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MATRIX REPRESENTATIONS OF FINITELY GENERATED GRASSMANN ALGEBRAS AND SOME CONSEQUENCES

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ABSTRACT

We prove that the m-generated Grassmann algebra can be embedded into a $2^{m-1} \times 2^{m-1}$ matrix algebra over a factor of a commutative polynomial algebra in m indeterminates. Cayley–Hamilton and standard identities for $n \times n$ matrices over the m-generated Grassmann algebra are derived from this embedding. Other related embedding results are also presented.

1. Introduction

Let K be a field (of characteristic zero) and consider the free associative K-algebra $K\langle x_1, \ldots, x_m \rangle$ generated by the (non-commuting) indeterminates $x_1, \ldots, x_m$. The elements of $K\langle x_1, \ldots, x_m \rangle$ are K-linear combinations of monomials of the form $x_{i_1} \cdots x_{i_k}$ with $i_1, \ldots, i_k \in \{1, \ldots, m\}$ (not necessarily different). The m-generated Grassmann (or exterior) algebra is defined as the factor K-algebra

$$E^{(m)} = K\langle x_1, \ldots, x_m \rangle \big/ I(x_1, \ldots, x_m),$$

where

$$I(x_1, \ldots, x_m) = (x_i x_j + x_j x_i \mid 1 \le i \le j \le m)_K$$

is the (two-sided) K-ideal of $K\langle x_1, \ldots, x_m \rangle$ generated by the polynomials $x_i x_j + x_j x_i$ for $1 \le i \le j \le m$. The usual notation for this Grassmann algebra is

$$E^{(m)} = K\langle v_1, \ldots, v_m \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j \le m \rangle,$$

where the cosets $v_i = x_i + I(x_1, \ldots, x_m)$ are the anticommuting generators of $E^{(m)}$. Clearly, any element of $E^{(m)}$ is a unique K-linear combination of monomials of the form $v_{i_1} \cdots v_{i_k}$ with $1 \le i_1 < \cdots < i_k \le m$. It follows that $\dim_K E^{(m)} = 2^m$.

The definition of the countably infinitely generated Grassmann algebra

$$E = K\langle v_1, \ldots, v_m, \ldots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j \rangle$$

is similar.

For a ring (or K-algebra) R let $\mathrm{M}_n(R)$ denote the full $n \times n$ matrix ring (K-algebra) over R with identity $I_n \in \mathrm{M}_n(R)$. Since any matrix $A \in \mathrm{M}_n(E)$ has a finite number of entries and each entry contains a finite number of generators

v_i from E, there exists an integer $m \geq 1$ such that $A \in \mathrm{M}_n(E^{(m)})$. It follows that

$$\mathrm{M}_n(E) = \bigcup_{m=1}^{\infty} \mathrm{M}_n(E^{(m)}).$$

The algebras $E^{(m)}$ and E play a fundamental role in many areas of mathematics.

Our main inspiration is Kemer's pioneering work [K] on the T-ideals of associative algebras, which revealed the importance of the identities satisfied by the full $n \times n$ matrix algebra $\mathrm{M}_n(E)$ and by the algebra $\mathrm{M}_{n,t}(E)$ of (n,t)-supermatrices (this is a certain K-subalgebra of $\mathrm{M}_n(E)$). The prime T-ideals of $K\langle x_1, \ldots, x_m, \ldots \rangle$ are exactly the T-ideals of the identities satisfied by $\mathrm{M}_n(K)$ for $n \geq 1$. The T-prime T-ideals are the prime T-ideals plus the T-ideals of the identities of $\mathrm{M}_n(E)$ for $n \geq 1$ and of $\mathrm{M}_{n,t}(E)$ for $n-1 \geq t \geq 1$. Another remarkable result is that, for n sufficiently large, any T-ideal contains the T-ideal of the identities satisfied by $\mathrm{M}_n(E)$. Thus the algebras $\mathrm{M}_n(E)$ and $\mathrm{M}_n(E^{(m)})$ served as the main motivation for the present work.

An additional motivation is the well-known embedding of the skew field $\mathbb{H}$ of the real quaternions into 4×4 real matrices:

$$a + bi + cj + dk \longmapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$. The above definition provides an injective $\mathbb{R}$-algebra homomorphism $\upsilon : \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$. Using the natural extension

$$\upsilon_n : \mathrm{M}_n(\mathbb{H}) \longrightarrow \mathrm{M}_n(\mathrm{M}_4(\mathbb{R})) \cong \mathrm{M}_{4n}(\mathbb{R}),$$

an $n \times n$ matrix over $\mathbb{H}$ can be viewed as a $4n \times 4n$ matrix over $\mathbb{R}$. For a quaternionic matrix $A \in \mathrm{M}_n(\mathbb{H})$, the Cayley–Hamilton identity for $\upsilon_n(A)$ yields the same identity (with real coefficients) of degree $4n$ for A itself.

A similar approach to get a Cayley–Hamilton identity of degree $2n$ for a matrix $A \in \mathrm{M}_n(E^{(2)})$ is based on embedding the two-generated exterior algebra $E^{(2)}$ into a 2×2 matrix algebra over a certain commutative ring (see [SzvW]).

In order to present a Cayley–Hamilton identity of degree $2^{m-1}n$ for a matrix in $\mathrm{M}_n(E^{(m)})$, first we consider the so-called constant trace (CT-) representations of an arbitrary K-algebra. In Section 2 we derive a Cayley–Hamilton identity for $n \times n$ matrices over any algebra having a CT-representation. Then in Section

3, using induction, we present a CT-representation (K-embedding)

$$\varepsilon^{(m)} : E^{(m)} \longrightarrow \mathrm{M}_{2^{m-1}}(K[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2)).$$

Notice that we cannot expect similar results for the infinitely generated Grassmann algebra E. Since E does not satisfy any of the standard identities, it follows that E does not embed into any full matrix algebra over a commutative ring.

Finally in Section 4 we use a certain factor of a skew polynomial algebra to give a broad generalization of the embedding process for $E^{(m)}$.

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2. CT-Representations and Cayley–Hamilton identities

Let ${}_K R$ be an arbitrary and ${}_K\Omega$ be a commutative (associative) algebra over K (notice that $K \subseteq \mathrm{Z}(R)$ and $K \subseteq \Omega$). For an integer $t \geq 1$, we consider representations of R over Ω which are injective K-algebra homomorphisms (K-embeddings) $\varepsilon : R \longrightarrow \mathrm{M}_t(\Omega)$. We call ε a constant trace (CT-) representation if $\mathrm{tr}(\varepsilon(r)) \in K$ for all $r \in R$ (here $\mathrm{tr}(\varepsilon(r))$ is the sum of the diagonal entries of the $t \times t$ matrix $\varepsilon(r) \in \mathrm{M}_t(\Omega)$).

Theorem 2.1: *Let $\varepsilon : R \longrightarrow \mathrm{M}_t(\Omega)$ be a CT-representation of R over Ω. If $A \in \mathrm{M}_n(R)$ is an $n \times n$ matrix, then A satisfies a Cayley–Hamilton identity of the form*

$$A^{tn} + c_1 A^{tn-1} + \cdots + c_{tn-1} A + c_{tn} I_n = 0,$$

where $c_i \in K$, $1 \leq i \leq tn$.

Proof. Let

$$\varepsilon_n : \mathrm{M}_n(R) \longrightarrow \mathrm{M}_n(\mathrm{M}_t(\Omega)) \cong \mathrm{M}_{tn}(\Omega)$$

be the natural extension of ε. For any matrix $A = [a_{i,j}]$ in $\mathrm{M}_n(R)$, the trace of the $tn \times tn$ matrix $B = \varepsilon_n(A)$ is the sum of the traces of the diagonal $t \times t$ blocks:

$$\mathrm{tr}(B) = \sum_{i=1}^{n} \mathrm{tr}(\varepsilon(a_{i,i})).$$

Since ε is a CT-representation, we have $\mathrm{tr}(\varepsilon(a_{i,i})) \in K$ for each $1 \leq i \leq n$. It follows that $\mathrm{tr}(B) \in K$. For the coefficients of the characteristic polynomial

$$\det(zI - B) = c_0 z^{tn} + c_1 z^{tn-1} + \cdots + c_{tn-1} z + c_{tn} \in \Omega[z]$$

of B, the following recursion holds: $c_0 = 1$ and

$$c_k = -\frac{1}{k}(c_{k-1}\mathrm{tr}(B) + c_{k-2}\mathrm{tr}(B^2) + \cdots + c_1 \mathrm{tr}(B^{k-1}) + c_0 \mathrm{tr}(B^k))$$

for $1 \leq k \leq tn$ (Newton formulae, see [R]). In view of

$$\mathrm{tr}(B^k) = \mathrm{tr}((\varepsilon_n(A))^k) = \mathrm{tr}(\varepsilon_n(A^k)) \in K,$$

we deduce that $c_i \in K$ for each $0 \leq i \leq tn$. Thus $\det(zI - B) \in K[z]$ and the Cayley–Hamilton identity for $B \in \mathrm{M}_{tn}(\Omega)$ is of the form

$$B^{tn} + c_1 B^{tn-1} + \cdots + c_{tn-1} B + c_{tn} I_n = 0.$$

It follows that

$$\begin{aligned}(\varepsilon_n(A))^{tn} + c_1(\varepsilon_n(A))^{tn-1} + \cdots + c_{tn-1}\varepsilon_n(A) + c_{tn} I_n \\ = \varepsilon_n(A^{tn} + c_1 A^{tn-1} + \cdots + c_{tn-1} A + c_{tn} I_n) = 0\end{aligned}$$

holds in $\mathrm{M}_{tn}(\Omega)$, thus the injectivity of ε_n gives the desired identity. ∎

3. CT-Representation of $E^{(m)}$

For $m = 1$ we have a natural isomorphism $E^{(1)} = K\left\langle v_1 \mid v_1^2 = 0 \right\rangle \cong K[z]/(z^2)$ of K-algebras. If $m = 2$, then the assignments

$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; v_1 \longmapsto \begin{bmatrix} z_1 & 0 \\ 0 & -z_1 \end{bmatrix}, \; v_2 \longmapsto \begin{bmatrix} 0 & z_2 \\ z_2 & 0 \end{bmatrix}$$

define the following CT-representation $\varepsilon^{(2)} : E^{(2)} \longrightarrow \mathrm{M}_2(K[z_1, z_2]/(z_1^2, z_2^2))$:

$$\varepsilon^{(2)}(c_0 + c_1v_1 + c_2v_2 + c_3v_1v_2) = \begin{bmatrix} c_0 + c_1z_1 + (z_1^2, z_2^2) & c_2z_2 + c_3z_1z_2 + (z_1^2, z_2^2) \\ c_2z_2 - c_3z_1z_2 + (z_1^2, z_2^2) & c_0 - c_1z_1 + (z_1^2, z_2^2) \end{bmatrix},$$

where $c_0, c_1, c_2, c_3 \in K$ and (z_1^2, z_2^2) is the ideal of the commutative polynomial ring $K[z_1, z_2]$ generated by the monomials z_1^2, z_2^2.

THEOREM 3.1: *For some integers $m, t \geq 2$, let $\varepsilon^{(m)} : E^{(m)} \longrightarrow \mathrm{M}_t(\Omega)$ be a CT-representation of $E^{(m)}$ over a commutative K-algebra Ω. Then the assignments*

$$1 \longmapsto \begin{bmatrix} I_t & 0 \\ 0 & I_t \end{bmatrix},\ v_i \longmapsto \begin{bmatrix} \varepsilon^{(m)}(v_i) & 0 \\ 0 & -\varepsilon^{(m)}(v_i) \end{bmatrix} \text{ for } 1 \leq i \leq m,$$

and

$$v_{m+1} \longmapsto \begin{bmatrix} 0 & \widehat{z}I_t \\ \widehat{z}I_t & 0 \end{bmatrix} \text{ (with } \widehat{z} = z + (z^2) \text{ in } \Omega[z]/(z^2))$$

define a CT-representation $\varepsilon^{(m+1)} : E^{(m+1)} \longrightarrow \mathrm{M}_{2t}(\Omega[z]/(z^2))$.

Proof. Any element of $E^{(m+1)}$ can be uniquely written as $g + hv_{m+1}$, where

$$g = \sum_{1 \leq i_1 < \ldots < i_k \leq m} c_{i_1,\ldots,i_k} v_{i_1} \cdots v_{i_k} \quad \text{and} \quad h = \sum_{1 \leq j_1 < \ldots < j_l \leq m} d_{j_1,\ldots,j_l} v_{j_1} \cdots v_{j_l}$$

are in $E^{(m)}$ with $c_{i_1,\ldots,i_k}, d_{j_1,\ldots,j_l} \in K$. For $1 \leq i_1 < \cdots < i_k \leq m$ and $1 \leq j_1 < \cdots < j_l \leq m$, our assignment gives

$$v_{i_1} \cdots v_{i_k} \longmapsto \begin{bmatrix} \varepsilon^{(m)}(v_{i_1} \cdots v_{i_k}) & 0 \\ 0 & (-1)^k \varepsilon^{(m)}(v_{i_1} \cdots v_{i_k}) \end{bmatrix}$$

and

$$v_{j_1} \cdots v_{j_l} v_{m+1} \longmapsto \begin{bmatrix} 0 & \varepsilon^{(m)}(v_{j_1} \cdots v_{j_l})\widehat{z} \\ (-1)^l \varepsilon^{(m)}(v_{j_1} \cdots v_{j_l})\widehat{z} & 0 \end{bmatrix}.$$

Thus

$$\varepsilon^{(m+1)}(g + hv_{m+1}) = \begin{bmatrix} \varepsilon^{(m)}(g_0 + g_1) & \varepsilon^{(m)}(h_0 + h_1)\widehat{z} \\ \varepsilon^{(m)}(h_0 - h_1)\widehat{z} & \varepsilon^{(m)}(g_0 - g_1) \end{bmatrix},$$

where $g = g_0 + g_1$ and $h = h_0 + h_1$ are the unique presentations as sums of an even and an odd element (with respect to the natural $\mathbb{Z}_2$-grading $E^{(m)} = E_0^{(m)} \oplus E_1^{(m)}$). Straightforward verification shows that

$$\varepsilon^{(m+1)} : E^{(m+1)} \longrightarrow \mathrm{M}_{2t}(\Omega[z]/(z^2))$$

is an injective homomorphism of K-algebras. In view of

$$\operatorname{tr}(\varepsilon^{(m+1)}(g + hv_{m+1})) = \operatorname{tr}(\varepsilon^{(m)}(g_0 + g_1)) + \operatorname{tr}(\varepsilon^{(m)}(g_0 - g_1)) \in K,$$

we deduce that $\varepsilon^{(m+1)}$ is a CT-representation of $E^{(m+1)}$. ∎

Corollary 3.2: *For any integer $m \geq 2$ there exists a CT-representation*

$$\varepsilon^{(m)} : E^{(m)} \longrightarrow \mathrm{M}_{2^{m-1}}(K[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2))$$

of $E^{(m)}$, where $(z_1^2, \ldots, z_m^2)$ is the ideal of $K[z_1, \ldots, z_m]$ generated by the monomials $z_1^2, \ldots, z_m^2$.

Proof. Starting from $\varepsilon^{(2)} : E^{(2)} \longrightarrow \mathrm{M}_2(K[z_1, z_2]/(z_1^2, z_2^2))$ and using

$$(K[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2))[z]/(z^2) \cong K[z_1, \ldots, z_m, z_{m+1}]/(z_1^2, \ldots, z_m^2, z_{m+1}^2),$$

iteration of the construction in Theorem 3.1 gives the desired CT-representation. ∎

Proposition 3.3: *If $\varepsilon : E^{(m)} \longrightarrow \mathrm{M}_t(\Omega)$ is a (not necessarily CT-) representation of $E^{(m)}$ for some integers $m, t \geq 2$ over a commutative K-algebra Ω, then $m \leq 2t - 1$.*

Proof. Let $S_m(x_1, \ldots, x_m) = \sum_{\pi \in \mathrm{Sym}(m)} \operatorname{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(m)}$ be the standard polynomial in $K\langle x_1, \ldots, x_m \rangle$. The well-known Amitsur–Levitzki theorem (see [R]) asserts that $S_{2t} = 0$ is a polynomial identity on $\mathrm{M}_t(\Omega)$. The existence of the embedding ε ensures that $S_{2t} = 0$ is also an identity on $E^{(m)}$. On the other hand,

$$S_m(v_1, \ldots, v_m) = m!\, v_1 \cdots v_m \neq 0$$

shows that $S_m = 0$ is not a polynomial identity on $E^{(m)}$. If $2t \leq m$, then $S_m = 0$ follows from $S_{2t} = 0$ (in any algebra). Thus we have $2t \nleq m$. ∎

Theorem 3.4 ("Cayley–Hamilton"): *If $A \in \mathrm{M}_n(E^{(m)})$ is an $n \times n$ matrix, then A satisfies an identity of the form*

$$A^{2^{m-1}n} + c_1 A^{(2^{m-1}n)-1} + \cdots + c_{(2^{m-1}n)-1} A + c_{2^{m-1}n} I_n = 0,$$

where $c_i \in K$, $1 \leq i \leq 2^{m-1}n$.

Proof. A combination of Theorem 2.1 and Corollary 3.2 gives the identity. ∎

Remark 3.5: In particular, we know the following about embeddability of $E^{(3)}$ into matrix algebras. By Corollary 3.2, $E^{(3)}$ admits a CT-embedding into a

4×4 matrix algebra. On the other hand, it has no CT-embedding into any 2×2 matrix algebra. The simple reason is that for $c_1, c_2 \in K$, the 2×2 Cayley–Hamilton identity

$$(v_1 + v_2v_3)^2 + c_1(v_1 + v_2v_3) + c_2 = 0$$

never holds in $E^{(3)}$, although such an identity should hold in $E^{(3)}$ by Theorem 2.1 if it CT-embeds into a 2×2 matrix algebra. We could not decide whether $E^{(3)}$ has a CT-embedding into a 3×3 matrix algebra.

For $m = 3$ and $t = 2$ the inequality $m \leq 2t - 1$ in Proposition 3.3 becomes equality, thus Proposition 3.3 does not contradict the existence of a non-CT-representation $\varepsilon : E^{(3)} \longrightarrow \mathrm{M}_2(\Omega)$. Indeed, the assignments

$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_1 \longmapsto \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}, v_2 \longmapsto \begin{bmatrix} y & 2y \\ -2y & -y \end{bmatrix}, v_3 \longmapsto \begin{bmatrix} -2z & -z \\ z & 2z \end{bmatrix}$$

define a K-embedding $E^{(3)} \longrightarrow \mathrm{M}_2(K[x,y,z]/(x^2,y^2,z^2))$. Since

$$v_1v_2v_3 \longmapsto \begin{bmatrix} 3xyz & 0 \\ 0 & 3xyz \end{bmatrix},$$

this embedding is not a CT-representation.

Remark 3.6: Any matrix $A \in \mathrm{M}_n(E)$ (here $\mathrm{char}(K) = 0$ is essential) satisfies left and right Cayley–Hamilton identities of the form

$$A^{n^2} + g_1A^{n^2-1} + \cdots + g_{n^2-1}A + g_{n^2}I_n = 0,$$
$$A^{n^2} + A^{n^2-1}h_1 + \cdots + Ah_{n^2-1} + I_nh_{n^2} = 0,$$

and a central Cayley–Hamilton identity of the form

$$A^{2n^2} + u_1A^{2n^2-1} + \cdots + u_{2n^2-1}A + u_{2n^2}I_n = 0,$$

where $g_i, h_i \in E$, $1 \leq i \leq n^2$ and $u_j \in E_0$, $1 \leq j \leq 2n^2$ (see [Sz1]).

THEOREM 3.7: *The standard identity $S_{2^mn} = 0$ of degree 2^mn is a polynomial identity on $\mathrm{M}_n(E^{(m)})$.*

Proof. Since the natural extension

$$(\varepsilon^{(m)})_n : \mathrm{M}_n(E^{(m)}) \longrightarrow \mathrm{M}_n(\mathrm{M}_{2^{m-1}}(K[z_1,\ldots,z_m]/(z_1^2,\ldots,z_m^2)))$$

of $\varepsilon^{(m)}$ (in Corollary 3.2) is a K-embedding and

$$\mathrm{M}_n(\mathrm{M}_{2^{m-1}}(K[z_1,\ldots,z_m]/(z_1^2,\ldots,z_m^2))) \cong \mathrm{M}_{2^{m-1}n}(K[z_1,\ldots,z_m]/(z_1^2,\ldots,z_m^2))$$

satisfies $S_{2^m n} = 0$ by the Amitsur–Levitzki theorem, the proof is complete. ∎

If $n = 1$, then Theorems 3.4 and 3.7 are far from being sharp.

Remark 3.8: Using Theorem 5.5 of Domokos [D] and the fact that $S_{m+2} = 0$ is an identity on $E^{(m)}$, we obtain that $S_{(m+1)n^2+1} = 0$ is an identity on $\mathrm{M}_n(E^{(m)})$.

The left regular representation of $E^{(m)}$ is an embedding

$$\lambda^{(m)} : E^{(m)} \longrightarrow \mathrm{End}_K(E^{(m)}) \cong \mathrm{M}_{2^m}(K)$$

of K-algebras, where $\mathrm{End}_K(E^{(m)})$ is the algebra of all K-linear maps of the 2^m-dimensional vector space ${}_K E^{(m)}$ and

$$\lambda^{(m)}(g) : E^{(m)} \longrightarrow E^{(m)}$$

is the left multiplication by $g \in E^{(m)}$. The size of the matrix algebra is 2^{m-1} in Corollary 3.2, half of the size 2^m provided by the above left regular representation. On the other hand, the base field K is replaced by a much bigger base ring (K-algebra) in Corollary 3.2. Since we cannot derive Theorems 3.4 and 3.7 from the regular representation, our half-sized embedding is really better in many ways than the regular representation.

If we keep K as the base field, then the next theorem gives a lower bound for the matrix size of any possible embedding of $E^{(m)}$.

THEOREM 3.9: *If $\lambda : E^{(m)} \longrightarrow \mathrm{M}_t(K)$ is an embedding of K-algebras, then* $3 \cdot 2^{m-2} = 2^{m-1} + 2^{m-2} \leq \lfloor \frac{t^2}{4} \rfloor + 1$.

Proof. Consider the K-subalgebra $E_0^{(m)}[v_m]$ of $E^{(m)}$ generated by the central $E_0^{(m)}$ and the generator v_m. Clearly, $E_0^{(m)}[v_m]$ is commutative, and any element of $E_0^{(m)}[v_m]$ is of the form $g_0 + h_0 v_m$, where $g_0 \in E_0^{(m)}$ and $h_0 \in E_0^{(m-1)}$. It follows that $\dim_K E_0^{(m)}[v_m] = 2^{m-1} + 2^{m-2}$. Since $\lambda(E_0^{(m)}[v_m]) \subseteq \mathrm{M}_t(K)$ is a commutative subalgebra of $\mathrm{M}_t(K)$, we can apply Schur's inequality (see [M]):

$$2^{m-1} + 2^{m-2} = \dim_K E_0^{(m)}[v_m] = \dim_K \lambda(E_0^{(m)}[v_m]) \leq \left\lfloor \frac{t^2}{4} \right\rfloor + 1. \qquad ∎$$

Remark 3.10: Since M. Domokos and M. Zubor recently proved (see [DZ]) that $\dim_K C \leq 3 \cdot 2^{2r-2}$ for any commutative K-subalgebra $C \subseteq E^{(2r)}$ of $E^{(2r)}$, the argument in the above proof cannot be improved if $m = 2r$ is even.

4. Skew polynomial rings and embeddings

For a ring (K-algebra) endomorphism $\sigma : R \longrightarrow R$, consider the skew polynomial ring (K-algebra) $R[w,\sigma]$ in the skew indeterminate w. The elements of $R[w,\sigma]$ are left polynomials of the form $f(w) = a_0 + a_1 w + \cdots + a_k w^k$ with $a_0, a_1, \ldots, a_k \in R$. Besides the obvious addition, we have the following multiplication rule in $R[w,\sigma]$: $wr = \sigma(r)w$ for all $r \in R$ and

$$(a_0 + a_1 w + \cdots + a_k w^k)(b_0 + b_1 w + \cdots + b_l w^l) = c_0 + c_1 w + \cdots + c_{k+l} w^{k+l},$$

where

$$c_m = \sum_{i+j=m, i\geq 0, j\geq 0} a_i \sigma^i(b_j).$$

If σ is an involution, then w^2 is a central element of $R[w,\sigma]$: we have $\sigma(\sigma(r)) = r$ and $w^2 r = w\sigma(r)w = \sigma(\sigma(r))w^2 = rw^2$ for all $r \in R$, moreover w^2 commutes with the powers of w. Thus the ideal (w^2) of $R[w,\sigma]$ generated by w^2 can be written as $(w^2) = R[w,\sigma]w^2 = w^2 R[w,\sigma]$. Consider the factor ring (K-algebra) $R[w,\sigma]/(w^2)$; then for any element $f(w) \in R[w,\sigma]$ there exists exactly one left polynomial of the form $r + sw \in R[w,\sigma]$ in the residue class $f(w) + (w^2)$. Hence the elements of $R[w,\sigma]/(w^2)$ can be represented by linear left polynomials with coefficients in R, and the multiplication in $R[w,\sigma]/(w^2)$ is the following:

$$(r + sw)(p + qw) = rp + (rq + s\sigma(p))w,$$

where $r, s, p, q \in R$.

If $R = E = E_0 \oplus E_1$ (or $R = E^{(m)} = E_0^{(m)} \oplus E_1^{(m)}$) is the natural $\mathbb{Z}_2$-grading, then $\sigma(g) = \sigma(g_0 + g_1) = g_0 - g_1$ defines a natural involution (here $g = g_0 + g_1$ is the unique presentation as a sum of an even and an odd element). It is easy to see that $E[w,\sigma]/(w^2) \cong E$ and $E^{(m)}[w,\sigma]/(w^2) \cong E^{(m+1)}$ as K-algebras.

We note that the idea of considering $R[w,\sigma]/(w^2)$ comes from [SSz].

THEOREM 4.1 ("Fundamental Embedding"): *For an involution $\sigma : R \longrightarrow R$, putting*

$$\mu(r + sw + (w^2)) = \begin{bmatrix} r + (z^2) & sz + (z^2) \\ \sigma(s)z + (z^2) & \sigma(r) + (z^2) \end{bmatrix}$$

(with $r, s \in R$) gives a K-embedding $\mu : R[w,\sigma]/(w^2) \longrightarrow \mathrm{M}_2(R[z]/(z^2))$.

Proof. We only have to prove the multiplicative property of μ:

$$\begin{aligned}\mu((r+sw+&(w^2))(p+qw+(w^2)))\\ &=\mu(rp+(rq+s\sigma(p))w+(w^2))\\ &=\begin{bmatrix} rp+(z^2) & (rq+s\sigma(p))z+(z^2)\\ \sigma(rq+s\sigma(p))z+(z^2) & \sigma(rp)+(z^2)\end{bmatrix}\\ &=\begin{bmatrix} r+(z^2) & sz+(z^2)\\ \sigma(s)z+(z^2) & \sigma(r)+(z^2)\end{bmatrix}\cdot\begin{bmatrix} p+(z^2) & qz+(z^2)\\ \sigma(q)z+(z^2) & \sigma(p)+(z^2)\end{bmatrix}\\ &=\mu(r+sw+(w^2))\mu(p+qw+(w^2)).\end{aligned}$$ ∎

The following is a broad generalization of Theorem 3.1.

THEOREM 4.2: *Let $\varepsilon : R \longrightarrow \mathrm{M}_t(\Omega)$ be a CT-representation of R over a commutative K-algebra Ω for some integer $t \geq 2$. If $\sigma : R \longrightarrow R$ is an involution, then there exists an induced CT-representation $\varepsilon^* : R[w,\sigma]/(w^2) \longrightarrow \mathrm{M}_{2t}(\Omega[z]/(z^2))$ of the factor $R[w,\sigma]/(w^2)$.*

Proof. Consider the natural extension

$$\overline{\varepsilon}_2 : \mathrm{M}_2(R[z]/(z^2)) \longrightarrow \mathrm{M}_2((\mathrm{M}_t(\Omega)[z])/(z^2)) \cong \mathrm{M}_{2t}(\Omega[z]/(z^2))$$

of $\overline{\varepsilon} : R[z]/(z^2) \longrightarrow (\mathrm{M}_t(\Omega)[z])/(z^2)$, where $\overline{\varepsilon}(r+sz+(z^2)) = \varepsilon(r)+\varepsilon(s)z+(z^2)$ for $r,s \in R$. Using the map $\mu : R[w,\sigma]/(w^2) \longrightarrow \mathrm{M}_2(R[z]/(z^2))$ in Theorem 4.1, the composition $\varepsilon^* = \overline{\varepsilon}_2 \circ \mu$ gives the induced CT-representation. Indeed, $\mathrm{tr}(\varepsilon^*(r+sw+(w^2))) = (\mathrm{tr}(\varepsilon(r))+(z^2))+(\mathrm{tr}(\varepsilon(\sigma(r)))+(z^2)) \in K+(z^2)$. ∎

Let $\sigma : R \longrightarrow R$ be a (K-algebra) endomorphism with $\sigma^t = 1$ (such a σ is an automorphism). Now w^t is a central element of $R[w,\sigma]$ and the ideal (w^t) of $R[w,\sigma]$ can be written as $(w^t) = R[w,\sigma]w^t = w^tR[w,\sigma]$. We close this section by mentioning the following generalization of Theorem 4.1 (see [Sz2]), which seems to have applications in Galois theory.

THEOREM 4.3: *For an endomorphism $\sigma : R \longrightarrow R$ with $\sigma^t = 1$, putting*

$$\mu(r_0+r_1w+\cdots+r_{t-1}w^{t-1}+(w^t)) = [\sigma^{i-1}(r_{j-i})z^{j-i}+(z^t)]_{t\times t}$$

gives an embedding $\mu : R[w,\sigma]/(w^t) \longrightarrow \mathrm{M}_t(R[z]/(z^t))$, where the difference $j-i \in \{0,1,\ldots,t-1\}$ is taken modulo t, and the element $\sigma^{i-1}(r_{j-i})z^{j-i}+(z^t)$ of the factor $R[z]/(z^t)$ is in the (i,j) position of the $t\times t$ matrix

$$[\sigma^{i-1}(r_{j-i})z^{j-i}+(z^t)]_{t\times t}.$$

The trace of $[\sigma^{i-1}(r_{j-i})z^{j-i} + (z^t)]_{t\times t}$ *is in* $R^\sigma + (z^t)$*, where*

$$R^\sigma = \{r \in R \mid \sigma(r) = r\}$$

is the fixed ring of σ.

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