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}|z|^{2|\alpha|}\Big)^{1/2} \\ &= \Big(\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2\Big)^{1/2}\Big(\sum_{m\leq C\delta L} \frac{\Gamma(m+L)}{m!\Gamma(L)}|z|^{2m}\Big)^{1/2} \\ &\leq \Big(\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2\Big)^{1/2}(1-|z|^2)^{-L/2} = \Big(\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2\Big)^{1/2}\sqrt{K_L(z,z)}.\end{aligned}$$

Now we shall see that, except for an event of small probability, (II) is bounded (if C is choosen appropiately). For $|z| \leq r$,

$$(II) \leq \sum_{|\alpha|>C\delta L} |a_\alpha| \Big(\frac{\Gamma(|\alpha|+L)}{\alpha!\Gamma(L)}\Big)^{1/2} \Big(\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}\Big)^{1/2} r^{|\alpha|}$$
$$\leq \sum_{|\alpha|>C\delta L} |a_\alpha| \Big(\frac{\Gamma(|\alpha|+L)}{|\alpha|!\Gamma(L)}\Big)^{1/2} r^{|\alpha|}.$$

Let $\beta > 0$ be such that $r = e^{-\beta}$ and consider $\gamma \in (0, \beta)$ and $\epsilon > 0$ such that $0 < \gamma < \gamma + \epsilon < \beta$. Define the following event:

$$A = \{|a_\alpha| \leq e^{\gamma|\alpha|},\ \forall \alpha : |\alpha| \geq C\delta L\}.$$

If A occurs, by the asymptotics (9),

$$(II) \leq \sum_{m>C\delta L} e^{\gamma m} \Big(\frac{\Gamma(m+L)}{m!\Gamma(L)}\Big)^{1/2} r^m \frac{\Gamma(m+n)}{\Gamma(n)m!}$$
$$\lesssim \frac{1}{\sqrt{\Gamma(L)}} \sum_{m>C\delta L} m^{\frac{L-1}{2}} m^{n-1} e^{\gamma m} r^m \leq \frac{1}{\sqrt{\Gamma(L)}} \sum_{m>C\delta L} m^{n+L/2} (e^\gamma r)^m.$$

Lemma 10: *Given $\epsilon > 0$ there exists $C > 0$ big enough so that for all $m > C\delta L$*

$$\frac{m^{n+L/2}}{\sqrt{\Gamma(L)}} \leq C e^{\epsilon m}.$$

Proof. It is enough to see that there exists a constant D such that for $x > C\delta L$

$$f(x) := \epsilon x - \Big(n + \frac{L}{2}\Big) \log x + \frac{1}{2} \log \Gamma(L) + D \geq 0.$$

Note that $\lim_{x\to\infty} f(x) = +\infty$ and that f is increasing for $x \geq \epsilon^{-1}(n + L/2)$. Choose C with $C\delta L > \epsilon^{-1}(n+L/2)$, so that f is increasing for $x > C\delta L$. Then, by Stirling's formula,

$$\begin{aligned} f(C\delta L) &= \epsilon C\delta L - \Big(n + \frac{L}{2}\Big) \log(C\delta L) + \frac{1}{2} \log \Gamma(L) + \log D \\ &= \epsilon C\delta L - \Big(n + \frac{L}{2}\Big) \log(C\delta) - n \log L + \frac{1}{2} \log \Big(\frac{2\pi}{L}\Big)^{1/2} - \frac{L}{2} + \mathrm{O}(1) \\ &= \Big[\epsilon C\delta - \frac{1}{2} \log(C\delta) - \frac{1}{2}\Big] L + \mathrm{o}(L). \end{aligned}$$

Choose C big enough so that the term in the brackets is positive, and therefore $f(x) > 0$ for $x > C\delta L$. ∎

Taking C as in this lemma we obtain

$$(II) \lesssim \sum_{m>C\delta L} e^{-[\beta-(\gamma+\epsilon)]m} \leq \frac{1}{1-e^{-[\beta-(\gamma+\epsilon)]}}.$$

Now we show that the event A has "big" probability. The variables $|a_\alpha|^2$ are independent exponentials, hence

$$\mathbb{P}[A] = \prod_{|\alpha|\geq C\delta L} 1 - \mathbb{P}[|a_\alpha| \geq e^{\gamma|\alpha|}] = \prod_{m\geq C\delta L} [1-e^{-e^{2\gamma m}}]^{\frac{\Gamma(n+m)}{\Gamma(n)m!}}.$$

Since $x = e^{-e^{2\gamma m}}$ is close to 0, we can use the estimate $\log(1-x) \simeq -x$. Thus, using (9) once more,

$$\log \mathbb{P}[A] = \sum_{m\geq C\delta L} \frac{\Gamma(n+m)}{\Gamma(n)m!} \log[1-e^{-e^{2\gamma m}}] \simeq - \sum_{m\geq C\delta L} m^{n-1} e^{-e^{2\gamma m}}.$$

There exists L_0 such that for all $L \geq L_0$ and $m \geq C\delta L$,

$$m^{n-1} e^{-e^{2\gamma m}} \leq e^{-e^{\gamma m}},$$

and therefore

$$\log \mathbb{P}[A] \geq - \sum_{m\geq C\delta L} e^{-e^{\gamma m}} \simeq -e^{-e^{\gamma C\delta L}}.$$

Choosing C big enough so that, in addition to the previous conditions,

$$\gamma C > \log \frac{1}{1-r^2},$$

we have

$$-e^{-e^{(2\gamma-\eta)C\delta L}} > -e^{-(1-r^2)^{-\delta L}}$$

and therefore

$$\mathbb{P}[A] \geq e^{-e^{-(1-r^2)^{-\delta L}}}.$$

So far we have proved that, after choosing γ appropriately, and under the event A,

$$|f_L(z)| \leq \Bigg(\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2\Bigg)^{1/2} \sqrt{K_L(z,z)} + C_r.$$

Therefore, the condition

$$\frac{|f_L(z)|^2}{K_L(z,z)} > e^{\delta L}$$

imposed in $\mathcal{E}_2$ implies that, for $|z| \leq r$ and L big,

$$\sum_{|\alpha|\leq C\delta L} |a_\alpha|^2 \geq \left(e^{\frac{\delta}{2}L} - \frac{C_r}{\sqrt{K_L(z,z)}}\right)^2 > \frac{1}{2} e^{\delta L}.$$

Let

$$M_L = \#\{\alpha : \ |\alpha| \leq C\delta L\} = \sum_{m \leq C\delta L} \frac{\Gamma(n+m)}{\Gamma(n)m!} \leq C\delta L \frac{\Gamma(n+C\delta L)}{\Gamma(n)(C\delta L)!} \simeq C^n \delta^n L^n.$$

Hence,

$$\begin{aligned}\mathbb{P}[A \cap \mathcal{E}_2] \leq & \mathbb{P}\Big[\Big\{ \sum_{|\alpha| \leq C\delta L} |a_\alpha|^2 \geq \frac{1}{2} e^{\delta L} \Big\}\Big] \\ \leq & \sum_{|\alpha| \leq C\delta L} \mathbb{P}\Big[|a_\alpha|^2 \geq \frac{e^{\delta L}}{2M_L}\Big] \\ = & M_L e^{-(\frac{e^{\delta L}}{2M_L})} \leq e^{-e^{\frac{\delta}{2}L}}.\end{aligned}$$

Using this last estimate and the bound for $\mathbb{P}[A]$, we have finally that

$$\mathbb{P}[\mathcal{E}_2] \leq e^{-e^{L\delta/2}}. \qquad ∎$$

It remains to prove Lemma 7. Before we proceed we need the following mean-value estimate of $\log|\hat{f}_L(\lambda)|^2$.

Lemma 11: *Let $\lambda \in \mathbb{B}_n$, $s > 0$ and consider the pseudo-hyperbolic ball $E(\lambda, s)$. Then*

$$\log|\hat{f}_L(\lambda)|^2 \leq \frac{1}{\mu(E(\lambda, s))} \int_{E(\lambda,s)} \log|\hat{f}_L(\xi)|^2 d\mu(\xi) + L\epsilon(n, s),$$

where

$$\epsilon(n, s) = \frac{n}{\mu(E(0, s))} \int_0^{\frac{s^2}{1-s^2}} x^{n-1} \log(1+x)dx \leq \frac{s^2}{1-s^2} = \mu(E(\lambda, s))^{1/n}.$$

Proof. By the subharmonicity of $\log|f_L(z)|^2$ we have

$$\begin{aligned}\log|\hat{f}_L(\lambda)|^2 \leq & \frac{1}{\mu(E(\lambda, s))} \int_{E(\lambda,s)} \log|f_L(\xi)|^2 d\mu(\xi) + L\log(1-|z|^2) \\ = & \frac{1}{\mu(E(\lambda, s))} \int_{E(\lambda,s)} \log|\hat{f}_L(\xi)|^2 d\mu(\xi) \\ & + L\Big[\log(1-|\lambda|^2) - \frac{1}{\mu(E(\lambda, s))} \int_{E(\lambda,s)} \log(1-|\xi|^2) d\mu(\xi)\Big].\end{aligned}$$

Identity (2) and the pluriharmonicity of $\log|1-\bar\lambda\cdot\xi|^2$ yield

$$\begin{aligned}\frac{1}{\mu(E(\lambda,s))}&\int_{E(\lambda,s)}\log(1-|\xi|^2)d\mu(\xi)\\ &=\frac{1}{\mu(B(0,s))}\int_{B(0,s)}\log(1-|\phi_\lambda(\xi)|^2)d\mu(\xi)\\ &=\log(1-|\lambda|^2)+\frac{1}{\mu(B(0,s))}\int_{B(0,s)}\log(1-|\xi|^2)d\mu(\xi).\end{aligned}$$

Changing into polar coordinates and performing the change of variable $x=\frac{r^2}{1-r^2}$ we get

$$\begin{aligned}\int_{B(0,s)}\log(1-|\xi|^2)d\mu(\xi)=&2n\int_0^s\log(1-r^2)\frac{r^{2n-1}}{(1-r^2)^{n+1}}\,dr\\ =&-n\int_0^{\frac{s^2}{1-s^2}}x^{n-1}\log(1+x)\,dx.\end{aligned}$$

This and the fact that $\mu(B(0,s))=\frac{s^{2n}}{(1-s^2)^n}$ ([Sto94] (4.4)) finish the proof. ∎

Proof of Lemma 7. According to Lemma 8(a), except for an exceptional event of probability $e^{-cL^{n+1}}$, there is $\lambda\in E:=E(z_0,s)$ such that

$$-L(\mu(E))^{1/n}<\log|\hat f_L(\lambda)|^2.$$

Therefore, using Lemma 11,

$$-L(\mu(E))^{1/n}<\frac{1}{\mu(E)}\int_E\log|\hat f_L(\xi)|^2d\mu(\xi)+L(\mu(E))^{1/n}.$$

Hence

$$0<\frac{1}{\mu(E)}\int_E\log|\hat f_L(\xi)|^2d\mu(\xi)+2L(\mu(E))^{1/n}.$$

Separating the positive and negative parts of the logarithm we obtain

$$\frac{1}{\mu(E)}\int_E\log^-|\hat f_L(\xi)|^2d\mu(\xi)\le\frac{1}{\mu(E)}\int_E\log^+|\hat f_L(\xi)|^2d\mu(\xi)+2L(\mu(E))^{1/n}.$$

Hence,

$$\frac{1}{\mu(E)}\int_E|\log|\hat f_L(\xi)|^2|d\mu(\xi)\le\frac{2}{\mu(E)}\int_E\log^+|\hat f_L(\xi)|^2d\mu(\xi)+2L(\mu(E))^{1/n}.$$

Again by Lemma 8, outside another exceptional event of probability $e^{-cL^{n+1}}$,

$$\frac{1}{\mu(E)}\int_E |\log|\hat{f}_L(\xi)|^2|d\mu(\xi) \leq 2\max_E \log^+|\hat{f}_L(\xi)|^2 + 2L(\mu(E))^{1/n} \leq 5L\mu(E)^{1/n}. \qquad \blacksquare$$

3. The hole theorem

Here we prove Theorem 4.

The upper bound is a direct consequence of the results in the previous section. Letting $U = B(0,r)$ and applying Corollary 3 with $\delta\mu(U)$ instead of δ we get

$$\mathbb{P}[Z_{f_L} \cap B(0,r) = \emptyset] \leq \mathbb{P}[|I_L(U) - L\mu(U)| > \delta L\mu(U)] \leq e^{-C_2 L^{n+1}}.$$

The method to prove the lower bound is by now standard (see, for example, [HKPV09, Theorem 7.2.3] and [ST04]): we shall choose three events forcing f_L to have a hole $B(0,r)$ and then we shall see that the probability of such events is at least $e^{-C_1 L^{n+1}}$. Our starting point is the estimate

$$|f_L(z)| \geq |a_0| - \Bigg|\sum_{0<|\alpha|\leq CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Bigg| - \Bigg|\sum_{|\alpha|>CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Bigg|,$$

where C will be chosen later on.

The first event is

$$E_1 := \{\ |a_0| \geq 1\},$$

which has probability

$$\mathbb{P}[E_1] = \mathbb{P}[|a_0|^2 \geq 1] = e^{-1}.$$

The second event corresponds to the tail of the power series of f_L. Let

$$E_2 := \Bigg\{\ |a_\alpha| \leq \sqrt{\frac{\alpha!\Gamma(n)}{\Gamma(n+|\alpha|)}}|\alpha|^n, \quad \forall\alpha:\ |\alpha| > CL\Bigg\}.$$

We shall see next that $\mathbb{P}[E_2]$ is big, and that under the event E_2 the tail of the power series of f_L is small.

Using (10) we have

$$\begin{aligned}\Big|\sum_{|\alpha|>CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big| &\leq \sum_{|\alpha|>CL}|a_\alpha|\Big[\frac{\Gamma(L+|\alpha|)}{\Gamma(L)\alpha!}\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}r^{2|\alpha|}\Big]^{1/2}\\ &\leq \sum_{m>CL}\Big[\frac{\Gamma(L+m)}{\Gamma(L)}r^{2m}\Big]^{1/2}\sum_{|\alpha|=m}|a_\alpha|\Big(\frac{\alpha^\alpha}{\alpha!|\alpha|^{|\alpha|}}\Big)^{1/2}.\end{aligned}$$

Thus, using Cauchy–Schwarz inequality and (11),

$$\Big|\sum_{|\alpha|>CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big| \leq \sum_{m>CL}\Big[\frac{\Gamma(L+m)}{\Gamma(L)m!}r^{2m}\Big]^{1/2}\Big(\sum_{|\alpha|=m}|a_\alpha|^2\Big)^{1/2}.$$

Using the asymptotics of the Gamma function (9), we estimate

$$\frac{\Gamma(m+L)}{\Gamma(L)m!}\simeq\frac{m^{L-1}}{\Gamma(L)}\leq\Big[\frac{m^{L/m}}{\Gamma(L)^{1/m}}\Big]^m.$$

Note that the function $g(x):=(x^L/\Gamma(L))^{1/x}$ is decreasing for $x\geq L$. Thus if $m>CL$, Stirling's formula yields

$$\frac{m^{L/m}}{\Gamma(L)^{1/m}}\leq\frac{(CL)^{1/C}}{\Gamma(L)^{1/(CL)}}=\frac{C^{1/C}L^{1/(2CL)}e^{1/C}}{(2\pi)^{1/(2CL)}}[1+\mathrm{o}(1)]\leq(eC)^{\frac{1}{C}}K^{\frac{1}{2C}},$$

where $K=\max_{x>0}x^{1/x}=e^{-1/e}$.

Let $h(C)=(eC)^{\frac{1}{C}}K^{\frac{1}{2C}}$ and note that $h(C)>1$ and $\lim_{C\to\infty}h(C)=1$. Hence, there exists C big enough so that $h(C)r^2\leq(1-\delta)^2$ and therefore

$$\begin{aligned}\Big|\sum_{|\alpha|>CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big| &\leq \sum_{m>CL}[h(C)r^2]^{m/2}\Big(\sum_{|\alpha|=m}|a_\alpha|^2\Big)^{1/2}\\ &\leq \sum_{m>CL}(1-\delta)^m\Big(\sum_{|\alpha|=m}|a_\alpha|^2\Big)^{1/2}.\end{aligned}$$

Under the event E_2,

$$\sum_{|\alpha|=m}|a_\alpha|^2\leq\sum_{|\alpha|=m}\frac{|\alpha|!\Gamma(n)}{\Gamma(n+|\alpha|)}|\alpha|^{2n}=m^{2n},$$

hence the tail of f_L is controlled by the tail of a convergent series and there exists C big enough so that

$$\Big|\sum_{|\alpha|>CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big| \le \sum_{m>CL}(1-\delta)^m m^{2n} < \frac{1}{4}.$$

Now we prove that the probability of E_2 is big. Since the variables a_α are independent we have, again by (9),

$$\begin{aligned}\mathbb{P}[E_2^c] &\le \sum_{|\alpha|>CL}\mathbb{P}\Big[|a_\alpha|^2 > \frac{|\alpha|!\Gamma(n)}{\Gamma(n+|\alpha|)}|\alpha|^{2n}\Big]\\ &= \sum_{m>CL}\mathbb{P}\Big[|\xi|^2 > \frac{m!\Gamma(n)}{\Gamma(n+m)}m^{2n}\Big]\frac{\Gamma(n+m)}{\Gamma(n)m!}\\ &\lesssim \sum_{m>CL}\mathbb{P}[|\xi|^2 > c_n m^{n+1}]m^{n-1} = \sum_{m>CL} e^{-c_n m^{n+1}} m^{n-1}.\end{aligned}$$

Thus for L big enough, $\mathbb{P}[E_2^c] \le 1/2$, and $\mathbb{P}[E_2] \ge 1/2$.

The third event takes care of the middle terms in the power series of f_L. Let

$$E_3 := \Big\{|a_\alpha|^2 < \frac{1}{16CL}\frac{|\alpha|!\Gamma(n)}{\Gamma(n+|\alpha|)}(1-r^2)^L \quad \forall\alpha : 0<|\alpha|\le CL\Big\}.$$

Using Cauchy–Schwarz inequality, (10) and(11) we get, as in previous computations,

$$\begin{aligned}&\Big|\sum_{0<|\alpha|\le CL} a_\alpha\Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big|\\ &\qquad\le\Big(\sum_{0<|\alpha|\le CL}|a_\alpha|^2\Big)^{1/2}\Big(\sum_{0<|\alpha|\le CL}\frac{\Gamma(|\alpha|+L)}{\Gamma(L)\alpha!}\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}r^{2|\alpha|}\Big)^{1/2}\\ &\qquad\le\Big(\sum_{0<|\alpha|\le CL}|a_\alpha|^2\Big)^{1/2}\Big(\sum_{0<m\le CL}\frac{\Gamma(m+L)}{\Gamma(L)m!}r^{2m}\Big)^{1/2}\\ &\qquad\le\Big(\sum_{0<|\alpha|\le CL}|a_\alpha|^2\Big)^{1/2}(1-r^2)^{-L/2}.\end{aligned}$$

Under the event E_3,

$$\sum_{0<|\alpha|\le CL}|a_\alpha|^2 \le \sum_{0<m\le CL}\frac{1}{16CL}(1-r^2)^L = \frac{1}{16}(1-r^2)^L,$$

and therefore

$$\Big|\sum_{0<|\alpha|\leq CL} a_\alpha \Big(\frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)}\Big)^{1/2} z^\alpha\Big| \leq \frac{1}{4}.$$

On the other hand,

$$\mathbb{P}[E_3] = \prod_{0<m\leq CL} [1-e^{-\frac{1}{16CL}\frac{m!\Gamma(n)}{\Gamma(m+n)}(1-r^2)^L}]^{\frac{\Gamma(n+m)}{m!\Gamma(n)}}.$$

Note that if L is big enough, then the term appearing in the exponential is small. Since $1-e^{-x}\geq x/2$ for $x\in(0,1/2)$, we get

$$\begin{aligned}\mathbb{P}[E_3] &\geq \prod_{0<m\leq CL} \Big[\frac{1}{32CL}\frac{m!\Gamma(n)}{\Gamma(m+n)}(1-r^2)^L\Big]^{\frac{\Gamma(n+m)}{m!\Gamma(n)}} \\ &= \Big[\frac{\Gamma(n)}{32CL}(1-r^2)^L\Big]^{\sum_{0<m\leq CL}\frac{\Gamma(n+m)}{m!\Gamma(n)}} \prod_{0<m\leq CL}\Big(\frac{m!}{\Gamma(m+n)}\Big)^{\frac{\Gamma(n+m)}{m!\Gamma(n)}}.\end{aligned}$$

Now we estimate each term of the product and the sum by the "worst" term. Denote $M=[CL]$. The exponent in the first factor is controlled by

$$\sum_{m=1}^{M}\frac{\Gamma(n+m)}{m!\Gamma(n)} \leq M\frac{\Gamma(n+M)}{M!\Gamma(n)} = \frac{\Gamma(n+M)}{\Gamma(M)\Gamma(n)} \leq M^n \leq (CL)^n.$$

Similarly, for the second factor we have

$$\begin{aligned}\prod_{m=1}^{M}\Big(\frac{m!}{\Gamma(m+n)}\Big)^{\frac{\Gamma(n+m)}{m!\Gamma(n)}} &\geq \Big(\frac{M!}{\Gamma(M+n)}\Big)^{M\frac{\Gamma(n+M)}{M!\Gamma(n)}} \geq \Big(\frac{M!}{\Gamma(M+n)}\Big)^{\frac{\Gamma(n+M)}{\Gamma(M)}} \\ &\geq \Big(\frac{\Gamma(CL+1)}{\Gamma(CL+n)}\Big)^{\frac{\Gamma(n+CL)}{\Gamma(CL)}}.\end{aligned}$$

Then, using again (9),

$$\begin{aligned}\log\mathbb{P}[E_3] \geq& (CL)^n\log\Big[\frac{\Gamma(n)}{32CL}(1-r^2)^L\Big] + \frac{\Gamma(n+CL)}{\Gamma(CL)}\log\Big[\frac{\Gamma(CL+1)}{\Gamma(CL+n)}\Big] \\ \gtrsim& (CL)^n\log\Big[\frac{\Gamma(n)}{32CL}(1-r^2)^L\Big] + (CL)^n\log(CL)^{1-n} \\ =& C^nL^n\Big[\log\frac{\Gamma(n)}{32C^n} - n\log L - L\log\frac{1}{1-r^2}\Big] \\ =& -C^nL^{n+1}\log\frac{1}{1-r^2}\Big[1+\frac{n\log L}{L\log\frac{1}{1-r^2}} - \frac{\log\frac{\Gamma(n)}{32C^n}}{L\log\frac{1}{1-r^2}}\Big] \\ =& -C^nL^{n+1}\log\frac{1}{1-r^2}[1+o(1)].\end{aligned}$$

Finally,

$$\mathbb{P}[E_2 \cap E_3 \cap \mathcal{C}] \geq e^{-C(n)\log(\frac{1}{1-r^2})L^{n+1}[1+o(1)]},$$

and under this event $|f_L(z)| \geq 1 - 1/4 - 1/4 > 0$.

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