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ON WEIGHTED COVERING NUMBERS AND THE LEVI–HADWIGER CONJECTURE

BY

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ABSTRACT

We define new natural variants of the notions of weighted covering and separation numbers and discuss them in detail. We prove a strong duality relation between weighted covering and separation numbers and prove a few relations between the classical and weighted covering numbers, some of which hold true without convexity assumptions and for general metric spaces. As a consequence, together with some volume bounds that we discuss, we provide a bound for the famous Levi–Hadwiger problem concerning covering a convex body by homothetic slightly smaller copies of itself, in the case of centrally symmetric convex bodies, which is qualitatively the same as the best currently known bound. We also introduce the weighted notion of the Levi–Hadwiger covering problem, and settle the centrally-symmetric case, thus also confirm the equivalent fractional illumination conjecture [19, Conjecture 7] in the case of centrally symmetric convex bodies (including the characterization of the equality case, which was unknown so far).

1. Introduction

1.1. Background and motivation. Covering numbers can be found in various fields of mathematics, including combinatorics, probability, analysis and geometry. They often participate in the solution of many problems in quite a natural manner.

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In the combinatorial world, the idea of fractional covering numbers is well-known and utilized for many years. In [2], the authors introduced the weighted notions of covering and separation numbers of convex bodies and shed new light on the relations between the classical notions of covering and separation, as well as on the relations between the classical and weighted notions. In this note we propose a variant of these numbers which is perhaps more natural and discuss these numbers in more detail, revealing more useful relations, as well as some applications. To state our main results, we need some definitions. The impatient reader may skip the following section and go directly to Section 1.3 where the main results are stated.

Apart from deepening our understanding of these notions, and revealing more useful relations, we also consider this work as a first step towards the functionalization of covering and separation numbers; in the past decade, various parts from the theory of convex geometry have been gradually extended to the realm of log-concave functions. Numerous results found their functional generalizations. One natural way to embed convex sets in $\mathbb{R}^n$ into the class of log-concave functions is to identify every convex set K with its characteristic function $\mathbb{1}_K$. Besides being independently interesting, such extensions may sometimes be applied back to the setting of convex bodies. For further reading, we refer the reader to [1, 3, 4, 14, 15]. Since covering numbers play a considerable part in the theory of convex geometry, their extension to the realm of log-concave functions seems to be an essential building block for this theory. Our results using functional covering numbers will be published elsewhere.

1.2. Definitions. Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. The **classical covering number** of K by T is defined to be the minimal number of translates of T such that their union covers K, namely

$$N(K,T) = \min\left\{ N : N \in \mathbb{N},\, \exists x_1, \ldots, x_N \in \mathbb{R}^n;\ K \subseteq \bigcup_{i=1}^{N} (x_i + T) \right\}.$$

Here and in the sequel we assume that the covered set K is compact and the covering set T has non-empty interior so that the covering number will be finite. However, one may remove these restrictions so long as we are content also with infinite outcomes.

A well-known variant of the covering number is obtained by considering only translates of T that are centered in K, namely

$$\overline{N}(K,T) = \min\left\{N : N \in \mathbb{N},\, \exists x_1,\ldots,x_N \in K;\ K \subseteq \bigcup_{i=1}^{N}(x_i+T)\right\}.$$

Clearly, $N(K,T) \le \overline{N}(K,T)$, and it is easy to check that for convex bodies[1] K and T, we have $\overline{N}(K,T-T) \le N(K,T)$. Furthermore, if T is a Euclidean ball then $N(K,T) = \overline{N}(K,T)$.

The classical notion of the separation number of T in K is closely related to covering numbers and is defined to be the maximal number of non-overlapping translates of T which are centered in K;

$$M(K,T) = \max\{M : N \in \mathbb{N},\, \exists x_1,\ldots,x_M \in K;\ (x_i+T)\cap(x_j+T) = \emptyset\, \forall i \neq j\}.$$

It is a standard equivalence relation that $N(K,T-T) \le M(K,T) \le N(K,T)$. We also define the less conventional

$$\overline{M}(K,T) = \max\{M : N \in \mathbb{N},\, \exists x_1,\ldots,x_M \in K;\ (x_i+T)\cap(x_j+T)\cap K = \emptyset\, \forall i \neq j\}.$$

Note that the condition $(x_i+T)\cap(x_j+T) = \emptyset$ is equivalent to $x_i - x_j \notin T - T$ which means that $M(K,T) = M(K,-T) = M(K,\frac{T-T}{2})$. Moreover, it is easily checked that for a convex K and for a centrally symmetric convex body L (i.e., $L = -L$) we have $M(K,L) = \overline{M}(K,L)$ and thus by the last remark $M(K,T) = \overline{M}(K,T)$ for any convex body T. In the sequel, we will define weighted counterparts for $M(K,T)$ and $\overline{M}(K,T)$ which will not necessarily be equal, even in the convex and centrally symmetric case.

In order to define the weighted versions, let $\mathbb{1}_A$ denote the indicator function of a set$A \subseteq \mathbb{R}^n$, equal to 1 if $x \in A$ and 0 if $x \notin A$.

Definition 1.1: A sequence of pairs $S = \{(x_i,\omega_i) : x_i \in \mathbb{R}^n,\, \omega_i \in \mathbb{R}^+\}_{i=1}^N$ of points and weights is said to be a **weighted covering of** K **by** T if for all $x \in K$ we have $\sum_{i=1}^N \omega_i \mathbb{1}_{x_i+T}(x) \ge 1$. The **total weight** of the covering is denoted by $\omega(S) = \sum_{i=1}^N \omega_i$. The **weighted covering number** of K by T is defined to be the infimal total weight over all weighted coverings of K by T and is denoted by $N_\omega(K,T)$.

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[1] By convex body we mean, here and in the sequel, a compact convex set with non-empty interior.

One may consider only coverings $S = \{(x_i, \omega_i) : x_i \in K, \omega_i \in \mathbb{R}^+\}_{i=1}^N$ with centers of T in K. The corresponding weighted covering number for such coverings, denoted here by $\overline{N}_\omega(K,T)$, is defined to be the infimal total weight over such coverings. Clearly, $\overline{N}_\omega(K,T) \le N_\omega(K,T)$. The weighted notions of covering and separation numbers corresponding to $\overline{N}(K,T)$ and $\overline{M}(K,T)$ were introduced in [2]. In this note, we shall focus on the weighted versions of $N(K,T)$ and $M(K,T)$.

Let us reformulate the above definitions in the language of measures. Note that the covering condition $\sum_{i=1}^N \omega_i \mathbb{1}_{x_i+T}(x) \ge 1$ for all $x \in K$ is equivalent to $\nu * \mathbb{1}_T \ge \mathbb{1}_K$, where $\nu = \sum_{i=1}^N \omega_i \delta_{x_i}$ is the discrete measure with masses ω_i centered at x_i and where $*$ stands for the convolution

$$(\nu * \mathbb{1}_T)(x) = \int_{\mathbb{R}^n} \mathbb{1}_T(x-y)d\nu(y).$$

Let $\mathcal{D}_+^n$ denote all non-negative discrete and finite measures on $\mathbb{R}^n$ and let $\mathrm{supp}(\nu) \subseteq \mathbb{R}^n$ denote the support of a measure ν on $\mathbb{R}^n$. Thus, the weighted covering numbers of K by T can be written as

$$N_\omega(K,T) = \inf\{\nu(\mathbb{R}^n) : \nu * \mathbb{1}_T \ge \mathbb{1}_K, \nu \in \mathcal{D}_+^n\}$$

and

$$\overline{N}_\omega(K,T) = \inf\{\nu(\mathbb{R}^n) : \nu * \mathbb{1}_T \ge \mathbb{1}_K, \nu \in \mathcal{D}_+^n \text{ with } \mathrm{supp}(\nu) \subseteq K\}.$$

It is natural to extend this notion of covering to general non-negative measures. Let $\mathcal{B}_+^n$ denote all non-negative Borel regular measures on $\mathbb{R}^n$.

Definition 1.2: Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subset \mathbb{R}^n$ be compact with non-empty interior. A non-negative measure $\mu \in \mathcal{B}_+^n$ is said to be a **covering measure** of K by T if $\mu * \mathbb{1}_T \ge \mathbb{1}_K$. The corresponding weighted covering number is defined by

$$N^*(K,T) = \inf\left\{\int_{\mathbb{R}^n} d\mu : \mu * \mathbb{1}_T \ge \mathbb{1}_K, \mu \in \mathcal{B}_+^n\right\}.$$

Clearly, $N^*(K,T) \le N_\omega(K,T)$. In Proposition 2.6, we show that the above infimum is actually a minimum, that is, there exists an optimal covering Borel measure of K by T. Note that the set of optimal covering measures is clearly convex.

The weighted notions of the separation are defined similarly; a measure $\mu \in \mathcal{B}_+^n$ is said to be T-separated if $\mu * \mathbb{1}_T \le 1$. The weighted separation numbers, corresponding to $N_\omega(K,T)$, $\overline{N}_\omega(K,T)$ and $N^*(K,T)$, are respectively defined by

$$M_\omega(K,T) = \sup\left\{ \int_K d\nu : \nu * \mathbb{1}_T \le 1,\ \nu \in \mathcal{D}_+^n \right\},$$
$$\overline{M}_\omega(K,T) = \sup\left\{ \int_K d\nu : \forall x \in K,\ (\nu * \mathbb{1}_T)(x) \le 1,\ \nu \in \mathcal{D}_+^n \right\}$$

and

$$M^*(K,T) = \sup\left\{ \int_K d\mu : \mu * \mathbb{1}_T \le 1, \mu \in \mathcal{B}_+^n \right\},$$

where again clearly $M_\omega(K,T) \le M^*(K,T)$.

1.3. Main results. Our first main result is a strong duality between weighted covering and separation numbers; it turns out that $N^*(K,T)$ and $M^*(K,-T)$ can be interpreted as the outcome of two dual problems in the sense of linear programming. Indeed, as in [2], this observation is a key ingredient in the proof of our first main result below, which states that the outcome of these dual problems is the same (we call this "strong duality").

Theorem 1.3: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be a compact with non-empty interior. Then*

$$M_\omega(K,T) = M^*(K,T) = N^*(K,-T).$$

Remark 1.4: While it is not clear, so far, whether strong duality also holds for fractional covering numbers with respect to discrete measures, namely whether $N_\omega(K,T) = M_\omega(K,-T)$, one may show that

$$\lim_{\delta \to 0^+} N_\omega(K, -(1+\delta)T) = \lim_{\delta \to 0} M_\omega(K, (1+\delta)T) \le M_\omega(K,T).$$

In particular, for almost every $t > 0$

$$M_\omega(K,tT) = M^*(K,tT) = N^*(K,-tT) = N_\omega(K,tT).$$

See discussion in Section 2.2, Remark 2.5.

As a consequence of Theorem 1.3, together with the well-known homothety equivalence between classical covering and separation numbers $N(K,T-T) \le M(K,T) \le N(K,T)$, we immediately get the following equivalence relation between the classical and weighted covering numbers (which has also appeared in [2] for the pair $\overline{M}_\omega, \overline{N}_\omega$).

Corollary 1.5: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$N(K, T - T) \leq N_\omega(K, T) \leq N(K, T).$$

We remark that Corollary 1.5 is actually implied by the weak duality

$$M^*(K, -T) \leq N^*(K, T)$$

which we prove in Proposition 2.1 below, the proof of which is relatively simple. Similarly, we shall prove in Proposition 2.1 that $\overline{M}_\omega(K, -T) \leq \overline{N}_\omega(K, T)$ providing an alternative short proof for the weak duality result in [2, Theorem 6].

For a centrally symmetric convex set T, Corollary 1.5 reads $N(K, 2T) \leq N_\omega(K, T) \leq N(K, T)$. Although this "constant homothety" equivalence of classical and weighted covering is useful, it turns out to be insufficient in certain situations. To that end, we introduce our second main result, in which the homothety factor 2 is replaced by a factor $1 + \delta$ with $\delta > 0$ arbitrarily close to 0. This gain is diminished by an additional logarithmic factor; such a result is a reminiscent of Lovász's [17] well-known inequality for fractional covering numbers of hypergraphs.

Theorem 1.6: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T_1, T_2 \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$N(K, T_1 + T_2) \leq \ln(4\overline{N}(K, T_2))(N_\omega(K, T_1) + 1) + \sqrt{\ln(4\overline{N}(K, T_2))(N_\omega(K, T_1) + 1)}.$$

We remark that for the proof of our application in Section 3 below, we shall use $T_1 = \delta T$ and $T_2 = (1 - \delta)T$ for $0 < \delta < 1$ and a single convex body T. It is also worth mentioning that Theorem 1.6 holds for $\overline{N}_\omega(K, T)$ and $\overline{N}(K, T)$ as well (with the exact same proof).

1.4. Additional inequalities. Let $\mathrm{Vol}(A)$ denote the Lebesgue volume of a set $A \subseteq \mathbb{R}^n$. The classical covering and separation numbers satisfy simple volume bounds. Such volume bounds also hold for the weighted case, and turn out to be quite useful.

THEOREM 1.7: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$\max\left\{\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(T)}, 1\right\} \leq N^*(K,T) \leq \frac{\mathrm{Vol}(K-T)}{\mathrm{Vol}(T)}.$$

Remark 1.8: Let us show, by using the above volume bounds, that classical and weighted covering numbers are not equal in general, even for centrally symmetric convex bodies such as a cube and a ball (for a simple 2-dimensional example, see the last part of Remark 2.7). Namely, we show that $N_\omega(K,T) \neq N(K,T)$, where $T = B_2^n$ is the unit ball in $\mathbb{R}^n$ and $K = [-R,R]^n$ for a large enough R. Indeed, it was shown in [11] that the lower limit of the density of covering a cube by balls, defined as the limit of the ratio $N([-R,R]^n, B_2^n) \cdot \mathrm{Vol}(B_2^n)/(2R)^n$, as R tends to infinity is bounded from below by $16/15 - \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$. However, by our volume bounds in Theorem 1.7, it follows that the weighted covering density $N_\omega([-R,R]^n, B_2^n) \cdot \mathrm{Vol}(B_2^n)/(2R)^n$ approaches 1 as $R \to \infty$. Note that by Proposition 2.1 below, this also means that $M(Q, B_2^n) \neq N(Q, B_2^n)$ for a large enough cube and dimension.

1.5. AN APPLICATION. A famous conjecture, known as the Levi–Hadwiger or the Gohberg–Markus covering problem, was posed in [16], [13] and [12]. It states that in order to cover a convex set by slightly smaller copies of itself, one needs at most 2^n copies.

CONJECTURE: *Let $K \subseteq \mathbb{R}^n$ be a convex body with non empty interior. Then there exists $0 < \lambda < 1$ such that*

$$N(K, \lambda K) \leq 2^n.$$

Equivalently, $N(K, \mathrm{int}(K)) \leq 2^n$. Moreover, equality holds if and only if K is a parallelotope.

This problem has drawn much attention over the years, but only little has been unraveled so far. We mention that Levi confirmed the conjecture for the plane, and that Lassak confirmed it for centrally symmetric bodies in $\mathbb{R}^3$. The currently best known general upper bound for $n \geq 3$ is

$$\binom{2n}{n}(n \ln n + n \ln\ln n + 5n)$$

and the best bound for centrally symmetric convex bodies is

$$2^n(n\ln n + n\ln\ln n + 5n),$$

both of which are simple consequences of Rogers' bound for the asymptotic lower densities for covering the whole space by translates of a general convex body; see [20]. For a comprehensive survey of this problem and the aforementioned results, see [8].

It is natural, after introducing weighted covering, to formulate the Levi–Hadwiger covering problem for the case of weighted covering.

Conjecture 1.9: *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\lim_{\lambda\to 1^-} N_\omega(K,\lambda K) \le 2^n.$$

Moreover, equality holds if and only if K is a parallelotope.

For centrally symmetric convex bodies, we verify Conjecture 1.9, including the equality case. We show

Theorem 1.10: *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\lim_{\lambda\to 1^-} N_\omega(K,\lambda K) \le \begin{cases} 2^n, & K=-K,\\ \binom{2n}{n}, & K\neq -K, \end{cases}$$

Moreover, for centrally symmetric K, $\lim_{\lambda\to 1^-} N_\omega(K,\lambda K) = 2^n$ if and only if K is a parallelotope.

It is worth mentioning that the classical covering problem of Levi–Hadwiger is equivalent to the problem of the illumination of a convex body (for surveys see [18, 6]) which asks how many directions are required to illuminate the entire boundary of a convex body K (a direction $u \in \mathbb{S}^{n-1}$ is said to illuminate a point b in the boundary of K if the ray emanating from b in direction u intersects the interior of K). A fractional version of the illumination problem was considered in [19], where it was proven that the fractional illumination number of a convex body K, denoted by $i^*(K)$, satisfies that $i^*(K) \le \binom{2n}{n}$ and that $i^*(K) \le 2^n$ for all centrally symmetric bodies (with parallelotopes attaining equality). It was further conjectured [19, Conjecture 7] that $i^*(K) \le 2^n$ for all convex bodies and that equality is attained only for parallelotopes. However, as no relation between fractional and usual illumination numbers was proposed, this result remained isolated. Also, it seems that the equality conditions were not analyzed.

In fact, one may verify that the proof of the equivalence between the illumination problem and the Levi–Hadwiger covering problem (see [7, Theorem 7]) carries over to the fractional setting and conclude that $i^*(K) = \lim_{\lambda\to 1^-} N_\omega(K, \lambda K)$. Thus, Theorem 1.10 actually verifies the aforementioned results about fractional illumination and also verifies [19, Conjecture 7] for the case of centrally symmetric convex bodies, including the equality hypothesis.

Combining the inequality in Theorem 1.6 with the volume inequality in Theorem 1.7, we prove the following bound for the classical Levi–Hadwiger problem, in the case of centrally symmetric convex bodies, which is the same as the aforementioned (best known) general bound of Rogers.

Corollary 1.11: *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body. Then for all $n \geq 3$,*

$$\lim_{\lambda\to 1^-} N(K, \lambda K) \leq 2^n(n\ln(n) + n\ln\ln(n) + 5n).$$

We remark that the above bound and Rogers' bound are asymptotically equivalent, and that in both cases the constant $5n$ above may be improved by performing more careful computations, improving and optimizing over various constants. We avoid such computations as they will not affect the order of magnitude of this bound, and complicate the exposition.

The remainder of this note is organized as follows. In Section 2.1 we show weak duality between weighted covering and separation numbers. In Section 2.2 we prove Theorem 1.3. In Section 2.3 we discuss the existence of optimal covering measures. In Section 2.4 we discuss the approximation of uniform covering measures by discrete covering measures. In Section 2.5 we prove Theorem 1.7. In Section 2.6 we prove Theorem 1.6. In Section 2.7 we discuss the weighted notions of covering and separation in the setting of general metric spaces. In Section 3 we discuss both the classical and weighted versions of the Levi–Hadwiger covering problems, proving Theorem 1.10 and Corollary 1.11.

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2. Weighted covering and separation

2.1. Weak duality.

Proposition 2.1: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$M^*(K,T) \leq N^*(K,-T) \quad \text{and} \quad \overline{M}_\omega(K,T) \leq \overline{N}_\omega(K,-T).$$

In particular, we also have that $M_\omega(K,T) \leq N_\omega(K-T)$.

Proof. Let μ be a covering measure of K by $-T$. Let ρ be a T-separated measure. By our assumptions we have that $\mathbb{1}_T * \rho \leq 1$ and $\mathbb{1}_{-T} * \mu \geq \mathbb{1}_K$. Thus

$$\begin{aligned}\int_K d\rho(x) &= \int \mathbb{1}_K(x) \cdot d\rho(x) \leq \int (\mathbb{1}_{-T} * \mu)(x) d\rho(x)\\ &= \int d\rho(x) \int \mathbb{1}_{-T}(x-y) d\mu(y)\\ &= \int d\mu(y) \int \mathbb{1}_T(y-x) d\rho(x)\\ &= \int (\mathbb{1}_T * \rho)(y) d\mu(y)\\ &\leq \int d\mu(y)\end{aligned}$$

and so $M^*(K,T) \leq N^*(K,-T)$. Similarly, by considering $\mathbb{1}_T * \rho \leq 1$ only on K and μ which must be supported only on K, the exact same inequality yields $\overline{M}_\omega(K,T) \leq \overline{N}_\omega(K,-T)$. ∎

2.2. Strong duality. In this section we prove Theorem 1.3. By Proposition 2.1 it is enough to show an inequality $M_\omega(K,T) \geq N_\omega(K,-T)$.

We start with the discretized versions of our weighted covering and separation notions. Let $\Lambda = \{x_i\}_{i=1}^d \subseteq \mathbb{R}^n$ be some finite set, which will be chosen later, and define

$$N_\omega(K,T,\Lambda) = \inf\left\{ \sum_{i=1}^N \omega_i : \exists (x_i, \omega_i)_{i=1}^N \subseteq (\Lambda, \mathbb{R}^+), \sum_{i=1}^N \omega_i \mathbb{1}_T(x-x_i) \geq 1_K(x) \forall x \in \Lambda \right\}$$

and

$$M_\omega(K,T,\Lambda) = \sup\Big\{ \sum_{i=1}^N \omega_i : \exists (x_i,\,\omega_i)_{i=1}^N \subseteq (\Lambda\cap K,\,\mathbb{R}^+),\ \sum_{i=1}^N \omega_i \mathbb{1}_T(x-x_i) \le 1\ \forall x\in\Lambda \Big\}.$$

In this setting, linear programming duality gives us an equality of the form

$$N_\omega(K,T,\Lambda) = M_\omega(K,-T,\Lambda). \tag{2.1}$$

Indeed, define the vectors $b, c \in \mathbb{R}^d$ by

$$c_i = \begin{cases} 1, & x_i \in K, \\ 0, & \text{otherwise}, \end{cases} \qquad b_i = 1$$

and the $d \times d$ matrix M by

$$M_{ij} = \begin{cases} 1, & x_i \in x_j + T, \\ 0, & \text{otherwise}. \end{cases}$$

Note that

$$M^T_{ij} = \begin{cases} 1, & x_i \in x_j - T, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product in $\mathbb{R}^d$. Then, in the language of vectors and matrices, the above discretized weighted covering and separation notions read

$$N_\omega(K,T,\Lambda) = \min\{\langle b,\,x\rangle : Mx \ge c,\ x \ge 0\},$$
$$M_\omega(K,-T,\Lambda) = \max\{\langle c,\,y\rangle : M^T y \le b,\ y \ge 0\},$$

which are equal by the well-known duality theorem of linear programming; see, e.g., [5].

Next, we shall use this observation with a specific family of sets $\Lambda(\delta)$. A set $\Lambda(\delta) \subseteq \mathbb{R}^n$ is said to be a δ-net of a set $A \subseteq \mathbb{R}^n$ if for every $x \in A$ there exists $y \in \Lambda(\delta)$ for which $|x-y| \le \delta$. In other words, $A \subseteq \Lambda + \delta B_2^n$. We shall make use of the two following simple lemmas, corresponding to [2, Lemmas 14–15].

Lemma 2.2: *Let $K \subseteq \mathbb{R}^n$ be compact, $T \subseteq \mathbb{R}^n$ compact with non-empty interior and let $\Lambda(\delta) \subseteq K$ be some δ-net for K. Then*

$$N_\omega(K, T+\delta B_2^n) \le N_\omega(K,T,\Lambda(\delta)).$$

Proof. Indeed, we have that

$$\begin{aligned} N_\omega(K, T+\delta B_2^n) \leq & N_\omega(K\cap\Lambda(\delta)+\delta B_2^n, T+\delta B_2^n) \\ \leq & N_\omega(K\cap\Lambda(\delta), T) \leq N_\omega(K,T,\Lambda(\delta)). \end{aligned}$$ ∎

LEMMA 2.3: *Let $K\subseteq\mathbb{R}^n$ be compact, $T\subseteq\mathbb{R}^n$ be compact with non-empty interior and let $\Lambda(\delta)\subseteq\mathbb{R}^n$ be some δ-net for $K+T$. Then*

$$M_\omega(K,T)\geq M_\omega(K,T+\delta B_2^n,\Lambda(\delta)).$$

Proof. Suppose that $\{(x_i,\omega_i)\}_{i=1}^M\subseteq(K\cap\Lambda(\delta),\mathbb{R}^+)$ satisfies the condition in the definition of $M_\omega(K,T+\delta B_2^n,\Lambda)$, namely for all $x\in\Lambda(\delta)$ we have that $\sum_{i=1}^N\omega_i\mathbb{1}_{T+\delta B_2^n}(x-x_i)\leq 1$. Then it is also weighted T-separated in the usual sense (that is, satisfying for all $x\in\mathbb{R}^n$ that $\sum_{i=1}^N\omega_i\mathbb{1}_T(x-x_i)\leq 1$). Indeed, otherwise we would have a point in $x\in\mathbb{R}^n$ such that $\sum_{i=1}^M\omega_i\mathbb{1}_T(x-x_i)>1$. Since $x_i\in K$, it follows that $x\in K+T$ and so there exists a point $y\in\Lambda(\delta)$ for which $y-x\in\delta B_2^n$, which means that $\sum_{i=1}^M\omega_i\mathbb{1}_{T+\delta B_2^n}(y-x_i)>1$, a contradiction to our assumption. ∎

Finally, to prove Theorem 1.3 we shall need the following continuity result for weighted covering numbers:

PROPOSITION 2.4: *Let $K\subseteq\mathbb{R}^n$ be compact and let $T\subseteq\mathbb{R}^n$ be compact with non-empty interior. Then*

$$\lim_{\delta\to 0^+}N^*(K,T+\delta B_2^n)=N^*(K,T).$$

Proof. Clearly we have that

$$\lim_{\delta\to 0}N^*(K,T+\delta B_2^n)\leq N^*(K,T).$$

For the opposite direction, let $\delta_k\underset{k\to\infty}{\longrightarrow}0$ and let f_k be a sequence of continuous functions satisfying $\mathbb{1}_T\leq f_k\leq\mathbb{1}_{T+\delta_k D}$ so that $f_k\underset{k\to\infty}{\longrightarrow}\mathbb{1}_T$ point-wise monotonically. Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of covering Borel regular measures of K by f_k (the definition is straightforward: replace $\mathbb{1}_T$ in the original definition by f_k) such that $\int_{\mathbb{R}^n}d\mu_k(x)=N^*(K,f_k)+\varepsilon_k$ with $0<\varepsilon_k\to 0$. By the well-known Banach–Alaoglu theorem and passing to a subsequence we may assume without loss of generality that $\mu_k\overset{w^*}{\longrightarrow}\mu$ for some non-negative regular Borel measure. We claim that μ is a covering measure of K by T. Indeed, let $x\in K$. For $k\geq l$

we have that

$$1 \leq (\mu_k * f_k)(x) \leq (\mu_k * f_l)(x).$$

By the weak* convergence of μ_k to μ, taking the limit $k \to \infty$ implies that $1 \leq (\mu * f_l)(x)$ and hence, by the monotone convergence theorem, taking the limit $l \to \infty$ implies that $1 \leq (\mu * \mathbb{1}_T)(x)$. Thus, μ is a covering measure of K by T. This means that

$$\lim_{k\to\infty} N^*(K, f_k) = \lim_{k\to\infty} \int_{\mathbb{R}^n} d\mu_k \geq \int_{\mathbb{R}^n} d\mu \geq N^*(K,T),$$

which in turn implies the equality $\lim_{\delta\to 0^+} N^*(K, T + \delta B_2^n) = N^*(K,T)$, as claimed. ∎

Proof of Theorem 1.3. We use Lemmas 2.2–2.3 together with (2.1) as follows; let $\Lambda(\delta_k)$ be a sequence of δ_k-nets for $K+T$ with $\delta_k \to 0^+$ such that $K \cap \Lambda(\delta_k)$ are δ_k-nets for K. For each k we have

$$\begin{aligned} M_\omega(K,T) &\geq M_\omega(K, T + \delta_k B_2^n, \Lambda(\delta_k)) \\ &= N_\omega(K, -(T + \delta_k B_2^n), \Lambda(\delta_k)) \\ &\geq N_\omega(K, -(T + 2\delta_k B_2^n)). \end{aligned} \tag{2.2}$$

Thus, by Proposition 2.4

$$M_\omega(K,T) \geq \lim_{k\to\infty} N^*(K, -T + 2\delta_k B_2^n) = N^*(K, -T).$$

Taking the above inequality into account together with Proposition 2.1, the proof is thus complete. ∎

Remark 2.5: In [2], Proposition 22 is analogous to Proposition 2.4 above with N_ω instead of N^*. We mention that replacing $T + \delta B_2^n$ by $(1+\delta)T$ is of no significance because any two bodies in fixed dimension are equivalent. The proof presented in [2] is not correct, as it is based on [2, Lemma 20] which contains an error.

Note, however, that since the function $N^*(K, tT)$ is monotone in $t > 0$, it is clearly continuous almost everywhere. This, combined with the reasoning in [2, Proof of Theorem 7] (or, similarly, the reasoning above for N^*), implies that for almost every $t > 0$ we have

$$M_\omega(K, tT) = N_\omega(K, -tT).$$

By taking the limit as $t \to 1^+$ we get that

$$\lim_{\delta\to 0^+} M_\omega(K,(1+\delta)T) = \lim_{\delta\to 0^+} N_\omega(K,-(1+\delta)T),$$

which, combined with Theorem 1.3, yields the following row of equalities (as N^*, and so also M_ω, are continuous), holding for all convex bodies $K,T \subset \mathbb{R}^n$:

$$M_\omega(K,T) = M^*(K,T) = N^*(K,-T) = \lim_{\delta\to 0^+} N_\omega(K,-(1+\delta)T).$$

2.3. Optimal measures.

Proposition 2.6: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then there exists a (non-empty) convex set $\mathcal{C} \subseteq \mathcal{B}_+^n$ of optimal regular Borel covering measures of K by T. That is, for every $\mu \in \mathcal{C}$ we have that $\mu * \mathbb{1}_T \geq \mathbb{1}_K$ and*

$$N^*(K,T) = \int_{\mathbb{R}^n} d\mu.$$

Proof. The proof follows the same lines as the proof of Proposition 2.4 Let $(\mu_k) \subseteq \mathcal{B}_+^n$ be a sequence of covering measures of K by T such that

$$\mu_k(\mathbb{R}^n) \underset{k\to\infty}{\longrightarrow} N^*(K,T).$$

By using the Banach-Alaoglu theorem and passing to a converging subsequence we may assume without loss of generality that $\mu_k \xrightarrow{w*} \mu$ for some measure $\mu \in \mathcal{B}_+^n$. Let us show that μ is a covering measure of K by T, that is $\mu * \mathbb{1}_T \geq \mathbb{1}_K$. Indeed, let $x \in K$ and let $f \geq \mathbb{1}_T$ be a compactly supported continuous function. Then $1 \leq (\mu_k * f)(x) \to (\mu * f)(x)$ as $k \to \infty$. Taking a monotone sequence (f_l) of compactly supported continuous functions satisfying $f_l \geq \mathbb{1}_T$ and point-wise converging to $\mathbb{1}_T$, it follows by the monotone convergence theorem that $(\mu * \mathbb{1}_T)(x) \geq 1$. To see that $\mu(\mathbb{R}^n) = N^*(K,T)$, note that $\mu(\mathbb{R}^n) \leq \lim_{k\to\infty} \mu_k(\mathbb{R}^n) = N^*(K,T)$ (lower semi-continuity of a norm with respect to weak* convergence). Clearly, we also have that $\mu(\mathbb{R}^n) \geq N^*(K,T)$.

Since the covering condition $\mu * \mathbb{1}_T \geq \mathbb{1}_K$ is preserved under convex combinations, as is the total measure, it follows that the set of optimal covering measures of K by T is convex. ∎

Remark 2.7: One might be tempted to ask whether there exists a measure which is simultaneously optimal-separating and optimal-covering; this turns out to be, in general, not correct. Indeed, one may consider the following example. Let T be the cross polytope in $\mathbb{R}^3$, that is, $\mathrm{conv}(\pm e_1, \pm e_2, \pm e_3)$, and let $K = \mathrm{conv}(e_1, e_2, e_3)$ (where $\mathrm{conv}(A)$ denotes the convex hull of A). That is, K is a two-dimensional triangle in $\mathbb{R}^3$. Clearly, $N(K,T) = N^*(K,T) = 1$. However, if there existed a measure μ which was both optimal-separating and optimal-covering, then in particular it would have had to be supported in K, therefore we would get that the weighted covering of K by the central section of T with the plane $(1,1,1)^\perp$ is also 1. This section, which can also be written as $L = \mathrm{conv}((e_i - e_j)/2 : i,j = 1,2,3)$, is the hexagon $\frac{K-K}{2}$. We claim, however, that $N^*(K,L) > 1$. Indeed, the vertex e_1, for example, is covered by the copies of L centered at the triangle $\mathrm{conv}(e_1, \frac{e_1+e_2}{2}, \frac{e_1+e_3}{2}) = \Delta_1$ and similarly define Δ_2, Δ_3. By the assumption of covering, $\mu(\Delta_i) \geq 1$. On the other hand, if it were true that $\mu(K) = 1$ we would get, for example, that

$$\mu\Big(\frac{e_1+e_2}{2}\Big) = \mu(\Delta_1 \cap \Delta_2) = \mu(\Delta_1) + \mu(\Delta_2) - \mu(\Delta_1 \cup \Delta_2) \geq 2 - 1 = 1.$$

As this would also apply to $\frac{e_1+e_3}{2}, \frac{e_2+e_3}{2}$, it is a contradiction. Note that this argument actually shows that $N^*(K,L) = \frac{3}{2}$ and further that the only optimal weighted covering of K by L is given by the measure $\frac{1}{2}\delta_{\frac{e_1+e_3}{2}} + \frac{1}{2}\delta_{\frac{e_2+e_3}{2}} + \frac{1}{2}\delta_{\frac{e_1+e_2}{2}}$. Moreover, note that K and L satisfy that $N_\omega(K,L) \neq N(K,L)$, hence providing a simple example for the fact that classical and weighted covering numbers are not equal in general. By Proposition 2.1, K and L also provide a simple example for the fact that classical covering and separation numbers are not equal in general.

2.4. A Glivenko–Cantelli class. In this section our goal is somewhat technical. We wish to use a uniform measure to bound $N_\omega(K,T)$, however it is not a member of $\mathcal{D}^n_+$. We claim that if we find some uniform covering measure of a set K by a convex set T (supported on some compact Borel set) with total mass m, then $N_\omega(K,T) \leq m$. This is because uniform measures can be approximated well by discrete ones, and requires a proof. To this end, we need to recall the definition of a Glivenko–Cantelli class. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. $\mathbb{R}^n$-valued random vectors having common distribution P. The empirical measure P_k is formed by placing mass $1/k$ at each of the points $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_k(\omega)$. A class $\mathcal{A}$ of Borel subsets $A \in \mathcal{A}$ of $\mathbb{R}^n$ is said to be a Glivenko–Cantelli class for P if

$$\sup_{A \in \mathcal{A}} |P_n(A) - P(A)| \xrightarrow{a.s.} 0.$$

In the following lemma, we will invoke a Glivenko–Cantelli theorem for the class $\mathcal{C}_n$ of convex subsets of $\mathbb{R}^n$. Namely, in [10, Example 14] it is shown that if a probability distribution P satisfies that $P(\partial K) = 0$ for all $K \in \mathcal{C}_n$, then $\mathcal{C}_n$ is a Glivenko–Cantelli class for P.

Lemma 2.8: *Let $K \subseteq \mathbb{R}^n$ and let $T \subseteq \mathbb{R}^n$ be a convex set. Let μ be a uniform measure on some compact Borel set $A \subseteq \mathbb{R}^n$, that is $d\mu = c\mathbb{1}_A dx$ for some $c > 0$. Suppose that μ is a covering measure of K by T. Then*

$$N_\omega(K,T) \le \mu(\mathbb{R}^n).$$

Proof. Let $\varepsilon > 0$. We need to show that there exists a finite discrete measure ν such that

$$(\nu * \mathbb{1}_T)(x) \ge 1$$

and $\nu(\mathbb{R}^n) \le \frac{1}{1-\varepsilon}\mu(\mathbb{R}^n)$. To this end, let $\mu_0 = \frac{1}{c\mathrm{Vol}(A)}\mu$ be the uniform probability measure on A, let $\xi_1, \xi_2, \dots$ be a sequence of i.i.d. $\mathbb{R}^n$-valued random vectors having common distribution μ_0, and let μ_n be the corresponding empirical measure. The assumption that μ is a covering measure of K by T is equivalent to the condition that $\mu_0(x+T) \ge \frac{1}{c\mathrm{Vol}(A)}$ for all $x \in K$. Since $\mu(\partial L) = 0$ for all $L \in \mathcal{C}_n$, it is implied by [10, Example 14] that $\mathcal{C}_n$ is a Glivenko–Cantelli class for μ_0 and so, for some $k > 1$,

$$\sup_{L \in \mathcal{C}_n} |\mu_0(L) - \mu_k(L)| < \frac{\varepsilon}{c\mathrm{Vol}(A)}$$

almost surely. In particular, there exists a discrete measure (one of the μ_k's) $\nu_0 = \sum_{i=1}^k \frac{1}{k}\delta_{x_i}$ for which

$$(\nu_0 * \mathbb{1}_T)(x) = v_0(x+T) \ge \frac{1-\varepsilon}{c\mathrm{Vol}(A)}$$

for all $x \in K$. Thus the measure $\nu = \frac{c\mathrm{Vol}(A)}{1-\varepsilon}\nu_0$ is a covering measure of K by T with $\nu(\mathbb{R}^n) = \frac{1}{1-\varepsilon}\mu(\mathbb{R}^n)$, as required. ∎

2.5. Volume bounds. In this section we divide the proof Theorem 1.7 into the following two propositions.

Proposition 2.9: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$N^*(K,T) \le \frac{\mathrm{Vol}(K-T)}{\mathrm{Vol}(T)}.$$

Additionally, if T is convex then

$$N_\omega(K,T) \le \frac{\mathrm{Vol}(K-T)}{\mathrm{Vol}(T)}.$$

Proof. By Theorem 1.3, it suffices to prove that $M^*(K,T) \le \frac{\mathrm{Vol}(K+T)}{\mathrm{Vol}(T)}$. Let $\mu \in \mathcal{B}_+^n$ be a T-separated measure, that is, $\mathbb{1}_T * \mu \le 1$. Then

$$\begin{aligned}\int_K \mathrm{Vol}(T)d\mu(x) &= \int_K d\mu(x)\int_{K+T}\mathbb{1}_T(y-x)dy = \int_{K+T} dy \int_K \mathbb{1}_T(y-x)d\mu(x)\\ &\le \int_{K+T}(\mathbb{1}_T * \mu)(y)dy \le \int_{K+T} dy = \mathrm{Vol}(K+T)\end{aligned}$$

and so $\mathrm{Vol}(T)M^*(K,T) \le \mathrm{Vol}(K+T)$ as claimed.

Alternatively, one may verify that the measure $\mu_0 = \mathbb{1}_{K-T}\frac{dx}{\mathrm{Vol}(T)}$ is a covering measure of K by T, from which the claim also follows. By Lemma 2.8, the latter argument implies that

$$N_\omega(K,T) \le \frac{\mathrm{Vol}(K-T)}{\mathrm{Vol}(T)}. \qquad \blacksquare$$

Proposition 2.10: *Let $K \subseteq \mathbb{R}^n$ be compact and let $T \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$\max\left\{\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(T)}, 1\right\} \le M_\omega(K,T).$$

Proof. By Theorem 1.3, it suffices to prove that

$$\max\left\{\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(T)}, 1\right\} \le N^*(K,T).$$

Let $\mu \in \mathcal{B}_+^n$ be a covering measure of K by T, that is, $\mu * \mathbb{1}_T \ge \mathbb{1}_K$. Then

$$\begin{aligned}\int \mathrm{Vol}(T)d\mu(x) &= \int d\mu(x)\int_{\mathbb{R}^n}\mathbb{1}_T(y-x)dy = \int_{\mathbb{R}^n} dy \int \mathbb{1}_T(y-x)d\mu(x)\\ &\ge \int \mathbb{1}_K(y)dy = \mathrm{Vol}(K)\end{aligned}$$

and so $N^*(K,T) \ge \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(T)}$. Moreover, let $x \in K$. Then

$$1 \le (\mu * \mathbb{1}_T)(x) = \int_{\mathbb{R}^n}\mathbb{1}_T(x-y)d\mu(y) \le \int_{\mathbb{R}^n} d\mu$$

and so $N^*(K,T) \ge 1$.

Alternatively, one may verify that the measure $\mu_0 = \mathbb{1}_K \frac{dx}{\mathrm{Vol}(T)}$ is T-separated in K, from which the claim also follows. The fact that $1 \le N^*(K,T)$ also follows from

$$1 \le M(K,T) \le M^*(K,T) = N^*(K,T). \qquad \blacksquare$$

2.6. An equivalence between classical and weighted covering. In this section we prove Theorem 1.6.

Proof of Theorem 1.6. Fix $\delta > 0$. Let $(x_i, \omega_i)_{i\in I}$ be a finite weighted discrete covering of K by T_1 with

$$\sum_{i\in I} \omega_i < N_\omega(K,T_1) + \varepsilon.$$

Without loss of generality we may assume that ω_i are rational numbers and moreover, by allowing repetitions of the covering points, we may assume that for all i, $\omega_i = \frac{1}{M}$ for some arbitrarily large $M \in \mathbb{N}$. Denote $N = \lfloor N_\omega(K,T_1)\rfloor$ and let $0 < \varepsilon < 1$ be small enough so that $N+1 \le N_\omega(K,T_1)+\varepsilon$. Our aim is to generate a classical covering of K by $T_1 + T_2$ from the above fractional covering by a random process, with cardinality not larger than

$$\ln(4\overline{N}(K,T_2))N_\omega(K,T_1) + \sqrt{\ln(4\overline{N}(K,T_2))N_\omega(K,T_1)}.$$

To this end, let S be an integer to be determined later and let $L > 1$ be some real number also to be determined later. Each point will be chosen independently with probability $p = \frac{S}{M}$. We claim that with positive probability, for $S = \ln(4\overline{N}(K,T_2))$ and $L = 1 + \frac{1}{\sqrt{S(N+1)}}$, the generated set is a covering of K by $T_1 + T_2$ and at the same time the cardinality of the generated set is not greater than

$$LS(N+1) \le LS\Big(\sum \omega_i + 1\Big) \le LS(N_\omega(K,T_1) + \varepsilon + 1).$$

First, we bound the probability that more than $LS(N+1)$ will turn out positive. Let X_i denote the Bernoulli random variable corresponding to x_i and let X denote their sum. Note that there are at most $M(N+1)$ trials as

$$\sum_{i\in I} \frac{1}{M} < N_\omega(K,T_1) + \varepsilon \le N+1.$$

Denote the cardinality of I by $|I|$. A standard Chernoff bound tells us that this probability can be bounded as follows. For any $t > 0$

$$\begin{aligned}\mathbb{P}(X \geq LS(N+1)) =& \mathbb{P}(e^{Xt} \geq e^{LSNt}) \\ \leq& \min_{t>0} \frac{\mathbb{E}(e^{tX_1} \cdots e^{tX_{|I|}})}{e^{LS(N+1)t}} \\ \leq& \min_{t>0} \frac{[pe^t + (1-p)]^{M(N+1)}}{e^{LS(N+1)t}} \\ =& \left(\frac{1}{L}\right)^{LS(N+1)} \left[\frac{(1-p)}{1-Lp}\right]^{(N+1)M(1-Lp)} \\ =& \left(\frac{1}{L}\right)^{LS(N+1)} \left[1 + \frac{S(L-1)}{M-LS}\right]^{(N+1)(M-LS)} \\ \simeq& \left(\frac{e^{L-1}}{L^L}\right)^{S(N+1)},\end{aligned}$$

where at the third equality the minimum is attained at $e^t = L \cdot \frac{1-p}{1-Lp}$ and the last step holds for sufficiently large M compared with S. Set $L = 1 + \xi$; then for $0 < \xi \leq 1$, one can verify that

$$\mathbb{P}(X \geq LS(N+1)) \leq \left(\frac{e^{L-1}}{L^L}\right)^{S(N+1)} \leq e^{-S(N+1)\xi/3}.$$

Next, we show that with sufficiently high probability our generated set is a covering of K by $T_1 + T_2$. To this end, pick a minimal covering $\{y_i\} \subseteq K$ (we insist the points of the net belong to K) of K by T_2. The cardinality of such a minimal net is $\overline{N}(K, T_2)$. If every point y_i is covered by a translate $x_j + T_1$, then the whole of K is covered by the translates $x_j + T_1 + T_2$ of our randomly generated set, as we desire. Let us consider one specific point $y_i = y$ and check the probability that it is covered by our randomly generated set. Since we insisted that $y \in K$ we know that

$$\sum_{\{i \in I : y \in x_i + T_1\}} \frac{1}{M} \geq 1,$$

which means that at least M of the original translates $x_i + T$ include y. Therefore, the probability that y is not covered is less than or equal to $(1 - \frac{S}{M})^M \leq e^{-S}$. Thus, the probability that one or more of the T_2-covering points $\{y_i\}$ is not covered is bounded from above by $\overline{N}(K, T_2)e^{-S}$.

To summarize the above, we bounded the probability that either K is not covered or the generated set consists of more than $LS(N+1)$ points by

$$e^{-S(N+1)\xi/3}+\overline{N}(K,T_2)e^{-S}$$

and so it is left to choose S and ξ so that this bound is less than 1. As one can verify, the choices $\xi=\frac{1}{\sqrt{S(N+1)}}$ and $S=\ln(4\overline{N}(K,T_2))$ satisfy this requirement. Thus, $N(K,T_1+T_2)$ is bounded by

$$\begin{aligned}LS(N+1)=&\left(1+\frac{1}{\sqrt{S(N+1)}}\right)\ln(4\overline{N}(K,T_2))(N+1)\\ =&\left(1+\frac{1}{\sqrt{\ln(4\overline{N}(K,T_2))(N+1)}}\right)\ln(4\overline{N}(K,T_2))(N+1)\\ \le&\ln(4\overline{N}(K,T_2))(N_\omega(K,T_1)+1)\\ &+\sqrt{\ln(4\overline{N}(K,T_2))(N_\omega(K,T_1)+1)}.\end{aligned}$$ ∎

2.7. The metric-space setting. The notions of covering and separation make sense also in the metric space setting. Let(X,d) be a metric space (with the induced metric topology), and $K\subset X$ some compact subset. We shall denote the ε-covering number of K by

$$N(K,\varepsilon)=\min\left\{N\in\mathbb{N}:\exists x_1,\dots,x_N\in\mathbb{R}^n;\ K\subseteq\bigcup_{i=1}^N B(x_i,\varepsilon)\right\},$$

where $B(x,\varepsilon)=\{y\in X:d(x,y)\le\varepsilon\}$. Similarly

$$\overline{N}(K,\varepsilon)=\min\left\{N\in\mathbb{N}:\exists x_1,\dots,x_N\in K;\ K\subseteq\bigcup_{i=1}^N B(x_i,\varepsilon)\right\}.$$

The corresponding notion of the separation number is defined to be the maximal number of non-overlapping ε-balls centered in K:

$$M(K,\varepsilon)=\max\{M\in\mathbb{N}:\exists x_1,\dots,x_M\in K,\ B(x_i,\varepsilon)\cap B(x_j,\varepsilon)=\emptyset\ \forall i\ne j\}.$$

In this case it makes sense also to define

$$\overline{M}(K,\varepsilon)=\max\{M\in\mathbb{N}:\exists x_1,\dots,x_M\in K,\ B(x_i,\varepsilon)\cap B(x_j,\varepsilon)\cap K=\emptyset\ \forall i\ne j\},$$

and one should note that in the case $K=X$ these notions of course coincide. Also note that the metric setting is inherently centrally symmetric. However, since we no longer work in a linear space, some of the arguments in the preceding sections need to be altered.

Let us define weighted covering and separation in the metric setting, and list the relevant theorems corresponding to those proved in previous sections which hold in this setting. We shall remark only on the parts of the proofs which are not identical to those from the linear realm.

Definition 2.11: Let (X,d) be a metric space and $K \subset X$ compact. A sequence of pairs $S = \{(x_i, \omega_i) : x_i \in X, \omega_i \in \mathbb{R}^+\}_{i=1}^N$ with $N \in \mathbb{N}$ points and weights is said to be a **weighted** ε**-covering** of K if for all $x \in K$, $\sum_{\{i:x\in B(x_i.\varepsilon)\}} \omega_i \geq 1$. The total weight of the covering is denoted by $\omega(S) = \sum_{i=1}^N \omega_i$. The weighted ε-covering number of K is defined to be the infimal total weight over all weighted ε-coverings of K and is denoted by $N_\omega(K,\varepsilon)$.

Similarly, we may define (in a slightly different language)

$$\overline{N}_\omega(K,\varepsilon) = \inf\left\{ \int_X d\nu : \forall x \int \mathbb{1}_{B(y,\varepsilon)}(x) d\nu(y) \geq \mathbb{1}_K(x), \nu \in \mathcal{D}_+(X) \text{ with } \operatorname{supp}(\nu) \subseteq K \right\}$$

where $\mathcal{D}_+(X)$ denotes all non-negative finite discrete measures on X. Let $\mathcal{B}_+(X)$ denote all non-negative Borel measures on X. The weighted covering number with respect to general measures is defined by

$$N^*(K,\varepsilon) = \inf\left\{ \int_X d\mu : \forall x \int \mathbb{1}_{B(y,\varepsilon)}(x) d\mu(y) \geq \mathbb{1}_K(x), \mu \in \mathcal{B}_+(X) \right\}.$$

The weighted notions of the separation number are defined similarly; a measure ρ is said to be ε-separated if for all $x \in X$, $\int \mathbb{1}_{B(y,\varepsilon)}(x) d\rho(y) \leq 1$ and ε-separated in K if $\int \mathbb{1}_{B(y,\varepsilon)}(x) d\rho(y) \leq 1$ for all $x \in K$. The weighted separation numbers, corresponding to $N_\omega(K,\varepsilon)$, $\overline{N}_\omega(K,\varepsilon)$ and $N^*(K,\varepsilon)$, are respectively defined by

$$M_\omega(K,\varepsilon) = \sup\left\{ \int_K d\rho : \forall x \in X \int \mathbb{1}_{B(y,\varepsilon)}(x) d\rho(y) \leq 1,\ \rho \in \mathcal{D}_+(X) \right\},$$

$$\overline{M}_\omega(K,\varepsilon) = \sup\left\{ \int_K d\rho : \forall x \in K \int \mathbb{1}_{B(y,\varepsilon)}(x) d\rho(y) \leq 1,\ \rho \in \mathcal{D}_+(X) \right\}$$

and

$$M^*(K,\varepsilon) = \sup\left\{ \int_K d\rho : \forall x \in X \int \mathbb{1}_{B(y,\varepsilon)}(x) d\rho(y) \leq 1,\ \rho \in \mathcal{B}_+(X) \right\}.$$

Our first result is a weak duality between weighted covering and separation numbers:

Theorem 2.12: *Let (X,d) be a metric space, $K \subseteq X$ compact and let $\varepsilon > 0$. Then*

$$M_\omega(K,\varepsilon) \le M^*(K,\varepsilon) \le N^*(K,\varepsilon) \le N_\omega(K,\varepsilon).$$

Proof. The first and last inequalities follow by definition, and so we should only prove the center inequality. To this end let μ be a weighted ε-covering measure of K and let ρ be a weighted ε-separated measure. By our assumptions we have that $\int \mathbb{1}_{B(y,\varepsilon)}(x)d\rho(y) \le 1$ and $\int \mathbb{1}_{B(y,\varepsilon)}(x)d\mu(y) \ge \mathbb{1}_K(x)$ for all $x \in X$. Thus

$$\begin{aligned}\int_K d\rho(x) = \int \mathbb{1}_K(x)\cdot d\rho(x) &\le \int\int \mathbb{1}_{B(y,\varepsilon)}(x)d\mu(y)d\rho(x)\\ &= \int d\rho(x)\int d\mu(y)\mathbb{1}_{B(y,\varepsilon)}(x)\\ &= \int d\mu(y)\int d\rho(x)\mathbb{1}_{B(x,\varepsilon)}(y)\\ &\le \int d\mu(y)\end{aligned}$$

and so $M^*(K,\varepsilon) \le N^*(K,\varepsilon)$. Similarly, one may show that

$$\overline{M}_\omega(K,\varepsilon) \le \overline{N}_\omega(K,\varepsilon). \qquad \blacksquare$$

As a corollary of Theorem 2.12, we immediately get the following equivalence relation between the classical and weighted covering numbers:

Corollary 2.13: *Let (X,d) be a metric space, $K \subseteq X$ compact and let $\varepsilon > 0$. Then*

$$N(K,2\varepsilon) \le N_\omega(K,\varepsilon) \le N(K,\varepsilon).$$

Proof. By Theorem 2.12, $M(K,\varepsilon) \le M_\omega(K,\varepsilon) \le N_\omega(K,\varepsilon) \le N(K,\varepsilon)$ and so we only need to verify the inequality

$$N(K,2\varepsilon) \le M(K,\varepsilon).$$

Indeed, let $(x_i)_{i=1}^N \subseteq K$ be ε-separated. Hence, for every $x \in K$ there exists some $i \in 1,\dots,N$ such that $B(x,\varepsilon) \cap B(x_i,\varepsilon) \ne \emptyset$ which by the triangle inequality means that $x \in B(x_i,2\varepsilon)$. Thus, $(x_i)_{i=1}^N$ is a 2ε-covering of K and so $N(K,2\varepsilon) \le M(K,\varepsilon)$, as needed. $\blacksquare$

3. The Levi–Hadwiger problem

In this section we prove Theorem 1.10 and Corollary 1.11. To this end, we shall need some preliminary results, and before that, some notation.

3.1. Preliminary results.

3.1.1. *A homothetic intersection.* Denote the segment between two vectors $x, y \in \mathbb{R}^n$ by $[x,y] = \{\lambda x + (1-\lambda)y : 0 \le \lambda \le 1\}$. Let ∂A denote the boundary of a set $A \subseteq \mathbb{R}^n$. We will need the following lemma, the proof of which was kindly shown to us by Rolf Schneider and is reproduced here.

Lemma 3.1 (Schneider): *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body. Let $a \in K$ and let p be the intersection point of ∂K with the ray emanating from 0 and passing through a. Assume that $(a+K)\cap K$ is homothetic to K. Then there exists a closed convex cone $C \subseteq \mathbb{R}^n$ (with vertex $\{0\}$) such that $K = (p-C)\cap(C-p)$.*

Proof. Denote the homothety h defined by $hK = K \cap (K+a)$. Since K is centrally symmetric, it follows that hK is symmetric about $\frac{a}{2}$, and since hK is homothetic to K it follows that $hK = \frac{a}{2} + \alpha K$, where $\alpha = \frac{p-a/2}{p}$. Thus $hK = \alpha(K-p)+p$, which means that $p = hp$ is the center of homothety of $h(x) = \alpha(x-p)+p$.

Define the cone

$$C_o = \{\lambda(p-y) : y \in \operatorname{int} K, \lambda \ge 0\}$$

and denote its closure by C. Let us prove that $(p-C_o)\cap(C_o-p) \subseteq K$; assume towards a contradiction that there exists $x \in (p-C_o)\cap(C_o-p)$ such that $x \notin K$. Let $y, z \in K$ be the points for which

$$[p,y] = K \cap [p,x], \quad [-p,z] = K \cap [-p,x]$$

and consider the quadrangle T in K with vertices $\pm p, y, z$. Since p is the center of homothety of h, the point $y' = \alpha(y-p)+p \in (hK)\cap[p,x]$ belongs to the boundary of hK. However, since the point $w = \alpha(y+p)-p$ is in the interior of T, it follows that, for some $\varepsilon > 0$, both $w+u \in T$ and $y'+u \in T$, where $u = \varepsilon\cdot(y-p)$ (see Figure 3.1). Since $y' = w+a$, it follows that $y'+u = (w+u)+a \in T+a$, and hence $y'+u \in K\cap(K+a)$, a contradiction to the fact that $[p,y'] = (hK)\cap[p,x]$.

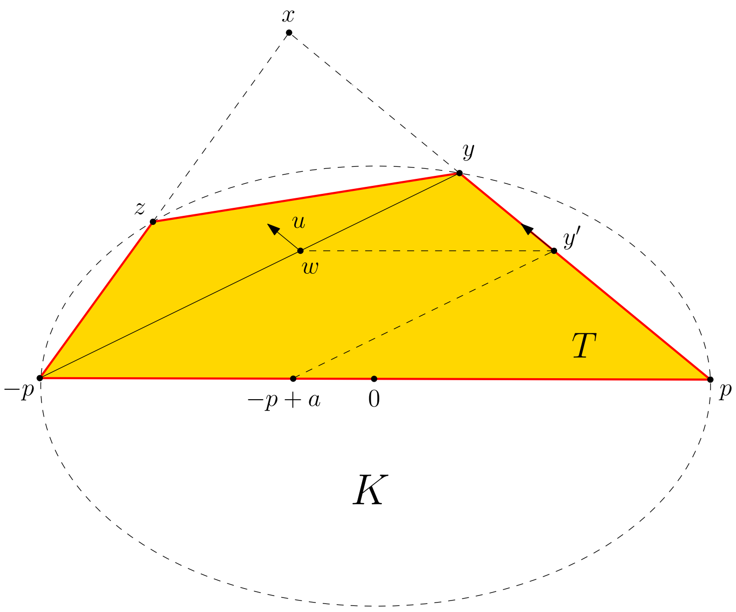

Figure 3.1. the vector $w+u$ belongs to K and so $y'+u \in K \cap (K+a)$.

We have proved that $(p-C_o)\cap(C_o-p) \subseteq K$ and hence $(p-C)\cap(C-p) \subseteq K$. The inclusion $K \subseteq (p-C)\cap(C-p)$ trivially holds, and thus

$$K = (p-C)\cap(C-p). \qquad \blacksquare$$

We remark that if K is not centrally symmetric, one may slightly adjust Lemma 3.1 and its proof in order to conclude the following lemma.

Lemma 3.2: *Let $K \subseteq \mathbb{R}^n$ be a convex body containing the origin in its interior. Let $a \in K$ and assume that the intersection point of ∂K with the ray emanating from 0 and passing through a is an exposed point of K, denoted by p. Let q denote the point in ∂K for which $0 \in (q,p)$. Assume that $(a+K)\cap K$ is homothetic to K. Then there exist closed convex cones $C_1, C_2 \subseteq \mathbb{R}^n$ (both with vertex $\{0\}$) such that $K = (p+C_1)\cap(q+C_2)$.*

The main difference between the proof of Lemma 3.1 and the proof of Lemma 3.2 is that, in the latter, in order to prove that p is the center of homothety of h, we need to use the assumption that p is an exposed point of K. This is done by using the exact same argument as in the equality case of Rogers–Shepard inequality in [21]. We shall not have use of Lemma 3.2 in this note, and we omit the proof's details.

3.1.2. *Covering a convex body by its interior.* It will be convenient to work with the weighted covering number of a set K by its interior $\mathrm{int}(K)$: $N_\omega(K, \mathrm{int}(K))$. The definition of this number is literally the same as for compact sets:

$$N_\omega(K, \mathrm{int}(K)) = \inf\{\nu(\mathbb{R}^n) : \nu * \mathbb{1}_{\mathrm{int}(K)} \geq \mathbb{1}_K, \nu \in \mathcal{D}^n_+\}.$$

We claim that covering a compact set, fractionally, by its interior is the limit of fractionally covering it by infinitesimally smaller homothetic copies of itself. More precisely, we prove the following.

Lemma 3.3: *Let $K \subseteq \mathbb{R}^n$ be compact with non-empty interior. Then*

$$N_\omega(K, \mathrm{int}(K)) = \lim_{\lambda \to 1^-} N_\omega(K, \lambda K).$$

Proof. Assume without loss of generality that $0 \in \mathrm{int}(K)$. The inequality

$$N_\omega(K, \mathrm{int}(K)) \leq \lim_{\lambda \to 1^-} N_\omega(K, \lambda K)$$

is straightforward by definition. For the opposite direction, let $\mu = \sum_{i=1}^N \alpha_i \delta_{x_i}$ be a covering measure of K by $\mathrm{int}(K)$, i.e., $\mu * \mathbb{1}_{\mathrm{int}(K)} \geq \mathbb{1}_K$, and denote the Euclidean open ball of radius $r > 0$ and centered at x by $B(x, r) \subseteq \mathbb{R}^n$. Note that if $x \in K$, $x \in \bigcap_{i\in A}(x_i + \mathrm{int}(K))$ for some set of indices A, then

$$B(x, r) \subseteq \bigcap_{i\in A}(x_i + \mathrm{int}(K))$$

for some open ball $B(x, r)$. Since $1 \leq (\mu * \mathbb{1}_{\mathrm{int}(K)})(x)$, it follows that for all $y \in B(x, r)$ we also have

$$1 \leq (\mu * \mathbb{1}_{\mathrm{int}(K)})(y) = \sum_{i=1}^N \alpha_i \mathbb{1}_{x_i+\mathrm{int}(K)}(y).$$

Hence, as K is compact, there exists $\delta > 0$ such that for all $x \in K$,

$$\begin{aligned} 1 &\leq \sum_{i=1}^N \alpha_i \mathbb{1}_{x_i+\mathrm{int}(K)}((1+\delta)x) \\ &= \sum_{i=1}^N \alpha_i \mathbb{1}_{\frac{1}{1+\delta}\mathrm{int}(K)}\left(x - \frac{x_i}{1+\delta}\right) = (\nu * \mathbb{1}_{\frac{1}{1+\delta}\mathrm{int}(K)})(x) \end{aligned}$$

where $\nu = \sum_{i=1}^N \alpha_i \delta_{\frac{x_i}{1+\delta}}$. Therefore, $1 \le \mu * \mathbb{1}_{\frac{1}{1+\delta}\mathrm{int}(K)} \le \mu * \mathbb{1}_{\lambda_0 K}$ for some $0 < \lambda_0 < 1$, and so

$$\lim_{\lambda \to 1^-} N_\omega(K, \lambda K) \le N_\omega(K, \mathrm{int}(K)),$$

from which the desired equality is implied. ∎

3.1.3. *Antipodal sets.* In this section we recall a beautiful result by Danzer and Grünbaum, which we will need to invoke later on. To state their result, recall that given a convex body $K \subseteq \mathbb{R}^n$, a set of points $A \subseteq K$ is said to be an antipodal set in K if for each distinct pair of points in A there is a pair of distinct parallel supporting hyperplanes of K, each containing one of the two points.

Danzer and Grünbaum [9] proved the following theorem.

Theorem 3.4 (Danzer and Grünbaum): *The maximal cardinality of an antipodal set in a convex body $K \subseteq \mathbb{R}^n$ is bounded from above by 2^n. Moreover, equality holds if and only if K is a parallelotope.*

3.2. Completing the proofs. We now prove the weighted version of the Levi–Hadwiger problem.

Proof of Theorem 1.10. Suppose first that K is not centrally symmetric. Then the volume inequality in Proposition 2.9 immediately implies that

$$\lim_{\lambda \to 1^-} N_\omega(K, \lambda K) \le \lim_{\lambda \to 1^-} \frac{\mathrm{Vol}(K - \lambda K)}{\mathrm{Vol}(\lambda K)} = \binom{2n}{n},$$

as required. Of course, in the symmetric case the same argument gives the bound 2^n. But we proceed differently so as to be able to analyze the equality case.

Suppose that K is centrally symmetric. Without loss of generality, we assume that K has non-empty interior and that an open ball $B(0, r)$ of radius $r > 0$ is contained in K. By Lemma 3.3, we may work with the weighted covering number of K by its interior $N_\omega(K, \mathrm{int}(K))$, and by Lemma 2.8 we may also consider uniform covering measures to bound $N_\omega(K, \mathrm{int}(K))$ from above. Indeed, consider the uniform measure μ on K with density $\frac{2^n}{\mathrm{Vol}(K)}$, that is

$$d\mu(y) = 2^n \frac{\mathbb{1}_K(y)}{\mathrm{Vol}(K)} dy. \tag{3.1}$$

Let us verify that μ is a covering measure of K by $\mathrm{int}(K)$. Indeed, let $x \in K$. Then

$$(\mu * \mathbb{1}_{\mathrm{int}(K)})(x) = \frac{2^n}{\mathrm{Vol}(K)} \int \mathbb{1}_{\mathrm{int}(K)}(y)\mathbb{1}_K(x-y)dy = 2^n \frac{\mathrm{Vol}(K \cap (x+K))}{\mathrm{Vol}(K)}.$$

Since

$$K \cap (x+K) \supseteq \frac{K}{2} + \frac{1}{2}[K \cap (2x+K)] \supseteq \frac{K+x}{2}, \tag{3.2}$$

it follows that

$$2^n \frac{\mathrm{Vol}(K \cap (x+K))}{\mathrm{Vol}(K)} \geq 2^n \frac{\mathrm{Vol}(K/2)}{\mathrm{Vol}(K)} = 1, \tag{3.3}$$

as required. This means that $N_\omega(K, \mathrm{int}(K)) \leq \mu(\mathbb{R}^n) = 2^n$. To address the equality case, assume that for some centrally symmetric convex body K we have $N_\omega(K, \mathrm{int}(K)) = 2^n$. In particular, for no $0 < c < 1$ is $c\mu$ (for μ given in (3.1)) a covering measure of K by $\mathrm{int}(K)$. Therefore, the inequality in (3.3) must be an equality for some $x \in K$. Indeed, if not, a standard compactness argument shows that there exists $c \in (0,1)$ such that for all $x \in K$,

$$c2^n \frac{\mathrm{Vol}(K \cap (x+K))}{\mathrm{Vol}(K)} \geq 1,$$

which means that $c\mu$ is a covering measure of K by $\mathrm{int}(K)$, a contradiction to the assumption $N_\omega(K, \mathrm{int}(K)) = 2^n$.

Next, note that the inequality (3.3) is strict if and only if at least one of the inclusions in (3.2) is strict and, moreover, the rightmost inclusion in (3.2) is strict as long as $x \in K$ is not an extremal point of K. Thus, the preceding two arguments imply that K has at least one extremal point $x_0 \in K$ for which $(x_0 + K) \cap K = \frac{K}{2} + \frac{1}{2}[K \cap (2x_0 + K)] = \frac{K+x_0}{2}$.

Our aim for the remaining part of the proof is to show that K actually has at least 2^n extremal points $x_1, \ldots, x_{2^n} \in K$ such that $(x_i + K) \cap K = \frac{K+x_i}{2}$ for all $i = 1, \ldots, 2^n$, and use the characterization given in Lemma 3.1 for K in order to deduce that $A = \{x_1, \ldots, x_{2^n}\}$ is an antipodal set of K. Finally, we shall invoke Theorem 3.4 to conclude that K is a parallelotope.

Assume that there exists exactly k extremal points of K $x_1, \ldots, x_k \in K$ such that

$$(x_i + K) \cap K = \frac{K}{2} + \frac{1}{2}[K \cap (2x_i + K)] = \frac{K+x_i}{2}$$

for all $i = 1, \dots, k$. Then, by using the same compactness argument as before, it follows that there exists $0<c<1$ such that for all $x \in K \setminus \{B(x_1, r), \dots, B(x_k, r)\}$,

$$((c\mu) * \mathbb{1}_{\mathrm{int}(K)})(x) = c\mu(x + \mathrm{int}(K)) \geq 1.$$

Since $B(0, r) \subseteq \mathrm{int}(K)$, we have that $B(x_i, r) \subseteq x_i + \mathrm{int}(K)$, and so it follows that the measure $\nu = c\mu + (1-c)\sum_{i=1}^k \delta_{x_i}$ is a covering measure of K by $\mathrm{int}(K)$. Therefore, the equality assumption $N_\omega(K, \mathrm{int}(K)) = 2^n$ implies that $\nu(\mathbb{R}^n) = c2^n + (1-c)k \geq 2^n$ which implies that $k \geq 2^n$. Concluding the above, there exist at least 2^n extremal points $A = \{x_1, \dots, x_{2^n}\}$ in K such that $K \cap (x_i + K) = \frac{K + x_i}{2}$ for all $i \in A$. By Lemma 3.1, for each $i \in A$ there exists a closed convex cone C_i such that $K = (x_i - C_i) \cap (C_i - x_i)$.

Let us next prove that if $x_j \neq x_i$, then x_j belongs to the boundary of $C_i - x_i$. Indeed, if x_j belonged to the interior of $C_i - x_i$, then it would have to belong to the boundary of $x_i - C_i$ as it belongs to ∂K. However, since $x_j \neq x_i$, there exists a segment $(a, b) \subseteq x_i - C_i$ on the ray emanating from x_i and passing through x_j which contains x_j. Together with the assumption that x_j belongs to the interior of $C_i - x_i$, it follows that there exists a segment $(a', b') \subseteq (a, b)$ which both contains x_j and is contained in $K = (x_i - C) \cap (C_i - x_i)$, a contradiction to the fact that x_j is an extremal point of K.

It remains to show that A is an antipodal set of K. Indeed, since x_j belongs to the boundary of $C_i - x_i$, the segment $[-x_i, x_j]$ is contained in the boundary of $C_i - x_i$ and so there exists a supporting hyperplane H of $C_i - x_i$ which contains both $-x_i$ and x_j. In particular, H supports K. In other words, there exists a vector $v \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in C_i - x_i$,

$$\langle x, v\rangle \leq \langle x_j, v\rangle = \langle -x_i, , v\rangle.$$

Hence, for all $x \in x_i - C_i$,

$$\langle x, v\rangle \geq \langle -x_j, v\rangle = \langle x_i, , v\rangle,$$

which means that

$$\begin{aligned} H' =& H + (x_i - x_j) = \{x + x_i - x_j \in \mathbb{R}^n : \langle x, v\rangle = \langle x_j, v\rangle\} \\ =& \{y \in \mathbb{R}^n : \langle y, v\rangle = \langle x_i, v\rangle\} \end{aligned}$$

contains x_i, supports $x_i - C_i$, and in particular supports K. Thus, we conclude that A is an antipodal set of K. By Theorem 3.4, the maximal cardinality of an antipodal set of a convex body is 2^n and equality holds only for parallelotopes, and thus K is a parallelotope. ∎

Proof of Corollary 1.11. Fix $0 < \delta < 1$ and let $n \geq 3$. By Theorem 1.6, for any $0 < \lambda < 1$

$$\begin{aligned} N(K, \lambda K) \leq & \ln(4\overline{N}(K, \delta\lambda K))(N_\omega(K, (1-\delta)\lambda K) + 1) \\ & + \sqrt{\ln(4\overline{N}(K, \delta\lambda K))(N_\omega(K, (1-\delta)\lambda K) + 1)} \\ \leq & \ln(4\overline{N}(K, \delta\lambda K))N_\omega(K, (1-\delta)\lambda K) \\ & + \sqrt{\ln(4\overline{N}(K, \delta\lambda K))N_\omega(K, (1-\delta)\lambda K)} + 2\ln(4\overline{N}(K, \delta\lambda K)). \end{aligned}$$

By Theorem 1.7, we have that

$$N_\omega(K, (1-\delta)\lambda K) \leq \frac{\mathrm{Vol}(K + (1-\delta)\lambda K)}{\mathrm{Vol}((1-\delta)\lambda K)} = \left(1 + \frac{1}{(1-\delta)\lambda}\right)^n.$$

By classical volume bounds we have that

$$\overline{N}(K, \delta\lambda K) \leq M\left(K, \frac{\delta}{2}\lambda K\right) \leq \frac{\mathrm{Vol}(K + \frac{\delta}{2}\lambda K)}{\mathrm{Vol}(\frac{\delta}{2}\lambda K)} = \left(1 + \frac{2}{\lambda\delta}\right)^n$$

and so

$$\begin{aligned} N(K, \lambda K) \leq & \left(1 + \frac{1}{(1-\delta)\lambda}\right)^n \left[n \ln(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\lambda\delta})\right] \\ & + \sqrt{(1 + \frac{1}{\left(1-\delta)\lambda\right.}\Big)^n \left[n \ln\left(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\lambda\delta}\right)\right]} \\ & + 2n \ln\left(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\lambda\delta}\right). \end{aligned}$$

Taking the limit $\lambda \to 1^-$ implies that

$$\begin{aligned} \lim_{\lambda \to 1^-} N(K, \lambda K) \leq & \left(1 + \frac{1}{(1-\delta)}\right)^n \left[n \ln\left(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\delta}\right)\right] \\ & + \sqrt{\left(1 + \frac{1}{(1-\delta)}\right)^n [n \ln\left(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\delta}\right)\Big]} \\ & + 2n \ln\left(4^{1/n} + \frac{2 \cdot 4^{1/n}}{\delta}\right). \end{aligned}$$

By plugging $\delta = \frac{1}{n\ln(n)}$ we get

$$\begin{aligned}\lim_{\lambda\to 1^-} N(K,\lambda K) \leq & \left(2+\frac{1}{n\ln n-1}\right)^n [n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n)] \\ & + \sqrt{\left(2+\frac{1}{n\ln n-1}\right)^n [n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n)]} \\ & + 2n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n).\end{aligned}$$

Since, for all $n \geq 3$,

$$\left(2+\frac{1}{n\ln n-1}\right)^n \leq 2^n e^{1/(2\ln n-2/n)} \leq 2^n\left(1+\frac{1}{\ln n-1/n}\right) \leq 2^n\left(1+\frac{2}{\ln n}\right)$$

and

$$n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n) \leq n\ln(4n\ln n) = n\ln 4 + n\ln n + n\ln\ln n,$$

it follows that

$$\begin{aligned}&\left(2+\frac{1}{n\ln n-1}\right)^n n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n) \\ &\qquad \leq 2^n\left(1+\frac{2}{n\ln n}\right)(n\ln n + n\ln\ln n + n\ln 4) \\ &\qquad \leq 2^n(n\ln n + n\ln\ln n + 3.1n).\end{aligned}$$

Moreover, one may also check that

$$\sqrt{\left(2+\frac{1}{n\ln n-1}\right)^n n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n)} \leq 2^n 0.5n$$

and that

$$2n\ln(4^{1/n}+2\cdot 4^{1/n} n\ln n) \leq 2^n 0.7n.$$

Thus, it follows that

$$\lim_{\lambda\to 1^-} N(K,\lambda K) \leq 2^n(n\ln n + n\ln\ln n + 5n). \qquad ■$$

References

[1] S. Artstein-Avidan, B. Klartag and V. Milman, *The Santaló point of a function, and a functional form of the Santaló inequality*, Mathematika **51** (2004), 33–48.

[2] S. Artstein-Avidan and O. Raz, *Weighted covering numbers of convex sets*, Advances in Mathematics **227** (2011), 730–744.

[3] K. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbf{R}^n$, Studia Mathematica **88** (1988), 69–84.

[4] F. Barthe, *On a reverse form of the Brascamp–Lieb inequality*, Inventiones Mathematicae **134** (1998), 335–361.

[5] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, Vol. 54, American Mathematical Society, Providence, RI, 2002.

[6] K. Bezdek, *The illumination conjecture and its extensions*, Periodica Mathematica Hungarica **53** (2006), 59–69.

[7] V. G. Boltjansky and I. Gohberg, *Results and Problems in Combinatorial Geometry*, Cambridge University Press, Cambridge, 1985.

[8] P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.

[9] L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee*, Mathematische Zeitschrift **79** (1962), 95–99.

[10] J. Elker, D. Pollard and W. Stute, *Glivenko–Cantelli theorems for classes of convex sets*, Advances in Applied Probability **11** (1979), 820–833.

[11] P. Erdös and C. A. Rogers, *The covering of n-dimensional space by spheres*, Journal of the London Mathematical Society **28** (1953), 287–293.

[12] I. Gohberg and A. Markus, *A problem on covering of convex figures by similar figures*, (in Russian), Izv. Mold. Fil. Akad. Nauk SSSR **10** (1960), no. 76, 87–90.

[13] H. Hadwiger, *Ungelöstes Probleme Nr. 20*, Elemente der Mathematik **12** (1957), 121.

[14] B. Klartag, *Marginals of geometric inequalities*, in *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics, Vol. 1910, Springer, Berlin–Heidelberg, 2007, pp. 133–166.

[15] B. Klartag and V. D. Milman, *Geometry of log-concave functions and measures*, Geometriae Dedicata **112** (2005), 169–182.

[16] F. W. Levi, *Überdeckung eines Eibereiches durch Parallelverschiebung seines offenen Kerns*, Archiv der Mathematik **6** (1955), 369–370.

[17] L. Lovász, *On the ratio of optimal integral and fractional covers*, Discrete Mathematics **13** (1975), 383–390.

[18] H. Martini and V. Soltan, *Combinatorial problems on the illumination of convex bodies*, Aequationes Mathematicae **57** (1999), 121–152.

[19] M. Naszódi, *Fractional illumination of convex bodies*, Contributions to Discrete Mathematics **4** (2009), 83–88.

[20] C. A. Rogers and C. Zong, *Covering convex bodies by translates of convex bodies*, Mathematika **44** (1997), 215–218.

[21] R. Schneider, *Convex Bodies: the Brunn–Minkowski Theory*, Encyclopedia of Mathematics and its Applications, Vol. 44, Cambridge University Press, Cambridge, 1993.