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ALGEBRAS HAVING BASES THAT CONSIST SOLELY OF UNITS

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ABSTRACT

Algebras having bases that consist entirely of units (called invertible algebras) are studied. Among other results, it is shown that all finite-dimensional algebras over a field other than the binary field $\mathbb{F}_2$ have this property. Invertible finite-dimensional algebras over $\mathbb{F}_2$ are fully characterized. Examples of invertible algebras are shown to include all (non-trivial) matrix rings over arbitrary algebras. In addition, various families of algebras, including group rings and crossed products, are characterized in terms of invertibility. Invertibility of infinite-dimensional algebras is also explored.

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1. Introduction and preliminaries

This paper is an introduction to the study of algebras (over not-necessarily commutative rings) that satisfy the condition of the title. Some years ago, as we started the study of this topic, we published the preliminary report [4]; the results presented here extend and complete those earlier results. While our paper is written in a mostly self-contained manner so that reading [4] is not a prerequisite, we make every effort, for historical accuracy, to point out the connections between the current results and those in the earlier paper whenever doing so seems appropriate.

In this paper, when we use the expression A **is an R-algebra** we deviate from the standard use of that terminology in two ways: one which narrows the net that we cast and another one that widens it. First, we do not allow a proper homomorphic image of the ring R to be contained in A; according to the definition we use in this paper, R itself is contained in A. The second difference is that R is not necessarily assumed to be contained in the center of A; in fact, we do not even assume that the ring R is commutative. While not a part of our definition of an algebra, a feature that for obvious reasons will be common to all algebras considered here is that they will be free as (left) R-modules. This expectation seems compatible with not allowing a proper homomorphic image of R (rather than R itself) to be embeddable in A. So, the setting is the following: we have a ring A that has a subring R such that A is a free left R-module. We assume commutativity of neither A nor R.

Readers should rest assured that we are not taking these liberties with terminology lightly. There are two main reasons for using the familiar terminology in our slightly different setting. First, we could not pass on the opportunity of pointing out that many natural examples which are not algebras in the traditional sense experience the phenomenon in which we are interested. Second, it will eventually be the case that for convenience we will focus only on algebras in the traditional sense of the expression. The reason for this is that the generality gained by relaxing our hypotheses does not come without a price and therefore, on several occasions, we must settle with focusing only on classical algebras for our results. So, for the most part, readers will do fine if they choose to ignore the fact that our original aim is a little more ambitious and pretend the paper is only about algebras in the classical sense if that makes them more comfortable.

For practical purposes, what we have done is simply to relax the usual definition of an algebra by removing the condition that R be commutative and by asking simply that when $r \in R$ acts on $a, b \in A$, then $r(ab) = (ra)b$ but not necessarily $r(ab) = a(rb)$. Technically speaking, every time we refer to bases, we should be saying left bases. In order to avoid unnecessary pedantry, we will refrain from the latter.

An important observation is that it is often the case that the set of inverses of a linearly independent set of units is not linearly independent. For example, consider $\frac{\mathbb{F}_2[x]}{\langle x^3+x+1\rangle}$. The set $\{1, x^2+1, x^2+x\}$ is linearly independent, but its set of inverses $\{1, x, x+1\}$ is not. Questions related to this notion are discussed in [3]. In the next paragraph we introduce a hierarchy of notions regarding the existence of bases that consist of units. One of those notions, that of invertible-2 algebras, pertains to the comments in this paragraph as it addresses an instance when the set of inverses of certain linearly independent families of units remain linearly independent.

The following concepts were introduced in [4]. Let A be an algebra over a ring R and let $\mathcal{B}$ be a basis for A over R; $\mathcal{B}$ is an **invertible** basis if each element of $\mathcal{B}$ is invertible in A. If $\mathcal{B}$ is an invertible basis such that $\mathcal{B}^{-1}$, the set of the inverses of the elements of $\mathcal{B}$, also constitutes a basis, then $\mathcal{B}$ is an **invertible-2** (I2) basis. An algebra with an invertible basis is an **invertible algebra** and an algebra with an I2 basis is an I2 **algebra**. Some similar notions in the literature include **k-good rings** (cf. [8]) and **S-rings** (cf. [7]). A ring is said to be k-good when every element is a sum of exactly k units. A ring is said to be an S-ring if every element is a sum of units. Of course when R is a division ring, A is invertible if and only if it is an S-ring. In general, though, there do not seem to be any other connections between invertible R-algebras and R algebras that are S-rings. There also do not seem to be any formal connections between invertible algebras and k-good rings.

Some of the results obtained in this paper and their relations to those presented in [4] are outlined in the remaining part of this introduction.

A result obtained in [4] was that, under certain commutativity requirements (the elements of R had to commute with the elements of the invertible basis $\mathcal{B}$), an invertible algebra A was indeed a group ring if the invertible basis $\mathcal{B}$ is closed under products. In Section 2, we extend this result and provide similar characterizations of skew group rings, twisted group rings and crossed products.

In [4], it is shown that over any ring R, the matrix ring $A = M_n(R)$ (for any $n \geq 1$) is an I2 R-algebra. Considering that R need not be commutative, this surprising result is perhaps the biggest motivation for the relaxation of the definition of algebra that we adopted in this project. In Section 3, we extend this result by showing that if B is any algebra over a ring R having an R-basis that includes the identity of B, then for any $n \geq 2$, $A = M_n(B)$ is an I2 algebra over R. Notice in particular that this holds even if B itself is not an invertible R-algebra.

An initial study in [4] of finite-dimensional algebras over a field F led to the realization that most of those algebras considered were indeed invertible. The exceptions were always in the case of $F = \mathbb{F}_2$ and related to having a quotient algebra isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2$, which is obviously not an invertible $\mathbb{F}_2$-algebra. Somehow, we did not dare to conjecture that this was always the case until that possibility was later suggested to us by Miodrag Iovanov in conversation. We have now proven that indeed almost all finite-dimensional algebras over a division ring D are I2 algebras. The exceptions are only $\mathbb{F}_2$ algebras which have a quotient isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2$. In addition, we have found characterizations of invertible semilocal D-algebras, and have shown that all local R-algebras are invertible for any ring R. These results are in Section 4.

Finally, because of the results of Section 4, it makes sense to shift our attention to infinite-dimensional algebras, and in light of the successes of Section 3, it seems reasonable to focus on infinite matrix rings of various types. We do so in Section 5.

2. A Classification of invertible algebras

In this section we present a hierarchy of invertible algebras in terms of additional properties of their invertible bases.

Definition 2.1: Let A be an algebra over a ring R and let $\mathcal{B}$ be an invertible basis for A over R.

(1) If for every $v \in \mathcal{B}$ there exists $\alpha \in U(R)$ such that $\alpha v^{-1} \in \mathcal{B}$, then $\mathcal{B}$ is **scalarly closed under inverses**(**SCUI**). An algebra with a SCUI basis is a **SCUI algebra**. If $\mathcal{B}$ is a SCUI basis such that for all $v \in \mathcal{B}$, $\alpha = 1$, then $\mathcal{B}$ is simply **closed under inverses**(**CUI**). An algebra with a CUI basis is an **CUI algebra**.

(2) If for all $v, w \in \mathcal{B}$ there exists $\alpha \in U(R)$ such that $\alpha vw \in \mathcal{B}$, then $\mathcal{B}$ is **scalarly closed under products**(**SCUP**). An algebra with a SCUP basis is a **SCUP algebra**. If $\mathcal{B}$ is a SCUP basis such that for all $v, w \in \mathcal{B}$, $\alpha = 1$, then $\mathcal{B}$ is simply **closed under products**(**CUP**). An algebra with a CUP basis is a **CUP algebra**.

For algebras and bases with the aforementioned properties, if 1 belongs to the basis, we indicate this by attaching a 1 to the shorthand version of their names. For example, a CUI1 basis $\mathcal{B}$ is a CUI basis with $1 \in \mathcal{B}$, and an algebra having a CUI1 basis is a CUI1 algebra.

CUI and CUI1 bases and algebras were respectively called **invertible-3** and **invertible-4** in [4].

Example 2.2: Consider the field extension of $\mathbb{C}$ over $\mathbb{R}$; $\mathcal{B} = \{1, i\}$ is an invertible basis for this field extension. Notice that $\mathcal{B} = \{1, i\}$ is both a SCUP and a SCUI basis.

PROPOSITION 2.3: *Let A be an algebra over a ring R with SCUP basis $\mathcal{B}$. Then $|\mathcal{B} \cap U(R)| = 1$, and A has a SCUP1 basis. If $\mathcal{B}$ is a CUP basis, then $1 \in \mathcal{B}$.*

Proof. Let $v \in \mathcal{B}$. Now, $1 = \sum_{i=1}^{k} \alpha_k v_k$ for some $\alpha_k \in R$ and $v_k \in \mathcal{B}$. Multiplying by v we get

$$v = \sum_{i=1}^{k} \alpha_k v_k v.$$

Since $\mathcal{B}$ is a SCUP basis, $v = \sum_{i=1}^{k} \alpha_k \beta_k^{-1} (\beta_k v_k v)$ for some $\beta_k \in U(R)$ with $\beta_k v_k v \in \mathcal{B}$. Since $v \in \mathcal{B}$, by the linear independence of $\mathcal{B}$, $v = \beta_l v_l v$ for some $1 \leq l \leq k$. Then $1 = \beta_l v_l$ and so $\beta_l^{-1} = v_l \in \mathcal{B} \cap U(R)$. Since $\beta_l^{-1} \in U(R)$, for $a \in R$, $a = a\beta_l \beta_l^{-1}$. So, $\mathcal{B} \cap U(R) = \{\beta_l^{-1}\} \subset R$, and $\{1\} \cup \mathcal{B} \setminus \{\beta_l^{-1}\}$ is a SCUP1 basis for A.

If $\mathcal{B}$ is also a CUP basis, then the β_k's in the above argument are all 1, in particular, $v_l = 1$. ∎

PROPOSITION 2.4: *Let A be an algebra over a ring R with SCUP basis $\mathcal{B}$. Then $\mathcal{B}$ is also a SCUI basis. If $\mathcal{B}$ is a CUP basis, then $\mathcal{B}$ is a CUI basis, and $\mathcal{B}$ forms a group under the multiplication in A.*

Proof. Let $v \in \mathcal{B}$. Note that $\mathcal{B}v$ is a basis for A. By Proposition 2.3 there exists some $\beta \in U(R) \cap \mathcal{B}$. Then $\beta = \sum_{i=1}^{k} \alpha_k v_k v$ for some $\alpha_k \in R$ and $v_k \in \mathcal{B}$. Since

$\mathcal{B}$ is a SCUP basis, and $\beta \in \mathcal{B}$, $\beta = \gamma v_l v$ for some $1 \leq l \leq k$ and $\gamma \in U(R)$. Then, $\gamma^{-1}\beta v^{-1} = v_l \in \mathcal{B}$.

When $\mathcal{B}$ is a CUP basis, the same proof with $\beta = \gamma = 1$ shows that $\mathcal{B}$ is a CUI basis. Then $vw^{-1} \in \mathcal{B}$ for any $v, w \in \mathcal{B}$, therefore $\mathcal{B}$ forms a group under multiplication in A. ∎

We now aim to characterize various types of crossed products in terms of invertible bases. In fact, we will point out that crossed products are SCUP1 algebras. We start by reminding the reader of the pertinent definitions (see [6] for more details).

Definition 2.5: Let R be a ring with 1 and let G be a group. Then a **crossed product** $R * G$ is an associative ring with $\overline{G}$, a copy of G, as an R-basis. Multiplication is determined by the following two rules ([6]):

(1) For $x, y \in G$ there exists a unit $\tau(x, y) \in U(R)$ such that $\bar{x}\bar{y} = \tau(x, y)\overline{xy}$. This action is called the twisting of the crossed product.
(2) For $x \in G$ there exists $\sigma_x \in Aut(R)$ such that for every $r \in R$, $\bar{x}r = \sigma_x(r)\bar{x}$. This action is called the skewing of the crossed product.

In order to characterize the various types of crossed products in terms of invertible bases, we need some terminology to describe the ways in which scalars interact with elements of a basis, We introduce this terminology in the following definitions.

Definition 2.6: Let A be an algebra over a ring R, and $\mathcal{B}$ an R-basis for A. If for all $v \in \mathcal{B}$ there exists $\sigma_v \in Aut(R)$ such that for all $r \in R$, $vr = \sigma_v(r)v$, then R **scalarly commutes** with $\mathcal{B}$. In the case that for all $v \in \mathcal{B}$, $\sigma_v = 1$, we naturally say R **commutes** with $\mathcal{B}$.

Proposition 2.7: *Let A be an algebra over a ring R.*

(1) *A is a crossed product if and only if A has a SCUP basis that scalarly commutes with R.*
(2) *A is a skew group ring if and only if A has a CUP basis that scalarly commutes with R.*
(3) *A is a twisted group ring if and only if A has a SCUP basis that commutes with R.*
(4) *A is a group ring if and only if A has a CUP basis that commutes with R.*

Proof. (1) Suppose A is a crossed product with basis $\bar{G}$. Then by definition of a crossed product, $\bar{G}$ is scalarly closed under products and R scalarly commutes with $\bar{G}$.

Now, let $\mathcal{B}$ be a SCUP basis for A over R that scalarly commutes with R. By Proposition 2.3, there exists $\alpha \in U(R)$ such that $\{\alpha\} = \mathcal{B} \cap U(R)$. By the proof of Proposition 2.3, $\mathcal{C} = \{1\} \cup \mathcal{B} \setminus \{\alpha\}$ is a SCUP basis with 1; $\mathcal{C}$ clearly scalarly commutes with R.

Since $\mathcal{C}$ is a SCUP basis, for $v, w \in \mathcal{C}$ there exists $\alpha_{v,w} \in U(R)$ and $z \in \mathcal{C}$ such that $vw = \alpha_{v,w} z$. Define $\star : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by $\star : (v, w) \mapsto z$ as above. Let $v, w, x \in \mathcal{C}$. Using the facts that $\mathcal{C}$ is a SCUP basis and it scalarly commutes with R along with the associativity of A, we have

$$\begin{aligned} v(wx) &=(vw)x, \\ v\alpha_{w,x}(w \star x) &=\alpha_{v,w}(v \star w)x, \\ \sigma_v(\alpha_{w,x})v(w \star x) &=\alpha_{v,w}\alpha_{v\star w,x}((v \star w) \star x), \\ \sigma_v(\alpha_{w,x})\alpha_{v,w\star x}(v \star (w \star x)) &=\alpha_{v,w}\alpha_{v\star w,x}((v \star w) \star x), \end{aligned}$$

for some $\alpha_{w,x}, \alpha_{v,w\star x}, \alpha_{v,w}, \alpha_{v\star w,x} \in \mathcal{C}$ and $\sigma_v \in Aut(R)$. Since

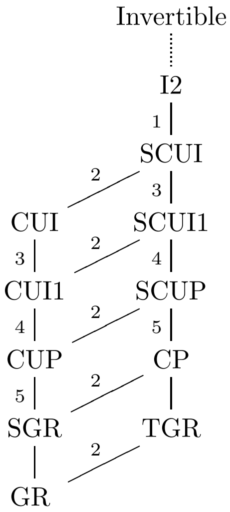
$$(v \star (w \star x)), ((v \star w) \star x) \in \mathcal{C}$$

and $\mathcal{C}$ is a basis,

$$v \star (w \star x) = (v \star w) \star x.$$

So, $\star$ is an associative binary operation on $\mathcal{C}$. The identity of A is clearly an identity of $(\mathcal{C}, \star)$. By Proposition 2.4, $\mathcal{C}$ is a SCUI basis so $(\mathcal{C}, \star)$ has inverses as well showing it is a group. Hence, A is a crossed product. (2)–(4) follow from (1). ∎

The following diagram shows the class inclusions of the various types of algebras summarized in this section. In the diagram, CP, SGR, TGR and GR stand for crossed product, skew group ring, twisted group ring and group ring, respectively. A solid line connecting two classes indicates a proper inclusion, and the number near the line references the specific example from Example 2.8 that shows the inclusion to be proper. A dotted line indicates that we do not know at this moment whether the inclusion is proper or the two notions are actually equivalent. For instance, an example of an invertible algebra which is not an I2 algebra has thus far remained elusive. We do have examples of invertible bases that are not I2, but we have not been able to confirm that for those algebras no invertible basis is I2. Example 5.1 is one of such a basis. There are only two dotted lines in the diagram, and in both cases we suspect the inclusion is proper.



The following examples demonstrate the proper inclusions in the above diagram.

Example 2.8:

(1) (An I2 algebra that is not a SCUI algebra) Consider $A = \frac{\mathbb{F}_3[x,y]}{\langle x^2, y^2, xy\rangle}$. Let $U(A)$ be the group of units of A. Then

$$U(A) = \{\alpha + \beta x + \gamma y \mid \alpha, \beta, \gamma \in \mathbb{F}_3, \alpha \neq 0\}.$$

Note that for $a = \alpha + \beta x + \gamma y \in U(A)$,

$$a^{-1} = (\alpha + \beta x + \gamma y)^{-1} = \alpha - \beta x - \gamma y.$$

An I2 $\mathbb{F}_3$-basis for A is $\{1 + x, 1 + y, 1 + x + y\}$. So, A is I2. To see that there is no invertible basis that is SCUI, we first observe the only elements that are their own inverses are ± 1. Let $\mathcal{B}$ be an invertible basis. Let $v \in \mathcal{B}$. If $\mathcal{B}$ is SCUI, then $\delta v^{-1} \in \mathcal{B}$. Since A has dimension 3, the other element of $\mathcal{B}$ (aside from v and δv^{-1}) must be its own inverse. Therefore, one of ± 1 is in $\mathcal{B}$. Without loss of generality, let $1 \in \mathcal{B}$. Refer back to how inverses look in A. If $v = \alpha + \beta x + \gamma y$ then $\delta v^{-1} = \delta\alpha - \delta\beta x - \delta\gamma y$. Forming the following linear combination we see the set $\{v, \delta v^{-1}, 1\}$ will be linearly dependent:

$$1(v) + \delta(\delta v^{-1}) + \alpha(1) = 0.$$

Therefore, $\mathcal{B}$ is not a basis for A.

(2) (A twisted group ring that is not a CUI algebra) The real quaternions, $\mathbb{H}$, are an algebra over the field of real numbers, $\mathbb{R}$. The basis $\mathcal{B} = \{1, i, j, k\}$ is clearly a SCUP basis for $\mathbb{H}$. By Proposition 2.7, since $\mathcal{B}$ commutes with $\mathbb{R}$, $\mathbb{H}$ is a twisted group ring. Now, suppose $\mathcal{A} = \{v_1, v_2, v_3, v_4\}$ is a CUI basis for $\mathbb{H}$ over $\mathbb{R}$. We have two cases, being either 1 or -1 is in $\mathcal{A}$ or neither are in $\mathcal{A}$. If $v_1 = 1$ or -1, then without loss of generality $v_2^{-1} = v_3$. This forces v_4 to be its own inverse. However, in a division ring the only elements that are their own inverses are 1 and -1, a contradiction. So, assume that neither 1 nor -1 belongs to $\mathcal{A}$. Note that since $\mathcal{A}$ is a CUI basis and considering that for $v = a + bi + cj + dk \in \mathbb{H}$, $v^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a - bi - cj - dk)$, it follows that no element of $\mathcal{A}$ can have $a = 0$. So, let $v_1 = a + bi + cj + dk$ with $a \neq 0$ and $v_2 = v_1^{-1}$. Then write $v_3 = \alpha + \beta i + \gamma j + \delta k$ with $\alpha \neq 0$, so $v_4 v_3^{-1} = \frac{1}{\alpha^2+\beta^2+\gamma^2+\delta^2}(\alpha - \beta i - \gamma j - \delta k)$. It is easy to see that the span of $\mathcal{A}$ has dimension at most three, contradicting our assumption that $\mathcal{A}$ is indeed a basis. Therefore $\mathbb{H}$ is not a CUI algebra over $\mathbb{R}$.

(3) (A CUI algebra that is not a CUI1 algebra) Consider $M_2(\mathbb{F}_2)$. By Proposition 3.6, it is a CUI algebra. By process of elimination there are 6 units in this algebra and it can be seen that no basis including the identity can be a CUI basis.

(4) (A CUI1 algebra that is not a CUP algebra) Consider $A = \frac{\mathbb{F}_2[x,y]}{\langle x^2, y^2, xy \rangle}$. Then $U(A) = \{1, 1 + x, 1 + y, 1 + x + y\}$. To form a CUI1 basis, $\mathcal{B}$, we must have $1 \in \mathcal{B}$. Therefore, we must have two of the other three remaining invertible elements. Since the square of any unit here is 1 and the product of any two non-identity units is the third one, we see that no CUI1 basis we form can be a CUP basis.

Another family of examples is $A = M_n(\mathbb{F}_2)$ for odd n. By Proposition 3.6, A is a CUI1 algebra. Assume A has a CUP basis $\mathcal{B}$. By Corollary 2.4, $\mathcal{B}$ forms a group under multiplication in A. Furthermore, $\mathbb{F}_2$ commutes with $\mathcal{B}$. By Proposition 2.7, A is a group algebra. But, A is simple, which would be a contradiction. So, A does not have a CUP basis.

(5) (A CUP algebra that is not a crossed product) Let C_2 be any subgroup of order 2 of S_3, A be the group algebra $\mathbb{F}_3 S_3$ and R be the group algebra $\mathbb{F}_3 C_2$. It is not hard to see that A is a CUP algebra over R, with the subgroup of order 3 of S_3 forming the CUP basis. Suppose now that A is a crossed product over R. Then $A = R * C_3$ where C_3 is the cyclic group of order 3. Then C_3 acts on R and, in particular, on its two primitive idempotents. A group of order 3 can act only trivially on a set of two elements, so these idempotents are fixed by C_3. Hence, R is central in A, a contradiction.

3. Rings of matrices

In [4] it is shown that over any ring R, the matrix ring $A = M_n(R)$ (for any $n \geq 1$) is an I2 R-algebra. Here we significantly extend that result.

Proposition 3.1: *Let A be an algebra over a ring R with a basis that includes 1. Then $M_n(A)$ is an I2 algebra over R for $n \geq 2$.*

Proof. Let $\mathcal{B}$ be a basis for A over R such that $1 \in \mathcal{B}$. For $1 \leq i, j \leq n$ let e_{ij} be the matrix with 1 in the i, j entry and 0 elsewhere. Denote by I_n the identity matrix and for $1 \leq k \leq n-1$ let $P_k = I_n - e_{kk} - e_{k+1,k+1} + e_{k,k+1} + e_{k+1,k}$, so P_k is the permutation matrix that is I_n with rows k and $k+1$ interchanged. For $b \in \mathcal{B}$ and $1 \leq i, j \leq n$, $i \neq j$ let $v_{ijb} = I_n + e_{ij} b$. For $b \in \mathcal{B}$ and $1 \leq i \leq n-1$ let $v_{iib} = P_i + e_{ii} b$. For $b \in \mathcal{B} \setminus \{1\}$, let $v_{nnb} = P_{n-1} + e_{nn} b$. Let $v_{nn1} = I_n$. Let $\mathcal{A} = \{v_{ijb} | 1 \leq i, j \leq n, b \in \mathcal{B}\}$. Notice that $\mathcal{A}$ consists of invertible elements. We will show $\mathcal{A}$ is a basis for $M_n(A)$ over R. For any $1 \leq i, j \leq n$ with $i \neq j$, $e_{ij} = v_{ij1} - v_{nn1}$. and for $2 \leq k \leq n$, $e_{kk} = v_{nn1} + e_{k,k-1} + e_{k-1,k} - v_{k-1,k-1,1}$. Since $e_{11} = v_{nn1} - \sum_{i=2}^{n} e_{ii}$, we have that e_{kk} is in the span of $\mathcal{A}$ for any k. So, for any $b \in \mathcal{B}$ and $1 \leq i, j \leq n$, $e_{ij} b$ is in the span of $\mathcal{A}$ since for $1 \leq k \leq n-1$; P_k is as well. Hence, $\mathcal{A}$ spans $M_n(A)$.

Now assume

$$\sum_{b \in \mathcal{B}, 1 \leq i, j \leq n} r_{ijb} v_{ijb} = 0$$

for some $r_{ijb} \in R$. For fixed $1 \leq i, j \leq n$, considering the (i, j)-th entry of the sum, we have $0 = \sum_{b \in \mathcal{B} \setminus \{1\}} r_{ijb} b + x \cdot 1$ for some $x \in R$. So, by the linear independence of $\mathcal{B}$, for any i, j and $b \in \mathcal{B} \setminus \{1\}$, $r_{ijb} = 0$. At this point, our original sum becomes

$$\sum_{1\leq i,j\leq n} r_{ij1}v_{ij1} = 0.$$

Now, for $2 \leq k \leq n$, considering the (k,k) and $(1,1)$ entries of the sum, we have

$$0 = \left[\sum_{1\leq i,j\leq n} r_{ij1}\right] - r_{k-1,k-1,1}$$

and

$$0 = \sum_{1\leq i,j\leq n} r_{ij1},$$

showing $r_{k,k,1} = 0$ when $1 \leq k \leq n-1$. Then for any $1 \leq i,j \leq n$ where $i \neq j$, considering the (i,j) entry, we see $r_{ij1} = 0$. Then we must have that $r_{111} = 0$. So, $\mathcal{A}$ is a basis for $M_n(A)$ over R. Finally, $v_{ijb}^{-1} = v_{i,j,-b}$ when $i \neq j$, $v_{kkb}^{-1} = P_k - e_{k+1,k+1}b$ for $1 \leq k \leq n-1$, $v_{nnb}^{-1} = v_{n-1,n-1,-b}$ when $b \neq 1$ and $v_{nn1}^{-1} = v_{nn1}$. In a similar fashion as above, it can be shown that $\mathcal{A}^{-1}$ is also a basis for $M_n(A)$ over R, therefore $M_n(A)$ is I2 over R. ∎

Corollary 3.2: *Invertibility is not a Morita invariant.*

Proof. This follows immediately from Proposition 3.1, as A need not be invertible over R for $M_n(A)$ to be invertible over R. ∎

Direct sums of invertible algebras are not always invertible; $\mathbb{F}_2 \oplus \mathbb{F}_2$ as an $\mathbb{F}_2$-algebra is an easy example. In [4] the authors define a nice $\mathbb{F}_2$-algebra to be one with an invertible basis such that the sum of some even number of basis elements is invertible. Interest in this notion comes from the fact that if A is such an $\mathbb{F}_2$-algebra and B is an invertible $\mathbb{F}_2$-algebra, then $A \oplus B$ is also invertible. Consequently we use the second property to extend this definition to R-algebras in general.

Definition 3.3: Let R be a ring and let A be an invertible R-algebra. Then A is **nice** if $A \oplus B$ is invertible over R for any invertible R-algebra B.

Proposition 3.4: *Let R be a ring and let A be an R-algebra with a basis that includes 1. Then for $n \geq 2$, $M_n(A)$ is nice. Furthermore, if B is an I2 R-algebra, then $M_n(A) \oplus B$ is also I2.*

Proof. Let $\mathcal{B}$ be an invertible basis of B, $\mathcal{A}$ be a basis with 1 for A and $\mathcal{V} = \{v_{ija} | 1 \leq i,j \leq n, a \in \mathcal{A}\}$ be the I2 basis for $M_n(A)$ from Proposition 3.1. Let

$b' \in \mathcal{B}$ and let S be an invertible matrix with only zeros and ones as its entries and zero diagonal. We claim that

$$\mathcal{C} = \{\mathcal{V} \times \{b'\}\} \cup \{\{I_n\} \times \mathcal{B} \setminus \{b'\}\} \cup \{(S, b')\}$$

is an invertible basis of $M_n(A) \oplus B$. Assume

$$\sum_{1 \leq i,j \leq n, a \in \mathcal{A}} r_{ijab'}(v_{ija}, b') + \sum_{b \in \mathcal{B}\setminus\{b'\}} r_{nn1b}(I_n, b) + r_S(S, b') = 0$$

for some $r_{ijab}, r_S \in R$. So,

$$\left[\sum_{1 \leq i,j \leq n, a \in \mathcal{A}} r_{ijab'} + r_S\right] b' + \sum_{b \in \mathcal{B}\setminus\{b'\}} r_{nn1b} b = 0 \tag{1}$$

which implies for $b \in \mathcal{B}\setminus\{b'\}$, $r_{nn1b} = 0$. Then $\sum_{1 \leq i,j \leq n, a \in \mathcal{A}} r_{ijab'} v_{ija} + r_S S = 0$. Since S does not have entries on its diagonal, it can be shown in a fashion similar to that used in the proof of Proposition 3.1 that this implies $r_{ijab'} = 0$ and $r_{ii1b'} = 0$ for $1 \leq i, j \leq n$ and $a \in \mathcal{A} \setminus \{1\}$. So, $\sum_{1 \leq i,j \leq n, i \neq j} r_{ij1b'} v_{ij1} + r_S S = 0$. Considering the $(1,1)$ entry in the above sum we have $\sum_{1 \leq i,j \leq n, i \neq j} r_{ij1b'} = 0$. From (1),

$$0 = \left[\sum_{1 \leq i,j \leq n, a \in \mathcal{A}} r_{ijab'} + r_S\right] = \left[\sum_{1 \leq i,j \leq n, i \neq j} r_{ij1b'} + r_S\right] = r_S.$$

Since v_{ij1} is the only possible basis vector left where the (i,j) entry is nonzero in this case, $r_{ij1b'} = 0$ when $i \neq j$. To see that $\mathcal{C}$ spans $M_n(A) \oplus B$ note the following. For $i \neq j$, $(e_{ij}, 0) = (v_{ij1}, b') - (I_n, b')$. Then for $2 \leq k \leq n$,

$$(e_{ii}, 0) = (I_n, b') - [(v_{i-1,i-1,1}, b') - (e_{i-1,i}, 0) - (e_{i,i-1}.0)].$$

Since S has no $(1,1)$ entry, $(0, b')$ can be produced from (S, b'). Then $(e_{11}, 0) = (I_n, b') - (0, b') - (\sum_{i=2}^{n} e_{ii}, 0)$. The spanning set result then easily follows. Finally, if $\mathcal{B}$ is I2 then of course $\mathcal{C}$ is as well. ∎

At this point we can characterize the invertible semilocal algebras over a division ring, but we save this for Proposition 4.2.

First introduced in [1] and [2], Leavitt path algebras are a subject of much current research. These algebras include direct sums of matrix rings. So, in light of Propositions 3.1 and 3.4, it is natural to wonder which Leavitt path algebras are invertible. This is the subject of [5].

The only condition on $M_n(A)$ assumed under Proposition 3.1 is that A has an R-basis with 1, but imposing a few additional conditions on R, A, or n can give results beyond I2 on $M_n(A)$.

PROPOSITION 3.5: *Let A be an algebra over a ring R with basis that includes 1. Assume n is even and $char(R) = 2$. Then $M_n(A)$ is a CUI algebra over R.*

Proof. Let $\mathcal{B}$ be a basis for A over R such that $1 \in \mathcal{B}$. For $1 \leq k \leq \frac{n}{2}$ let $P_{2k-1} = P_{2k} = I_n - e_{2k-1,2k-1} - e_{2k,2k} + e_{2k-1,2k} + e_{2k,2k+1}$, the permutation matrix that is I_n with rows $2k-1$ and $2k$ interchanged. For $b \in \mathcal{B}$ and $1 \leq i \leq n$ let $v_{iib} = P_i + e_{ii}b$. For $b \in \mathcal{B}$ and $1 \leq i, j \leq n$, $i \neq j$ let $v_{ijb} = I_n + e_{ij}b$. Let $\mathcal{A} = \{v_{ijb} | 1 \leq i, j \leq n, b \in \mathcal{B}\}$. We will show $\mathcal{A}$ is a basis for $M_n(A)$ over R. First note $I_n = \sum_{k=1}^{\frac{n}{2}} v_{2k-1,2k-1,1} - v_{2k,2k,1}$, showing I_n is in the span of $\mathcal{A}$. So, for any i, j such that $i \neq j$ and any $b \in \mathcal{B}$, $e_{ij}b$ is in the span of $\mathcal{A}$. For any k that is odd, $e_{kk} = I_n - [v_{k+1,k+1,1} - e_{k+1,k} - e_{k,k+1}]$ and for any k that is even, $e_{kk} = I_n - [v_{k-1,k-1,1} - e_{k-1,k} - e_{k,k-1}]$. So, for any k, $e_{kk}b$ is in the span of $\mathcal{A}$. Hence, $\mathcal{A}$ spans $M_n(A)$.

Now assume

$$\sum_{b \in \mathcal{B}, 1 \leq i,j \leq n} r_{ijb} v_{ijb} = 0$$

for some $r_{ijb} \in R$. For fixed $1 \leq i, j \leq n$, considering the (i, j)-th entry of the sum, we have $0 = \sum_{b \in \mathcal{B} \setminus \{1\}} r_{ijb} b + x \cdot 1$ for some $x \in R$. So, by the linear independence of $\mathcal{B}$, for any i, j and $b \in \mathcal{B} \setminus 1$, $r_{ijb} = 0$. Also, if $|i - j| > 1$ or $(i, j) = (2k, 2k+1)$ for some k or $(i, j) = (2k+1, 2k)$ for some k, then $r_{ij1} = 0$. At this point, our original sum becomes

$$0 = \sum_{1 \leq k \leq \frac{n}{2}} r_{2k-1,2k-1,1} v_{2k-1,2k-1,1} + r_{2k,2k,1} v_{2k,2k,1} + r_{2k,2k-1,1} v_{2k,2k-1,1} + r_{2k-1,2k,1} v_{2k-1,2k,1}.$$

Now, for $1 \leq k_1, k_2 \leq n$ where k_1 is odd and k_2 is even, considering the $(k_1 + 1, k_1 + 1)$-th, $(k_2 - 1, k_2 - 1)$-th and the $(2, 2)$ entries of the sum, we have

$$0 = \sum_{1 \leq i,j \leq n} r_{ij1} - r_{k_1,k_1,1},$$

$$0 = \sum_{1 \leq i,j \leq n} r_{ij1} - r_{k_2,k_2,1}$$

and

$$0 = \sum_{1 \le i,j \le n} r_{ij1} - r_{111},$$

showing $r_{k_1,k_1,1} = r_{k_2,k_2,1} = r_{111}$. Also, for any non-diagonal entry $(2k, 2k-1)$ we have $r_{2k,2k-1,1} + r_{2k-1,2k-1,1} + r_{2k,2k,1} = 0$, so

$$r_{2k,2k-1,1} = -2r_{2k-1,2k-1,1} = -2r_{111}$$

and similarly, $r_{2k-1,2k,1} = -2r_{111}$. Consider the $(2,2)$ entry in the sum. We have

$$\begin{aligned} 0 =& \left[\sum_{1 \le k \le \frac{n}{2}} r_{2k-1,2k-1,1} + r_{2k,2k,1} + r_{2k,2k-1,1} + r_{2k-1,2k,1} \right] - r_{111} \\ =& -(n+1)r_{111} \\ =& (n+1)r_{111}. \end{aligned}$$

Since $char(R) = 2$ and n is even, $r_{111} = 0$. So, $\mathcal{A}$ is a basis for $M_n(A)$ over R. Finally, $v_{ijb}^{-1} = v_{ijb}$ when $i \neq j$ and $v_{2k-1,2k-1,b}^{-1} = v_{2k,2k,b}$ for $1 \le k \le \frac{n}{2}$, so $\mathcal{A}$ is a CUI basis. ∎

Proposition 3.6: *For all n, $M_n(\mathbb{F}_2)$ is a CUI algebra over $\mathbb{F}_2$. For odd n, $M_n(\mathbb{F}_2)$ is a CUI1 algebra over $\mathbb{F}_2$.*

Proof. Assume n is even. For $1 \le k \le \frac{n}{2}$ let

$$P_{2k-1} = P_{2k} = I_n - e_{2k-1,2k-1} - e_{2k,2k} + e_{2k-1,2k} + e_{2k,2k+1},$$

the permutation matrix that is I_n with rows $2k-1$ and $2k$ interchanged. For $1 \le i, j \le n$, if $i = j$ let $v_{ij} = v_{ii} = P_i + e_{ii}$, and if $i \neq j$ let $v_{ij} = I_n + e_{ij}$. Let $\mathcal{A} = \{v_{ij|1 \le i,j \le n}\}$. We will show $\mathcal{A}$ is a basis for $M_n(A)$ over R. First note $I_n = \sum_{k=1}^{\frac{n}{2}} v_{2k-1,2k-1} - v_{2k,2k}$, so I_n is in the span of $\mathcal{A}$. So, for any i, j with $i \neq j$, e_{ij} is in the span of $\mathcal{A}$. For any k that is odd,

$$e_{kk} = I_n - [v_{k+1,k+1} - e_{k+1,k} - e_{k,k+1}],$$

and for any k that is even,

$$e_{kk} = I_n - [v_{k-1,k-1} - e_{k-1,k} - e_{k,k-1}].$$

So, for any k, e_{kk} is in the span of $\mathcal{A}$. Hence, $\mathcal{A}$ spans $M_n(\mathbb{F}_2)$. Then, since $\dim M_n(\mathbb{F}_2) = |\mathcal{A}| = n^2$, $\mathcal{A}$ is a basis. Finally, $v_{ij}^{-1} = v_{ij}$ when $i \neq j$ and $v_{2k-1,2k-1}^{-1} = v_{2k,2k}$ for $1 \le k \le \frac{n}{2}$, so $\mathcal{A}$ is a CUI algebra.

Now assume n is odd. For $1 \leq k \leq \frac{n-1}{2}$ let

$$P_{2k-1} = P_{2k} = I_n - e_{2k-1,2k-1} - e_{2k,2k} + e_{2k-1,2k} + e_{2k,2k+1},$$

the permutation matrix that is I_n with rows $2k-1$ and $2k$ interchanged. For $1 \leq i, j \leq n$, if $i = j$ and $i < n$ let $v_{ij} = v_{ii} = P_i + e_{ii}$, let $v_{nn} = I_n$, and if $i \neq j$ let $v_{ij} = I_n + e_{ij}$. Using arguments similar to those used in the case when n is even, it can be shown that $\mathcal{A} = \{v_{ij} | 1 \leq i, j \leq n\}$ is a CUI1 basis for $M_n(\mathbb{F}_2)$ over $\mathbb{F}_2$. ∎

Proposition 3.7: *Let R be a ring such that 2 is invertible in R. Then $M_n(R)$ is a CUI1 algebra over R.*

Proof. Let $v_{nn} = I_n$. For $1 \leq i, j \leq n$ where $(i,j) \neq (n,n)$, if $i = j$ let $v_{ij} = v_{ii} = v_{nn} - 2e_{ii}$, if $i > j$ let $v_{ij} = v_{jj} + e_{ij}$ and if $i < j$ let $v_{ij} = v_{ii} + e_{ij}$. Clearly, $\mathcal{B} = \{v_{ij|1 \leq i,j \leq n}\}$ is a spanning set.

Now assume

$$\sum_{1 \leq i,j \leq n} r_{ij} v_{ij} = 0$$

for some $r_{ij} \in R$. For fixed $1 \leq i, j \leq n$ such that $i \neq j$, considering the (i,j)-th entry of the sum, we see $r_{ij} = 0$. Now for $1 \leq k \leq n-1$, considering the (k,k) and (n,n) entries in the sum, we have that $0 = 2r_{kk} + \sum_{1 \leq i \leq n, i \neq k} r_{ii}$ and $0 = r_{kk} + \sum_{1 \leq i \leq n, i \neq k} r_{ii}$, showing $r_{kk} = 0$. So, $r_{nn} = 0$ as well so $\mathcal{B}$ is a basis. Finally, it is easy to see the square of any basis element is the identity, hence $\mathcal{B}$ is a CUI1 basis. ∎

Example 3.8: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ is a CUI1 basis for $M_2(\mathbb{F}_3)$.

4. (Almost) all finite-dimensional algebras over a division ring are invertible-2

In the majority of this section, we restrict our attention to algebras over a division ring D. Characterizations of invertible semilocal D-algebras are given. While it is not known whether all invertible algebras are also I2, we show that this is the case for all finite-dimensional D-algebras. Finally, we show that all local R-algebras are invertible. Before proceeding to our main results, we provide Proposition 3.5 from [4] which is needed to prove Proposition 4.2. In this section, for a ring R, we will denote the Jacobson radical of R by $J(R)$.

LEMMA 4.1 (Proposition 3.5 in [4]): *Let R be a ring such that 1 is the sum of two units and let A and B be invertible algebras over R. Then $A \oplus B$ is an invertible algebra over R.*

In [4] it was assumed that the algebras in the above lemma were finite-dimensional, but the proof holds without this assumption.

PROPOSITION 4.2: *Let D be a division ring and let A be a semilocal D-algebra. If $D \neq \mathbb{F}_2$, then A is invertible. If $D = \mathbb{F}_2$, then A is invertible if and only if A does not admit an algebra epimorphism to $\mathbb{F}_2 \oplus \mathbb{F}_2$.*

Proof. First observe that if there exists an algebra epimorphism $f : A \to \mathbb{F}_2 \oplus \mathbb{F}_2$, then $f(J(A)) \subseteq J(\mathbb{F}_2 \oplus \mathbb{F}_2) = 0$. So, A admits an algebra epimorphism to $\mathbb{F}_2 \oplus \mathbb{F}_2$ if and only if $A/J(A)$ does as well. In this case $f(A)$ is not invertible, hence neither is A.

Assume then that A does not admit such an algebra epimorphism, and let $J = J(A)$. Since A is semilocal, we have an algebra isomorphism

$$A/J \cong \bigoplus_{i=1}^{m} M_{n_i}(D_i)$$

for some division rings D_i. Clearly, $\bigoplus_{i=1}^{m} M_{n_i}(D_i)$ admits an algebra epimorphism to $\mathbb{F}_2 \oplus \mathbb{F}_2$ if and only if $D = \mathbb{F}_2$ and $M_{n_i}(D_i) = M_{n_j}(D_j) = \mathbb{F}_2$ for some $i \neq j$. By assumption $A/J \cong \bigoplus_{i=1}^{m} M_{n_i}(D_i)$ does not admit such an algebra epimorphism, so there are no such distinct $M_{n_i}(D_i)$ and $M_{n_j}(D_j)$, hence A/J is I2 over D by Proposition 3.4 and Lemma 4.1. It is not hard to see that such an I2 basis for A/J can be chosen that includes $\overline{1}$. So, choose $\{1, u_2, \ldots, u_m\}$ from $A \setminus J$ such that their images in A/J give an I2 basis for A/J over D. Since the image of every u_i is invertible in A/J, every u_i is invertible in A. Then for any basis $\{b_i | i \in \mathcal{I}\}$ of J,

$$\mathcal{G} = \{1, u_2, \ldots, u_m\} \cup \bigcup_{i \in \mathcal{I}} \{1 - b_i\}$$

is a set of units spanning A over D, and since D is a division ring $\mathcal{G}$ contains a basis of A. ∎

When A is finite-dimensional over D, we can conclude more.

Proposition 4.3: *Let D be a division ring and let A be a finite-dimensional D-algebra. If $D \neq \mathbb{F}_2$, then A is I2. If $D = \mathbb{F}_2$, then A is I2 if and only if A does not admit an algebra epimorphism to $\mathbb{F}_2 \oplus \mathbb{F}_2$.*

Proof. Our algebra A is semilocal, so in light of Proposition 4.2 we need only show that A is I2 when A is invertible. Assume then that A is invertible, and let $J = J(A)$. Since A is artinian, J is a nilpotent ideal, so let l be the nilpotency of J. It is not hard to see that J^i/J^{i+1} is a finite-dimensional vector space over D for $1 \leq i \leq l-1$. Given i, let $d(i) = \dim_D(J^i/J^{i+1})$, and choose $b_{i1}, \ldots, b_{i,d(i)} \in J^i$ such that their images in J^i/J^{i+1} give a basis for J^i/J^{i+1} over D. Then the set $\bigcup_{\substack{1\leq i\leq l-1\\1\leq j\leq d(i)}} b_{ij}$ spans J over D, and is a basis for J since its cardinality is precisely $\dim_D(J)$. Then choosing $\{1, u_2, \ldots, u_m\}$ from $A \setminus J$ as in the proof of Proposition 4.2,

$$\mathcal{B} = \{1, u_2, \ldots, u_m\} \cup \bigcup_{\substack{1\leq i\leq l-1\\1\leq j\leq d(i)}} \{1 - b_{ij}\}$$

spans A over D. Since $m = \dim_D(A/J) = \dim_D(A) - \dim_D(J)$, we have $|\mathcal{B}| = \dim_D(A)$, therefore $\mathcal{B}$ is a basis for A over D. We claim that $\mathcal{B}$ is I2.

Since J is nilpotent with nilpotency l, $(1 - b_{ij})^{-1} = 1 + c_{ij}$ where $c_{ij} = b_{ij} + b_{ij}^2 + \cdots + b_{ij}^l$. So,

$$\mathcal{B}^{-1} = \{1, u_2^{-1}, \ldots, u_m^{-1}\} \cup \bigcup_{\substack{1\leq i\leq l-1\\1\leq j\leq d(i)}} \{1 + c_{ij}\}.$$

Notice that $b_{ij} \equiv c_{ij}$ modulo J^{i+1} for every i, j and the arguments used on $\bigcup_{\substack{1\leq i\leq l-1\\1\leq j\leq d(i)}} b_{ij}$ also show that $\bigcup_{\substack{1\leq i\leq l-1\\1\leq j\leq d(i)}} c_{ij}$ is a basis for J over D. Since the images of $\{1, u_2^{-1}, \ldots, u_m^{-1}\}$ also give a basis for A/J, the same arguments used on $\{1, u_2, \ldots, u_m\}$ also show that $A \setminus J$ is also spanned by $\mathcal{B}^{-1}$. Finally, since $|\mathcal{B}^{-1}| = |\mathcal{B}|$, the set $\mathcal{B}^{-1}$ is a basis for A over D. ∎

The last result of this section may appear redundant in light of Proposition 4.2, but now R need not be a division ring.

Proposition 4.4: *Let R be a ring and let A be a free local R-algebra. Then A is invertible.*

Proof. Let $\mathcal{B}$ be a basis for A over R. Since A is local, there exists $v \in \mathcal{B} \cap U(A)$. It is not hard to see $v^{-1}\mathcal{B}$ is also an R-basis of A which includes 1. So, assume that $1 \in \mathcal{B}$. Since A is local, for any $x \in A$, either x or $1-x$ is a unit. Define a map $f : \mathcal{B} \to U(A)$ by $f(x) = x$ if $x \in U(A)$ and $f(x) = 1-x$ otherwise. Let $\mathcal{B}' = f(\mathcal{B})$. Given any $b \in \mathcal{B} \setminus U(A)$, we have $b = 1-(1-b)$, so $\mathcal{B}$ is in the span of $\mathcal{B}'$. Assume then that $r \cdot 1 + \sum_{i=1}^{n} r_i f(b_i) = 0$ for some $b_i \in \mathcal{B} \setminus \{1\}$ and $r, r_i \in R$. Then

$$\begin{aligned} 0 =& r + \sum_{i=1}^{n} r_i f(b_i) \\ =& r + \sum_{i=1}^{m} r_i b_i + \sum_{i=m+1}^{n} r_i(1-b_i) \\ =& \left[r + \sum_{i=m+1}^{n} r_i \right] \cdot 1 + \sum_{i=1}^{m} r_i b_i - \sum_{i=m+1}^{n} r_i b_i \end{aligned}$$

for some $1 \le m \le n$. Since $\mathcal{B}$ is a basis with 1, $r = r_i = 0$ for $1 \le i \le n$. Hence, $\mathcal{B}'$ is a basis. By construction $\mathcal{B}' \subset U(A)$, so $\mathcal{B}'$ is an invertible basis. ∎

5. Some invertible infinite-dimensional algebras

Having characterized which finite-dimensional algebras over a division ring were invertible in the previous section, it seems natural to transfer our study to infinite-dimensional invertible algebras. However, in this section we revert back to algebras over arbitrary rings, and not just division rings.

There are already several well known classes of infinite-dimensional invertible algebras. As stated before, group rings provide an example of an invertible algebra, and when the group is infinite, we have an infinite-dimensional invertible algebra. Also, any infinite division ring extension is an infinite-dimensional invertible algebra. The next example has come to be a rather important infinite-dimensional invertible algebra. Its significance arises from the fact that it provides an example of an invertible basis that is not I2.

Example 5.1: Let F be a field. Consider the F-algebra $F(x)$, the field of rational functions over F. Notice that

$$\mathcal{G} = \left\{ \frac{x^n}{f(x)} | n \in \{0,1,2,\ldots\}, f(x) \in F[x] \setminus \{0\} \right\}$$

generates $F(x)$. However, $\mathcal{G}^{-1} \subset F[x, x^{-1}]$. As a consequence, $\mathcal{G}^{-1}$ does not generate $F(x)$, so any basis $\mathcal{B}$ contained in $\mathcal{G}$ is an invertible basis that is not I2.

Our central result in this section, Proposition 5.3, is a generalization of Proposition 3.3 of [4]. Before presenting Proposition 5.3, we give Proposition 3.3 of [4] here since it is used in the proof of Proposition 5.3.

Lemma 5.2 (Proposition 3.3 in [4]): *Let A_0, A_1, A_2 be rings such that A_2 is an invertible algebra over A_1 with invertible A_1-basis $\mathcal{B}_2$ and A_1 is an invertible algebra over A_0 with invertible A_0-basis $\mathcal{B}_1$. Then A_2 is an invertible A_0-algebra and $\mathcal{B}_1\mathcal{B}_2$ is an invertible A_0-basis of A_2.*

Proposition 5.3: *Let $A_0, A_1, \ldots$ be a chain of rings such that for $i \geq 1$, A_i is an invertible algebra over A_{i-1}. Then $A = \bigcup_{i\geq 0} A_i$ is an invertible A_0-algebra.*

Proof. For $i \geq 1$, let $\mathcal{B}_i$ be an invertible basis which includes 1 for A_i over A_{i-1} (it is not hard to see that any invertible algebra has such a basis) and let $\mathcal{C}_i = \mathcal{B}_1\mathcal{B}_2\cdots\mathcal{B}_i$. Since $1 \in B_i$, we have a chain $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots$. Let $\mathcal{C} = \bigcup_{i\geq 1}\mathcal{C}_i$ and let $\alpha \in A$. Then there exists some k such that $\alpha \in A_k$. Clearly, $\mathcal{C}_k$ generates α over A_0, so $\mathcal{C}$ is a generating set. Assume $\sum_{i=1}^{n}\gamma_i v_i = 0$ for some $n \in \mathbb{N}$, $v_i \in \mathcal{C}$ and $\gamma_i \in A_0$. Then there exists some l such that $\{v_i\}_{i=1}^{n} \subseteq \mathcal{C}_l$. By Lemma 5.2, $\mathcal{C}_l$ is a basis for A_l over A_0. Therefore, we must have $\gamma_i = 0$ for all i. Thus $\mathcal{C}$ is a linearly independent set and therefore a basis for A over A_0. ∎

Corollary 5.4: *Let $A_0, A_1, \ldots$ be a chain of rings such that for $i \geq 1$, A_i is an algebra over A_{i-1} with an I2 A_{i-1}-basis that commutes with all elements of A_{i-1}. Then $A = \bigcup_{i\geq 0} A_i$ is an I2 A_0-algebra.*

Proof. For $i \geq 1$, let $\mathcal{B}_i$ be an I2 A_{i-1}-basis which includes 1 and commutes with A_{i-1} for A_i. (It is not hard to see here that an I2 basis that has this commuting property can produce a basis with 1 having the same properties.) Let $\mathcal{C}_i = \mathcal{B}_1\mathcal{B}_2\cdots\mathcal{B}_i$. Since $1 \in B_i$, we have a chain $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots$. Let $\mathcal{C} = \bigcup_{i\geq 1}\mathcal{C}_i$. By the proof of Proposition 5.3, $\mathcal{C}$ is a an invertible basis for A over A_0. Now let $\mathcal{D}_i = \mathcal{B}_1^{-1}\mathcal{B}_2^{-1}\cdots\mathcal{B}_i^{-1}$ and $\mathcal{D} = \bigcup_{i\geq 1}\mathcal{D}_i$. Since $\mathcal{B}_i^{-1}$ is an invertible A_{i-1}-basis, again by the proof of Proposition 5.3, $\mathcal{D}$ is an invertible basis for A over A_0. Let $v \in \mathcal{C}$. Then $v = v_1v_2\cdots v_l$ for some l and $v_i \in \mathcal{B}_i$. So, $v^{-1} = v_l^{-1}\cdots v_2^{-1}v_1^{-1} \in \mathcal{C}^{-1}$ and by the commuting property of the $\mathcal{B}_i$ we have $v^{-1} = (v_1v_2\cdots v_l)^{-1} = (v_l\cdots v_2v_1)^{-1} = v_1^{-1}v_2^{-1}\cdots v_l^{-1} \in \mathcal{D}$. A similar argument shows $\mathcal{D} \subset \mathcal{C}^{-1}$, so $\mathcal{D} = \mathcal{C}^{-1}$. So, A is an I2 A_0-algebra. ∎

Remark 5.5: In Corollary 5.4, if I2 is replaced by CUI1, SCUI1 or SCUP, using similar arguments, the result holds. It will not hold necessarily for CUI or SCUI, since the bases used need 1 in them and these classes of algebras strictly contain CUI1 and SCUI1, respectively.

Next we consider a specific subring of the ring of row and column finite matrices over any ring. We make use of Proposition 5.3 to show this ring is, in fact, invertible over its base ring.

COROLLARY 5.6: *Let R be a ring and let*

$$A = \left\{ \begin{bmatrix} X & 0 & 0 & \dots \\ 0 & X & 0 & \dots \\ 0 & 0 & X & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} | X \in M_n(R), n \geq 1 \right\}.$$

Then A is an invertible R-algebra.

Proof. Let $\{p_1, p_2, \dots\}$ be the set of prime numbers in some ordering. Let $n_0 = 1$ and for $i \geq 1$ let $n_i = (p_1 \cdot p_2 \cdots p_i)^i$ and $q_i = \frac{n_i}{n_{i-1}}$. For $i \geq 0$ let

$$A_i = \left\{ \begin{bmatrix} X & 0 & 0 & \dots \\ 0 & X & 0 & \dots \\ 0 & 0 & X & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} | X \in M_{n_i}(R) \right\}.$$

Then

$$A_i = \left\{ \begin{bmatrix} X & 0 & 0 & \dots \\ 0 & X & 0 & \dots \\ 0 & 0 & X & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} | X \in M_{q_i}(M_{n_{i-1}}(R)) \right\}$$

when $i \geq 1$. Viewing A_i this way, we see $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_{i \geq 0} A_i \subseteq A$. To show the reverse inclusion, let $B \in A$. There exists $k \in \mathbb{N}$ such that B consists of the same $k \times k$ matrix down the diagonal, so $k = \prod_{j=1}^{r} p_j^{m_j}$ for some r and m_j's. Let $t = \max\{r, m_1, \dots, m_r\}$. Since $k | n_t$, $B \in A_t$ and $A \subseteq \bigcup_{i \geq 0} A_i$. This proves $A = \bigcup_{i \geq 0} A_i$. For $i \geq 0$, $A_i \cong M_{n_i}(R)$, so by Proposition 3.1, $M_{n_{i+1}}(R) = M_{q_{i+1}}(M_{n_i}(R))$ is invertible over $M_{n_i}(R)$. Hence, for $i \geq 0$, A_{i+1} is invertible over A_i. The conclusion follows from Proposition 5.3. ∎

Our next proposition handles a family of matrices we nickname "kite matrices" (for hopefully obvious reasons). This provides another class of infinite matrix rings that are invertible algebras.

Proposition 5.7: *Let R be a ring and let*

$$A = \left\{ \begin{bmatrix} X & 0 & 0 & \dots \\ 0 & r & 0 & \dots \\ 0 & 0 & r & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} | n \in \mathbb{N}, X \in M_n(R), r \in R \right\}.$$

Then A is an invertible algebra over R.

Proof. We denote the identity matrix by I. For $i, j \in \mathbb{N}$ with $i \neq j$, define $v_{ij} = I + e_{ij}$. For $i \in \mathbb{N}$ let $v_{ii} = I - e_{ii} + e_{i,i+1} + e_{i+1,i}$. Let $\mathcal{B} = \{I\} \cup \{v_{ij} | i, j \in \mathbb{N}\}$. We will show $\mathcal{B}$ is a basis for A over R. For any $i, j \in \mathbb{N}$ such that $i \neq j$ we have $e_{ij} = v_{ij} - I$ and $e_{ii} = I - v_{ii} + e_{i,i+1} + e_{i+1,i}$. So, for any $i, j \in \mathbb{N}$, e_{ij} is in the span of $\mathcal{B}$. Also, we can generate the "tails" of the elements as $r(I - \sum_{i=1}^{n} e_{ii})$ for $r \in R$ and any n. Hence, $\mathcal{B}$ spans A over R.

Now assume

$$rI + \sum_{i,j \in \mathbb{N}} r_{ij} v_{ij} = 0$$

for some $r, r_{ij} \in R$ where only a finite number of the r_{ij}'s are nonzero. Then there exists an n such that for $i, j > n$, $r_{ij} = 0$. So,

$$rI + \sum_{1 \leq i,j \leq n} r_{ij} v_{ij} = 0.$$

Now, for $1 \leq k \leq n$, considering the (k, k) and $(n+1, n+1)$ entries of the sum, we have

$$rI + \sum_{1 \leq i,j \leq n} r_{ij} - r_{kk} = 0$$

and

$$rI + \sum_{1 \leq i,j \leq n} r_{ij} = 0,$$

showing $r_{kk} = 0$ for $1 \leq k \leq n$. Then, for $1 \leq i, j \leq n$, considering the (i, j) entry of the sum, we have $r_{ij} = 0$. Finally, $r = 0$. Hence, $\mathcal{B}$ is a basis for A over R. It is not hard to see that the elements in $\mathcal{B}$ are invertible, so A is an invertible algebra over R. ∎

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