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DERIVED STRING TOPOLOGY AND THE EILENBERG–MOORE SPECTRAL SEQUENCE

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ABSTRACT

Let M be a simply-connected closed manifold of dimension m. Chas and Sullivan have defined (co)products on the homology of the free loop space $H_*(LM)$. Félix and Thomas have extended the loop (co)products to those of simply-connected Gorenstein spaces over a field. We describe these loop (co)products in terms of the torsion and extension functors by developing string topology in appropriate derived categories.

In Algebraic Topology, one of the most important tools for computing the (co)homology of the space of free loops on a space is the (co)homological Eilenberg–Moore spectral sequence. Consider, over any field, the homological Eilenberg–Moore spectral sequence converging to $H_*(LM)$. Our description of the loop product enables one to conclude that this spectral sequence is multiplicative with respect to the Chas–Sullivan loop product and that its E_2-term is the Hochschild cohomology of $H^*(M)$. This gives a new method to compute the loop products on $H_*(LS^m)$ and $H_*(L\mathbb{C}P^r)$, the free loop space homology of spheres and complex projective spaces.

1. Introduction

To any topological space X is associated its space of free loops LX. Let M be a simply-connected closed oriented manifold of dimension m. In [5], Chas and Sullivan have defined a product called the **loop product** on the homology of LM

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-m}(LM)$$

and a coproduct called the **loop coproduct**

$$H_n(LM) \to H_{n-m}(LM \times LM)$$

such that the shifted homology of the free loop space $\mathbb{H}_*(LM) := H_{*+m}(LM)$ is a graded algebra while $H_{*-m}(LM)$ is a graded coalgebra. This was the beginning of String Topology.

In [12], Félix and Thomas generalized the loop (co)products of Chas and Sullivan from manifolds to the spaces that they studied previously and what they call Gorenstein spaces. The formal definition of Gorenstein space states a condition for the Ext functor for the differential graded algebra of singular cochains of a space in a field. These spaces form a class much more general than that of manifolds and it includes Poincaré duality spaces and Borel constructions, in particular, the classifying spaces of connected Lie groups; see [10, 33, 22].

In the remainder of this section, our main results are surveyed.

We describe explicitly the loop (co)products for a Gorenstein space in terms of the differential torsion product and the extension functors; see Theorems 2.3, 2.5 and 2.14. The key idea of the consideration comes from the general setting in [12] for defining string operations mentioned above. Thus our description of the loop (co)product with derived functors fits **derived string topology**, namely the framework of string topology due to Félix and Thomas.

In Algebraic Topology, there are two powerful spectral sequences to compute the (co)homology of a space namely the Leray–Serre spectral sequence and the Eilenberg–Moore spectral sequence (EMSS). Both have advantages and disadvantages. The differentials in the EMSS are often simple (even trivial). But usually, you have extension problems.

In [8], Cohen, Jones and Yan have shown that the Leray–Serre spectral sequence converging to $H_*(LM)$ is multiplicative with respect to the loop product. Here we show that over any field $\mathbb{K}$, the homological Eilenberg–Moore spectral sequence converging to $H_*(LM)$ enjoys also a multiplicative structure corresponding to the loop product. Its E_2-term is represented by the Hochschild cohomology ring of $H^*(M;\mathbb{K})$ with coefficients in itself,

$$HH^*(H^*(M;\mathbb{K}), H^*(M;\mathbb{K}));$$

see Theorem 2.11. This was announced by McClure [28, Theorem B].

It is conjectured that there is an isomorphism of graded algebras between the loop homology of M and the Hochschild cohomology of the singular cochains on M. But over $\mathbb{F}_p$, even in the case of a simply-connected closed orientable manifold, there is no complete written proof of such an isomorphism of algebras (see [14, p. 237] for details). Anyway, even if we assume such isomorphism, it is not clear that the spectral sequence obtained by filtering Hochschild cohomology is isomorphic to the EMSS although these two spectral sequences have the same E_2 and E_∞-term.

It is worth stressing that our main result on the multiplicativity of the EMSS, Theorem 2.11, is applicable, not only to Poincaré duality spaces, but to each space in the more wide class of Gorenstein spaces. Note also that the EMSS is compatible with the loop coproduct; see Theorem 2.8.

Let M be a simply-connected space whose cohomology is of finite dimension and is generated by a single element (e.g., M is a sphere or a complex projective space). Then explicit calculations of the EMSS made in the sequel [20] to this paper yield that the loop homology of M is isomorphic to the Hochschild cohomology of $H^*(M;\mathbb{K})$ as an algebra. This illustrates computability of our spectral sequence in Theorem 2.11.

Let A be a commutative Gorenstein algebra. We define a generalized cup product on the Hochschild cohomology $HH^*(A, A^\vee)$ of A with coefficients in its dual, $A^\vee$; see Example 5.7. If M is a simply-connected Gorenstein space, then the algebra of polynomial differential forms $A_{PL}(M)$ is a commutative Gorenstein algebra and we show that over $\mathbb{Q}$, $HH^*(A_{PL}(M), A_{PL}(M)^\vee)$ is isomorphic as algebras to $H_*(LM)$; see Theorem 2.17. Thus, when M is a Poincaré duality space, we recover the isomorphism of algebras

$$\mathbb{H}_*(LM;\mathbb{Q}) \cong HH^*(A_{PL}(M), A_{PL}(M))$$

of Félix and Thomas; see Corollary 2.18.

With the aid of the torsion functor descriptions of the loop (co)products, we see that the composite (the loop product) $\circ$ (the loop coproduct) is trivial for a simply-connected Poincaré duality space; see Theorem 2.13.

It is also important to mention that in the Appendices, we have paid attention to signs and extended the properties of shriek maps on Gorenstein spaces given in [12], in order to prove that the loop product is associative and commutative for Poincaré duality space.

2. Derived string topology and main results

The goal of this section is to state our results in detail. The proofs are found in Sections 3 to 7.

We begin by recalling the most prominent result on shriek maps due to Félix and Thomas, which supplies string topology with many homological and homotopical algebraic tools. Let $\mathbb{K}$ be a field of arbitrary characteristic. In what follows, we denote by $C^*(M)$ and $H^*(M)$ the normalized singular cochain algebra of a space M with coefficients in $\mathbb{K}$ and its cohomology, respectively. For a differential graded algebra A, let D(Mod-A) and D(A-Mod) be the derived categories of right A-modules and left A-modules, respectively. Unless otherwise explicitly stated, it is assumed that a space has the homotopy type of a CW-complex whose homology with coefficients in an underlying field is of finite type.

Consider a pull-back diagram $\mathcal{F}$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

in which p is a fibration over a simply-connected Poincaré duality space M of dimension m with the fundamental class ω_M and N is a Poincaré duality space of dimension n with the fundamental class ω_N.

Theorem 2.1 ([23],[12, Theorems 1 and 2]): *With the notation above there exist unique elements*

$$f^! \in \mathrm{Ext}^{m-n}_{C^*(M)}(C^*(N), C^*(M)) \quad \textit{and} \quad g^! \in \mathrm{Ext}^{m-n}_{C^*(E)}(C^*(X), C^*(E))$$

such that $H^*(f^!)(\omega_N) = \omega_M$, and in $\mathrm{D}(\mathrm{Mod}\text{-}C^*(M))$ *the following diagram is commutative:*

$$\begin{tikzcd} C^*(X) \arrow[r, "g^!"] & C^{*+m-n}(E) \\ C^*(N) \arrow[u, "q^*"] \arrow[r, "f^!"'] & C^{*+m-n}(M). \arrow[u, "p^*"'] \end{tikzcd}$$

Let A be a differential graded augmented algebra over $\mathbb{K}$. We call A a **Gorenstein algebra** of dimension m if

$$\dim \mathrm{Ext}^*_A(\mathbb{K}, A) = \begin{cases} 0 & \text{if } * \neq m, \\ 1 & \text{if } * = m. \end{cases}$$

A path-connected space M is called a $\mathbb{K}$-**Gorenstein space** (simply, Gorenstein space) of dimension m if the normalized singular cochain algebra $C^*(M)$ with coefficients in $\mathbb{K}$ is a Gorenstein algebra of dimension m. We write $\dim M$ for the dimension m.

The result [10, Theorem 3.1] yields that a simply-connected Poincaré duality space, for example a simply-connected closed orientable manifold, is Gorenstein. The classifying space BG of connected Lie group G and the Borel construction $EG \times_G M$ for a simply-connected Gorenstein space M with $\dim H^*(M;\mathbb{K}) < \infty$ on which G acts are also examples of Gorenstein spaces; see [10, 33, 22]. Observe that, for a closed oriented manifold M, $\dim M$ coincides with the ordinary dimension of M and that for the classifying space BG of a connected Lie group,

$\dim BG = -\dim G$. Thus the dimensions of Gorenstein spaces may become negative.

The following theorem enables us to generalize the above result concerning shriek maps on a Poincaré duality space to that on a Gorenstein space.

THEOREM 2.2 ([12, Theorem 12]): *Let X be a simply-connected $\mathbb{K}$-Gorenstein space of dimension m whose cohomology with coefficients in $\mathbb{K}$ is of finite type. Then*

$$\mathrm{Ext}^*_{C^*(X^n)}(C^*(X), C^*(X^n)) \cong H^{*-(n-1)m}(X),$$

where $C^*(X)$ is considered a $C^*(X^n)$-module via the diagonal map $\Delta : X \to X^n$.

We denote by $\Delta^!$ the map in $\mathrm{D}(\text{Mod-}C^*(X^n))$ which corresponds to a generator of $\mathrm{Ext}^{(n-1)m}_{C^*(X^n)}(C^*(X), X^*(X^n)) \cong H^0(X)$. Then, for a Gorenstein space X of dimension m and a fibre square

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "\Delta"'] & X^n, \end{tikzcd}$$

there exists a unique map $g^!$ in $\mathrm{Ext}^{(n-1)m}_{C^*(E)}(C^*(E'), C^*(E))$ which fits into the commutative diagram in $\mathrm{D}(\text{Mod-}C^*(X^n))$

$$\begin{tikzcd} C^*(E') \arrow[r, "g^!"] & C^*(E) \\ C^*(X) \arrow[r, "\Delta^!"'] \arrow[u, "(p')^*"] & C^*(X^n). \arrow[u, "p^*"'] \end{tikzcd}$$

We remark that the result follows from the same proof as that of Theorem 2.1.

Let $K \xleftarrow{f} A \xrightarrow{g} L$ be a diagram in the category of differential graded algebras (henceforth called DGA's). We consider K and L right and left modules over A via maps f and g, respectively. Then the differential torsion product $\mathrm{Tor}_A(K, L)$ is denoted by $\mathrm{Tor}_A(K, L)_{f,g}$ when the actions are emphasized.

We recall here the Eilenberg–Moore map. Consider the pull-back diagram $\mathcal{F}$ mentioned above, in which p is a fibration and M is a simply-connected space. Let $\varepsilon : F \to C^*(E)$ be a left semi-free resolution of $C^*(E)$ in $C^*(M)$-Mod, the category of left $C^*(M)$-modules. Then the Eilenberg–Moore map

$$EM : \mathrm{Tor}^*_{C^*(M)}(C^*(N), C^*(E))_{f^*,p^*} = H(C^*(N) \otimes_{C^*(M)} F) \longrightarrow H^*(X)$$

is defined by

$$EM(x \otimes_{C^*(M)} u) = q^*(x) \smile (g^*\varepsilon(u)) \quad \text{for } x \otimes_{C^*(M)} u \in C^*(N) \otimes_{C^*(M)} F.$$

We see that the map EM is an isomorphism of graded algebras with respect to the cup products; see [17] for example. In particular, for a simply-connected space M, consider the commutative diagram,

$$\begin{tikzcd} LM \arrow[r] \arrow[d, "ev_0"'] & M^I \arrow[d, "p=(ev_0,ev_1)"'] & M \arrow[l, "\sigma"', "\simeq"] \arrow[dl, "\Delta"] \\ M \arrow[r, "\Delta"'] & M \times M & \end{tikzcd}$$

where ev_i stands for the evaluation map at i and $\sigma : M \overset{\simeq}{\hookrightarrow} M^I$ for the inclusion of the constant paths. We then obtain the composite EM' :

$$H^*(LM) \underset{\cong}{\xleftarrow{EM}} \mathrm{Tor}^*_{C^*(M^{\times 2})}(C^*M, C^*M^I)_{\Delta^*,p^*} \underset{\cong}{\xrightarrow{\mathrm{Tor}_1(1,\sigma^*)}} \mathrm{Tor}^*_{C^*(M^{\times 2})}(C^*M, C^*M)_{\Delta^*,\Delta^*}.$$

Our first result states that the torsion functor $\mathrm{Tor}^*_{C^*(M^{\times 2})}(C^*(M), C^*(M))_{\Delta^*,\Delta^*}$ admits (co)products which are compatible with EM'.

In order to describe such a result, we first recall the definition of the loop product on a simply-connected Gorenstein space. Consider the diagram

$$\begin{tikzcd} LM \arrow[d, "ev_0"'] & LM \times_M LM \arrow[l, "Comp"'] \arrow[r, "q"] \arrow[d] & LM \times LM \arrow[d, "(ev_0,ev_1)"] \\ M \arrow[r, equal] & M \arrow[r, "\Delta"'] & M \times M, \end{tikzcd} \tag{2.1}$$

where the right-hand square is the pull-back of the diagonal map Δ, q is the inclusion and $Comp$ denotes the concatenation of loops. By definition the composite

$$q^! \circ (Comp)^* : C^*(LM) \to C^*(LM \times_M LM) \to C^*(LM \times LM)$$

induces the dual to the loop product Dlp on $H^*(LM)$; see [12, Introduction]. We see that $C^*(LM)$ and $C^*(LM \times LM)$ are $C^*(M \times M)$-modules via the map $ev_0 \circ \Delta$ and (ev_0, ev_1), respectively. Moreover, since $q^!$ is a morphism of $C^*(M \times M)$-modules, it follows that so is $q^! \circ (Comp)^*$. The proof of Theorem 2.1 states that the map $q^!$ is obtained extending the shriek map $\Delta^!$, which is first given, in the derived category $\mathrm{D}(\text{Mod-}C^*(M \times M))$. This fact allows us to formulate $q^!$ in terms of differential torsion functors.

Theorem 2.3: *Let M be a simply-connected Gorenstein space of dimension m. Consider the comultiplication $\widetilde{(Dlp)}$ given by the composite*

$$
\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M))_{\Delta^*,\Delta^*} \arrow[r, "\mathrm{Tor}_{p_{13}^*}(1,1)"] \arrow[dd, dotted] & \mathrm{Tor}^*_{C^*(M^3)}(C^*(M),C^*(M))_{((1\times\Delta)\circ\Delta)^*,((1\times\Delta)\circ\Delta)^*} \\
 & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M),C^*(M^2))_{(\Delta^2\circ\Delta)^*,\Delta^{2*}} \arrow[u, "\cong"', "\mathrm{Tor}_{(1\times\Delta\times 1)^*}(1,\Delta^*)"] \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"] \\
(\mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M))^{\otimes 2}_{\Delta^*,\Delta^*})^{*+m} \arrow[r, "\cong", "\widetilde{\top}"'] & \mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2),C^*(M^2))_{\Delta^{2*},\Delta^{2*}}.
\end{tikzcd}
$$

See Remark 2.4 below for the definition of $\widetilde{\top}$. Then the composite EM' :

$$
H^*(LM) \xrightarrow[\cong]{EM^{-1}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M^I))_{\Delta^*,p^*}
$$
$$
\xrightarrow[\cong]{\mathrm{Tor}_1(1,\sigma^*)} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M))_{\Delta^*,\Delta^*}
$$

is an isomorphism which respects the dual to the loop product Dlp and the comultiplication $\widetilde{(Dlp)}$ defined here.

Remark 2.4: The isomorphism $\widetilde{\top}$ in Theorem 2.3 is the canonical map defined by [17, p. 26] or by [27, p. 255] as the composite

$$
\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M))^{\otimes 2} \arrow[r, "\top"] & \mathrm{Tor}^*_{C^*(M^2)^{\otimes 2}}(C^*(M)^{\otimes 2},C^*(M)^{\otimes 2}) \arrow[d, "\mathrm{Tor}_\gamma(\gamma,\gamma)"] \\
\mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2),C^*(M^2)) \arrow[r, "\cong", "\mathrm{Tor}_{EZ^\vee}(EZ^\vee,EZ^\vee)"'] & \mathrm{Tor}^*_{(C_*(M^2)^{\otimes 2})^\vee}((C_*(M)^{\otimes 2})^\vee,(C_*(M)^{\otimes 2})^\vee)
\end{tikzcd}
$$

where $\top$ is the $\top$-product of Cartan–Eilenberg [4, XI. Proposition 1.2.1] or [26, VIII.Theorem 2.1], $EZ : C_*(M)^{\otimes 2} \overset{\simeq}{\to} C_*(M^2)$ denotes the Eilenberg–Zilber quasi-isomorphism and $\gamma : \mathrm{Hom}(C_*(M),\mathbb{K})^{\otimes 2} \to \mathrm{Hom}(C_*(M)^{\otimes 2},\mathbb{K})$ is the canonical map.

It is worth mentioning that this theorem gives an intriguing decomposition of the cup product on the Hochschild cohomology of a commutative algebra; see Lemma 5.4 below.

The loop coproduct on a Gorenstein space is also interpreted in terms of torsion products. In order to recall the loop coproduct, we consider the commutative diagram

(2.2)
$$\begin{tikzcd} LM \times LM & LM \times_M LM \arrow[l, "q"'] \arrow[r, "Comp"] \arrow[d] & LM \arrow[d, "l"] \\ & M \arrow[r, "\Delta"'] & M \times M, \end{tikzcd}$$

where $l : LM \to M \times M$ is a map defined by $l(\gamma) = (\gamma(0), \gamma(\frac{1}{2}))$. By definition, the composite

$$Comp^! \circ q^* : C^*(LM \times LM) \to C^*(LM \times_M LM) \to C^*(LM)$$

induces the dual to the loop coproduct $Dlcop$ on $H^*(LM)$.

Note that we apply Theorem 2.1 to (2.2) in defining the loop coproduct. On the other hand, applying Theorem 2.1 to the diagram (2.1), the loop product is defined.

Theorem 2.5: *Let M be a simply-connected Gorenstein space of dimension m. Consider the multiplication defined by the composite*

$$\begin{tikzcd} (\mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*})^{\otimes 2} \arrow[r, "\cong", "\widetilde{\top}"'] \arrow[dd, dotted] & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2), C^*(M^2))_{\Delta^{2*},\Delta^{2*}} \arrow[d, "\mathrm{Tor}_1(\Delta^*,1)"] \\ & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^2))_{(\Delta^2\circ\Delta)^*,\Delta^{2*}} \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"] \\ \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*} & \mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2), C^*(M^2))_{\gamma'^*,\Delta^{2*}} \arrow[l, "\cong"', "\mathrm{Tor}_{\alpha^*}(\Delta^*,\Delta^*)"] \end{tikzcd}$$

where the maps $\alpha : M^2 \to M^4$ and $\gamma' : M^2 \to M^4$ are defined by

$$\alpha(x, y) = (x, y, y, y) \quad \textit{and} \quad \gamma'(x, y) = (x, y, y, x).$$

See Remark 2.4 above for the definition of $\widetilde{\top}$. Then the composite EM' :

$$H^*(LM) \xrightarrow[\cong]{EM^{-1}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M^I))_{\Delta^*,p^*}$$
$$\xrightarrow[\cong]{\mathrm{Tor}_1(1,\sigma^*)} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*}$$

is an isomorphism which respects the dual to the loop coproduct $Dlcop$ and the multiplication defined here.

Remark 2.6: A relative version of the loop product is also in our interest. Let $f : N \to M$ be a map. Then by definition, the relative loop space $L_f M$ fits into the pull-back diagram

$$\begin{tikzcd} L_f M \arrow[r] \arrow[d] & M^I \arrow[d, "{(ev_0, ev_1)}"] \\ N \arrow[r, "{(f,f)}"'] & M \times M, \end{tikzcd}$$

where ev_t denotes the evaluation map at t. We may write $L_N M$ for the relative loop space $L_f M$ in case there is no danger of confusion. Suppose further that M is simply-connected and has a base point. Let N be a simply-connected Gorenstein space. Then the diagram

$$\begin{tikzcd} L_N M & L_N M \times_N L_N M \arrow[l, "Comp"'] \arrow[r, "q"] \arrow[d] & L_N M \times L_N M \arrow[d, "{(ev_0, ev_1)}"] \\ & N \arrow[r, "\Delta"'] & N \times N \end{tikzcd}$$

gives rise to the composite

$$q^! \circ (Comp)^* : C^*(L_N M) \to C^*(L_N M \times_N L_N M) \to C^*(L_N M \times L_N M)$$

which, by definition, induces the dual to the relative loop product $Drlp$ on the cohomology $H^*(L_N M)$ with degree $\dim N$; see [14, 16] for case that N is a smooth manifold. Since the diagram above corresponds to the diagram (2.1), the proof of Theorem 2.3 permits one to conclude that $Drlp$ has also the same description as in Theorem 2.3, where $C^*(N)$ is put instead of $C^*(M)$ in the left-hand variables of the torsion functors in the theorem.

As for the loop coproduct, we cannot define its relative version in natural way because of the evaluation map l of loops at $\frac{1}{2}$; see the diagram (2.2). Indeed the point $\gamma(\frac{1}{2})$ for a loop γ in $L_N M$ is not necessarily in N.

The associativity of Dlp and $Dlcop$ on a Gorenstein space is an important issue. We describe here an algebra structure on the shifted homology $H_{-*+d}(L_N M) = (H^*(L_N M)^\vee)^{*-d}$ of a simply-connected Poincaré duality space N of dimension d with a map $f : N \to M$ to a simply-connected space.

We define a map $m : H_*(L_N M) \otimes H_*(L_N M) \to H_*(L_N M)$ of degree d by

$$m(a \otimes b) = (-1)^{d(|a|+d)}((Drlp)^\vee)(a \otimes b)$$

for a and $b \in H_*(L_NM)$; see [8, sign of Proposition 4] or [39, Definition 3.2]. Moreover, put $\mathbb{H}_*(L_NM) = H_{*+d}(L_NM)$. Then we establish the following proposition.

Proposition 2.7: *Let N be a simply-connected Poincaré duality space. Then the shifted homology $\mathbb{H}_*(L_NM)$ is an associative algebra with respect to the product m. Moreover, if $M = N$, then the shifted homology $\mathbb{H}_*(LM)$ is graded commutative.*

As mentioned below, the loop product on L_NM is not commutative in general.

We call a bigraded vector space V a *bimagma* with shifted degree (i,j) if V is endowed with a multiplication $V \otimes V \to V$ and a comultiplication $V \to V \otimes V$ of degree (i,j).

Let K and L be objects in Mod-A and A-Mod, respectively. Consider a torsion product of the form $\mathrm{Tor}_A(K,L)$ which is the homology of the derived tensor product $K \otimes^{\mathbb{L}}_A L$. The external degree of the bar resolution of the second variable L filters the torsion products. Indeed, we can regard the torsion product $\mathrm{Tor}_A(K,L)$ as the homology $H(M \otimes_A B(A,A,L))$ with the bar resolution $B(A,A,L) \to L$ of L. Then the filtration $\mathcal{F} = \{F^p\mathrm{Tor}_A(K,L)\}_{p\leq 0}$ of the torsion product is defined by

$$F^p\mathrm{Tor}_A(K,L) = \mathrm{Im}\{i^* : H(M \otimes_A B^{\leq p}(A,A,L)) \to \mathrm{Tor}_A(K,L)\}.$$

Thus the filtration $\mathcal{F} = \{F^p\mathrm{Tor}_{C^*(M^2)}(C^*(M), C^*(M^I))\}_{p\leq 0}$ induces a filtration of $H^*(LM)$ via the Eilenberg–Moore map for a simply-connected space M.

By adapting differential torsion functor descriptions of the loop (co)products in Theorems 2.3 and 2.5, we can give the EMSS a bimagma structure.

Theorem 2.8: *Let M be a simply-connected Gorenstein space of dimension d. Then the Eilenberg–Moore spectral sequence*

$$\{E_r^{*,*}, d_r\}$$

converging to $H^*(LM;\mathbb{K})$ admits loop (co)products which is compatible with those in the target; that is, each term $E_r^{*,*}$ is endowed with a comultiplication

$$Dlp_r : E_r^{p,q} \to \bigoplus_{s+s'=p,t+t'=q+d} E_r^{s,t} \otimes E_r^{s',t'}$$

and a multiplication

$$Dlcop_r : E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'+d}$$

which are compatible with differentials in the sense that

$Dlp_r d_r = (-1)^d (d_r \otimes 1 + 1 \otimes d_r) Dlp_r$ *and* $Dlcop_r(d_r \otimes 1 + 1 \otimes d_r) = (-1)^d d_r Dlcop_r$.

Here $(d_r \otimes 1 + 1 \otimes d_r)(a \otimes b)$ *means* $d_r a \otimes b + (-1)^{p+q} a \otimes d_r b$ *if* $a \in E_r^{p,q}$. *Note the unusual sign* $(-1)^d$. *Moreover, the* E_∞*-term* $E_\infty^{*,*}$ *is isomorphic to* $\mathrm{Gr}H^*(LM;\mathbb{K})$ *as a bimagma with shifted degree* $(0,d)$.

If the dimension of the Gorenstein space is non-positive, unfortunately the loop product and the loop coproduct in the EMSS are trivial and the only information that Theorem 2.8 gives is the following corollary.

Corollary 2.9: *Let* M *be a simply-connected Gorenstein space of dimension* d. *Assume that* d *is negative or that* d *is null and* $H^*(M)$ *is not concentrated in degree* 0. *Consider the filtration given by the cohomological Eilenberg–Moore spectral sequence converging to* $H^*(LM;\mathbb{K})$. *Then the dual to the loop product and that to the loop coproduct increase both the filtration degree of* $H^*(LM)$ *by at least one.*

Remark 2.10: a) Let M be a simply-connected closed oriented manifold. We can choose a map $\Delta^! : C^*(M) \to C^*(M \times M)$ so that $H(\Delta^!) w_M = w_{M\times M}$; that is, $\Delta^!$ is the usual shriek map in the cochain level. Then the map Dlp and $Dlcop$ coincide with the dual to the loop product and to the loop coproduct in the sense of Chas and Sullivan [5], Cohen and Godin [9], respectively. Indeed, this fact follows from the uniqueness of the shriek map and the comments in three paragraphs in the end of [12, p. 421]. Thus the Eilenberg–Moore spectral sequence in Theorem 2.8 converges to $H^*(LM;\mathbb{K})$ as an algebra and a coalgebra.

b) Let M be the classifying space BG of a connected Lie group G. Since the homotopy fibre of $\Delta : BG \to BG\times BG$ in (2.1) and (2.2) is homotopy equivalent to G, we can choose the shriek map $\Delta^!$ described in Theorems 2.5 and 2.3 as the integration along the fibre. Thus $q^!$ also coincides with the integration along the fibre; see [12, Theorems 6 and 13]. This yields that the bimagma structure in $\mathrm{Gr}H^*(LBG;\mathbb{K})$ is induced by the loop product and coproduct in the sense of Chataur and Menichi [7].

c) Let M be the Borel construction $EG \times_G X$ of a connected compact Lie group G acting on a simply-connected closed oriented manifold X. In [2], Behrend, Ginot, Noohi and Xu defined a loop product and a loop coproduct on the homology $H_*(L\mathfrak{X})$ of free loop of a stack $\mathfrak{X}$. Their main example of stack is the quotient stack $[X/G]$ associated to a connected compact Lie group G acting smoothly on a closed oriented manifold X. Although Félix and Thomas did not prove it, we believe that their loop (co)products for the Gorenstein space $M = EG\times_G X$ coincide with the loop (co)products for the quotient stack $[X/G]$ of [2].

The following theorem is the main result of this paper.

Theorem 2.11: *Let N be a simply-connected Gorenstein space of dimension d. Let $f : N \to M$ be a continuous map to a simply-connected space M. Then the Eilenberg–Moore spectral sequence is a right-half plane cohomological spectral sequence $\{\mathbb{E}_r^{*,*}, d_r\}$ converging to the Chas–Sullivan loop homology $\mathbb{H}_*(L_N M)$ as an algebra with*

$$\mathbb{E}_2^{*,*} \cong HH^{*,*}(H^*(M); \mathbb{H}_*(N))$$

as a bigraded algebra; that is, there exists a decreasing filtration

$$\{F^p\mathbb{H}_*(L_N M)\}_{p\geq 0}$$

of $(\mathbb{H}_*(L_N M), m)$ such that $\mathbb{E}_\infty^{*,*} \cong Gr^{*,*}\mathbb{H}_*(L_N M)$ as a bigraded algebra, where

$$Gr^{p,q}\mathbb{H}_*(L_N M) = F^p\mathbb{H}_{-(p+q)}(L_N M)/F^{p+1}\mathbb{H}_{-(p+q)}(L_N M).$$

Here the product on the $\mathbb{E}_2$-term is the cup product (see Definition 5.1 (1)) induced by

$$(-1)^d\overline{H(\Delta^!)^\vee} : \mathbb{H}_*(N) \otimes_{H^*(M)} \mathbb{H}_*(N) \to \mathbb{H}_*(N).$$

Suppose further that N is a Poincaré duality space. Then the $\mathbb{E}_2$-term is isomorphic to the Hochschild cohomology $HH^{*,*}(H^*(M); H^*(N))$ with the cup product as an algebra.

Taking N to be the point, we obtain the following well-known corollary.

Corollary 2.12 (cf. [27, Corollary 7.19]): *Let M be a pointed topological space. Then the Eilenberg–Moore spectral sequence $E_2^{*,*} = \mathrm{Ext}^{*,*}_{H^*(M)}(\mathbb{K}, \mathbb{K})$ converging to $H_*(\Omega M)$ is a spectral sequence of algebras with respect to the Pontryagin product.*

When $M = N$ is a closed manifold, Theorem 2.11 has been announced by McClure in [28, Theorem B]. But the proof has not appeared. Moreover, McClure claimed that when $M = N$, the Eilenberg–Moore spectral sequence is a spectral sequence of BV-algebras. We have not yet been able to prove this very interesting claim.

We compare here the two spectral sequences converging the loop homology to two spectral sequences converging toward the Hochschild cohomology of the singular cochain on a space.

The homological Leray–Serre type	The cohomological Eilenberg–Moore type
$E^2_{-p,q} = H^p(M; H_q(\Omega M)) \Rightarrow \mathbb{H}_{-p+q}(LM)$ as an algebra, where M is a simply-connected closed oriented manifold; see [8].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M)) \Rightarrow \mathbb{H}_{-p-q}(LM)$ as an algebra, where M is a simply-connected Poincaré duality space; see Theorem 2.11.
$E^2_{p,q} = H^{-p}(M) \otimes \mathrm{Ext}^{-q}_{C^*(M)}(\mathbb{K}, \mathbb{K}) \Rightarrow HH^{-p-q}(C^*(M); C^*(M))$ as an algebra, where M is a simply-connected space whose cohomology is locally finite; see [35].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M)) \Rightarrow HH^{p+q}(C^*(M); C^*(M))$ as a BV-algebra, where M is a simply-connected Poincaré duality space; see [19].

Observe that each spectral sequence in the table above converges strongly to the target.

It is important to remark that, for a fibration $N \to X \to M$ of closed orientable manifolds, Le Borgne [25] has constructed a spectral sequence converging to the loop homology $\mathbb{H}_*(LX)$ as an algebra with $E_2 \cong \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$ under an appropriate assumption; see also [6] for applications of the spectral sequence. We refer the reader to [29] for spectral sequences concerning a generalized homology theory in string topology.

We focus on a global nature of the loop (co)product. Drawing on the torsion functor description of the loop product and the loop coproduct mentioned in Theorems 2.3 and 2.5, we have the following result.

THEOREM 2.13: *Let M be a simply-connected Poincaré duality space. Then the composite* (*the loop product*) $\circ$ (*the loop coproduct*) *is trivial.*

When M is a connected closed oriented manifold, the triviality of this composite was first proved by Tamanoi [41, Theorem A]. Tamanoi has also shown that this composite is trivial when M is the classifying space BG of a connected Lie group G [40, Theorem 4.4].

The same argument as in the proof of [41, Theorem A] deduces that if string operations on a Poincaré duality space give rise to a 2-dimensional TQFT, then all operations associated to surfaces of genus at least one vanish.

We are aware that the description of the loop coproduct in Theorem 2.5 has no opposite arrow such as $\mathrm{Tor}_{(1\times\Delta\times 1)^*}(1,\Delta^*)$ in Theorem 2.3. This is a key to the proof of Theorem 2.13. Though we have not yet obtained the same result as Theorem 2.13 on a more general Gorenstein space, some obstruction for the composite to be trivial can be found in a hom-set, namely the extension functor, in an appropriate derived category; see Remark 4.5. This small but significant result also asserts an advantage of derived string topology.

We may describe the loop product in terms of the extension functor.

Theorem 2.14: *Let M be a simply-connected Poincaré duality space. Consider the multiplication defined by the composite*

$$\begin{tikzcd}
\mathrm{Ext}^*_{C^*(M^2)}(C^*(M),C^*(M))^{\otimes 2}_{\Delta^*,\Delta^*} \arrow[rr, "\widetilde{\vee}", "\cong"'] \arrow[dd, dotted] & & \mathrm{Ext}^*_{C^*(M^4)}(C^*(M^2),C^*(M^2))_{\Delta^{2*},\Delta^{2*}} \arrow[d, "\mathrm{Ext}_1(1,\Delta^*)"] \\
& & \mathrm{Ext}^*_{C^*(M^4)}(C^*(M^2),C^*(M))_{\Delta^{2*},(\Delta^2\circ\Delta)^*} \\
\mathrm{Ext}^*_{C^*(M^2)}(C^*(M),C^*(M))_{\Delta^*,\Delta^*} & & \mathrm{Ext}^*_{C^*(M^3)}(C^*(M),C^*(M))_{((1\times\Delta)\circ\Delta)^*,((1\times\Delta)\circ\Delta)^*}. \arrow[ll, "\mathrm{Ext}_{p_{13}^*}(1,1)"'] \arrow[u, "\mathrm{Ext}_{(1\times\Delta\times 1)^*}(\Delta^*,1)"', "\cong"]
\end{tikzcd}$$

See Remark 2.15 below for the definition of $\widetilde{\vee}$. Then there is an explicit isomorphism $\Phi^\vee : \mathrm{Ext}^{-p}_{C^*(M^2)}(C^*M,C^*M)_{\Delta^*,\Delta^*} \to H_{p+m}(LM)$ which respects the multiplication defined here and the loop product.

Remark 2.15: The isomorphism $\widetilde{\vee}$ in Theorem 2.14 is the composite

$$\begin{tikzcd}
\mathrm{Ext}^*_{C^*(M^2)}(C^*M,C^*M)^{\otimes 2} \arrow[r, "\vee"] & \mathrm{Ext}^*_{C^*(M^2)^{\otimes 2}}(C^*(M)^{\otimes 2},C^*(M)^{\otimes 2}) \arrow[d, "\mathrm{Ext}_1(1,\gamma)"] \\
\mathrm{Ext}^*_{(C_*(M^2)^{\otimes 2})^\vee}((C_*(M)^{\otimes 2})^\vee,(C_*(M)^{\otimes 2})^\vee) \arrow[r, "\cong", "\mathrm{Ext}_\gamma(\gamma,1)"'] \arrow[d, "\mathrm{Ext}_{EZ^\vee}(EZ^\vee,1)"', "\cong"] & \mathrm{Ext}^*_{C^*(M^2)^{\otimes 2}}(C^*(M)^{\otimes 2},(C_*(M)^{\otimes 2})^\vee) \\
\mathrm{Ext}^*_{C^*(M^4)}(C^*(M^2),(C_*(M)^{\otimes 2})^\vee) & \mathrm{Ext}^*_{C^*(M^4)}(C^*(M^2),C^*(M^2)) \arrow[l, "\mathrm{Ext}_1(1,EZ^\vee)"', "\cong"]
\end{tikzcd}$$

where $\vee$ is the $\vee$-product of Cartan–Eilenberg [4, XI. Proposition 1.2.3] or [26, VIII.Theorem 4.2], $EZ : C_*(M)^{\otimes 2} \xrightarrow{\simeq} C_*(M^2)$ denotes the Eilenberg–Zilber quasi-isomorphism and $\gamma : \mathrm{Hom}(C_*(M), \mathbb{K})^{\otimes 2} \to \mathrm{Hom}(C_*(M)^{\otimes 2}, \mathbb{K})$ is the canonical map.

Remark 2.16: We believe that the multiplication on

$$\mathrm{Ext}^*_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*}$$

defined in Theorem 2.14 coincides with the Yoneda product.

Denote by $A(M)$ the functorial commutative differential graded algebra $A_{PL}(M)$; see [11, Corollary 10.10]. Let $\varphi : A(M)^{\otimes 2} \xrightarrow{\simeq} A(M^2)$ be the quasi-isomorphism of algebras given by [11, Example 2 pp. 142–3]. Note that the composite $\Delta^* \circ \varphi$ coincides with the multiplication of $A(M)$. Note also that we have an Eilenberg–Moore isomorphism EM for the functor $A(M)$; see [11, Theorem 7.10].

Replacing the singular cochains over the rationals $C^*(M;\mathbb{Q})$ by the commutative algebra $A_{PL}(M)$ in Theorem 2.3, we obtain the following theorem.

Theorem 2.17 (Compare with [13]): *Let N be a simply-connected Gorenstein space of dimension n and $N \to M$ a continuous map to a simply-connected space M. Let Φ be the map given by the commutative square*

$$\begin{tikzcd}
H^{p+n}(A(L_N M)) \arrow[d, dotted, "\Phi"'] & \mathrm{Tor}^{A(M^2)}_{-p-n}(A(N), A(M^I))_{\Delta^*,p^*} \arrow[l, "EM"', "\cong"] \arrow[d, "\cong"', "Tor^1(1,\sigma^*)"] \\
HH_{-p-n}(A(M), A(N)) \arrow[r, "\mathrm{Tor}^{\varphi}(1,1)", "\cong"'] & \mathrm{Tor}^{A(M^2)}_{-p-n}(A(N), A(M))_{\Delta^*,\Delta^*}.
\end{tikzcd}$$

Then the dual $HH^{-p-n}(A(M), A(N)^\vee) \xrightarrow{\Phi^\vee} H_{p+n}(L_N M;\mathbb{Q})$ to Φ is an isomorphism of graded algebras with respect to the loop product $Dlp^\vee$ and the generalized cup product on Hochschild cohomology induced by

$$(\Delta_{A(N)})^\vee : A(N)^\vee \otimes A(N)^\vee \to A(N)^\vee$$

(see Example 5.7).

Corollary 2.18: *Let N be a simply-connected Poincaré duality space of dimension n. Let $N \to M$ be a continuous map to a simply-connected space M. Then $HH^{-p}(A(M), A(N))$ is isomorphic as graded algebras to $H_{p+n}(L_N M;\mathbb{Q})$*

with respect to the loop product $Dlp^\vee$ and the cup product on Hochschild cohomology induced by the morphism of algebras $A(f) : A(M) \to A(N)$ (See Remark 5.2 and Definition 5.1(1))

Remark 2.19: When $M = N$ is a Poincaré duality space, such an isomorphism of algebras between Hochschild cohomology and Chas–Sullivan loop space homology was first proved in [13] (See also [32] over $\mathbb{R}$). But here our isomorphism is explicit since we do not use a Poincaré duality DGA model for $A(M)$ given by [24]. In fact, as explained in [13], such an isomorphism is an isomorphism of BV-algebras, since Φ is compatible with the circle action and Connes boundary map. Here the BV-algebra on $HH^*(A(M), A(M))$ is given by [30, Theorem 18 or Proof of Corollary 20].

In the forthcoming paper [21], we discuss the loop (co)products on the classifying space BG of a Lie group G by looking at the integration along the fibre $(Comp)^! : H^*(LBG \times_{BG} LBG) \to H^*(LBG)$ of the homotopy fibration $G \to LBG \times_{BG} LBG \to BG$. In a sequel [22], we intend to investigate duality on extension groups of the (co)chain complexes of spaces. Such discussion enables one to deduce that Noetherian H-spaces are Gorenstein. In adding, the loop homology of a Noetherian H-space is considered.

The rest of this paper is organized as follows. Section 3 is devoted to proving Theorems 2.3, 2.5, 2.8 and Corollary 2.9. Theorem 2.13 is proved in Section 4. In section 5, we decompose the cup product of the Hochschild cohomology of a commutative algebra using the Ext functor. Section 6 proves Theorems 2.11, 2.14 and 2.17 and Corollary 2.18. We prove Proposition 2.7 and discuss the associativity and commutativity of the loop product on Poincaré duality space in Section 7. In the last three sections (Appendices 8, 9 and 10), shriek maps on Gorenstein spaces are considered and their important properties, which we use in the body of the paper, are described.

3. Proofs of Theorems 2.3, 2.5 and 2.8

In order to prove Theorem 2.3, we consider two commutative diagrams

(3.1)
$$\begin{tikzcd}
& LM \times LM \arrow[rr, "i"] \arrow[dd, "p\times p"] & & M^I \times M^I \arrow[dd, "p\times p = p^2"] \\
LM \times_M LM \arrow[ur, "q"] \arrow[rr, "j"] \arrow[dd] & & M^I \times_M M^I \arrow[ur, "\widetilde{q}"] \arrow[dd, "{ev_0, ev_1, ev_1 = u}"] & \\
& M \times M \arrow[rr, "\Delta\times\Delta"] & & M^4 \\
M \arrow[ur, "\Delta"] \arrow[rr, "(1\times\Delta)\circ\Delta = v"'] & & M^3 \arrow[ur, "1\times\Delta\times 1 = w"'] &
\end{tikzcd}$$

and

(3.2)
$$\begin{tikzcd}
& LM \times_M LM \arrow[rr, "j"] \arrow[dl, "Comp"'] \arrow[dd] & & M^I \times_M M^I \arrow[dl, "Comp = c"] \arrow[dd, "{(ev_0, ev_1 = ev_0, ev_1) = u}"] \\
LM \arrow[rr, "k"] \arrow[dd, "p"'] & & M^I \arrow[dd, "p"] & \\
& M \arrow[rr, "(1\times\Delta)\circ\Delta"] \arrow[dl, equal] & & M^3 \arrow[dl, "p_{13}"] \\
M \arrow[rr, "\Delta"'] & & M \times M &
\end{tikzcd}$$

in which front and back squares are pull-back diagrams. Observe that the left- and right-hand-side squares in (3.1) are also pull-back diagrams. Here Δ and k denote the diagonal map and the inclusion, respectively. Moreover, $p_{13} : M^3 \to M^2$ is the projection defined by $p_{13}(x,y,z) = (x,z)$ and $Comp : LM \times_M LM \to LM$ stands for the concatenation of loops. The cube (3.2) first appeared in [13, p. 320].

Proof of Theorem 2.3. Consider the diagram

(3.3)
$$\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M^I))_{\Delta^*, p^*} \arrow[r, "\mathrm{Tor}_{p_{13}^*}(1, c^*)"] \arrow[d, "EM"', "\cong"] & \mathrm{Tor}^*_{C^*(M^3)}(C^*(M), C^*(M^I \times_M M^I))_{v^*, u^*} \arrow[ddl, "EM_1"', "\cong"] \\
H^*(LM) \arrow[d, "Comp^*"'] \arrow[dd, bend right, "Dlp"'] & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^I \times M^I))_{(wv)^*, p^{2*}} \arrow[u, "\cong", "\mathrm{Tor}_{w^*}(1, \widetilde{q}^*)"'] \arrow[d, "\mathrm{Tor}_1(\Delta^!, 1)"] \arrow[dl, "EM_2", "\cong"'] \\
H^*(LM \times_M LM) \arrow[d, "H(q^!)"'] & \mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2), C^*(M^I \times M^I))_{\Delta^{2*}, p^{2*}}. \arrow[dl, "EM_3", "\cong"'] \\
H^{*+m}(LM \times LM) &
\end{tikzcd}$$

where EM and EM_i denote the Eilenberg–Moore maps.

The diagram (3.2) is a morphism of pull-backs from the back face to the front face. Therefore the naturality of the Eilenberg–Moore map yields that the upper-left triangle is commutative.

We now consider the front square and the right-hand side square in the diagram (3.1). The squares are pull-back diagrams and hence we have a large pull-back one connecting them. Therefore the naturality of the Eilenberg–Moore map shows that the triangle in the center of the diagram (3.3) is commutative. Thus it follows that the map $\mathrm{Tor}_{w^*}(1, \widetilde{q}^*)$ is an isomorphism.

Let $\varepsilon : \mathbb{B} \overset{\simeq}{\twoheadrightarrow} C^*(M)$ be a right $C^*(M^2)$-semifree resolution of $C^*(M)$. By [12, Proof of Theorem 2 or Remark p. 429], the following square is commutative in the derived category of right $C^*(LM \times LM)$-modules:

$$\begin{tikzcd}
C^*(LM \times_M LM) \arrow[r, "q^!"] & C^*(LM \times LM) \\
\mathbb{B} \otimes_{C^*(M^2)} C^*(LM \times LM) \arrow[u, "EM_4"', "\simeq"] \arrow[r, "\Delta^! \otimes 1_{C^*(LM \times LM)}"'] & C^*(M^2) \otimes_{C^*(M^2)} C^*(LM \times LM) \arrow[u, equal, "EM_5"]
\end{tikzcd}$$

By taking homology, we obtain that the top square in the following diagram commutes:

(3.4)

$$\begin{tikzcd}
H^*(LM \times_M LM) \arrow[r, "H^*(q^!)"] & H^{*+m}(LM \times LM) \\
\mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(LM \times LM)) \arrow[u, "EM_4"', "\cong"] \arrow[r, "\mathrm{Tor}_1(\Delta^!,1)"'] & \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(LM \times LM)) \arrow[u, equal, "EM_5"] \\
\mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^I \times M^I)) \arrow[u, "\mathrm{Tor}^*_{(\Delta\times\Delta)^*}(1,i^*)"'] \arrow[r, "\mathrm{Tor}_1(\Delta^!,1)"'] \arrow[uu, bend left=60, "EM_2"] & \mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2), C^*(M^I \times M^I)) \arrow[u, "\mathrm{Tor}^*_{(\Delta\times\Delta)^*}(1,i^*)"] \arrow[uu, bend right=60, "EM_3"']
\end{tikzcd}$$

The bottom square obviously commutes. We now consider the left-hand square and the back square in the diagram (3.1). The squares are pull-back diagrams and hence we have a large pull-back one connecting them. Therefore the naturality of the Eilenberg–Moore map shows that the left-hand side in (3.4) is commutative. The same argument or the definition of the Eilenberg–Moore map shows that the right-hand side in (3.4) is commutative.

So finally, the lower square in (3.3) is commutative.

The usual proof [17, p. 26] that the Eilenberg–Moore isomorphism EM is an isomorphism of algebras with respect to the cup product gives the following commutative square:

$$\begin{tikzcd}
H^*(LM)^{\otimes 2} \arrow[d, "\times"', "\cong"] & \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M^I))^{\otimes 2}_{\Delta^*, p^*} \arrow[l, "\cong"', "EM^{\otimes 2}"] \arrow[d, "\cong"', "\widetilde{\top}"] \\
H^*(LM \times LM) & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2), C^*(M^I \times M^I))_{\Delta^{2*}, p^{2*}}. \arrow[l, "EM_3"', "\cong"]
\end{tikzcd}$$

This square is the top square in [27, p. 255].

Consider the commutative diagram of spaces where the three composites of the vertical morphisms are the diagonal maps

$$\begin{tikzcd}
M \arrow[r, equal] \arrow[d, "\sigma"', "\simeq"] \arrow[dd, bend right=60, "\Delta"'] & M \arrow[r, "\Delta"] \arrow[d, "\sigma'"', "\simeq"] & M \times M \arrow[d, "\sigma\times\sigma"', "\simeq"] \arrow[dd, bend left=60, "\Delta\times\Delta"] \\
M^I \arrow[d, "p"'] & M^I \times_M M^I \arrow[l, "c"] \arrow[r, "\widetilde{q}"'] \arrow[d, "u"] & M^I \times M^I \arrow[d, "p\times p"'] \\
M \times M & M^3 \arrow[l, "p_{13}"] \arrow[r, "w"'] & M^4.
\end{tikzcd}$$

Using the homotopy equivalence σ, σ' and σ^2, we have the result. ∎

We decompose the maps, which induce the loop coproduct, with pull-back diagrams. Let $l : LM \to M \times M$ be a map defined by $l(\gamma) = (\gamma(0), \gamma(\frac{1}{2}))$. We define a map $\varphi : LM \to LM$ by $\varphi(\gamma)(t) = \gamma(2t)$ for $0 \le t \le \frac{1}{2}$ and $\varphi(\gamma)(t) = \gamma(1)$ for $\frac{1}{2} \le t \le 1$. Then φ is homotopic to the identity map and fits into the commutative diagram

(3.5)
$$\begin{tikzcd}
& LM \arrow[rr, "j"] \arrow[dl, "\varphi"'] \arrow[dd, "ev_0"] & & M^I \arrow[dl, "\beta"] \arrow[dd, "{ev_0, ev_1 = p}"] \\
LM \arrow[rr, "\simeq"] \arrow[dd, "l"'] & & M^I \times M^I \arrow[dd, "p\times p"] & \\
& M \arrow[rr, "\Delta"] \arrow[dl, "\Delta"'] & & M \times M \arrow[dl, "\alpha"] \\
M \times M \arrow[rr, "\gamma'"'] & & M^4. &
\end{tikzcd}$$

Here the maps $\alpha : M^2 \to M^4$, $\beta : M^I \to M^I \times M^I$ and $\gamma' : M^2 \to M^4$ are defined by $\alpha(x, y) = (x, y, y, y)$, $\beta(r) = (r, c_{r(1)})$ with the constant loop $c_{r(1)}$ at $r(1)$ and $\gamma'(x, y) = (x, y, y, x)$, respectively. We consider moreover the two pull-back squares

$$\begin{tikzcd} LM \times_M LM \arrow[r, "Comp"] \arrow[d] & LM \arrow[r] \arrow[d, "l"'] & M^I \times M^I \arrow[d, "p\times p"] \\ M \arrow[r, "\Delta"'] & M \times M \arrow[r, "\gamma'"'] & M^4 \end{tikzcd} \tag{3.6}$$

and the commutative cube

$$\begin{tikzcd} & LM \times_M LM \arrow[rr, "i\circ q"] \arrow[dl, "q"'] \arrow[dd] & & M^I \times M^I \arrow[dl, equal] \arrow[dd, "p^2"] \\ LM \times LM \arrow[rr, "i"] \arrow[dd] & & M^I \times M^I \arrow[dd, "p^2"] & \\ & M \arrow[rr, "(\Delta\times\Delta)\circ\Delta"] \arrow[dl, "\Delta"'] & & M^4 \arrow[dl, equal] \\ M \times M \arrow[rr, "\Delta\times\Delta"'] & & M^4 & \end{tikzcd} \tag{3.7}$$

in which front and back squares are also pull-back diagrams.

Proof of Theorem 2.5. We see that the diagrams (3.5), (3.6) and (3.7) give rise to a commutative diagram

$$\begin{tikzcd} H^*(LM \times LM) \arrow[d, "q^*"'] \arrow[dd, bend right=60, "Dlcop"'] & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2), C^*((M^I)^2))_{\Delta^{2*},p^{2*}} \arrow[l, "EM"', "\cong"] \arrow[d, "\mathrm{Tor}_1(\Delta^*,1)"] \\ H^*(LM \times_M LM) \arrow[d, "Comp^!"'] & \mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^I \times M^I))_{(wv)^*,p^{2*}} \arrow[l, "\cong"', "EM"] \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"] \\ H^{*+m}(LM) \arrow[d, "\varphi^*=id"'] & \mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2), C^*(M^I \times M^I))_{\gamma'^*,p^{2*}}. \arrow[l, "EM"', "\cong"] \arrow[d, "\mathrm{Tor}_{\alpha^*}(\Delta^*,\beta^*)"', "\cong"] \\ H^{*+m}(LM) & \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M), C^*(M^I))_{\Delta^*,p^*}. \arrow[l, "\cong"', "EM"] \end{tikzcd}$$

In fact, the diagrams (3.5) and (3.7) give morphisms of pull-backs from the back face to the front face. Therefore the naturality of the Eilenberg–Moore map yields that the top and the bottom squares are commutative.

Using the diagram (3.6), the same argument as in the proof of Theorem 2.3 enables us to conclude that the middle square is commutative.

Since the following diagram of spaces

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d, "\sigma"', "\simeq"] \arrow[dd, bend right=60, "\Delta"'] & M \times M \arrow[d, "\sigma\times\sigma"', "\simeq"] \arrow[dd, bend left=60, "\Delta\times\Delta"] \\
M^I \arrow[r, "\beta"] \arrow[d, "p"'] & M^I \times M^I \arrow[d, "p\times p"'] \\
M \times M \arrow[r, "\alpha"] & M^4
\end{tikzcd}$$

is commutative, the theorem follows. ∎

By considering the free loop fibration $\Omega M \overset{\widetilde{\eta}}{\to} LM \overset{ev_0}{\to} M$, we define for Gorenstein space (see Example 8.2) an intersection morphism

$$H(\widetilde{\eta}_!) : H_{*+m}(LM) \to H_*(\Omega M)$$

generalizing the one defined by Chas and Sullivan [5]. Using the following commutative cube:

$$\begin{tikzcd}
& LM \arrow[rr] \arrow[dd, "ev_0" near start] & & M^I \arrow[dd, "p"] \\
\Omega M \arrow[ur, "\widetilde{\eta}"] \arrow[rr, crossing over] \arrow[dd] & & PM \arrow[ur, "\widetilde{\eta\times 1}"'] & \\
& M \arrow[rr, "\Delta" near start] & & M \times M \\
* \arrow[ur, "\eta"] \arrow[rr] & & * \times M \arrow[ur, "\eta\times 1"'] \arrow[from=uu, crossing over, "ev_1" near start] &
\end{tikzcd}$$

where all the faces are pull-backs, we obtain similarly the following theorem.

Theorem 3.1: *Let M be a simply-connected Gorenstein with generator ω_M in $\mathrm{Ext}^m_{C^*(M)}(\mathbb{K}, C^*(M))$. Then the dual of the intersection morphism $H(\widetilde{\eta}^!)$ is given by the commutative diagram*

$$\begin{tikzcd}
H^*(\Omega M) \arrow[dd, "H(\tilde{\eta}^!)"'] & \mathrm{Tor}^*_{C^*(M)}(\mathbb{K}, C^*(PM))_{\eta*,ev_1{}^*} \arrow[l, "EM"', "\cong"] \arrow[r, "\mathrm{Tor}^*_1(1,\eta^*)", "\cong"'] & \mathrm{Tor}^*_{C^*(M)}(\mathbb{K},\mathbb{K})_{\eta*,\eta^*} \\
& \mathrm{Tor}^*_{C^*(M^2)}(\mathbb{K}, C^*(M^I))_{\eta*,p^*} \arrow[ul, "\cong", "EM"'] \arrow[u, "\cong", "\mathrm{Tor}^*_{(\eta\times1)^*}(1,(\widetilde{\eta\times1})^*)"'] \arrow[r, "\mathrm{Tor}^*_1(1,\sigma^*)", "\cong"'] \arrow[d, "\mathrm{Tor}^*_1(\omega_M,1)"] & \mathrm{Tor}^*_{C^*(M^2)}(\mathbb{K}, C^*M)_{\eta*,\Delta^*} \arrow[u, "\cong", "\mathrm{Tor}^*_{(\eta\times1)^*}(1,\eta^*)"'] \arrow[d, "\mathrm{Tor}^*_1(\omega_M,1)"] \\
H^{*+m}(LM) & \mathrm{Tor}^*_{C^*(M^2)}(C^*(M),C^*(M^I))_{\Delta*,p^*} \arrow[l, "\cong"', "EM"] \arrow[r, "\mathrm{Tor}^*_1(1,\sigma^*)", "\cong"'] & \mathrm{Tor}^*_{C^*(M^2)}(C^*M,C^*M)_{\Delta*,\Delta^*}.
\end{tikzcd}$$

Let $\widehat{\mathcal{F}}$ be the pull-back diagram in the front of (3.1). Let $\widetilde{\mathcal{F}}$ denote the pull-back diagram obtained by combining the front and the right-hand-side squares in (3.1). Then a map inducing the isomorphism $\mathrm{Tor}_{w^*}(1, \widetilde{q}^*)$ gives rise to a morphism $\{f_r\} : \{\widetilde{E}_r, \widetilde{d}_r\} \to \{\widehat{E}_r, \widehat{d}_r\}$ of spectral sequences, where $\{\widehat{E}_r, \widehat{d}_r\}$ and $\{\widetilde{E}_r, \widetilde{d}_r\}$ are the Eilenberg–Moore spectral sequences associated with the fibre squares $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$, respectively. In order to prove Theorem 2.8, we need the following lemma.

Lemma 3.2: *The map f_2 is an isomorphism.*

Proof. We identify f_2 with the map

$$\mathrm{Tor}_{H^*(w)}(1, H^*(\Delta)) : \mathrm{Tor}_{H^*(M^4)}(H^*M, H^*(M \times M)) \to \mathrm{Tor}_{H^*(M^3)}(H^*M, H^*M)$$

up to isomorphism between the E_2-term and the torsion product. Thus, in order to obtain the result, it suffices to apply part (1) of Lemma 5.3 for the algebra $H^*(M)$ and the module $H^*(M)$. ∎

We are now ready to give the EMSS (co)multiplicative structures.

Proof of Theorem 2.8. Gugenheim and May [17, p. 26] have shown that the map $\widetilde{\top}$ induces a morphism of spectral sequences from $E_r \otimes E_r$ to the Eilenberg–Moore spectral sequence converging to $H^*(LM \times LM)$. In fact, $\widetilde{\top}$ induces an isomorphism of spectral sequences. All the other maps between torsion products in Theorems 2.3 and 2.5 preserve the filtrations. Thus in view of Lemma 3.2, we have Theorem 2.8.

In fact, the shriek map $\Delta^!$ is in $\mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$. Then we have $d\Delta^! = (-1)^m \Delta^! d$. Let $\{\widehat{E}_r^{*,*}, \widehat{d}_r\}$ and $\{\widetilde{E}_r^{*,*}, \widetilde{d}_r\}$ be the EMSS's converging to $\mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*((M^I))^2)$ and $\mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2), C^*((M^I)^2))$, respectively. Let $\{f_r\} : \{\widehat{E}_r^{*,*}, \widehat{d}_r\} \to \{\widetilde{E}_r^{*,*}, \widetilde{d}_r\}$ be the morphism of spectral sequences which gives rise to $\mathrm{Tor}_1(\Delta^!, 1)$. Recall the map

$$\Delta^! \otimes 1 : \mathbb{B} \otimes_{C^*(M^4)} \mathbb{B}' \to C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}'$$

in the proof of Theorem 2.3. It follows that, for any $b \otimes b' \in \mathbb{B} \otimes_{C^*(M^4)} \mathbb{B}'$,

$$\begin{aligned}(\Delta^! \otimes 1)d(b \otimes b') =& \Delta^! \otimes 1(db \otimes b' + (-1)^{\deg b} b \otimes db') \\ =& \Delta^! db \otimes b' + (-1)^{\deg b} \Delta^! b \otimes db' \\ =& (-1)^m d\Delta^! b \otimes b' + (-1)^{\deg b} \Delta^! b \otimes db'.\end{aligned}$$

On the other hand, we see that

$$\begin{aligned} d(\Delta^! \otimes 1)(b \otimes b') =& d(\Delta^!(b \otimes b')) \\ =& d\Delta^! b \otimes b' + (-1)^{\deg b + m} \Delta^! b \otimes db' \end{aligned}$$

and hence $(\Delta^! \otimes 1)d = (-1)^m d(\Delta^! \otimes 1)$. This implies that $f_r \widehat{d_r} = (-1)^m \widetilde{d_r} f_r$. The fact yields the compatibility of the multiplication with the differential of the spectral sequence.

The same argument does work well to show the compatibility of the comultiplication with the differential of the EMSS. ∎

Proof of Corollary 2.9. Since $H^*(\Delta^!)$ is $H^*(M^2)$-linear, it follows that

$$H^*(\Delta^!) \circ H^*(\Delta)(x) = H^*(\Delta^!)(1) \cup x.$$

If $d < 0$ then $H^*(\Delta^!)(1) = 0$.

If $d = 0$ then $H^*(\Delta^!)(1) = \lambda 1$ where $\lambda \in \mathbb{K}$, and so the composite

$$H^*(\Delta^!) \circ H^*(\Delta)$$

is the multiplication by the scalar λ. Let m be a non-trivial element of positive degree in $H^*(M)$. Then we see that

$$0 = H^*(\Delta^!) \circ H^*(\Delta)(m \otimes 1 - 1 \otimes m) = \lambda(m \otimes 1 - 1 \otimes m).$$

Therefore $\lambda = 0$.

So in both cases, we have proved that $H^*(\Delta^!) \circ H^*(\Delta) = 0$. Since $H^*(\Delta)$ is surjective, $H^*(\Delta^!) : H^*(M) \to H^{*+d}(M^2)$ is trivial. In particular, the induced maps $\mathrm{Tor}^*_{H^*(M^4)}(H^*(\Delta^!), H^*(M^I \times_M M^I))$ and $\mathrm{Tor}^*_{H^*(M^4)}(H^*(\Delta^!), H^*((M^I)^2)$ are trivial. Then it follows from Theorems 2.3 and 2.5 that both the comultiplication and the multiplication on the E_2-term of the EMSS, which correspond to the duals to loop product and loop coproduct on $H^*(LM)$, are null. Therefore, $E_\infty^{*,*} \cong \mathrm{Gr} H^*(LM)$ is equipped with a trivial coproduct and a trivial product. The conclusion then follows. ∎

Remark 3.3: It follows from Corollary 2.9 that under the hypothesis of Corollary 2.9 the two composites

$$H^*(M) \otimes H^*(M) \xrightarrow{p^* \otimes p^*} H^*(LM) \otimes H^*(LM) \xrightarrow{Dlcop} H^*(LM)$$

and

$$H_*(LM) \otimes H_*(LM) \xrightarrow{\text{Loop product}} H_*(LM) \xrightarrow{p_*} H_*(M)$$

are trivial. This can also be proved directly since we have the commuting diagram

$$\begin{tikzcd} H^*(LM \times_M LM) \arrow[r, "H(Comp^!)"] & H^*(LM \times LM) \\ H^*(M) \arrow[u] \arrow[r, "H(\Delta^!)"'] & H^*(M \times M) \arrow[u, "(p\times p)^*"'] \end{tikzcd}$$

and since $p_* : H_*(LM) \to H_*(M)$ is a morphism of graded algebras with respect to the loop product and to the intersection product $H(\Delta_!)$. As we saw in the proof of Corollary 2.9, under the hypothesis of Corollary 2.9, $H^*(\Delta^!)$ and its dual $H_*(\Delta_!)$ are trivial.

4. Proof of Theorem 2.13

The following Lemma is interesting on his own since it gives a very simple proof of a result of Klein (see Remark 4.3 below).

Lemma 4.1: *Let M be an oriented simply-connected Poincaré duality space of dimension m. Let $M \twoheadrightarrow B$ be a fibration. Denote by $M \times_B M$ the pull-back over B. Then for all $p \in \mathbb{Z}$, $\mathrm{Ext}^{-p}_{C^*(B)}(C^*(M), C^*(M))$ is isomorphic to $H_{p+m}(M \times_B M)$ as a vector space.*

Remark 4.2: A particular case of Lemma 4.1 is the isomorphism of graded vector spaces $\mathrm{Ext}^{-p}_{C^*(M\times M)}(C^*(M), C^*(M)) \cong H_{p+m}(LM)$ underlying the isomorphism of algebras given in Theorem 2.14. Note yet that in the proof of Lemma 4.1, we consider right $C^*(B)$-modules, and that in the proof of Theorem 2.14, we need left $C^*(M^2)$-modules; see Section 6.

Remark 4.3: Let F be the homotopy fibre of $M \to B$. In [18, Theorem B], Klein shows in terms of spectra that $\mathrm{Ext}^{-p}_{C_*(\Omega B)}(C_*(F), C_*(F)) \cong H_{p+m}(M \times_B M)$ and so, using the Yoneda product [18, Theorem A], $H_{*+m}(M \times_B M)$ is a graded algebra.

The isomorphism above and that in Lemma 4.1 make us aware of duality on the extension functors of (co)chain complexes of spaces. As mentioned in the Introduction, this is one of the topics in [22].

Proof of Lemma 4.1. The Eilenberg–Moore map gives an isomorphism

$$H_{p+m}(M \times_B M) \cong \mathrm{Ext}^{-p-m}_{C^*(B)}(C^*(M), C_*(M)).$$

The cap with a representative σ of the fundamental class $[M] \in H_m(M)$ gives a quasi-isomorphism of right-$C^*(M)$-modules of upper degre $-m$,

$$\sigma \cap - : C^*(M) \overset{\simeq}{\to} C_{m-*}(M), x \mapsto \sigma \cap x.$$

Therefore, we have an isomorphism

$$\mathrm{Ext}^*_{C^*(B)}(C^*(M), \sigma\cap -) : \mathrm{Ext}^{-p}_{C^*(B)}(C^*(M), C^*(M)) \to \mathrm{Ext}^{-p-m}_{C^*(B)}(C^*(M), C_*(M)).$$

This completes the proof. ∎

Proof of Theorem 2.13. Theorems 2.3 and 2.5 allow us to describe part of the composite $H^*(LM \times_M LM) \overset{H(q^!)}{\to} H^{*+m}(LM \times LM) \overset{Dlcop}{\to} H^{*+m}(LM \times LM)$ in terms of the following composite of appropriate maps between torsion functors:

$$\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^2))_{(wv)^*,\Delta^{2*}} \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"] & \\
\mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M^2), C^*(M^2))_{\Delta^{2*},\Delta^{2*}}. \arrow[d, "\mathrm{Tor}_1(\Delta^*,1)"] & \\
\mathrm{Tor}^{*+m}_{C^*(M^4)}(C^*(M), C^*(M^2))_{(wv)^*,\Delta^{2*}} \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"] & \\
\mathrm{Tor}^{*+2m}_{C^*(M^4)}(C^*(M^2), C^*(M^2))_{\gamma'^*,\Delta^{2*}} \arrow[d, "\mathrm{Tor}_1(\Delta^*,1)"'] \arrow[dr, bend left, "\mathrm{Tor}_{\alpha^*}(\Delta^*,\Delta^*)", "\cong"'] & \\
\mathrm{Tor}^{*+2m}_{C^*(M^4)}(C^*(M), C^*(M^2))_{\Delta^*\alpha^*,\Delta^{2*}} \arrow[r, "\mathrm{Tor}_{\alpha^*}(1,\Delta^*)"'] & \mathrm{Tor}^{*+2m}_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*}.
\end{tikzcd}$$

By virtue of Lemma 4.1, we see that $\mathrm{Ext}^{2m}_{C^*(M^4)}(C^*(M), C^*(M))_{(wv)^*,\Delta^*\alpha^*}$ is isomorphic to $H_{-m}(M^{S^1 \vee S^1 \vee S^1}) = \{0\}$. Then the composite

$$C^*(M) \overset{\Delta^!}{\to} C^*(M \times M) \overset{\Delta^*}{\to} C^*(M) \overset{\Delta^!}{\to} C^*(M \times M) \overset{\Delta^*}{\to} C^*(M)$$

is null in $\mathrm{D}(\text{Mod-}C^*(M^4))$. Therefore the composite $Dlcop \circ H(q^!)$ is trivial and hence $Dlcop \circ Dlp := Dlcop \circ H(q^!) \circ comp^*$ is also trivial. ∎

Remark 4.4: Instead of using Lemma 4.1, one can show that

$$\mathrm{Ext}^{2m}_{C^*(M^4)}(C^*(M), C^*(M))_{(wv)^*,\Delta^*\alpha^*} = \{0\}$$

as follows: Consider the cohomological Eilenberg–Moore spectral sequence with

$$\mathbb{E}_2^{p,*} \cong \mathrm{Ext}^p_{H^*(M^4)}(H^*(M), H^*(M))$$

converging to $\mathrm{Ext}^*_{C^*(M^4)}(C^*(M), C^*(M))_{(wv)^*,\Delta^*\alpha^*}$. Then we see that

$$\mathbb{E}_1^{p,*} = \mathrm{Hom}(H^*(M) \otimes H^+(M^4)^{\otimes p}, H^*(M)).$$

Therefore, since M^4 is simply-connected and $H^{>m}(M) = \{0\}$, $\mathbb{E}_r^{p,q} = \{0\}$ if $q > m - 2p$ (compare with Remark 6.1). Therefore

$$\mathrm{Ext}^{p+q}_{C^*(M^4)}(C^*(M), C^*(M))_{(wv)^*,\Delta^*\alpha^*} = \{0\} \quad \text{if } p+q > m.$$

Remark 4.5: Let M be a Gorenstein space of dimension m. The proof of Theorem 2.12 shows that if the composite

$$\Delta^* \circ \Delta^! \circ \Delta^* \circ \Delta^! \in \mathrm{Ext}^{2m}_{C^*(M^4)}(C^*(M), C^*(M))_{(wv)^*,\Delta^*\alpha^*}$$

is the zero element. Then *Dlcop* $\circ$ *Dlp* is trivial.

Remark 4.6: In the proof of Theorem 2.13, it is important to work in the derived category of $C^*(M^4)$-modules: Suppose that M is the classifying space of a connected Lie group of dimension $-m$. Then since m is negative, the composite $\Delta^! \circ \Delta^*$ is null. In fact

$$\Delta^! \circ \Delta^* \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M^2), C^*(M^2))_{1^*,1^*} \cong H^m(M^2) = \{0\}.$$

But in general, *Dlcop* is not trivial; see [12, Theorem D] and [21]. Therefore the composite $\Delta^* \circ \Delta^! \circ \Delta^* \in \mathrm{Ext}^m_{C^*(M^4)}(C^*(M^2), C^*(M))_{\Delta^{2*},\Delta^*\alpha^*}$ is also not trivial.

5. The generalized cup product on the Hochschild cohomology

After recalling (defining) the (generalized) cup product on the Hochschild cohomology, we give an extension functor description of the product. The result plays an important role in proving our main theorem, Theorem 2.11.

Definition 5.1: Let A be a (differential graded) algebra. Let M be an A-bimodule. Recall that we have a canonical map [30, p. 283]

$$\otimes_A : HH^*(A, M) \otimes HH^*(A, M) \to HH^*(A, M \otimes_A M).$$

(1) Let $\bar{\mu}_M : M \otimes_A M \to M$ be a morphism of A-bimodules of degree d. Then the **cup product** $\cup$ on $HH^*(A, M)$ is the composite

$$HH^p(A, M) \otimes HH^q(A, M) \xrightarrow{\otimes_A} HH^{p+q}(A, M \otimes_A M) \xrightarrow{HH^{p+q}(A,\bar{\mu}_M)} HH^{p+q+d}(A, M).$$

(2) Let $\varepsilon : Q \overset{\simeq}{\to} M \otimes_A M$ be a $A \otimes A^{op}$-projective (semi-free) resolution of $M \otimes_A M$. Let $\bar{\mu}_M \in \mathrm{Ext}^d_{A\otimes A^{op}}(M \otimes_A M, M) = H^d(\mathrm{Hom}_{A\otimes A^{op}}(Q, M))$. Then the **generalized cup product** $\cup$ on $HH^*(A, M)$ is the composite

$$HH^*(A,M)^{\otimes 2} \xrightarrow{\otimes_A} HH^*(A, M \otimes_A M) \xrightarrow{HH^*(A,\varepsilon)^{-1}} HH^*(A,Q) \xrightarrow{HH^*(A,\bar{\mu}_M)} HH^*(A,M).$$

Remark 5.2: Let M be an associative (differential graded) algebra with unit 1_M. Let $h : A \to M$ be a morphism of (differential graded) algebras. Then

$$a \cdot m \star b := h(a)mh(b)$$

defines an A-bimodule structure on M such that the multiplication of M, $\mu_M : M \otimes M \to M$ induces a morphism of A-bimodules $\bar{\mu}_M : M \otimes_A M \to M$.

Conversely, let M be an A-bimodule equipped with an element $1_M \in M$ and a morphism of A-bimodules $\bar{\mu}_M : M \otimes_A M \to M$ such that

$$\bar{\mu}_M \circ (\bar{\mu}_M \otimes_A 1) = \bar{\mu}_M \circ (1 \otimes_A \bar{\mu}_M)$$

and such that the two maps $m \mapsto \bar{\mu}_M(m \otimes_A 1)$ and $m \mapsto \bar{\mu}_M(1 \otimes_A m)$ coincide with the identity map on M. Then the map $h : A \to M$ defined by $h(a) := a \cdot 1_M$ is a morphism of algebras.

Lemma 5.3: *Let A be a commutative algebra. Let M be a A-module. Let B be an $A^{\otimes 2}$-module. Let $\mu : A^{\otimes 2} \to A$ denote the multiplication of A. Let $q : B \otimes B \twoheadrightarrow B \otimes_A B$ be the quotient map. Then*

1. $\mathrm{Tor}_*^{1\otimes\mu\otimes 1}(1,\mu) : \mathrm{Tor}_*^{A^{\otimes 4}}(M, A \otimes A) \overset{\cong}{\to} \mathrm{Tor}_*^{A^{\otimes 3}}(M, A)$ *is an isomorphism,*
2. $\mathrm{Hom}_{1\otimes\mu\otimes 1}(q, 1) : \mathrm{Hom}_{A^{\otimes 3}}(B \otimes_A B, M) \overset{\cong}{\to} \mathrm{Hom}_{A^{\otimes 4}}(B \otimes B, M)$ *is an isomorphism and*
3. $\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(q, 1) : \mathrm{Ext}^*_{A^{\otimes 3}}(B \otimes_A B, M) \overset{\cong}{\to} \mathrm{Ext}^*_{A^{\otimes 4}}(B \otimes B, M)$ *is also an isomorphism.*

Proof. (1) Consider the bar resolution $\xi : B(A, A, A) \overset{\simeq}{\to} A$ of A. Since the complex $B(A, A, A)$ is a semifree A-module, it follows from [11, Theorem 6.1] that $\xi \otimes_A \xi : B(A, A, A) \otimes_A B(A, A, A) \to A \otimes_A A = A$ is a quasi-isomorphism and hence it is a projective resolution of A as an $A^{\otimes 3}$-module. We moreover have a commutative diagram

$$\begin{tikzcd} B(A,A,A)\otimes B(A,A,A) \arrow[r, "\xi\otimes\xi"] \arrow[d, "q"'] & A\otimes A \arrow[d, "\mu"] \\ B(A,A,A)\otimes_A B(A,A,A) \arrow[r, "\xi\otimes_A\xi"'] & A \end{tikzcd}$$

in which q is the natural projection and the first row is a projective resolution of $A\otimes A$ as an $A^{\otimes 4}$-module. It is immediate that q is a morphism of $A^{\otimes 4}$-modules with respect to the morphism of algebras $1\otimes\mu\otimes 1 : A^{\otimes 4}\to A^{\otimes 3}$. Then $\mathrm{Tor}_{1\otimes\mu\otimes 1}(1,\mu)$ is induced by the map

$$1\otimes q : M\otimes_{A^{\otimes 4}} B(A,A,A)\otimes B(A,A,A)\to M\otimes_{A^{\otimes 3}} B(A,A,A)\otimes_A B(A,A,A).$$

Since A is commutative, it follows that both the source and target of $1\otimes q$ are isomorphic to $W := M\otimes B(\mathbb{K},A,\mathbb{K})\otimes B(\mathbb{K},A,\mathbb{K})$ as a vector space. As a linear map, $1\otimes q$ coincides with the identity map on W up to isomorphism.

(2) By the universal property of the quotient map $q : B\otimes B \twoheadrightarrow B\otimes_A B$, $\mathrm{Hom}_{1\otimes\mu\otimes 1}(q,1)$ is an isomorphism.

(3) Let $\varepsilon : P \overset{\simeq}{\twoheadrightarrow} B$ be an $A^{\otimes 2}$-projective (semi-free) resolution of B. We have a commutative square of $A^{\otimes 4}$-modules

$$\begin{tikzcd} P\otimes P \arrow[r, "\varepsilon\otimes\varepsilon", "\simeq"'] \arrow[d, "q'"'] & B\otimes B \arrow[d, "q"] \\ P\otimes_A P \arrow[r, "\varepsilon\otimes_A\varepsilon", "\simeq"'] & B\otimes_A B \end{tikzcd}$$

Therefore $\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(q,1)$ is induced by $\mathrm{Hom}_{1\otimes\mu\otimes 1}(q',1)$ which is an isomorphism by (2). ∎

The isomorphism $\mathrm{Tor}_*^{1\otimes\mu\otimes 1}(1,\mu)$ of part (1) of Lemma 5.3 is used in the proof of Theorem 2.8 while the similar isomorphisms of parts (2) and (3) are used in Lemma 5.4 below.

The following lemma gives an interesting decomposition of the cup product of the Hochschild cohomology of a commutative (possible differential graded) algebra.

Lemma 5.4: *Let A be a commutative (differential graded) algebra. Let M be a A-module. Let $\mu : A^{\otimes 2}\to A$ denote the multiplication of A. Let $\eta : \mathbb{K}\to A$ be the unit of A. Then:*

(1) *Let $\mu_M \in Hom_{A^{\otimes 4}}(M^{\otimes 2}, M)$. Then μ_M induced a quotient map*

$$\bar{\mu}_M : M \otimes_A M \to M$$

and the cup product $\cup$ of the Hochschild cohomology of A with coefficients in M, $HH^*(A, M) = \mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu}$ is given by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} \otimes \mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} \arrow[r, "\otimes"] \arrow[dd, "\cup"'] & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, M^{\otimes 2})_{\mu^{\otimes 2},\mu^{\otimes 2}} \arrow[d, "\mathrm{Ext}^*_1(1,\mu_M)"] \\
& \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, M)_{\mu^{\otimes 2},\mu\circ\mu^{\otimes 2}} \arrow[d, "\cong"', "\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(\mu,1)^{-1}"] \\
\mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} & \mathrm{Ext}^*_{A^{\otimes 3}}(A, M)_{\mu\circ(\mu\otimes 1),\mu\circ(\mu\otimes 1)} \arrow[l, "\mathrm{Ext}^*_{1\otimes\eta\otimes 1}(1,1)"]
\end{tikzcd}$$

(2) *Let $\varepsilon : R \overset{\simeq}{\rightrightarrows} M \otimes M$ be a $A^{\otimes 4}$-projective (semi-free) resolution of $M \otimes M$. Let $\mu_M \in \mathrm{Ext}_{A^{\otimes 4}}(M^{\otimes 2}, M) = H(\mathrm{Hom}_{A^{\otimes 4}}(R, M))$. Let $\bar{\mu}_M$ be $\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(q, 1)^{-1}(\mu_M)$. Then the generalized cup product $\cup$ of the Hochschild cohomology of A with coefficients in M, $HH^*(A, M) = \mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu}$, is given by the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} \otimes \mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} \arrow[r, "\otimes"] \arrow[dddd, "\cup"'] & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, M^{\otimes 2})_{\mu^{\otimes 2},\mu^{\otimes 2}} \arrow[d, "\cong"', "(\mathrm{Ext}^*_1(1,\varepsilon))^{-1}"] \\
& \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, R) \arrow[d, "\mathrm{Ext}^*_1(1,\mu_M)"] \\
& \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, M)_{\mu^{\otimes 2},\mu\circ\mu^{\otimes 2}} \arrow[d, "\cong"', "\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(\mu,1)^{-1}"] \\
& \\
\mathrm{Ext}^*_{A^{\otimes 2}}(A, M)_{\mu,\mu} & \mathrm{Ext}^*_{A^{\otimes 3}}(A, M)_{\mu\circ(\mu\otimes 1),\mu\circ(\mu\otimes 1)} \arrow[l, "\mathrm{Ext}^*_{1\otimes\eta\otimes 1}(1,1)"]
\end{tikzcd}$$

As mentioned at the beginning of this section, Lemma 5.4 (1) contributes toward proving Theorem 2.11. Moreover, in view of part (2) of the lemma, we prove Theorem 2.17.

Proof of Lemma 5.4. (1) Consider the bar resolution $\xi : B(A, A, A) \overset{\simeq}{\rightrightarrows} A$ of A. Since the complex $B(A, A, A)$ is a semi-free A-module, it follows from [11, Theorem 6.1] that $\xi \otimes_A \xi : B(A, A, A) \otimes_A B(A, A, A) \to A \otimes_A A = A$ is a quasi-isomorphism and hence it is a projective resolution of A as an $A^{\otimes 3}$-module. We moreover have a commutative diagram

$$\begin{tikzcd} B(A,A,A)\otimes B(A,A,A) \arrow[r, "\xi\otimes\xi"] \arrow[d, "q"'] & A\otimes A \arrow[d, "\mu"] \\ B(A,A,A)\otimes_A B(A,A,A) \arrow[r, "\xi\otimes_A\xi"'] & A \end{tikzcd}$$

in which q is the natural projection and the first row is a projective resolution of $A\otimes A$ as an $A^{\otimes 4}$-module. It is immediate that q is a morphism of $A^{\otimes 4}$-modules with respect to the morphism of algebras $1\otimes\mu\otimes 1 : A^{\otimes 4}\to A^{\otimes 3}$. Then $\mathrm{Tor}_{1\otimes\mu\otimes 1}(1,\mu)$ is induced by the map

$$1\otimes q : M\otimes_{A^{\otimes 4}} B(A,A,A)\otimes B(A,A,A)\to M\otimes_{A^{\otimes 3}} B(A,A,A)\otimes_A B(A,A,A).$$

Since A is commutative, it follows that both the source and target of $1\otimes q$ are isomorphic to $W := M\otimes B(\mathbb{K},A,\mathbb{K})\otimes B(\mathbb{K},A,\mathbb{K})$ as a vector space. As a linear map, $1\otimes q$ coincides with the identity map on W up to isomorphism.

(2) By the universal property of the quotient map $q : B\otimes B \twoheadrightarrow B\otimes_A B$, $\mathrm{Hom}_{1\otimes\mu\otimes 1}(q,1)$ is an isomorphism.

(3) Let $\varepsilon : P \overset{\simeq}{\twoheadrightarrow} B$ be an $A^{\otimes 2}$-projective (semi-free) resolution of B. We have a commutative square of $A^{\otimes 4}$-modules

$$\begin{tikzcd} P\otimes P \arrow[r, "\varepsilon\otimes\varepsilon", "\simeq"'] \arrow[d, "q'"'] & B\otimes B \arrow[d, "q"] \\ P\otimes_A P \arrow[r, "\varepsilon\otimes_A\varepsilon", "\simeq"'] & B\otimes_A B \end{tikzcd}$$

Therefore $\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(q,1)$ is induced by $\mathrm{Hom}_{1\otimes\mu\otimes 1}(q',1)$ which is an isomorphism by (2).

(4) Let A be any algebra and M be any A-bimodule. Let $\xi : \mathbb{B}\overset{\simeq}{\twoheadrightarrow} A$ be an $A\otimes A^{op}$-projective (semi-free) resolution (for example, the double bar resolution). Let $c : \mathbb{B}\to\mathbb{B}\otimes_A\mathbb{B}$ be a morphism of A-bimodules such that the diagram of A-bimodules

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\xi"] \arrow[d, "c"] & A \arrow[d, "\cong"'] \\ \mathbb{B}\otimes_A\mathbb{B} \arrow[r, "\xi\otimes_A\xi"'] & A\otimes_A A \end{tikzcd}$$

is homotopy commutative. The cup product of f and $g\in\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B},M)$ is the composite $\bar{\mu}_M\circ(f\otimes_A g)\circ c\in\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B},M)$ [36, p. 134].

Suppose now that A is commutative and that the A-bimodule structure on M comes from the multiplication μ of A and an A-module structure on M. The following diagram of complexes gives two different decompositions of the cup product on $\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}, M)$:

$$\begin{tikzcd}
\mathrm{Hom}_{A^{\otimes 2}}(\mathbb{B}, M)\otimes \mathrm{Hom}_{A^{\otimes 2}}(\mathbb{B}, M) \arrow[r, "\otimes"] \arrow[d, "\otimes_A"] & \mathrm{Hom}_{A^{\otimes 4}}(\mathbb{B}\otimes\mathbb{B}, M\otimes M) \arrow[d, "\mathrm{Hom}_1(1,\mu_M)"] \\
\mathrm{Hom}_{A^{\otimes 3}}(\mathbb{B}\otimes_A\mathbb{B}, M\otimes_A M) \arrow[dr, "\mathrm{Hom}_1(1,\bar{\mu}_M)"] & \mathrm{Hom}_{A^{\otimes 4}}(\mathbb{B}\otimes\mathbb{B}, M) \\
\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}\otimes_A\mathbb{B}, M) \arrow[d, "\mathrm{Hom}_1(c,1)"] & \mathrm{Hom}_{A^{\otimes 3}}(\mathbb{B}\otimes_A\mathbb{B}, M) \arrow[l, "\mathrm{Hom}_{1\otimes\eta\otimes 1}(1,1)"] \arrow[u, "\cong"', "\mathrm{Hom}_{1\otimes\mu\otimes 1}(q,1)"] \\
\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}, M) &
\end{tikzcd}$$

(5) Let $\varepsilon : P \overset{\simeq}{\twoheadrightarrow} M$ be a surjective $A\otimes A^{op}$-projective (semi-free) resolution of M. Tnen μ_M can be considered as an element of $\mathrm{Hom}_{A^{\otimes 4}}(P\otimes P, M)$. By lifting, there exists

$$\mu_P \in \mathrm{Hom}_{A^{\otimes 4}}(P\otimes P, P)$$

such that $\varepsilon\circ\mu_P = \mu_M$. By (2), there exists $\bar{\mu}_P$ such that $\bar{\mu}_P\circ q = \mu_P$. We can take $\bar{\mu}_M = \varepsilon\circ\bar{\mu}_P$.

It is now easy to check that the isomorphism

$$HH^*(A,\varepsilon) : HH^*(A,P) \overset{\cong}{\twoheadrightarrow} HH^*(A,M)$$

transports the cup product on $HH^*(A,P)$ defined using $\bar{\mu}_P$ to the generalized cup product on $HH^*(A,M)$ defined using $\bar{\mu}_M$. We now check that the isomorphism $HH^*(A,\varepsilon) : HH^*(A,P) \overset{\cong}{\twoheadrightarrow} HH^*(A,M)$ transports the composite

$$\begin{tikzcd}
\mathrm{Ext}^*_{A^{\otimes 2}}(A,P)_{\mu,\mu}\otimes \mathrm{Ext}^*_{A^{\otimes 2}}(A,P)_{\mu,\mu} \arrow[r, "\otimes"] & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, P^{\otimes 2})_{\mu^{\otimes 2},\mu^{\otimes 2}} \arrow[d, "\mathrm{Ext}^*_1(1,\mu_P)"] \\
& \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2}, P)_{\mu^{\otimes 2},\mu\circ\mu^{\otimes 2}} \arrow[d, "\cong"', "\mathrm{Ext}^*_{1\otimes\mu\otimes 1}(\mu,1)^{-1}"] \\
\mathrm{Ext}^*_{A^{\otimes 2}}(A,P)_{\mu,\mu} & \mathrm{Ext}^*_{A^{\otimes 3}}(A,P)_{\mu\circ(\mu\otimes 1),\mu\circ(\mu\otimes 1)} \arrow[l, "\mathrm{Ext}^*_{1\otimes\eta\otimes 1}(1,1)"']
\end{tikzcd}$$

into the composite

$$\begin{array}{ccc}
\mathrm{Ext}^*_{A^{\otimes 2}}(A,M)_{\mu,\mu}\otimes \mathrm{Ext}^*_{A^{\otimes 2}}(A,M)_{\mu,\mu} & \xrightarrow{\otimes} & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2},M^{\otimes 2})_{\mu^{\otimes 2},\mu^{\otimes 2}} \\
 & & \cong\downarrow (\mathrm{Ext}^*_1(1,\varepsilon\otimes\varepsilon))^{-1} \\
 & & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2},P\otimes P) \\
 & & \downarrow \mathrm{Ext}^*_1(1,\mu_M) \\
 & & \mathrm{Ext}^*_{A^{\otimes 4}}(A^{\otimes 2},M)_{\mu^{\otimes 2},\mu\circ\mu^{\otimes 2}} \\
 & & \cong\downarrow \mathrm{Ext}^*_{1\otimes\mu\otimes 1}(\mu,1)^{-1} \\
\mathrm{Ext}^*_{A^{\otimes 2}}(A,M)_{\mu,\mu} & \xleftarrow[\mathrm{Ext}^*_{1\otimes\eta\otimes 1}(1,1)]{} & \mathrm{Ext}^*_{A^{\otimes 3}}(A,M)_{\mu\circ(\mu\otimes 1),\mu\circ(\mu\otimes 1)}
\end{array}$$

By applying (4) to μ_P, we have proved (5). ∎

Theorem 5.5 (Compare with [12, Theorem 12]): *Let B be a simply-connected commutative Gorenstein cochain algebra of dimension m such that for any $i\in\mathbb{N}$, $H^i(B)$ is finite-dimensional. Then* $\mathrm{Ext}^{*+m}_{B\otimes B}(B,B\otimes B)\cong H^*(B)$.

Proof. The proof of [12, Theorem 12] for the strongly homotopy commutative algebra $C^*(X)$ obviously works in the case of a commutative algebra B. ∎

Remark 5.6: In [1, Theorem 2.1 i) iv)] Avramov and Iyengar have shown a related result in the non-graded case: Let S be a commutative algebra over a field $\mathbb{K}$, which is the quotient of a polynomial algebra $\mathbb{K}[x_1,\dots,x_d]$ or, more generally, which is the quotient of a localization of $\mathbb{K}[x_1,\dots,x_d]$. Then S is Gorenstein if and only if the graded S-module $\mathrm{Ext}^*_{S\otimes S}(S,S\otimes S)$ is projective of rank 1.

Example 5.7 (The generalized cup product of a Gorenstein algebra): Let $A\to B$ be a morphism of commutative differential graded algebras where B satisfies the hypotheses of Theorem 5.5. Let $\Delta_B: B\to B\otimes B$ be a generator of $\mathrm{Ext}^m_{B\otimes B}(B,B\otimes B)\cong\mathbb{K}$. By taking duals, we obtain the following element of $\mathrm{Ext}^m_{A\otimes A}(B^\vee\otimes B^\vee,B^\vee)$:

$$(\Delta_B)^\vee: B^\vee\otimes B^\vee\to(B\otimes B)^\vee\to B^\vee.$$

By (3) of Lemma 5.3, $(\Delta_B)^\vee$ induces an element $\bar{\mu}_{B^\vee}\in\mathrm{Ext}^m_{A\otimes A}(B^\vee\otimes_A B^\vee,B^\vee)$. Therefore, by (2) of Definition 5.1, we have a generalized cup product

$$HH^p(A,B^\vee)\otimes HH^q(A,B^\vee)\overset{\cup}{\to} HH^{p+q+m}(A,B^\vee).$$

In the case $A=B$, of course, we believe that $HH^*(A,A^\vee)$ equipped with this generalized cup product and Connes coboundary is a non-unital BV-algebra.

6. Proofs of Theorems 2.11, 2.14 and 2.17 and Corollary 2.18

We first prove Theorem 2.17. The same argument is applicable when proving Theorem 2.11.

Proof of Theorem 2.17. Step 1: The polynomial differential functor $A(X)$ extends to a functor $A(X,Y)$ for pairs of spaces $Y \subset X$. The two natural short exact sequences [11, p. 124] $\to A(X,Y) \to A(X) \to A(Y) \to 0$ and $0 \to C^*(X,Y) \to C^*(X) \to C^*(Y) \to 0$ are naturally weakly equivalent [11, pp. 127–8]. Therefore all the results of Félix and Thomas given in [12] with the singular cochains algebra $C^*(X)$ are valid with $A(X)$ (for example, the description of the shriek map of an embedding $N \hookrightarrow M$ at the level of singular cochains given on page 419 of [12]). In particular, our Theorem 2.3 is valid when we replace $C^*(X)$ by $A(X)$. (Note also that a proof similar to the proof of Theorem 8.3 or 8.6 shows that the dual of the loop product on $A(L_NM)$ is isomorphic to the dual of the loop product defined on $C^*(L_NM)$.) This means the following: Let $\Delta^!_A$ be a generator of $\mathrm{Ext}^n_{A(N^2)}(A(N), A(N^2)) \cong \mathbb{Q}$ given by [12, Theorem 12]. Then the composite $\mathrm{Tor}^1(1,\sigma^*) \circ EM^{-1}$ is an isomorphism of algebras between the dual of the loop product Dlp on $H^*(A(L_NM))$ and the coproduct defined by the composite on the left column of the following diagram.

Step 2: We have chosen $\Delta^!_A$ and $\Delta_{A(N)}$ such that the composite $\varphi \circ \Delta_{A(N)}$ is equal to $\Delta^!_A$ in the derived category of $A(N)^{\otimes 2}$-modules. Therefore the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Tor}_*^{A(M^2)}(A(N), A(M)) \arrow[d, "\mathrm{Tor}^{A(p_{13})}(1,1)"'] & \mathrm{Tor}_*^{A(M)^{\otimes 2}}(A(N), A(M)) \arrow[l, "\mathrm{Tor}^{\varphi}(1,1)"', "\cong"] \arrow[d, "\mathrm{Tor}^{1\otimes\eta\otimes 1}(1,1)"] \\
\mathrm{Tor}_*^{A(M^3)}(A(N), A(M)) & \mathrm{Tor}_*^{A(M)^{\otimes 3}}(A(N), A(M)) \arrow[l, "\mathrm{Tor}^{\varphi\circ(1\otimes\varphi)}(1,1)"', "\cong"] \\
\mathrm{Tor}_*^{A(M^4)}(A(N), A(M^2)) \arrow[u, "\mathrm{Tor}^{A(1\times\Delta\times 1)}(1,A(\Delta))", "\cong"'] \arrow[d, "\mathrm{Tor}^1(\Delta^!_A,1)"'] & \mathrm{Tor}_*^{A(M)^{\otimes 4}}(A(N), A(M)^{\otimes 2}) \arrow[l, "\mathrm{Tor}^{\varphi}(1,\varphi)"', "\cong"] \arrow[u, "\mathrm{Tor}^{1\otimes\mu\otimes 1}(1,\mu)"', "\cong"] \arrow[d, "\mathrm{Tor}^1(\Delta_{A(N)},1)"] \\
\mathrm{Tor}_*^{A(M^4)}(A(N^2), A(M^2)) \arrow[d, "\cong"'] & \mathrm{Tor}_*^{A(M)^{\otimes 4}}(A(N)^{\otimes 2}, A(M)^{\otimes 2}) \arrow[l, "\mathrm{Tor}^{\varphi}(\varphi,\varphi)"', "\cong"] \arrow[d, "\cong"'] \\
(\mathrm{Tor}_*^{A(M^2)}(A(N), A(M)))^{\otimes 2} & (\mathrm{Tor}_*^{A(M)^{\otimes 2}}(A(N), A(M)))^{\otimes 2} \arrow[l, "\cong"', "(\mathrm{Tor}^{\varphi}(1,1))^{\otimes 2}"]
\end{tikzcd}$$

Step 3: Dualizing and using the natural isomorphism

$$\mathrm{Ext}^*_B(Q, P^\vee) \overset{\cong}{\to} \mathrm{Tor}^*_B(P, Q)^\vee$$

for any differential graded algebra B, right B-module P and left B-module Q, we see that the dual of Φ is an isomorphism of algebras with respect to the loop product and to the long composite given by the diagram of (2) of Lemma 5.4 when $A := A(M)$, $M := A(N)^\vee$ and $\mu_M := (\Delta_{A(N)})^\vee$.

Step 4: We apply part (2) of Lemma 5.4 to see that this long composite coincides with the generalized cup product of the Gorenstein algebra $B := A(N)$. ∎

Proof of Theorem 2.11. Let $\{E_r^{*,*}, d_r\}$ denote the spectral sequence described in Theorem 2.8. Then we define a spectral sequence $\{\mathbb{E}_r^{*,*}, d_r^\vee\}$ by

$$\mathbb{E}_r^{p,q+d} := (E_r^{*,*\vee})^{p,q} = (E_r^{-p,-q})^\vee.$$

The decreasing filtration $\{F^pH^*(L_NM)\}_{p\le 0}$ of $H^*(L_NM)$ induces the decreasing filtration $\{F^pH_*(L_NM)\}_{p\ge 0}$ of $H_*(L_NM)$ defined by

$$F^pH_*(L_NM) = (H^*(L_NM)/F^{-p}H^*(L_NM))^\vee.$$

By definition, the Chas–Sullivan loop homology (the shift homology) $\mathbb{H}_*(L_NM)$ is given by $\mathbb{H}_{-(p+q)}(L_NM) = (H^*(L_NM)^\vee)^{p+q-d}$. By Proposition 2.7, the product m on $\mathbb{H}_*(L_NM)$ is defined by

$$m(a\otimes b) = (-1)^{d(|a|-d)}(Dlp)^\vee(a\otimes b)$$

for $a\otimes b \in (H^*(L_NM)^\vee)^*\otimes(H^*(L_NM)^\vee)^*$. Then we see that

$$\begin{aligned}\mathbb{E}_\infty^{p,q} \cong& F^p(H^*(L_NM)^\vee)^{p+q-d}/F^{p+1}(H^*(L_NM)^\vee)^{p+q-d}\\ =& F^p\mathbb{H}_{-(p+q)}(L_NM)/F^{p+1}\mathbb{H}_{-(p+q)}(L_NM).\end{aligned}$$

The composite $\widetilde{(Dlp)}$ in Theorem 2.3 which gives rise to Dlp on $H^*(L_NM)$ preserves the filtration of the EMSS $\{\widetilde{E_r^{*,*}, d_r}\}$; see Remark 2.6. As mentioned in the proof of Theorem 2.8, the map $\widetilde{(Dlp)}$ induces the morphism

$$(Dlp)_r : E_r^{*,*} \to E_r^{*,*}\otimes E_r^{*,*}$$

of spectral sequences of bidegree $(0, d)$. Define $m_r : \mathbb{E}_r^{*,*}\otimes\mathbb{E}_r^{*,*} \to \mathbb{E}_r^{*,*}$ by

$$m_r(a\otimes b) = (-1)^{d(|a|+d)}((Dlp)_r)^\vee(a\otimes b),$$

where $|a| = p+q$ if $a \in (E_r^{*,*\vee})^{p,q}$. Then a straightforward computation enables us to deduce that $m_r(d_r^\vee a\otimes b + (-1)^{|a|+d}a\otimes d_r^\vee b) = d_r^\vee\circ m_r(a\otimes b)$ for any r.

Note that $|a| + d = p + q + d$ is the total degree of a in $\mathbb{E}_r^{*,*}$. It turns out that $\{\mathbb{E}_r, d_r^\vee\}$ is a spectral sequence of algebras converging to $\mathbb{H}_{-*}(L_N M)$ as an algebra.

It remains now to identify the $\mathbb{E}_2$-term with the Hochschild cohomology. We proceed as in the proof of Theorem 2.17 replacing the polynomial differential functor A by singular cohomology H^*. The product Dlp_2 is given by the composite

$$\begin{tikzcd}
\mathrm{Tor}_*^{H^*(M^2)}(H^*(N), H^*(M)) \arrow[r, "\mathrm{Tor}^{H^*(p_{13})}(1,1)"'] & \mathrm{Tor}_*^{H^*(M^3)}(H^*(N), H^*(M)) \\
& \mathrm{Tor}_*^{H^*(M^4)}(H^*(N), H^*(M^2)) \arrow[u, "\cong"', "\mathrm{Tor}^{H^*(1\times\Delta\times 1)}(1,H^*(\Delta))"] \arrow[d, "\mathrm{Tor}^1(\Delta^!,1)"'] \\
(\mathrm{Tor}_*^{H^*(M^2)}(H^*(N), H^*(M)))^{\otimes 2} & \mathrm{Tor}_*^{H^*(M^4)}(H^*(N^2), H^*(M^2)) \arrow[l, "\cong"']
\end{tikzcd}$$

Dualizing and using the natural isomorphism

$$\mathrm{Ext}_B^*(Q, P^\vee) \overset{\cong}{\to} \mathrm{Tor}_B^*(P, Q)^\vee$$

for any graded algebra B, right B-module P and left B-module Q, we see that $Dlp_2^\vee$, the dual of Dlp_2, is the long composite given by the diagram of Lemma 5.4 (1) when $A := H^*(M)$, $M := H_*(N)$ and $\mu_M := H(\Delta^!)^\vee$. By Lemma 5.4 (1), we obtain an isomorphism of algebras $u : HH^*(H^*(M), H_*(N)) \overset{\cong}{\to} (E_2^{*,*})^\vee$ with respect to $Dlp_2^\vee$ and the cup product induced by

$$\bar{\mu}_M = \overline{H(\Delta^!)^\vee} : H_*(N) \otimes_{H^*(M)} H_*(N) \to H_*(N).$$

Using Example 10.3 (ii) and Example 10.5 (i), we finally obtain an isomorphism of algebras $\mathbb{E}_2^{*,*} \cong HH^*(H^*(M), \mathbb{H}_*(N))$ with respect to m_2 and the cup product induced by

$$\bar{\mu}_{\mathbb{M}} : \mathbb{H}_*(N) \otimes_{H^*(M)} \mathbb{H}_*(N) \to \mathbb{H}_*(N).$$

Note that $\bar{\mu}_{\mathbb{M}}$ coincides with $\overline{H(\Delta^!)^\vee}$ only up to the multiplication by $(-1)^d$.

Suppose further that N is a Poincaré duality space of dimension d. Consider the two squares

$$\begin{tikzcd}
H_{d-p}(N) \otimes H_{d-q}(N) \arrow[r, "\times"] & H_{2d-p-q}(N \times N) \arrow[r, "H(\Delta^!)^\vee"] & H_{d-p-q}(N) \\
H^p(N) \otimes H^q(N) \arrow[r, "\times"] \arrow[u, "-\cap[N]\otimes-\cap[N]"] & H^{p+q}(N \times N) \arrow[r, "H(\Delta)"] \arrow[u, "-\cap[N]\times[N]"'] & H^{p+q}(N) \arrow[u, "-\cap[N]"']
\end{tikzcd}$$

The right square is the diagram (2) of Proposition 9.1. Therefore the right square commutes by Corollary 9.5. By [3, VI.5.4 Theorem], we see that

$$\begin{aligned}(\alpha\times\beta)\cap([N]\times[N]) =&(-1)^{|\alpha||[N]|}(\alpha\cap[N])\times(\beta\cap[N])\\ &= \times\circ(-\cap[N])\otimes(-\cap[N])(\alpha\otimes\beta).\end{aligned}$$

This means that the left square commutes. Therefore, we have proved that the isomorphism of lower degree d, $\theta_{H^*(N)} := -\cap[N] : H^*(N)\to H_{d-*}(N)$, is a morphism of algebras with respect to the cup product and the composite of $H(\Delta^!)^\vee$ and the homological cross product. By naturality of the cup product on Hochschild cohomology defined (Remark 5.2) by a morphism of algebras, this implies that the morphism

$$HH^*(1,-\cap[N]) : HH^*(H^*(M),H^*(N)) \overset{\cong}{\to} HH^*(H^*(M),H_*(N))$$

is an isomorphism of algebras of lower degree d. We see that the composite

$$\zeta = u\circ HH^*(1,-\cap[N]) : HH^*(H^*(M),H^*(N)) \overset{\cong}{\to} \mathbb{E}_2^{*,*}$$

is an isomorphism of algebras; see Example 10.3 (i) and (ii). This completes the proof. ∎

Remark 6.1: For the EMSS $\{E_r^{*,*},d_r\}$ described in Theorem 2.8, we see that $E_r^{p,q}=0$ if $q<-2p$ since M is simply-connected. This implies that $\mathbb{E}_r^{p,q}=0$ if $q>-2p+d$.

Proof of Corollary 2.18. Denote by $\int_N : C^*(N)\overset{\simeq}{\to} A(N)$ a quasi-isomorphism of complexes which coincides in homology with the natural equivalence of algebras between the singular cochains and the polynomial differential forms [11, Corollary 10.10]. Let $\psi_N : C_*(N)\hookrightarrow C^*(N)^\vee$ be the canonical inclusion of the complex $C_*(N)$ into its bidual defined in [14, 7.1] or [31, Property 57 i)] by $\psi_N(c)(\varphi)=(-1)^{|c||\varphi|}\varphi(c)$ for $c\in C_*(N)$ and $\varphi\in C^*(N)$. Consider the diagram of complexes

$$\begin{tikzcd}
C_*(N)\otimes C_*(N) \arrow[rr, "EZ"] \arrow[d, "\psi_N\otimes\psi_N"'] & & C_*(N\times N) \arrow[d, "\psi_{N\times N}"] \\
C^*(N)^\vee\otimes C^*(N)^\vee \arrow[r] \arrow[d, "\int_N^\vee\otimes\int_N^\vee"'] & (C^*(N)\otimes C^*(N))^\vee \arrow[d, "(\int_N\otimes\int_N)^\vee"'] & C^*(N\times N)^\vee \arrow[l, "AW^{\vee\vee}"'] \arrow[d, "\int_{N\times N}^\vee"] \\
A(N)^\vee\otimes A(N)^\vee \arrow[r] & (A(N)\otimes A(N))^\vee & A(N\times N)^\vee \arrow[l, "\varphi^\vee"]
\end{tikzcd}$$

where EZ and AW are the Eilenberg–Zilber and Alexander–Whitney maps. The bottom left square commutes by naturality of the horizontal maps. The top rectangle commutes in homology since by [3, VI.5.4 Theorem],

$$< \alpha \times \beta, a \times b >= (-1)^{|\beta||a|} < \alpha, a >< \beta, b >$$

for all α, $\beta \in H^*(N)$ and a, $b \in H_*(N)$. The bottom right square commutes in homology since using $H(\int_N)$, $H(\varphi)$ can be identified with the cohomological cross product [11, Example 2 pp. 142–3] and [37, Chap. 5 Sec. 6, 14 Corollary].

Let $\theta_N : A(N) \overset{\simeq}{\to} A(N)^\vee$ be a quasi-isomorphim of upper degree $-n$ right $A(N)$-linear such that the image of the fundamental class $[N]$ by the composite $C_*(N) \overset{\psi_N}{\to} C^*(N)^\vee \overset{\int_N^\vee}{\to} A(N)^\vee$ is the class of $\theta_N(1)$. Let

$$\theta_{N\times N} : A(N \times N) \overset{\simeq}{\to} A(N \times N)^\vee$$

be a quasi-isomorphim of upper degree $-2n$ right $A(N^2)$-linear such that the image of the fundamental class $[N] \times [N]$ by the composite

$$C_*(N \times N) \overset{\psi_{N\times N}}{\to} C^*(N \times N)^\vee \overset{\int_{N\times N}^\vee}{\to} A(N \times N)^\vee$$

is the class of $\theta_{N\times N}(1)$. Using the previous commutative diagram, the classes $\varphi^\vee \circ \theta_{N\times N}(1)$ and $\theta_N(1) \otimes \theta_N(1)$ are equal.

Consider the diagram in the derived category of $A(N)^{\otimes 2}$-modules

$$\begin{tikzcd}
A(N) \otimes A(N) \arrow[rr, "\varphi"] \arrow[d, "\theta_N\otimes\theta_N"'] & & A(N \times N) \arrow[r, "A(\Delta)"] \arrow[d, "\theta_{N\times N}"'] & A(N) \arrow[d, "\theta_N"] \\
A(N)^\vee \otimes A(N)^\vee \arrow[r] & (A(N) \otimes A(N))^\vee & A(N \times N)^\vee \arrow[l, "\varphi^\vee"] \arrow[r, "\Delta_A^{!\vee}"'] & A(N)^\vee
\end{tikzcd}$$

By Corollary 9.5, the right square commutes in the derived category of $A(N\times N)$-modules. The left rectangle commutes up to homotopy of $A(N)\otimes A(N)$-modules since the classes $\varphi^\vee \circ \theta_{N\times N}(1)$ and $\theta_N(1) \otimes \theta_N(1)$ are equal.

Finally, since $\theta_N : A(N) \overset{\simeq}{\to} A(N)^\vee$ is a morphism of algebras of upper degree $-n$ in the derived category of $A(N)^{\otimes 2}$-modules, by example 10.5(ii), $HH^*(1, \theta_N) : HH^*(A(M), A(N)) \overset{\cong}{\to} HH^*(A(M), A(N)^\vee)$ is an isomorphism of algebras of upper degree $-n$. ∎

Proof of Theorem 2.14. The cap with a representative σ of the fundamental class $[M] \in H_m(M)$ gives a quasi-isomorphism of right-$C^*(M)$-modules of upper degre $-m$,

$$\sigma\cap - : C^*(M) \overset{\cong}{\to} C_{m-*}(M), x \mapsto \sigma \cap x.$$

Let $\Phi : H^{*+m}(LM) \overset{\cong}{\to} Tor^*_{C^*(M^{\times 2})}(C_*(M), C^*(M))$ be the composite of the isomorphisms

$$\begin{tikzcd}
H^{p+m}(LM) \arrow[r, "EM^{-1}", "\cong"'] & Tor^{p+m}_{C^*(M^2)}(C^*(M), C^*(M^I))_{\Delta^*,p^*} \arrow[r, "Tor_1(1,\sigma^*)", "\cong"'] & Tor^{p+m}_{C^*(M^2)}(C^*(M), C^*(M))_{\Delta^*,\Delta^*} \arrow[d, "\cong"', "Tor_1(\sigma\cap -,1)"] \\
& & Tor^p_{C^*(M^2)}(C_*(M), C^*(M)).
\end{tikzcd}$$

Let $\varepsilon : \mathbb{B} \overset{\simeq}{\to} C^*(M)$ be a right $C^*(M^2)$-semi-free resolution of $C^*(M)$. By 1) of Proposition 9.1 and Corollary 9.5, $\Delta^!$ fits into the following homotopy commutative diagram of right $C^*(M^2)$-modules:

$$\begin{tikzcd}
\mathbb{B} \arrow[rr, "\Delta^!"] \arrow[d, "\varepsilon"', "\simeq"] & & C^{*+m}(M^2) \arrow[d, "\simeq"', "\sigma^2\cap -"] \\
C^*(M) \arrow[r, "\sigma\cap -", "\simeq"'] & C_{m-*}(M) \arrow[r, "\Delta_*"] & C_{m-*}(M^2)
\end{tikzcd}$$

Here $(\sigma^2 \cap -)(x) = EZ(\sigma\otimes\sigma)\cap x$. By applying the functor

$$\mathrm{Tor}^*_{C^*(M^4)}(-, C^*(M^2)),$$

we obtain the commutative square

$$\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^4)}(C^*(M), C^*(M^2)) \arrow[r, "\mathrm{Tor}_1(\sigma\cap -,1)"] \arrow[d, "\mathrm{Tor}_1(\Delta^!,1)"'] & \mathrm{Tor}^*_{C^*(M^4)}(C_*(M), C^*(M^2)) \arrow[d, "\mathrm{Tor}_1(\Delta_*,1)"] \\
\mathrm{Tor}^*_{C^*(M^4)}(C^*(M^2), C^*(M^2)) \arrow[r, "\mathrm{Tor}_1(\sigma^2\cap -,1)"'] & \mathrm{Tor}^*_{C^*(M^4)}(C_*(M^2), C^*(M^2)).
\end{tikzcd}$$

Therefore, using Theorem 2.3, Φ is an isomorphism of coalgebras with respect to the dual of the loop product and to the following composite:

$$\begin{tikzcd}
\mathrm{Tor}^*_{C^*(M^2)}(C_*M, C^*M) \arrow[r, "\mathrm{Tor}_{p_{13}^*}(1,1)"] & \mathrm{Tor}^*_{C^*|!(M^3)}(C_*M, C^*M) \\
\mathrm{Tor}^*_{C^*(M^4)}(C_*(M^2), C^*(M^2)) & \mathrm{Tor}^*_{C^*(M^4)}(C_*(M), C^*(M^2)) \arrow[l, "\mathrm{Tor}_1(\Delta_*,1)"] \arrow[u, "\cong", "\mathrm{Tor}_{(1\times\Delta\times 1)^*}(1,\Delta^*)"'] \\
\mathrm{Tor}^*_{C^*(M^4)}(C_*(M)^{\otimes 2}, C^*(M^2)) \arrow[u, "\cong", "\mathrm{Tor}_1(EZ,1)"'] \arrow[r, "\cong", "\mathrm{Tor}_{EZ^\vee}(1,EZ^\vee)"'] & \mathrm{Tor}^*_{(C_*(M^2)^{\otimes 2})^\vee}(C_*(M)^{\otimes 2}, (C_*(M)^{\otimes 2})^\vee) \\
\mathrm{Tor}^*_{C^*(M^2)}(C_*(M), C^*(M))^{\otimes 2} \arrow[r, "\cong", "\top"'] & \mathrm{Tor}^*_{C^*(M^2)^{\otimes 2}}(C_*(M)^{\otimes 2}, C^*(M)^{\otimes 2}). \arrow[u, "\mathrm{Tor}_\gamma(1,\gamma)", "\cong"']
\end{tikzcd}$$

Dualizing and using the natural isomorphism

$$\operatorname{Ext}^*_B(Q, P^\vee) \overset{\cong}{\to} \operatorname{Tor}^*_B(P,Q)^\vee$$

for any differential graded algebra B, right B-module P and left B-module Q, we see that the dual of Φ, $\Phi^\vee$, is an isomorphism of algebras with respect to the loop product and to the multiplication defined in Theorem 2.14. ∎

7. Associativity of the loop product on a Poincaré duality space

In this section, by applying the same argument as in the proof of [41, Theorem 2.2], we shall prove the associativity of the loop products.

Proof of Proposition 2.7. We prove the proposition in the case where $N = M$. The same argument as in the proof permits us to conclude that the loop homology $\mathbb{H}_*(L_N M)$ is associative with respect to the relative loop products.

Let M be a simply-connected Gorenstein space of dimension d. In order to prove the associativity of the dual to Dlp, we first consider the diagram

$$\begin{tikzcd}
LM \times LM & (LM \times_M LM) \times LM \arrow[l, "Comp\times 1"'] \arrow[r, "q\times 1"] & LM \times LM \times LM \\
LM \times_M LM \arrow[u, "q"] \arrow[d, "Comp"'] & LM \times_M LM \times_M LM \arrow[l, "Comp\times_M 1"'] \arrow[r, "q\times_M 1"] \arrow[u, "1\times_M q"'] \arrow[d, "1\times_M Comp"'] & LM \times (LM \times_M LM) \arrow[u, "1\times q"'] \arrow[d, "1\times Comp"'] \\
LM & LM \times_M LM \arrow[l, "Comp"] \arrow[r, "q"'] & LM \times LM,
\end{tikzcd}$$

for which the lower left-hand-side square is homotopy commutative and the other three squares are strictly commutative. Consider the corresponding diagram

$$\begin{tikzcd}
H^*(LM\times LM) \arrow[r, "(Comp\times 1)^*"] & H^*((LM\times_M LM)\times LM) \arrow[r, "\varepsilon'\alpha H(q^!)\otimes 1"] & H^*(LM\times LM\times LM) \\
H^*(LM\times_M LM) \arrow[u, "H(q^!)"] \arrow[r, "(Comp\times_M 1)^*"'] & H^*(LM\times_M LM\times_M LM) \arrow[u, "H((1\times_M q)^!)"'] \arrow[r, "H((q\times_M 1)^!)"] & H^*(LM\times(LM\times_M LM)) \arrow[u, "\varepsilon\alpha' 1\otimes H(q^!)"'] \\
H^*(LM) \arrow[u, "Comp^*"] \arrow[r, "Comp^*"'] & H^*(LM\times_M LM) \arrow[u, "(1\times_M Comp)^*"] \arrow[r, "H(q^!)"'] & H^*(LM\times LM).
\end{tikzcd}$$

The lower left square obviously commutes. By Theorem 8.5, the upper left square and the lower right square are commutative. We now show that the upper right square commutes.

By Theorem 8.6, we see that

$$H((q\times 1)^!) = \alpha H(q^!)\otimes 1 \quad \text{and} \quad H((1\times q)^!) = \alpha' 1\otimes H(q^!)$$

where α and $\alpha' \in \mathbb{K}^*$. By virtue of [12, Theorem C], in $\mathrm{D}(\text{Mod-}C^*(LM^{\times 3}))$,

$$\varepsilon'(\Delta\times 1)^!\circ\Delta^! = \varepsilon(1\times\Delta)^!\circ\Delta^!$$

where $(\varepsilon,\varepsilon') \neq (0,0) \in \mathbb{K}\times\mathbb{K}$. Therefore the uniqueness of the shriek map implies that

$$\varepsilon'(q\times 1)^!\circ(1\times_M q)^! = \varepsilon(1\times q)^!\circ(q\times_M 1)^!$$

in $\mathrm{D}(\text{Mod-}C^*(LM^{\times 3}))$; see [12, Theorem 13].

So, finally, we have proved that

$$\varepsilon'\alpha(Dlp\otimes 1)\circ Dlp = \varepsilon\alpha'(1\otimes Dlp)\circ Dlp.$$

Suppose that M is a Poincaré duality space of dimension d. By part (2) of Theorem 8.6, $\alpha = 1$ and $\alpha' = (-1)^d$. Since

$$\begin{aligned}\varepsilon(\omega_M\times\omega_M\times\omega_M) =&\varepsilon H((\Delta\times 1)^!)\circ H(\Delta^!)(\omega_M)\\ =&\varepsilon' H((1\times\Delta)^!)\circ H(\Delta^!)(\omega_M) = \varepsilon'(\omega_M\times\omega_M\times\omega_M),\end{aligned}$$

we see that $\varepsilon = \varepsilon'$. Therefore $(Dlp\otimes 1)\circ Dlp = (-1)^d(1\otimes Dlp)\circ Dlp$. Thus Example 10.3(i) together with Lemma 10.6 (i) and (ii) yields that the product $m : \mathbb{H}_*(LM)\otimes\mathbb{H}_*(LM)\to\mathbb{H}_*(LM)$ is associative.

We prove that the loop product is graded commutative. Consider the commutative diagram

$$\begin{tikzcd}
& LM\times_M LM \arrow[rr, "q"] \arrow[dl, "T"'] \arrow[dd] & & LM\times LM \arrow[dl, "T"] \arrow[dd, "p\times p"] \\
LM\times_M LM \arrow[rr, "q"] \arrow[dd, "p"'] & & LM\times LM \arrow[dd, "p\times p"] & \\
& M \arrow[rr, "\Delta"] \arrow[dl, equal] & & M\times M \arrow[dl, "T"] \\
M \arrow[rr, "\Delta"'] & & M\times M. &
\end{tikzcd}$$

By Theorem 8.3 below, $H(q^!)\circ T^* = \varepsilon T^*\circ H(q^!)$. Since $Comp\circ T$ is homotopic to $Comp$, $Dlp = \varepsilon T^*\circ Dlp$. If M is a Poincaré duality space with orientation class $\omega_M \in H^d(M)$, then $T^*(\omega_M\otimes\omega_M) = (-1)^{d^2}(\omega_M\otimes\omega_M)$. Therefore by part (a) of Remark 8.4, $\varepsilon = (-1)^d$. By Example 10.3(i) together with Lemma 10.6(i), we see that the product m is graded commutative. This completes the proof. ∎

Remark 7.1: The commutativity of the loop homology $\mathbb{H}_*(L_N M)$ does not follow from the proof of Proposition 2.7. In general, $Comp \circ T$ is not homotopic to $Comp$ in $L_N M$. As mentioned in the Introduction, the relative loop product is not necessarily commutative; see [34].

8. Appendix: Properties of shriek maps

In this section, we extend the definitions and properties of shriek maps on Gorenstein spaces given in [12]. These properties are used in Section 7.

Definition 8.1: A pull-back diagram,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

satisfies Hypothesis (H) (compare with the hypothesis (H) described in [12, p. 418]) if $p : E \twoheadrightarrow M$ is a fibration, for any $n \in \mathbb{N}$, $H^n(E)$ is of finite dimension and

$$\begin{cases} N \text{ is an oriented Poincaré duality space of dimension } n, \\ M \text{ is a 1-connected oriented Poincaré duality space of dimension } m, \end{cases}$$

or $f : B^r \to B^t$ is the product of diagonal maps $B \to B^{n_i}$, the identity map of B, the inclusion $\eta : * \to B$ for a simply-connected $\mathbb{K}$-Gorenstein space B.

Let n be the dimension of N or r times the dimension of B. Let m be the dimension of M or t times the dimension of B. It follows from [12, Lemma 1 and Corollary p. 448] that $H^q(N) \cong \mathrm{Ext}^{q+m-n}_{C^*(M)}(C^*(N), C^*(M))$. By definition, a shriek map $f^!$ for f is a generator of

$$\mathrm{Ext}^{\leq m-n}_{C^*(M)}(C^*(N), C^*(M)).$$

Moreover, there exists a unique element $g^! \in \mathrm{Ext}^{m-n}_{C^*(E)}(C^*(X), C^*(E))$ such that $g^! \circ C^*(q) = C^*(p) \circ f^!$ in the derived category of $C^*(M)$-modules; see Theorem 2.1.

Here we have extended the definitions of shriek maps due to Félix and Thomas in order to include the following example and the case $(\Delta \times 1)^!$ that we use in the proof of Proposition 2.7.

Example 8.2 (Compare with [12, pp. 419–420] where M is a Poincaré duality space): Let $F \overset{\widetilde{\eta}}{\to} E \overset{p}{\to} M$ be a fibration over a simply-connected Gorenstein space M with generator $\eta^! = \omega_M \in \mathrm{Ext}^m_{C^*(M)}(\mathbb{K}, C^*(M))$. By definition, $H(\widetilde{\eta}^!) : H^*(F) \to H^{*+m}(E)$ is the dual to the intersection morphism.

Let G be a connected Lie group. Then its classifying space BG is an example of Gorenstein space of negative dimension. Let F be a G-space. It is not difficult to see that our intersection morphism of $F \to F \times_G EG \to BG$ coincides with the integration along the fibre of the principal G-fibration $G \to F \times EG \to F \times_G EG$ for an appropriate choice of the generator $\eta^!$; see the proof of [12, Theorem 6].

Suppose now that $F \overset{\widetilde{\eta}}{\to} E \overset{p}{\to} M$ is a monoidal fibration. With the properties of shriek maps given in this section, generalizing [12, Theorem 10] (see also [16, Proposition 10]) in the Gorenstein case, one can show that the intersection morphism $H(\widetilde{\eta}_!) : H_{*+m}(E) \to H_*(F)$ is multiplicative if in the derived category of $C^*(M \times M)$-modules

$$\Delta^! \circ \omega_M = \omega_M \times \omega_M.$$

The generator $\Delta^! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$ is defined up to a multiplication by a scalar. If we could prove that $\Delta^! \circ \omega_M$ is always not zero, we would have a unique choice for $\Delta^!$ satisfying (11.1). Then we would have solved the "up to a constant problem" mentioned in [12, Q1 p. 423].

We now describe a generalized version of [12, Theorem 3]. We consider the following commutative diagram;

$$\begin{tikzcd}
 & X \arrow[rr, "g"] \arrow[dl, "v"'] \arrow[dd, "q"] & & E \arrow[dl, "k"'] \arrow[dd, "p"] \\
X' \arrow[rr, "g'"] \arrow[dd, "q'"'] & & E' \arrow[dd, "p'"] & \\
 & N \arrow[rr, "f"] \arrow[dl, "u"'] & & M \arrow[dl, "h"] \\
N' \arrow[rr, "f'"'] & & M' &
\end{tikzcd}$$

in which the back and the front squares satisfies Hypothesis (H).

THEOREM 8.3 (Compare with [12, Theorem 3]): *With the above notations, suppose that* $m' - n' = m - n$.

(1) *If h is a homotopy equivalence, then in the derived category of $C^*(M')$-modules,* $f^! \circ C^*(u) = \varepsilon C^*(h) \circ f'^!$, *where* $\varepsilon \in \mathbb{K}$.

(2) *If in the derived category of $C^*(M')$-modules, $f^! \circ C^*(u) = \varepsilon C^*(h) \circ f'^!$, then in the derived category of $C^*(E')$-modules, $g^! \circ C^*(v) = \varepsilon C^*(k) \circ g'^!$. In particular,*

$$H^*(g^!) \circ H^*(v) = \varepsilon H^*(k) \circ H^*(g'^!).$$

Remark 8.4: (a) In (1), if N' and M' are oriented Poincaré duality spaces, the constant ε is given by

$$H^n(f^!) \circ H^n(u)(\omega_{N'}) = \varepsilon H^{m'}(h) \circ H^{n'}(f'^!)(\omega_{N'}).$$

In fact, this is extracted from the uniqueness of the shriek map described in [12, Lemma 1].

(b) In [12, Theorem 3], it is not useful that v and k are homotopy equivalence. But in [12, Theorem 3], the homotopy equivalences u and h should be orientation preserving in order to deduce $\varepsilon = 1$.

(c) If the bottom square is the pull-back along a smooth embedding f' of compact oriented manifolds and a smooth map h transverse to N', then by [29, Proposition 4.2],

$$f^! \circ C^*(u) = C^*(h) \circ f'^! \quad \text{and} \quad H^*(g^!) \circ H^*(v) = H^*(k) \circ H^*(g'^!).$$

Proof of Theorem 8.3. The proofs of (1) and (2) follow from the proof of [12, Theorem 3]. But we review this proof, in order to explain that Theorem 8.3 is valid in the Gorenstein case and that we don't need to assume as in [12, Theorem 3] that u, k and v are homotopy equivalence.

(1) Since h is a homotopy equivalence,

$$\mathrm{Ext}^*_{C^*(M')}(C^*(N'), C^*(h)) :$$
$$\mathrm{Ext}^*_{C^*(M')}(C^*(N'), C^*(M')) \to \mathrm{Ext}^*_{C^*(M')}(C^*(N'), C^*(M))$$

is an isomorphism. By definition [12, Theorem 1 and p. 449], the shriek map $f'^!$ is a generator of $\mathrm{Ext}^{m'-n'}_{C^*(M')}(C^*(N'), C^*(M')) \cong \mathbb{K}$. Then $C^*(h) \circ f'^!$ is a generator of

$$\mathrm{Ext}^{m'-n'}_{C^*(M')}(C^*(N'), C^*(M)).$$

So since $f^! \circ C^*(u)$ is in $\mathrm{Ext}^{m-n}_{C^*(M')}(C^*(N'), C^*(M))$, we have (1).

(2) Let P be any $C^*(E')$-module. Since X' is a pull-back, a straightforward generalization of [12, Theorem 2] shows that

$$\mathrm{Ext}^*_{C^*(p')}(C^*(q'), P) : \mathrm{Ext}^*_{C^*(E')}(C^*(X'), P) \to \mathrm{Ext}^*_{C^*(M')}(C^*(N'), P)$$

is an isomorphism. Take $P := C^*(E)$. Consider the following cube in the derived category of $C^*(M')$-modules:

$$\begin{tikzcd}
& C^*(X) \arrow[rr, "g^!"] & & C^*(E) \\
C^*(X') \arrow[ur, "C^*(v)"] \arrow[rr, "g'^!"] & & C^*(E') \arrow[ur, "C^*(k)"'] & \\
& C^*(N) \arrow[uu, "C^*(q)"] \arrow[rr, "f^!"'] & & C^*(M). \arrow[uu, "C^*(p)"'] \\
C^*(N') \arrow[uu, "C^*(q')"] \arrow[ur, "C^*(u)"] \arrow[rr, "f'^!"'] & & C^*(M') \arrow[uu, "C^*(p')"] \arrow[ur, "C^*(h)"'] &
\end{tikzcd}$$

Since in $\mathrm{Ext}^*_{C^*(M')}(C^*(N'), C^*(E))$ the elements

$$g^! \circ C^*(v) \circ C^*(q') \quad \text{and} \quad \varepsilon C^*(k) \circ g'^! \circ C^*(q')$$

are equal, the assertion (2) follows. ∎

When u and h are the identity maps, Theorem 8.3 gives [12, Theorem 4] (compare with [16, Lemma 4]) and the following variant for Gorenstein spaces:

Theorem 8.5 (Naturality of shriek maps with respect to pull-backs): *Consider the two pull-back squares*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "v"'] & E \arrow[d, "k"] \\
X' \arrow[r, "g'"] \arrow[d, "q'"'] & E' \arrow[d, "p'"] \\
B^r \arrow[r, "\Delta"'] & B^t
\end{tikzcd}$$

where $\Delta : B^r \to B^t$ is the product of diagonal maps of a simply-connected $\mathbb{K}$-Gorenstein space B and p' and $p' \circ k$ are two fibrations. Then in the derived category of $C^*(E')$-modules, $g^! \circ C^*(v) = C^*(k) \circ g'^!$.

Theorem 8.6 (Products of shriek maps): *Let*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\
N \arrow[r, "f"] & M
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "q'"'] & E' \arrow[d, "p'"] \\
N' \arrow[r, "f'"] & M'
\end{tikzcd}$$

be two pull-back diagrams satisfying Hypothesis (H). Let $\times$ denote the cross product. Then:

(1) *The square*

$$\begin{tikzcd} H^*(N\times N') \arrow[r, "H^*((f\times f')^!)"] & H^*(M\times M') \\ H^*(N)\otimes H^*(N') \arrow[u, "\times"] \arrow[r, "\varepsilon H^*(f^!)\otimes H^*(f'^!)"'] & H^*(M)\otimes H^*(M') \arrow[u, "\times"'] \end{tikzcd}$$

is commutative for some $\varepsilon \in \mathbb{K}^*$.

(2) *Suppose that N, N', M and M' are Poincaré duality spaces oriented by $\omega_N \in H^n(N)$, $\omega_{N'} \in H^{n'}(N')$, $\omega_M \in H^m(M)$ and $\omega_{M'} \in H^{m'}(M')$. If we orient $N\times N'$ by $\omega_N \times \omega_{N'}$ and $M\times M'$ by $\omega_M\times\omega_{M'}$, then $\varepsilon = (-1)^{(m'-n')n}$.*

(3) *The square*

$$\begin{tikzcd} H^*(X\times X') \arrow[r, "H^*((g\times g')^!)"] & H^*(E\times E') \\ H^*(X)\otimes H^*(X') \arrow[u, "\times"] \arrow[r, "\varepsilon H^*(g^!)\otimes H^*(g'^!)"'] & H^*(E)\otimes H^*(E') \arrow[u, "\times"'] \end{tikzcd}$$

is commutative.

Remark 8.7: Here $g^! \otimes g'^!$ denotes the $C^*(E)\otimes C^*(E')$-linear map defined by

$$(g^!\otimes g'^!)(a\otimes b) = (-1)^{|g'^!||a|} g^!(a)\otimes g'^!(b).$$

Therefore, Theorem 8.6 (2) implies that

$$H^*((g\times g')^!)(a\times b) = (-1)^{(m'-n')(n+|a|)} H^*(g^!)(a)\times H^*(g'^!)(b).$$

The signs of [3, VI.14.3] are different from that mentioned here.

Proof of Theorem 8.6. (1) Let

$$EZ^\vee : C^*(M\times M') \overset{\simeq}{\to} (C_*(M)\otimes C_*(M'))^\vee$$

be the quasi-isomorphism of algebras dual to the Eilenberg–Zilber morphism. By definition [12, Theorem 1 and p. 449], $(f\times f')^!$ is a generator of

$$\mathrm{Ext}^{\leq m+m'-n-n'}_{C^*(M\times M')}(C^*(N\times N'), C^*(M\times M')).$$

Let h be the image of $(f\times f')^!$ by the composite of isomorphisms

$$\begin{array}{c}\mathrm{Ext}^*_{C^*(M\times M')}(C^*(N\times N'),C^*(M\times M'))\\ {\scriptstyle\mathrm{Ext}^*_{Id}(Id,EZ^\vee)}\Big\downarrow\cong\\ \mathrm{Ext}^*_{C^*(M\times M')}(C^*(N\times N'),(C_*(M)\otimes C_*(M'))^\vee)\\ {\scriptstyle\mathrm{Ext}^*_{EZ^\vee}(EZ^\vee,Id)}\Big\uparrow\cong\\ \mathrm{Ext}^*_{(C_*(M)\otimes C_*(M'))^\vee}((C_*(N)\otimes C_*(N'))^\vee,(C_*(M)\otimes C_*(M'))^\vee).\end{array}$$

Let

$$\Theta : C^*(M)\otimes C^*(M')\overset{\simeq}{\rightarrow}(C_*(M)\otimes C_*(M'))^\vee$$

be the quasi-isomorphism of algebras sending the tensor product of cochains $\varphi\otimes\varphi'$ to the form denoted again $\varphi\otimes\varphi'$ defined by

$$(\varphi\otimes\varphi')(a\otimes b)=(-1)^{|\varphi'||a|}\varphi(a)\varphi'(b).$$

Since $f^!\otimes f'^!$ is a generator of

$$\begin{aligned}&\mathrm{Ext}^{\leq m+m'-n-n'}_{C^*(M)\otimes C^*(M')}(C^*(N)\otimes C^*(N'),C^*(M)\otimes C^*(M'))\\ &\qquad\cong\mathrm{Ext}^{\leq m-n}_{C^*(M)}(C^*(N),C^*(M))\otimes\mathrm{Ext}^{\leq m'-n'}_{C^*(M')}(C^*(N'),C^*(M')),\end{aligned}$$

the image of h by the composite of isomorphisms

$$\begin{array}{c}\mathrm{Ext}^*_{(C_*(M)\otimes C_*(M'))^\vee}((C_*(N)\otimes C_*(N'))^\vee,(C_*(M)\otimes C_*(M'))^\vee)\\ \cong\Big\downarrow{\scriptstyle\mathrm{Ext}^*_{\Theta}(\Theta,Id)}\\ \mathrm{Ext}^*_{C^*(M)\otimes C^*(M')}(C^*(N)\otimes C^*(N'),(C_*(M)\otimes C_*(M'))^\vee)\\ \cong\Big\uparrow{\scriptstyle\mathrm{Ext}^*_{Id}(Id,\Theta)}\\ \mathrm{Ext}^*_{C^*(M)\otimes C^*(M')}(C^*(N)\otimes C^*(N'),C^*(M)\otimes C^*(M'))\end{array}$$

is an element $\varepsilon(f^!\otimes f'^!)$, where ε is a non-zero constant.

So, finally, we obtain an element

$$h\in\mathrm{Ext}^{m+m'-n-n'}_{(C_*(M)\otimes C_*(M'))^\vee}((C_*(N)\otimes C_*(N'))^\vee,(C_*(M)\otimes C_*(M'))^\vee)$$

such that in the derived category of $C^*(M\times M')$-modules

$$\begin{tikzcd} C^*(N\times N') \arrow[r, "(f\times f')^!"] \arrow[d, "EZ^\vee"'] & C^*(M\times M') \arrow[d, "EZ^\vee"] \\ (C_*(N)\otimes C_*(N'))^\vee \arrow[r, "h"'] & (C_*(M)\otimes C_*(M'))^\vee \end{tikzcd}$$

and in the derived category of $C^*(M)\otimes C^*(M')$-modules

$$\begin{tikzcd}
(C_*(N)\otimes C_*(N'))^\vee \arrow[r, "h"] & (C_*(M)\otimes C_*(M'))^\vee \\
C^*(N)\otimes C^*(N') \arrow[u, "\Theta"] \arrow[r, "\varepsilon f^!\otimes f'^!"'] & C^*(M)\otimes C^*(M') \arrow[u, "\Theta"']
\end{tikzcd}$$

are commutative squares. Applying cohomology, we obtain (1).

(2) By (1),

$$\begin{aligned}
\omega_M\times\omega_{M'} =& H^*((f\times f')^!)(\omega_N\times\omega_{N'})\\
=&\varepsilon(-1)^{(m'-n')n}H^*(f^!)(\omega_N)\times H^*(f'^!)(\omega_{N'})\\
=&\varepsilon(-1)^{(m'-n')n}\omega_M\times\omega_{M'}.
\end{aligned}$$

(3) Consider the following cube in the derived category of $C^*(M\times M')$-modules:

$$\begin{tikzcd}
& (C_*(X)\otimes C_*(X'))^\vee \arrow[rr, "k"] & & (C_*(E)\otimes C_*(E'))^\vee \\
C^*(X\times X') \arrow[ur, "EZ^\vee"] \arrow[rr, "(g\times g')^!"' near start] & & C^*(E\times E') \arrow[ur, "EZ^\vee"'] & \\
& (C_*(N)\otimes C_*(N'))^\vee \arrow[uu, "(C_*(q)\otimes C_*(q'))^\vee"' near start] \arrow[rr, "h"'] & & (C_*(M)\otimes C_*(M'))^\vee \arrow[uu, "(C_*(p)\otimes C_*(p'))^\vee"'] \\
C^*(N\times N') \arrow[uu, "C^*(q\times q')"] \arrow[ur, "EZ^\vee"] \arrow[rr, "(f\times f')^!"'] & & C^*(M\times M') \arrow[uu, "C^*(p\times p')"' near start] \arrow[ur, "EZ^\vee"'] &
\end{tikzcd}$$

with k defined below. Since

$$\begin{aligned}
&\mathrm{Ext}^*_{C^*(p\times p')}(C^*(q\times q'),C^*(E\times E')):\\
&\qquad\mathrm{Ext}^*_{C^*(E\times E')}(C^*(X\times X'),C^*(E\times E'))\\
&\qquad\qquad\to\mathrm{Ext}^*_{C^*(M\times M')}(C^*(N\times N'),C^*(E\times E'))
\end{aligned}$$

is an isomorphism, it follows that the maps

$$\begin{aligned}
&\mathrm{Ext}^*_{C^*(p\times p')}(C^*(q\times q'),(C_*(E)\otimes C_*(E'))^\vee):\\
&\qquad\mathrm{Ext}^*_{C^*(E\times E')}(C^*(X\times X'),(C_*(E)\otimes C_*(E'))^\vee)\\
&\qquad\qquad\to\mathrm{Ext}^*_{C^*(M\times M')}(C^*(N\times N'),(C_*(E)\otimes C_*(E'))^\vee)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ext}^*_{(C_*(p)\otimes C_*(p'))^\vee}((C_*(q)\otimes C_*(q'))^\vee,(C_*(E)\otimes C_*(E'))^\vee):\\
&\qquad\mathrm{Ext}^*_{(C_*(E)\otimes C_*(E'))^\vee}((C_*(X)\otimes C_*(X'))^\vee,(C_*(E)\otimes C_*(E'))^\vee)\\
&\qquad\quad\to \mathrm{Ext}^*_{(C_*(M)\otimes C_*(M'))^\vee}((C_*(N)\otimes C_*(N'))^\vee,(C_*(E)\otimes C_*(E'))^\vee)
\end{aligned}$$

are also isomorphisms. Let k be the image of $(C_*(p)\otimes C_*(p'))^\vee\circ h$ by the inverse of the isomorphism

$$\mathrm{Ext}^*_{(C_*(p)\otimes C_*(p'))^\vee}((C_*(q)\otimes C_*(q'))^\vee,(C_*(E)\otimes C_*(E'))^\vee).$$

Since $EZ^\vee\circ(g\times g')^!$ and $k\circ EZ^\vee$ have the same image by

$$\mathrm{Ext}^*_{C^*(p\times p')}(C^*(q\times q'),(C_*(E)\otimes C_*(E'))^\vee),$$

they coincide in the derived category of $C^*(E\times E')$-modules and hence we have proved the commutativity of the top square in the previous cube.

The same proof shows the commutativity of the top square of the following cube in the derived category of $C^*(M)\otimes C^*(M')$-modules:

$$\begin{tikzcd}
 & (C_*(X)\otimes C_*(X'))^\vee \arrow[rr, "k"] & & (C_*(E)\otimes C_*(E'))^\vee \\
C^*(X)\otimes C^*(X') \arrow[ur, "\Theta"] \arrow[rr, "\varepsilon g^!\otimes g'^!"] & & C^*(E)\otimes C^*(E') \arrow[ur, "\Theta"'] & \\
 & (C_*(N)\otimes C_*(N'))^\vee \arrow[uu, "(C_*(q)\otimes C_*(q'))^\vee"] \arrow[rr, "h"] & & (C_*(M)\otimes C_*(M'))^\vee \arrow[uu, "(C_*(p)\otimes C_*(p'))^\vee"'] \\
C^*(N)\otimes C^*(N') \arrow[uu, "C^*(q)\otimes C^*(q')"] \arrow[ur, "\Theta"] \arrow[rr, "\varepsilon f^!\otimes f'^!"'] & & C^*(M)\otimes C^*(M') \arrow[uu, "C^*(p)\otimes C^*(p')"'] \arrow[ur, "\Theta"'] &
\end{tikzcd}$$

The proof is now the same as in (1) by applying the cohomology functor to the top squares of the previous two commutative cubes. ∎

9. Appendix: Shriek maps and Poincaré duality

In this section, we compare precisely the shriek map defined by Félix and Thomas in [12, Theorem A] with various shriek maps defined by Poincaré duality.

We first recall the cap products used in the body of the present paper. Let $-\cap\sigma : C^*(M)\to C_{m-*}(M)$ be the cap product given in [3, VI.5], where

$\sigma \in C_m(M)$. Observe that $-\cap\sigma$ is defined by

$$(-\cap\sigma)(x) = (-1)^{|m||x|}x\cap\sigma.$$

By [3, VI.5.1 Proposition (iii)], the map $-\cap\sigma$ is a morphism of left $C^*(M)$-modules. The sign $(-1)^{|m||x|}$ makes $-\cap\sigma$ a left $C^*(M)$ linear map in the sense that $f(xm) = (-1)^{|f||x|}xf(m)$ as quoted in [11, p. 44].

We denote by $\sigma\cap - : C^*(M) \to C_{m-*}$ the cap product described in [31, §7]. The map $\sigma\cap -$ is a right $C^*(M)$-module map ([38, Proposition 2.1.1]). Moreover, we see that $x\cap\sigma = (-1)^{m|x|}\sigma\cap x$ in homology for any $x\in H^*(M)$.

Proposition 9.1: *Let N and M be two oriented Poincaré duality space of dimensions n and m. Let $[N]\in H_n(N)$, $[M]\in H_n(M)$ and $\omega_N \in H^n(N)$, $\omega_M\in H^m(M)$ such that the Kronecker products*

$$<\omega_M,[M]>=1=<\omega_N,[N]>.$$

Let $f : N \to M$ be a continous map. Let $f^!$ be the unique element of $\mathrm{Ext}^{m-n}_{C^*(M)}(C^*(N),C^*(M))$ such that $H(f^!)(\omega_N)=\omega_M$. Then:

(1) *The diagram in the derived category of right-$C^*(M)$ modules*

$$\begin{tikzcd} C^*(N) \arrow[r, "f^!"] \arrow[d, "{[N]\cap-}"'] & C^{*+m-n}(M) \arrow[d, "{[M]\cap-}"] \\ C_{n-*}(N) \arrow[r, "C_*(f)"'] & C_{n-*}(M) \end{tikzcd}$$

commutes up to the sign $(-1)^{m+n}$.

(2) *Let $\psi_N : C_*(N) \hookrightarrow C^*(N)^\vee$ be the canonical inclusion of the complex $C_*(N)$ into its bidual. The diagram of left $H^*(M)$-modules*

$$\begin{tikzcd} H^*(M)^\vee \arrow[r, "H^*(f^!)^\vee"] & H^{*+n-m}(N)^\vee \\ H_*(M) \arrow[u, "H(\psi_M)"] & H_{*+n-m}(N) \arrow[u, "H(\psi_N)"'] \\ H^{m-*}(M) \arrow[u, "{-\cap[M]}"] \arrow[r, "H^*(f)"'] & H^{m-*}(N) \arrow[u, "{-\cap[N]}"'] \end{tikzcd}$$

commutes up to the sign $(-1)^{n(m-n)}$.

(3) *Let* $\int_N : H(A(N)) \overset{\cong}{\to} H(C^*(N))$ *be the isomorphism of algebras induced by the natural equivalence between the rational singular cochains and the polynomial differential forms* [11, Corollary 10.10]. *Let* $\theta_N : A(N) \overset{\simeq}{\to} A(N)^\vee$ *be a morphism of* $A(N)$*-modules such that the class of* $\theta_N(1)$ *is the fundamental class of* N, $[N] \in H_n(A(N)^\vee) \cong H_n(N;\mathbb{Q})$. *Let* $f_A^!$ *be the unique element of* $\operatorname{Ext}^{m-n}_{A(M)}(A(N), A(M))$ *such that* $H(f_A^!)(\int_N^{-1}\omega_N) = \int_M^{-1}\omega_M$. *Then in the derived category of* $A(M)$*-modules, the diagram*

$$\begin{tikzcd} A(M)^\vee \arrow[r, "(f_A^!)^\vee"] & A(N)^\vee \\ A(M) \arrow[u, "\theta_M"] \arrow[r, "A(f)"'] & A(N) \arrow[u, "\theta_N"'] \end{tikzcd}$$

commutes also with the sign $(-1)^{n(m-n)}$.

Remark 9.2: Part (1) of the previous proposition is already in [12, p. 419] but without sign and with left-$C^*(M)$ modules. In particular, they should have defined their maps $-\cap[M] : C^*(M) \to C_{m-*}(M)$ by

$$(-\cap[M])(x) = (-1)^{|x|m} x \cap [M]$$

in order to have a left-$C^*(M)$ linear map [15, p. 283].

Note that the diagram of part (2) of Proposition 9.1 is commutative at the cochain level as shown by our proof below.

Proof. (1) By [12, Lemma 1], it suffices to show that the diagram commutes in homology on the generator $\omega_M \in H^m(M)$. Let $\varepsilon_M : H_0(M) \overset{\cong}{\to} \mathbb{K}$ be the augmentation. It is well-known that $\varepsilon_M(\omega_M \cap [M]) = <\omega_M, [M]> = 1$. Therefore $\varepsilon_M([M] \cap \omega_M) = (-1)^m \varepsilon_M(\omega_M \cap [M]) = (-1)^m$. On the other hand, $\varepsilon_N([N] \cap H(f^!)(\omega_M)) = \varepsilon_N([N] \cap \omega_N) = (-1)^n$.

(2) Since $H^*(f^!)$ is right $H^*(M)$-linear, its dual is left $H^*(M)$-linear. By $H^*(M)$-linearity, it suffices to check that the diagram commutes on $1 \in H^*(M)$. By [14, 7.1] or [31, Property 57 i)], $\psi_M : C_*(M) \hookrightarrow C^*(M)^\vee$ is defined by $\psi_M(c)(\varphi) = (-1)^{|c||\varphi|}\varphi(c)$ for $c \in C_*(M)$ and $\varphi \in C^*(M)$. Therefore $\psi_N([N])(\omega_N) = (-1)^{n^2}$. And

$$\begin{aligned} ((f^!)^\vee \circ \psi_M)([M])(\omega_N) =& (-1)^{m|f^!|}(\psi_M([M]) \circ f^!)(\omega_N) \\ =& (-1)^{m^2-mn}\psi_M([M])(\omega_M) = (-1)^{mn}. \end{aligned}$$

So $((f^!)^\vee \circ \psi_M)([M]) = (-1)^{n(m-n)}\psi_N([N])$. Observe that the map ψ_M is left $C^*(M)$-linear; see [14, p. 250].

(3) The isomorphism $\mathrm{Ext}^q_{A(M)}(A(M), A(N)^\vee) \cong H_{-q}(A(N)^\vee)$ maps any element φ to $H(\varphi)(1)$. Therefore, it suffices to check that the diagram commutes in homology on $1 \in H^0(A(M))$. Consider the cube

$$\begin{tikzcd}
& H(A(M)^\vee) \arrow[rr, "H((f^!_A)^\vee)"] & & H(A(N)^\vee) \\
H^*(M)^\vee \arrow[ur, "\int_M^\vee"] \arrow[rr, "H((f^!)^\vee)" near end] & & H^*(N)^\vee \arrow[ur, "\int_N^\vee"'] & \\
& H(A(M)) \arrow[uu, "H(\theta_M)" near start] \arrow[rr, "H(A(f))"' near start] \arrow[dl, "\int_M"'] & & H(A(N)) \arrow[uu, "H(\theta_N)"'] \arrow[dl, "\int_N"] \\
H^*(M) \arrow[uu, "H(\psi_M)\circ[M]\cap-"] \arrow[rr, "H^*(f)"'] & & H^*(N) \arrow[uu, "H(\psi_N)\circ[N]\cap-"' near end] &
\end{tikzcd}$$

The bottom face commutes by naturality of the isomorphism $\int_N$. The left face and right faces commute on 1 by definition of $\theta_M(1)$ and $\theta_N(1)$. The top face is the dual of the following square:

$$\begin{tikzcd}
H(A(N)) \arrow[r, "H^*(f^!_A)"] \arrow[d, "\int_N"'] & H(A(M)) \arrow[d, "\int_M"] & \\
H(C^*(N)) \arrow[r, "H^*(f^!)"'] & H(C^*(M)) \arrow[r, "H(\psi_M([M]))"'] & \mathbb{K}
\end{tikzcd}$$

Since $H(\psi_M([M])) \circ H^*(f^!) \circ \int_N = H(\psi_M([M])) \circ \int_M \circ H^*(f^!_A)$, by Lemma 9.3 below, the square is commutative. Therefore, the top face of the previous cube is commutative.

The front face is exactly the diagram in part 2) of this proposition. Therefore, the back face commutes on 1 up to the same sign $(-1)^{n(m-n)}$. ∎

The following lemma is a cohomological version of [12, Lemma 1].

Lemma 9.3: *Let P be a right $H^*(M)$-module. Then the map*

$$\Psi : \mathrm{Hom}^q_{H^*(M)}(P, H^*(M)) \to (P^{q-m})^\vee$$

mapping φ to the composite $H(\psi_M([M])) \circ \varphi$ is an isomorphism.

Remark 9.4: This lemma holds also with Ext instead of Hom and with $H_*(M)$, $C_*(M)$ or [12, Lemma 1] $C^*(M)$ instead of $H^*(M)$.

We give the proof of Lemma 9.3, since we don't understand all the proof of [12, Lemma 1].

Proof. Since the form

$$H(\psi_M([M])) : H(C^*(M)) \to \mathbb{K}, \quad \alpha \mapsto (-1)^{m|\alpha|} < \alpha, [M] >$$

coincides with $H^m(C^*(M)) \overset{[M]\cap -}{\to} H_0(C^*(M)) \overset{\varepsilon}{\cong} \mathbb{K}$, the map Ψ coincides with the composite of

$$\begin{aligned}\mathrm{Hom}^q_{H^*(M)}(P, H(\psi_M) \circ [M] \cap -) :&\mathrm{Hom}^q_{H^*(M)}(P, H^*(M)) \\ &\overset{\cong}{\to} \mathrm{Hom}^{q-m}_{H^*(M)}(P, H^*(M)^\vee)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{H^*(M)}(P, \mathrm{Hom}(H^*(M), \mathbb{K})) \cong&\mathrm{Hom}_{\mathbb{K}}(P \otimes_{H^*(M)} H^*(M), \mathbb{K}) \\ \cong&\mathrm{Hom}_{\mathbb{K}}(P, \mathbb{K}). \qquad \blacksquare\end{aligned}$$

Corollary 9.5: *Let N be an oriented Poincaré duality space. Let*

$$[N] \in H_n(N) \quad \textit{and} \quad \omega_N \in H^n(N)$$

such that the Kronecker product $< \omega_N, [N] >= 1$. Let $\Delta^!$ be the unique element of $\mathrm{Ext}^n_{C^*(N\times N)}(C^*(N), C^*(N \times N))$ such that $H(\Delta^!)(\omega_N) = \omega_N \times \omega_N$. (This is the $\Delta^!$ considered throughout this paper, since we orient $N \times N$ with the cross product $\omega_N \times \omega_N$: see part (2) of Theorem 8.6). Let $\Delta^!_A$ be the unique element of $Ext^n_{A(N\times N)}(A(N), A(N \times N))$ such that $H(\Delta^!_A)(\int_N^{-1} \omega_N) = \int_{N\times N}^{-1} \omega_N \times \omega_N$. Then in the case of $\Delta^!$ and of $\Delta^!_A$, all the diagrams of Proposition 9.1 commute exactly[1].

Proof. By [3, VI.5.4 Theorem], the cross products in homology and cohomology and the Kronecker product satisfy

$$< \omega_N \times \omega_N, [N] \times [N] >= (-1)^{|\omega_N||[N]|} < \omega_N, [N] >< \omega_N, [N] >= (-1)^n.$$

Therefore, by Proposition 9.1, the diagram of part 1) commutes up to the sign $(-1)^{2n+n}(-1)^n = +1$. The diagrams of parts (2) and (3) commute up to the sign $(-1)^{n(2n-n)}(-1)^n = +1$. $\blacksquare$

[1] As we see in the proof, this is lucky!

10. Appendix: Signs and degree shifting of products

Let A be a graded vector space equipped with a morphism $\mu_A : A \otimes A \to A$ of degree $|\mu_A|$. Let B be another graded vector space equipped with a morphism $\mu_B : B \otimes B \to B$ of degree $|\mu_B|$.

Definition 10.1: The multiplication μ_A is **associative** if

$$\mu_A \circ (\mu_A \otimes 1) = (-1)^{|\mu_A|} \mu_A \circ (1 \otimes \mu_A).$$

The multiplication μ_A is **commutative** if

$$\mu_A(a \otimes b) = (-1)^{|\mu_A|+|a||b|} \mu_A(b \otimes a)$$

for all a, $b \in A$. A linear map $f : A \to B$ is a **morphism of algebras of degree** $|f|$ if $f \circ \mu_A = (-1)^{|f||\mu_A|} \mu_B \circ (f \otimes f)$ (in particular, $|\mu_A| = |\mu_B| + |f|$).

Proposition 10.2: (i) *The composite $g \circ f$ of two morphisms of algebras f and g of degrees $|f|$ and $|g|$ is a morphism of algebras of degree $|f|+|g|$. The inverse f^{-1} of an isomorphism f of algebras of degree $|f|$ is a morphism of algebras of degree $-|f|$.*

(ii) *Let $f : A \to B$ be an isomorphism of algebras of degree $|f|$. Then μ_A is commutative if and only if μ_B is commutative. And μ_A is associative if and only if μ_B is associative.*

Example 10.3: (i) (Compare with [39, Remark 3.6 and proof of Proposition 3.5]) Let A be a lower graded vector space equipped with a morphism $\mu_A : A \otimes A \to A$ of lower degree $-d$ associative and commutative in the sense of Definition 10.1. Denote by $\mathbb{A} = s^{-d}A$ the d-desuspension [11, p. 41] of $A : \mathbb{A}_i = A_{i+d}$. Let $\mu_{\mathbb{A}} : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ be the morphism of degree 0 given by

$$\mu_{\mathbb{A}}(a \otimes b) = (-1)^{d(d+p)} \mu_A(a \otimes b)$$

for $a \in A_p$ and $b \in A_q$. Then the map $s^{-d} : A \to \mathbb{A}$, $a \mapsto a$, is an isomorphism of algebras of lower degree $-d$ and $\mu_{\mathbb{A}}$ is commutative and associative in the usual graded sense.

(ii) Let $f : A \to B$ be a morphism of algebras of degree $|f|$ in the sense of Definition 10.1 with respect to the multiplications μ_A and μ_B. Then the composite $\mathbb{A} \stackrel{(s^{|\mu_A|})^{-1}}{\to} A \stackrel{f}{\to} B \stackrel{s^{|\mu_B|}}{\to} \mathbb{B}$ is a morphism of algebras of degree 0 with respect to the multiplications $\mu_{\mathbb{A}}$ and $\mu_{\mathbb{B}}$.

The following proposition explains that the generalized cup product (Definition 5.1) is natural with respect to morphism of algebras of any degree (Definition 10.1).

PROPOSITION 10.4: *Let A be an algebra. Let M and N be two A-bimodules. Let $\bar{\mu}_M \in \mathrm{Ext}^*_{A\otimes A^{op}}(M\otimes_A M, M)$ and $\bar{\mu}_N \in \mathrm{Ext}^*_{A\otimes A^{op}}(N\otimes_A N, N)$. Let $f\in \mathrm{Ext}^*_{A\otimes A^{op}}(M,N)$ such that in the derived category of A-bimodules*

$$f\circ\bar{\mu}_M = (-1)^{|f||\bar{\mu}_M|}\bar{\mu}_N\circ(f\otimes_A f). \tag{1}$$

Then $HH^*(A,f): HH^*(A,M)\to HH^*(A,N)$ is a morphism of algebras of degree $|f|$.

Proof. Consider the diagram

$$\begin{tikzcd}
HH^*(A,M)^{\otimes 2} \arrow[r, "\otimes_A"] \arrow[d, "HH*(A{,}f)^{\otimes 2}"'] & HH^*(A,M\otimes_A M) \arrow[r, "HH^*(A{,}\bar{\mu}_M)"] \arrow[d, "HH*(A{,}f\otimes_A f)"] & HH^*(A,M) \arrow[d, "HH*(A{,}f)"] \\
HH^*(A,N)^{\otimes 2} \arrow[r, "\otimes_A"'] & HH^*(A,N\otimes_A N) \arrow[r, "HH^*(A{,}\bar{\mu}_N)"'] & HH^*(A,N)
\end{tikzcd}$$

The left square commutes exactly since for g, $h\in \mathrm{Hom}_{A\otimes A^{op}}(B(A,A,A),M)$,

$$(f\otimes_A f)\circ(g\otimes_A h)\circ c = (-1)^{|f||g|}((f\circ g)\otimes_A(f\circ h))\circ c.$$

By equation (1), the right square commutes up to the sign $(-1)^{|f||\bar{\mu}_M|}$. ∎

Example 10.5: (i) Let A be an algebra. Let M be an A-bimodule. Let $\bar{\mu}_M\in \mathrm{Ext}^d_{A\otimes A^{op}}(M\otimes_A M, M)$. Denote by $\mathbb{M} := s^{-d}M$ the d-desuspension of the A-bimodule M: for $a,b\in A$ and $m\in M$, $a(s^{-d}m)b := (-1)^{d|a|}s^{-d}(amb)$ [26, X.(8.4)]. Then the map $s^{-d}: M\to\mathbb{M}$ is an isomorphism of A-bimodules of degree d. Consider $\bar{\mu}_{\mathbb{M}}\in \mathrm{Ext}^0_{A\otimes A^{op}}(\mathbb{M}\otimes_A\mathbb{M},\mathbb{M})$ such that in the derived category of A-bimodules, $s^{-d}\circ\bar{\mu}_M = (-1)^{d|\bar{\mu}_M|}\bar{\mu}_{\mathbb{M}}\circ(s^{-d}\otimes_A s^{-d})$. Then $HH^*(A,s^{-d}): HH^*(A,M)\to HH^*(A,\mathbb{M})$ is a morphism of algebras of lower degree d. In particular, by Example 10.3 (ii), the composite

$$\mathbb{HH}^*(\mathbb{A},\mathbb{M}) := s^{-d}HH^*(A,M)\overset{s^d}{\to} HH^*(A,M) \overset{HH^*(A,s^{-d})}{\to} HH^*(A,\mathbb{M})$$

is an isomorphism of algebras of degree 0.

(ii) Let A be a commutative algebra. Let M and N be two A-modules. Let $\mu_M\in \mathrm{Ext}^*_{A^{\otimes 4}}(M\otimes M,M)$ and $\mu_N\in\mathrm{Ext}^*_{A^{\otimes 4}}(N\otimes N,N)$. Let $f\in \mathrm{Ext}^*_A(M,N)$

such that in the derived category of $A^{\otimes 4}$-modules,

$$f \circ \mu_M = (-1)^{|f||\mu_M|}\mu_N \circ (f \otimes f).$$

From Lemma 5.3(3) and Proposition 10.4, since

$$f \circ \bar{\mu}_M \circ q = (-1)^{|f||\bar{\mu}_M|}\bar{\mu}_N \circ (f \otimes_A f) \circ q,$$

$HH^*(A,f) : HH^*(A,M) \to HH^*(A,N)$ is a morphism of algebras of degree $|f|$.

Lemma 10.6: (i) *For a commutative diagram of graded $\mathbb{K}$-modules*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

the square

$$\begin{tikzcd} A^\vee & & B^\vee \arrow[ll, "f^\vee"'] \\ C^\vee \arrow[u, "h^\vee"] & & D^\vee \arrow[ll, "(-1)^{|f||k|+|g||h|}g^\vee"] \arrow[u, "k^\vee"'] \end{tikzcd}$$

is commutative.

(ii) *Let $f : A \to B$ and $g : C \to D$ be maps of graded $\mathbb{K}$-modules. Then the square*

$$\begin{tikzcd} A^\vee \otimes C^\vee \arrow[r, "f^\vee \otimes g^\vee"] \arrow[d, "\cong"'] & B^\vee \otimes D^\vee \arrow[d, "\cong"] \\ (A \otimes C)^\vee \arrow[r, "(f \otimes g)^\vee"'] & (B \otimes D)^\vee \end{tikzcd}$$

is commutative.

Proof. (i) By definition [42, 0.1 (7)], $f^\vee(\varphi) = (-1)^{|\varphi||f|}\varphi \circ f$. Therefore

$$(g \circ h)^\vee = (-1)^{|g||h|}h^\vee \circ g^\vee. \qquad \blacksquare$$

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