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DESINGULARIZATION PRESERVING STABLE SIMPLE NORMAL CROSSINGS

BY

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ABSTRACT

The subject is partial resolution of singularities. Given an algebraic variety X (not necessarily equidimensional) in characteristic zero (or, more generally, a pair (X, D), where D is a divisor on X), we construct a functorial desingularization of all but *stable simple normal crossings (stable-snc)* singularities, by smooth blowings-up that preserve such singularities. A variety has stable simple normal crossings at a point if, locally, its irreducible components are smooth and transverse in some smooth embedding variety. We also show that our main assertion is false for more general simple normal crossings singularities.

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1. Introduction

The subject of this article is partial resolution of singularities. Let X denote a (reduced) algebraic variety X over a field of characteristic zero and let D denote a $\mathbb{Q}$-Weil divisor on X. Our main result (see Theorem 1.8) asserts that we can resolve all but *stable simple normal crossings* singularities of (X, D) by a finite sequence of blowings-up, each of which is an isomorphism over the stable simple normal crossings points of its target. See Definitions 1.1, 1.4, and Lemma 1.2. The theorem is functorial (Remarks 1.9) and is obtained by an algorithm. Theorem 1.8 is false for more general normal crossings singularities; see Example 1.10. (Of course, a weaker desingularization result may hold in this case.) We do not assume that X is equidimensional; in particular, we do not define simple normal crossings singularities in a way that they are necessarily hypersurface singularities, as in [8]. Our main theorem generalizes [8]; simple normal crossings hypersurface singularities are necessarily stable. The proof of the theorem follows the philosophy of [6] that the desingularization invariant of [3] and [5] can be used together with natural geometric information to compute local normal forms of singularities.

Pairs (X, D) appear naturally in algebraic geometry when considering, for example, boundaries of non-closed varieties, markings on varieties in moduli problems, or loci of indeterminacy of rational mappings.

In birational geometry, partial desingularization is sometimes needed to resolve all singularities except those in some class to be preserved or which cannot be eliminated. For example, in order to simultaneously resolve the singularities of curves in a parametrized family, one needs to allow special fibres that have normal crossings singularities. Likewise, log resolution of singularities of a divisor produces a divisor with simple normal crossings, and stable simple normal crossings singularities appear when studying the higher-dimensional analogues of stable curves. There are recent applications of [8] by Fujino to several interesting results in the log minimal model program; e.g., [9], [10], [11]. See also [6] and [12].

Definition 1.1: A (reduced) algebraic variety X has a **stable simple normal crossings** (**stable-snc**) singularity at a point a (or X is **stable-snc at** a) if the irreducible components $X^{(i)}$ of X are smooth at a, and are transverse at a in some smooth embedding variety Z of a neighbourhood of a in X (i.e., the sum of the codimensions in Z of the tangent spaces of the $X^{(i)}$ at a equals the codimension of the intersection of the tangent spaces).

Note that, if X and Z are as in the preceding definition, then Z is necessarily a minimal local embedding variety. We say that X has a **simple normal crossings** (**snc**) singularity at a if there is a smooth local embedding variety Z at a with a system of regular coordinates in which each $X^{(i)}$ is a coordinate subspace. (This is a more general notion than "X is locally isomorphic to a simple normal crossings divisor", often used as the definition.)

LEMMA 1.2: *Let X denote an algebraic variety, and let $X^{(i)}$ denote the irreducible components of X. Let $a \in X$. Assume that each $X^{(i)}$ is smooth at a. Then the following conditions are equivalent:*

(1) *X is stable-snc at a.*
(2) *If Z is a smooth local embedding variety of X at a (of any possible dimension), then Z admits a system of regular coordinates*

$$(x_1, \ldots, x_p, z_1, \ldots, z_r, w_1, \ldots, w_s)$$

at a, with respect to which each

$$X^{(i)} = (\{x_k = 0\}_{k \in I_i}, z_1 = \cdots = z_r = 0),$$

for some partition $\bigcup_{i=1}^m I_i$ of $\{1, \ldots, p\}$.
(3) *X is snc at a and there is a smooth local embedding variety in which any two components $X^{(i)}$ are transverse at a.*
(4) *The intersection of the $X^{(i)}$ is smooth (as a scheme) at a, and X admits a smooth local embedding variety Z at a in which the sum of the codimensions of the $X^{(i)}$ equals the codimension of their intersection. (See also (3.1).)*

(5) *X admits a smooth local embedding variety at a in which the $X^{(i)}$ are smooth and in general position.*

Remark 1.3: A local embedding variety satisfying any of the conditions (3)–(5) is necessarily minimal. It follows from Lemma 1.2 that, if X is stable-snc at a, then the conditions (3)–(5) and the transversality property of Definition 1.1 are satisfied in any minimal embedding variety of X at a.

Definition 1.4: Let X denote a (reduced) algebraic variety and let $X^{(i)}$ denote the irreducible components of X. Let D denote a $\mathbb{Q}$-Weil divisor on X, i.e., D is a finite linear combination of reduced, irreducible subvarieties of X, each of codimension one in any $X^{(i)}$ that contains it. We say that (X, D) has (or is) **stable simple normal crossings** (**stable-snc**) at a point a if there is a local embedding $X \hookrightarrow Z$ at a, where Z is smooth and admits a regular system of coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r, w_1, \ldots, w_s)$ at a in which

(1) each $X^{(i)} := (\{x_k = 0\}_{k \in I_i}, z_1 = \cdots = z_r = 0)$, for some partition $\bigcup_{i=1}^m I_i$ of $\{1, \ldots, p\}$;
(2) $D = \sum_{j=1}^k \alpha_j (y_j = 0)|_X$ (locally at a), for some $\alpha_j \in \mathbb{Q}$.

We also say that the pair (X, D) is **stable-snc** if it is stable-snc at every point.

It follows that, if (X, D) is stable-snc at a, then any smooth local embedding variety at a admits a regular coordinate system as in Definition 1.4.

Observe that in Definition 1.4 we do not assume *a priori* that D arises from the intersection with X of a divisor on Z, though of course this property is satisfied if (X, D) is stable-snc.

Example 1.5: Consider

$$X := (x = y = 0) \cup (y = z = 0) \cup (x = z = 0) \subset \mathbb{A}^3_{x,y,z}.$$

Then X is snc at the origin but not stable-snc. On the other hand,

$$Y := (x = y = 0) \cup (y = z = 0)$$

is stable-snc.

Example 1.6: If $X = (xy = 0) \subset \mathbb{A}^3$ and $D = a_1 D_1 + a_2 D_2$, where $D_1 = (x = z = 0)$ and $D_2 = (y = z = 0)$, then the pair (X, D) is stable-snc if and only if $a_1 = a_2$.

Definition 1.7: **Transform** of a pair (X, D). Consider a sequence of blowings-up

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t, \tag{1.1}$$

where each σ_{j+1} has smooth centre $C_j \subset X_j$. Write $\widetilde{D}_0 := D$ and, for each $j = 0, 1, \ldots$, set $\widetilde{D}_{j+1} :=$ the birational transform of $\widetilde{D}_j$ plus the **exceptional divisor** $\sigma_{j+1}^{-1}(C_j)$ of σ_{j+1}.

Theorem 1.8: *Let X denote a (reduced) algebraic variety in characteristic zero and let D denote a $\mathbb{Q}$-Weil divisor on X. Then there is a sequence of blowings-up* (1.1) *such that*

(1) *$(X_t, \widetilde{D}_t)$ has only stable-snc singularities;*
(2) *each σ_{j+1} is an isomorphism over the locus of stable-snc points of $(X_j, \widetilde{D}_j)$.*

Remarks 1.9: (1) In the special case that $D = 0$, each $\widetilde{D}_j$ is the exceptional divisor of the morphism $\sigma_1 \circ \cdots \circ \sigma_j$, so that condition (1) of Theorem 1.8 is a stronger assertion than "X_t is stable-snc".

(2) In the special case that X is smooth, we say that D is a **simple normal crossings** or **snc** divisor on X if (X, D) is stable-snc (i.e., Definition 1.4 is satisfied with $p = 0$ at every point of X). This means that the components of D are smooth and intersect transversely. Theorem 1.8 in this case provides **log resolution of singularities** of D by a sequence of blowings-up (1.1) such that each σ_{j+1} is an isomorphism over the snc locus of $\widetilde{D}_j$. This is proved in [1, Thm. 1.5]. Earlier versions can be found in [15], [3, Sect. 12], [12] and [6, Thm. 3.1].

(3) The desingularization morphism of Theorem 1.8 is functorial in the category of pairs (X, D) with a fixed ordering on the components of X, and with respect to étale morphisms (or smooth morphisms; cf. [7, §6.3]) that preserve the number of irreducible components of X and D at every point. If $D = 0$, then the sequence of blowings-up is independent of the ordering of the components of X. Note that desingularization preserving only snc or stable-snc singularities cannot be functorial with respect to étale morphisms in general (as in the case of functorial resolution of singularities), because a normal crossings point becomes snc after an étale morphism.

The following example shows that Theorem 1.8 does not hold for more general snc singularities.

Example 1.10: Consider

$$X := (z = x = 0) \cup (z = y = 0) \cup (z + xw = x + y = 0) \subset \mathbb{A}^4_{w,x,y,z}.$$

Then X is snc at every point except the origin ($w = x = y = z = 0$), so the only blowing-up permissible as the first in the sequence (1.1) in Theorem 1.8 has centre the origin. In the "w-chart" with coordinates $(w, x/w, y/w, z/w)$, the strict transform X' of X is given by the same equations as X, and the exceptional divisor $D' = (w = 0)$. Therefore, (X', D') is snc except at 0, and the non-snc singularity at 0 cannot be eliminated by continuing to blow up only non-snc points.

Theorem 1.8 follows from a stronger version, Theorem 1.16 below, for which it will be convenient to work with triples (X, D, E) that distinguish the birational transforms of D from the exceptional divisors. In this notation, (X, D) has the same meaning as in Definition 1.4, and E is an ordered snc divisor on X in the sense of Definition 1.11 following (usually with all coefficients $a_k = 1$).

Definition 1.11: Let Z denote a smooth variety. An (**ordered**) **snc divisor** E on Z is a finite linear combination $\sum a_k H_k$ of (ordered) subvarieties H_k, where each $a \in Z$ admits a coordinate neighbourhood in which every H_k is a coordinate hypersurface. We identify the **support** of E, $\operatorname{supp} E := \sum H_k$, with the (ordered) set of smooth hypersurfaces $\{H_k\}$. The H_k are called the **components** of E.

Let X denote a variety. An (**ordered**) **snc divisor** E on X is a finite linear combination $\sum a_k H_k$ of (ordered) subvarieties H_k, such that each $a \in X$ admits a neighbourhood U and an embedding $X|_U \hookrightarrow Z$, Z smooth, where $E|_U$ is induced by an (ordered) snc divisor E_Z on Z (and each nonempty $H_k|_U$ is the restriction of a component of E_Z). Note that the **components** H_k of E need not be irreducible (or reduced). When all $a_k = 1$, we again identify E with the (ordered) set of smooth hypersurfaces $\{H_k\}$.

We also assume that E is a Weil divisor (as in Definition 1.4).

Remark 1.12: The latter assumption only excludes the possibility that a component of E contain an irreducible component of X. This possibility does not arise, in any case, for the exceptional divisor of a sequence of blowings-up as given by our main theorems. If we were to allow it, the proofs of Theorems 1.8 and 1.16 would simply require an additional step to separate and blow up such irreducible components of X (which contain no stable-snc points of (X, E)).

Let X denote a variety and let E denote an snc divisor on X. Consider a sequence of blowings-up

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t, \tag{1.2}$$

where each σ_{j+1} has smooth centre $C_j \subset X_j$. Write $E_0 = E$ and, for each $j = 0, 1, \ldots$, set $E_{j+1} :=$ the birational transform of E_j (with the induced ordering) plus the exceptional divisor $\sigma_{j+1}^{-1}(C_j)$ of σ_{j+1} (as the last element).

Definition 1.13: A smooth blowing-up $\sigma : X' \to X$ (i.e., a blowing-up with smooth centre $C \subset X$) is **admissible** (or **admissible** for (X, E)) if C is **snc with respect to** E (where the latter means that, for each $a \in C$, there is a neighbourhood U of a in X and an embedding $X|_U \hookrightarrow Z$ as above, where Z has a coordinate system in which C is a coordinate subspace and each component of E_Z is a coordinate hyperplane). The sequence of blowings-up (1.2) is **admissible** if each σ_{j+1} is admissible for (X_j, E_j). We will speak of j as a "year" in the "history" of blowings-up (1.2).

It follows from Definition 1.13 that, if E_j is snc and σ_{j+1} is admissible, then E_{j+1} is snc.

Definition 1.14: We say that (X, D, E) has (or is) **stable simple normal crossings** (**stable-snc**) at a point $a \in X$ if $(X, D + E)$ is stable-snc at a. We say that (X, D, E) is **stable-snc** if it is stable-snc at every point.

Definition 1.15: **Transform** of a triple (X, D, E). Consider a sequence of blowings-up (1.2) that is admissible for (X, E). Write $D_0 = D$ and $E_0 = E$. For each $j = 0, 1, \ldots$, set $D_{j+1} :=$ the birational transform of D_j, and define E_{j+1} as above.

Comparing this notation with that of Definition 1.7, note that, if $E = 0$, then $\widetilde{D}_j = D_j + E_j$, for each j. The notation of Definitions 1.7 and 1.15 will be used throughout the article. Superscripts will be used to denote irreducible components of varieties.

THEOREM 1.16: *Let X denote a (reduced) variety in characteristic zero. Let D denote a $\mathbb{Q}$-Weil divisor and E an ordered simple normal crossings divisor on X. Then there is a sequence of admissible smooth blowings-up* (1.2) *such that*

(1) (X_t, D_t, E_t) *has only stable-snc singularities;*
(2) *each σ_{j+1} is an isomorphism over the locus of stable-snc points of (X_j, D_j, E_j).*

Moreover, the association of the desingularization sequence (1.2) *to* (X, D, E) *is functorial in the category of triples* (X, D, E) *with a fixed ordering on the components of X, and with respect to étale (or smooth) morphisms that preserve the number of irreducible components of X at every point. (In the category of such triples with $D = 0$, an ordering of the components of X is not necessary for functoriality.)*

Theorem 1.8 is a consequence of Theorem 1.16.

An ordering on the set of irreducible components of X in the functoriality assertion in Theorem 1.16 is needed because of the inductive nature of our proof; see Section 8. We have no reason to believe that this is an intrinsic requirement. For example, we can remove the dependence on an ordering in the simpler setting of [8] (unpublished).

To prove Theorem 1.16, we construct the sequence of blowings-up in two main parts. We first make the transform of (X, E) stable-snc, and then perform further blowings-up to make the transform of (X, D, E) stable-snc. Comparing this article with previous papers, the first part is rather analogous to [1], while the second is closer to [8]. Nevertheless, the main new arguments here are for the first part; the second part differs from [8] in a more technical way.

The paper does not rely on technical details of a proof of resolution of singularities, but does use several basic notions and constructions concerning the desingularization algorithm and the desingularization invariant. The desingularization algorithm is used in [1], [6], [8] mainly in the case of a hypersurface or (weak desingularization of) an ideal. For desingularization of more general varieties as treated here, the notion of presentation of an invariant (of origin in [3]) is a useful tool that will be recalled in Section 2 below, with examples needed for the paper. Given a local invariant that admits a presentation, one can functorially construct a sequence of blowings-up along which the invariant never increases and eventually decreases [4], [5, Thm. 7.1].

In Section 2, we also briefly recall several other ideas from resolution of singularities that we use; in particular, the notions of maximal contact and the monomial part (or divisorial part) of an ideal with respect to an exceptional divisor, as well as the inductive construction based on these notions that is used to define the desingularization invariant. These techniques are used in Sections 2, 3 and 4. They are relatively elementary, but the desingularization algorithm involves a delicate recursion that is reflected in Sections 3 and 4. We will provide references to the expository *Crash course on the desingularization invariant* [6, Appendix]. See also the summary of desingularization techniques in [1, Sect. 2]. We have tried to make the article as self-contained as possible, though we give detailed references to [8] for certain arguments that are used in the same way here.

Beyond Theorem 1.16, a number of techniques in this paper may be of interest in other applications, in particular, other partial desingularization problems. In Section 2.4, for example, we give an algorithm for simultaneous desingularization of a finite collection of closed subvarieties of a given variety.

2. Presentation of an invariant

We will consider several local invariants of algebraic varieties X with values in partially ordered sets. These invariants $\iota = \iota_X$ provide different measures of singularity, and the desingularization algorithm for an associated marked ideal (a *presentation* of the invariant ι; see §2.2) is used to decrease the maximum values of ι. The desingularization algorithm for a marked ideal $\underline{\mathcal{I}}$ prescribes a sequence of blowings-up, determined by the maximum loci of a *desingularization invariant* $\mathrm{inv}_{\underline{\mathcal{I}}}$ [5]. A presentation of ι can be used to extend ι to a desingularization invariant inv_ι (§2.3), and the resulting desingularization algorithm (blowings-up determined by the maximum loci of inv_ι) reduces the invariant ι to its value at a general point of X.

In §§2.4 and 2.5, we will illustrate these ideas by constructing presentations for two local invariants that intervene in our proofs of Theorems 1.8 and 1.16. The first is used to prove that any algebraic variety can be transformed to a variety all of whose irreducible components are smooth, by a sequence of blowings-up that preserve points where all components are already smooth (Theorem 2.4). This will be the first step in the proof of our main result Theorem 1.8; the approach is different from that of [1] and [8], so that Theorem 1.8 involves an algorithm that differs from those of the latter, even in the special case that X is a hypersurface. In the following sections, we will remark certain simplifications of the remaining steps, relative to [1] and [8], that result from the use of Theorem 2.4.

Let Λ denote a partially ordered set, and let ι denote a **local invariant** with values in Λ. This means that, given an algebraic variety X, there is a function $\iota = \iota_X : X \to \Lambda$ such that, for all $a \in X$, $\iota(a)$ is an invariant of the local étale isomorphism class of X at a.

We will assume that ι satisfies the following three properties:

(1) ι is upper-semicontinuous; in particular, for all $a \in X$, $(\iota(x) \geq \iota(a)) := \{x \in X : \iota(x) \geq \iota(a)\}$ is closed;
(2) ι is **infinitesimally upper-semicontinuous**; i.e., for any smooth blowing-up $\sigma : X' \to X$ such that ι is locally constant on the centre of σ, $\iota(a') \leq \iota(a)$ whenever $a' \in X'$ and $a = \sigma(a')$;
(3) any non-increasing sequence in the value set of ι stabilizes.

Properties (1) and (2) are needed for the notion of a presentation of ι. Property (3) is needed to guarantee the termination of a desingularization algorithm based on the invariant ι.

An important example of a local invariant that satisfies the properties above is the **Hilbert–Samuel function** $\iota(a) = H_{X,a}$ of the local ring $\mathcal{O}_{X,a}$ (see Section 7 and [5, §1.3]). The Hilbert–Samuel function $H_{X,a} \in \mathbb{N}^{\mathbb{N}}$. The latter is partially ordered as follows: if $H_1, H_2 \in \mathbb{N}^{\mathbb{N}}$, then $H_1 \leq H_2$ if $H_1(k) \leq H_2(k)$, for all $k \in \mathbb{N}$.

Definition 2.1: Given a local invariant ι and a variety X with snc divisor E, we say that a sequence of blowings-up (1.2) of X is **admissible** for (X, ι) or for ι (or ι-admissible) if (1.2) is admissible in the sense of Definition 1.13, and ι is locally constant on each centre of blowing up C_j.

2.1. Marked ideals. The desingularization invariant is calculated using marked ideals [6, Def. A.5]—collections of data that are computed iteratively on **maximal contact** subspaces of increasing codimension [6, Def. A.11]. A **marked ideal** $\underline{\mathcal{I}}$ is a quintuple $(Z, N, E, \mathcal{I}, d)$, where $Z \supset N$ are smooth varieties, $E = \sum_{i=1}^{s} H_i$ is an snc divisor on Z that is transverse to N, $\mathcal{I} \subset \mathcal{O}_N$ is an ideal, and $d \in \mathbb{N}$. We will sometimes call N a "maximal contact subspace" by abuse of language, since it typically arises in this way.

The **cosupport** $\operatorname{cosupp}\underline{\mathcal{I}}$ of the marked ideal $\underline{\mathcal{I}}$ is defined as $\{x \in N : \operatorname{ord}_x \mathcal{I} \geq d\}$. A blowing-up σ of Z is **admissible** for $\underline{\mathcal{I}}$ if the centre C of σ is snc with E and $C \subset \operatorname{cosupp}\underline{\mathcal{I}}$. A presentation of an invariant ι (Definition 2.3 following) is a marked ideal whose admissible blowings-up correspond to those that are admissible for the invariant, in the sense of Definition 2.1.

Remark 2.2 (Equivalence of marked ideals): We say that two marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ (with the same ambient variety Z and the same normal crossings divisor E) are **equivalent** if they have the same sequences of *test transformations* (i.e., every test sequence for one is a test sequence for the other). **Test transformations** are transformations of a marked ideal by morphisms of three possible kinds: admissible blowings-up, projections from products with an affine line, and **exceptional blowings-up** [5, Defns. 2.5]. In particular, if $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ are equivalent, then they have the same cosupport and their transforms [6, §A.4] by any sequence of admissible blowings-up have the same cosupport. The remaining two types of test transformations are used to prove functoriality properties of the desingularization invariant and algorithm. We refer the reader to [5, §2] for definitions; we do not need these notions explicitly here.

A **resolution of singularities** of a marked ideal $\underline{\mathcal{I}}$ is a sequence of admissible blowings-up (1.2) after which $\operatorname{cosupp}\underline{\mathcal{I}}' = \emptyset$, where $\underline{\mathcal{I}}'$ is the transform of $\underline{\mathcal{I}}$. Resolution of singularities of a marked ideal [5] is functorial with respect to somewhat smaller equivalence classes of marked ideals, called semicoherent equivalence classes in [3], [4]. (This point was neglected in [5]. It is pointed out by Nobile in [14].) Two marked ideals $\underline{\mathcal{I}}$, $\underline{\mathcal{J}}$ as above are **semicoherent equivalent** if $\underline{\mathcal{I}}|_U$ and $\underline{\mathcal{J}}|_U$ are equivalent, for every open subset U of Z. Such a notion is needed because the desingularization algorithm involves associating **coefficient ideals** to certain marked ideals $\underline{\mathcal{I}}$ [6, §A.5]. Coefficient ideals exist only locally in general; semicoherent equivalence is a way to ensure that the appropriate equivalence class of a coefficient ideal depends only on that of $\underline{\mathcal{I}}$.

2.2. Presentation.

Definition 2.3: Let ι denote a local invariant satisfying properties (1) and (2) above. A **presentation** of ι at $a \in X$ is a marked ideal $\underline{\mathcal{I}} = (Z, N, 0, \mathcal{I}, d)$, where there is an étale open neighbourhood V of a and an embedding $X|_V \hookrightarrow Z$ such that

(1) for every open subset U of Z, a sequence of blowings-up over U is admissible for $\underline{\mathcal{I}}|_U$ if and only if each centre lies in $(\iota = \iota(a))$ (the locus of points where $\iota = \iota(a)$);
(2) the semicoherent equivalence class of $\underline{\mathcal{I}}$ is uniquely determined by ι and $(Z, X|_V)$.

For example, the Hilbert–Samuel function $H_{X,\cdot}$ admits a presentation at any point. In fact, given $a \in X$, there is a presentation $\underline{\mathcal{I}}$ of $H_{X,\cdot}$ at a, as above, where $N = Z =$ a minimal embedding variety of $X|_V$ at a (see [3, Ch. III]). Note that a presentation as defined here is called a "semicoherent presentation" in [3], [4].

In general, even a simple local invariant need not admit a presentation at a point of an arbitrary algebraic variety X. For example, does the *local embedding dimension* $e_{X,a}$ admit a presentation?

The purpose of a presentation is that, according to Definition 2.3, we can decrease the invariant ι over a given point a by resolution of singularities of a corresponding presentation. When ι decreases, we choose a new presentation and repeat the process. Of course, when ι decreases, we have not only the transform of X but also an exceptional divisor; in general, therefore, we have to consider a variety together with a simple normal crossings divisor ("boundary") E.

We can extend the invariant ι to track also the birational transforms of the snc divisor E. Consider a sequence of (X, E)-admissible blowings-up (1.2). Let us write E_j^1 for successive birational transforms of E, so that each $E_j = E_j^1 + \mathcal{E}_j^1$, where $\mathcal{E}_j^1$ denotes the exceptional divisor of the morphism given by the first j blowings-up. If $a \in X$, let $s(a)$ denote the number of components of E at a. Likewise, if $a \in X_j$, for any j, let $s(a)$ or $s_1(a)$ denote the number of components at a of E_j^1. We consider the invariant (ι, s), where such pairs are ordered lexicographically, defined over an ι-admissible sequence of blowings-up (1.2). An admissible sequence (1.2) is called **(ι, s)-admissible** if (ι, s) is locally constant on each centre of blowing up. If $a \in X_j$, we will write $E(a)$, $E^1(a)$ or $\mathcal{E}^1(a)$ to denote the set of components at a of E_j, E_j^1 or $\mathcal{E}_j^1$, respectively.

Suppose that $\underline{\mathcal{I}} = (Z, N, 0, \mathcal{I}, d)$ is a presentation of ι at $a \in X$, and that (near a) E is induced by an snc divisor on Z (which, for simplicity, we also denote E). For each component H of E, let $\mathcal{I}_H$ denote the ideal of H in $\mathcal{O}_Z$ and consider the marked ideal $(\mathcal{I}_H|_N, 1) := (Z, N, 0, \mathcal{I}_H|_N, 1)$. We introduce the **boundary** marked ideal $\mathcal{B} := \sum_{H \ni a} (\mathcal{I}_H|_N, 1)$ (see [3, Def. A.8 and §A.9]), and define $\underline{\mathcal{I}}^1 := \underline{\mathcal{I}} + \underline{\mathcal{B}}$. The equivalence class of the marked ideal $\underline{\mathcal{I}}^1$ depends only on that of $\underline{\mathcal{I}}$ and on E, so that $\underline{\mathcal{I}}^1$ is a presentation of (ι, s) at a in the sense of an obvious generalization of Definition 2.3.

2.3. Desingularization invariant. We define a **desingularization invariant** $\mathrm{inv} = \mathrm{inv}_\iota$ extending the invariant ι by

$$\mathrm{inv}(a) = (\iota(a), s(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a)),$$

where $\mathrm{inv}_{\underline{\mathcal{I}}^1}$ is the desingularization invariant $\mathrm{inv}_{\underline{\mathcal{I}}^1}$ for the marked ideal $\underline{\mathcal{I}}^1$ (see [6, App. A] and [5]. The desingularization invariant inv is defined recursively over a sequence (1.2) of inv-admissible blowings-up: for each j, if inv is defined over $X = X_0 \leftarrow \cdots \leftarrow X_j$ and σ_{j+1} is inv-admissible (i.e., σ_{j+1} is admissible for (X_j, E_j) and inv is locally constant on the centre C_j of σ_{j+1}), then inv extends to X_{j+1}, and the properties (1)–(3) analogous to those of ι above are satisfied by inv in the appropriate sense. The maximum locus of inv provides a global smooth centre of blowing up.

The desingularization invariant $\mathrm{inv}_{\underline{\mathcal{J}}}$ of a marked ideal $\underline{\mathcal{J}}$ depends only on the semicoherent equivalence class of $\underline{\mathcal{J}}$ and the dimension of the maximal contact subspace N. In order to get a well-defined semicontinuous invariant inv_ι, it is necessary to choose N in a way that $\dim N$ has a canonical value; e.g., in a way that $\dim N$ depends only on ι at a, or $\dim N$ is locally constant on $\{x : \iota(x) = \iota(a)\}$. This is an important issue in §§2.4, 2.5 below.

Some of the technology of the desingularization invariant will be used in Sections 3 and 4. Consider a sequence (1.2) of inv_ι-admissible blowings-up. Let $a \in X_j$. The desingularization invariant $\mathrm{inv} = \mathrm{inv}_\iota = (\iota(a), s(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a))$ is a sequence $(\nu_1(a), s_1(a), \ldots, \nu_q(a), s_q(a), \nu_{q+1}(a))$, where $\nu_1(a) = \iota(a)$, $s_1(a) = s(a)$ and $\mathrm{inv}_{\underline{\mathcal{I}}^1}(a) = (\nu_2(a), s_2(a), \ldots, \nu_{q+1}(a))$; each $s_k(a)$ is a nonnegative integer counting the number of elements of a certain block $E^k(a)$ of E_j at a, $\nu_{k+1}(a)$ is a positive rational number, $1 \leq k < q$, and $\nu_{q+1}(a)$ is either 0 or ∞. The successive pairs $(\nu_{k+1}(a), s_{k+1}(a))$, $k \geq 1$, are calculated using marked ideals $\underline{\mathcal{I}}^k = (Z, N^k, \mathcal{E}^k, \mathcal{I}^k, d^k)$, where $N^1 = N$ and N^{k+1} had codimension 1 in N^k, and where $\mathcal{E}^1(a) = E(a) \backslash E^1(a)$ and $\mathcal{E}^{k+1}(a) := \mathcal{E}^k(a) \backslash E^{k+1}(a)$, $k > 0$.

We also introduce truncations of inv. Let $\mathrm{inv}_{k+1}(a)$ denote the truncation of $\mathrm{inv}(a)$ after $s_{k+1}(a)$ (i.e., after the $(k+1)$st pair), and let $\mathrm{inv}_{k+1/2}(a)$ denote the truncation of $\mathrm{inv}(a)$ after $\nu_{k+1}(a)$.

Given $a \in X_j$, let a_i denote the image of a in X_i, $i \leq j$. (We will speak of **year** i in the history of blowings-up.) The **year of birth** of $\mathrm{inv}_{k+1/2}(a)$ (or $\mathrm{inv}_{k+1}(a)$) denotes the smallest i such that $\mathrm{inv}_{k+1/2}(a) = \mathrm{inv}_{k+1/2}(a_i)$ (respectively, $\mathrm{inv}_{k+1}(a) = \mathrm{inv}_{k+1}(a_i)$).

In terms of inv and its truncations, the terms $s_k(a)$ have the following meaning. Let i denote the birth-year of $\mathrm{inv}_{1/2}(a) = \nu_1(a)$, and let $E^1(a)$ denote the collection of components at a of the birational transform of E_i. Then $s_1(a) = \#E^1(a)$. We define $s_{k+1}(a)$, in general, by induction on k: Let i denote the year of birth of $\mathrm{inv}_{k+1/2}(a)$ and let $E^{k+1}(a)$ denote the components at a of the birational transform of $\mathcal{E}_i^k$. Set $s_{k+1}(a) := \#E^{k+1}(a)$. This definition is consistent with that of $s_1(a) = s(a)$ above because, in the year of birth of $\mathrm{inv}_{1/2}(a) = \iota(a)$, a new presentation of ι is chosen and the construction is started again. See [5] or [6, Appendix] for further details.

We write $\mathcal{I}^k$ as the product $\mathcal{M}(\underline{\mathcal{I}}^k) \cdot \mathcal{R}(\underline{\mathcal{I}}^k)$ of its **monomial** and **residual parts**. $\mathcal{M}(\underline{\mathcal{I}}^k)$ is the monomial part with respect to $\mathcal{E}^k$; i.e., the product of the ideals $\mathcal{I}_H$, $H \in \mathcal{E}^k$, each to the power $\mathrm{ord}_{H,a}\mathcal{I}^k$, where $\mathrm{ord}_{H,a}$ denotes the order along H at a. We set

$$\nu_{k+1}(a) := \frac{\mathrm{ord}_a \mathcal{R}(\underline{\mathcal{I}}^k)}{d^k} \quad \text{and} \quad \mu_{H,k+1}(a) := \frac{\mathrm{ord}_{H,a}\mathcal{I}^k}{d^k},\ H \in \mathcal{E}^k;$$

both are invariants of the equivalence class of $\underline{\mathcal{I}}^k$ and $\dim N^k$ (see [6, Def. 5.10]). The marked ideals $\underline{\mathcal{I}}^k$ are constructed iteratively (on the maximal contact subspaces N^k of decreasing dimension); the construction terminates when $\nu_{k+1}(a) = 0$ or ∞.

The passage from $\underline{\mathcal{I}}^k$ to $\underline{\mathcal{I}}^{k+1}$ actually involves two steps: first, from $\underline{\mathcal{I}}^k$ to a **companion ideal** $\underline{\mathcal{G}}(\underline{\mathcal{I}}^k)$ defined using the product decomposition of $\mathcal{I}^k$ above, and secondly, from $\underline{\mathcal{G}}(\underline{\mathcal{I}}^k)$ to $\underline{\mathcal{I}}^{k+1}$ as the **coefficient ideal plus boundary**. For more details, see [6, Appendix] and [1, Sect. 2].

2.4. Simultaneous desingularization of a collection of varieties.

Theorem 2.4: *Let X denote a (reduced) algebraic variety X. Then there is a finite sequence of admissible smooth blowings-up* (1.1) *such that*

1. *every irreducible component of X_t is smooth;*
2. *each σ_{j+1} is an isomorphism over the locus of points where all components of X_j are smooth.*

Moreover, given an snc divisor E on X, there is a sequence of smooth blowings-up as above which is admissible for (X, E). The association of the desingularization sequence to (X, E) is functorial with respect to étale morphisms that preserve the number of irreducible components of X at every point.

Remark 2.5: Consider two local invariants ι_1, ι_2 with values in partially-ordered sets Λ_1, Λ_2, respectively. Given a variety X, we have $(\iota_1, \iota_2) : X \to \Lambda_1 \times \Lambda_2$. There are two natural partial orders on $\Lambda_1 \times \Lambda_2$: (1) the **product order**, $(\lambda_1, \lambda_2) \geq (\kappa_1, \kappa_2)$ if $\lambda_1 \geq \kappa_1$ and $\lambda_2 \geq \kappa_2$, and (2) the lexicographic order. Clearly, for either order, (ι_1, ι_2) is semicontinuous, and infinitesimally semicontinuous, and any non-increasing sequence in its value set stabilizes. The maximal loci of (ι_1, ι_2) with respect to the two orders coincide locally at a point of X, but not necessarily globally.

Suppose that $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2$ are presentations of ι_1, ι_2 (respectively) at a point $a \in X$. Assume that $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2$ have common ambient variety Z and common maximal contact subvariety N. Then $\underline{\mathcal{I}}_1 + \underline{\mathcal{I}}_2$ is a presentation of (ι_1, ι_2), with respect to either order, but the desingularization algorithms based on (ι_1, ι_2), for the two orders, need not coincide: the invariant tells us in what order to assemble the local centres of blowing up given by presentations, and this depends on the partial order on $\Lambda_1 \times \Lambda_2$.

Proof of Theorem 2.4. Let $X^{(1)}, \ldots, X^{(m)}$ denote the irreducible components of X. Consider the local invariant $\iota_{X,a} := (H_{X,a}, H_{X^{(1)},a}, \ldots, H_{X^{(m)},a})$, $a \in X$, given by the Hilbert–Samuel functions of the local rings of X and the $X^{(i)}$ at a (with $H_{X^{(i)},a} = 0$ if $a \notin X^{(i)}$).

We consider $(H, H_1, \ldots, H_m) \in (\mathbb{N}^{\mathbb{N}})^{m+1}$ as a pair

$$(H, (H_1, \ldots, H_m)) \in \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^m,$$

and we use the product order on $\{(H_1, \ldots, H_m) \in (\mathbb{N}^{\mathbb{N}})^m\}$, but the lexicographic order on pairs in $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^m$.

Given $a \in X$, there is a local embedding $X|_U \hookrightarrow Z$ such that E is induced by an snc divisor on Z, and the Hilbert–Samuel functions $H_{X,\cdot}$ and $H_{X^{(i)},\cdot}$, $i = 1, \ldots, m$, admit presentations $\underline{\mathcal{I}} = (Z, N, 0, \mathcal{I}, d)$ and $\underline{\mathcal{I}}^{(i)} = (Z, N, 0, \mathcal{I}^{(i)}, d_i)$, $i = 1, \ldots, m$, where N is a minimal embedding variety for X at a. Then

$$\underline{\mathcal{H}} := \underline{\mathcal{I}} + \sum_{\{i:\, a \in X^{(i)}\}} \underline{\mathcal{I}}^{(i)}$$

is a presentation of $\iota_{X,\cdot}$ at a. We can extend $\iota = \iota_{X,\cdot}$ to a desingularization invariant $\mathrm{inv}_\iota = (\iota_{X,a}, s(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a))$, as above, where $\underline{\mathcal{I}}^1 = \underline{\mathcal{H}} + \underline{\mathcal{B}}$. Since we are using the product order on $(\mathbb{N}^{\mathbb{N}})^m$, inv_ι and the resulting desingularization algorithm do not depend on the ordering of the components $X^{(i)}$.

We modify this desingularization algorithm by making a selection from the sequence of centres of blowings-up, in the following way. At each step, consider the locus of points W where all components of (the transform of) X are smooth. Of course, W is open in X. Moreover, the maximum locus of inv_ι in $X\backslash W$ is closed in X, because the Hilbert–Samuel function distinguishes smooth from singular points, so that $\iota_{X,\cdot}$ distinguishes points where all components are smooth from points where at least one component is singular. Therefore, at each step, we can blow up the maximum locus of inv_ι in $X\backslash W$, and eventually $W = X$. ∎

Remark 2.6: More general families of varieties can also be simultaneously desingularized as in Theorem 2.4. See Step 3 of the proof of Theorem 5.1 for another application of the idea above.

2.5. Presentation of the number of irreducible components. Let X denote a reduced algebraic variety. Assume that all irreducible components $X^{(i)}$ of X are smooth. For all $a \in X$ let $\kappa(a) = \kappa_X(a)$ denote the number of irreducible components of X at a.

Let $a \in X$. Consider a local embedding $X|_U \hookrightarrow Z$ at a, and a smooth subvariety N of Z containing $\bigcap_{\{i:a\in X^{(i)}\}} X^{(i)}$ (restricted to U). For each i, let $\underline{\mathcal{I}}^{(i)}$ denote the marked ideal $\underline{\mathcal{I}}^{(i)} = (Z, N, 0, \mathcal{I}^{(i)}|_N, 1)$, where $\mathcal{I}^{(i)}$ is the ideal of $X^{(i)}$ in $\mathcal{O}_Z$. Define $\underline{\mathcal{I}}^N_{\Pi(X)} := \sum_{\{i:a\in X^{(i)}\}} \underline{\mathcal{I}}^{(i)}$. Clearly, $\operatorname{cosupp} \underline{\mathcal{I}}^N_{\Pi(X)}$ is the constant locus $(\kappa(x) = \kappa(a))$ of κ if U is small enough.

Consider a blowing-up $\sigma : X' \to X$ over U, with smooth centre in $\operatorname{cosupp} \underline{\mathcal{I}}^N_{\Pi(X)}$. Then the transform of each marked ideal $\underline{\mathcal{I}}^{(i)}$ is given by the ideal $u^{-1}\cdot\sigma^*(I^{(i)})|_{N'}$, where u is (a local generator of the ideal of) the exceptional divisor of σ, and N' is the strict transform of N. Since $X^{(i)}$ is smooth, $u^{-1}\cdot\sigma^*(I^{(i)})$ defines the strict transform of $X^{(i)}$. Therefore, the transform of the marked ideal $\underline{\mathcal{I}}^N_{\Pi(X)}$ equals $\underline{\mathcal{I}}^{N'}_{\Pi(X')}$, where X' is the strict transform of X. It is then easy to see that $\underline{\mathcal{I}}^N_{\Pi(X)}$ is a presentation of the invariant κ at a.

We can use the marked ideal $\underline{\mathcal{I}}^N_{\Pi(X)}$ to extend κ to a desingularization invariant $\mathrm{inv}^N_\kappa(a) = (\kappa(a), s(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a))$, as above. (In particular, $\underline{\mathcal{I}}^1 = \underline{\mathcal{I}}^N_{\Pi(X)}$ plus boundary, where we are allowing a given snc divisor E.)

Remark 2.7: Recall that the desingularization invariant $\mathrm{inv}_{\underline{\mathcal{J}}}$ of a marked ideal $\underline{\mathcal{J}}$ depends only on the semicoherent equivalence class of $\underline{\mathcal{J}}$ and the dimension of its maximal contact subvariety N. We have therefore written inv^N_κ to note the dependence on the dimension of the subvariety N involved in the marked ideal $\underline{\mathcal{I}}^N_{\Pi(X)}$. Note that $\mathrm{cosupp}\,\underline{\mathcal{I}}^N_{\Pi(X)}$ is the intersection of all components of X at a. In order to get a global desingularization algorithm based on the invariant κ, we need to choose N in a way that $\dim N$ has a canonical value. We can achieve this simply by taking $N = X^{(i)}$, for any i such that $X^{(i)}$ is of minimal dimension among the components of X at a. With N chosen in this way, the equivalence class of the marked ideal $\underline{\mathcal{I}}^N_{\Pi(X)}$ plus boundary depends only on X and E at a, and $\mathrm{inv}_\kappa := \mathrm{inv}^N_\kappa$ is globally semicontinuous.

In Section 3, we will use inv_κ to give a characterization of the condition stable-snc.

3. Characterization of stable-snc singularities of a variety with snc divisor

Consider an algebraic variety X with snc divisor E. Assume that all irreducible components of X are smooth. The main purpose of this section is to characterize stable-snc singularities of (X_j, E_j), over a sequence (1.2) of admissible blowings-up (see Theorem 3.9). This section is a generalization of [1, Sect. 3], but a presentation of the invariant $\kappa = \kappa_X$ (§2.5) plays a new role, and the assumption of smooth irreducible components allows some simplification.

Recall the following geometric characterization of stable-snc singularities of X, from Lemma 1.2. Let $a \in X$ and let Z denote a smooth local embedding variety of X at a. Let $X^{(1)}, \dots, X^{(m)}$ denote the irreducible components of X at a and let c_i denote the codimension of $X^{(i)}$ in Z, for each i. Then X is stable-snc at a if and only if (the scheme-theoretic intersection) $\bigcap_{i=1}^m X_i$ is smooth and of codimension

$$c = \sum_{i=1}^{m} c_i - (m-1)(\dim Z_a - e_{X,a}) \tag{3.1}$$

at a, where $e_{X,a}$ denotes the minimal embedding dimension of X at a.

Example 3.1: Let $X = X^{(1)} \cup X^{(2)} \subset \mathbb{A}^5$, where $X^{(1)} = (x = y = 0)$ and $X^{(2)} = (x + uz = y + ut = 0)$. Then $\mathbb{A}^5$ is a minimal embedding variety at the origin, and $X^{(1)} \cap X^{(2)} = (x = y = uz = ut = 0)$. Since $X^{(1)} \cap X^{(2)}$ is not smooth at 0, X is not stable-snc at 0. On the other hand, $X^{(1)} \cap X^{(2)}$ coincides with $(x = y = z = t = 0)$ at a nonzero point a of the latter, so that X is stable-snc at a, by (3.1).

The following definition describes the special values (denoted $\mathrm{inv}_{c,s}$) that inv_κ can take at a stable-snc point in any year j of a history of inv_κ-admissible blowings-up (see Lemma 3.5).

Definition 3.2: Consider $c = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$, and $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$. Set

$$\mathrm{inv}_{c,s} := (m, s_1, 1, s_2, \ldots, 1, s_d, 1, 0, \ldots, 1, 0, \infty),$$

where the total number of pairs (before ∞) is

$$r := |c| + |s| - \max\{c_i\}, \quad |c| := \sum_{i=1}^{m} c_i, \ |s| := \sum_{k=1}^{d} s_k$$

(cf. [1, Def. 2.1]).

The s_k in Definition 3.2 will represent the sizes of certain blocks of exceptional divisors. The c_i will eventually be the codimensions of the components of X in a local minimal embedding variety. The term $\max\{c_i\}$ appears in the expression for r because we are using a presentation of κ with maximal contact variety $N =$ a component of X of smallest dimension (see Remark 2.7).

Theorem 3.9 shows, in particular, that in year zero (i.e., before any blowings-up), stable-snc singularities can be characterized using the invariant inv_κ together with the dimensions of a minimal embedding variety and the irreducible components of X. The first example following shows that inv_κ alone is not enough to characterize stable-snc, while the second shows that we cannot replace inv_κ by the desingularization invariant inv_X based on the Hilbert–Samuel function.

Examples 3.3: (1) Consider $X = (xyz = 0)$ and $Y = (x = yz = 0) \cup (z = 0)$ in $\mathbb{A}^3$. In each case, κ has a presentation at 0 given by the marked ideal $(\mathbb{A}^3, (z = 0), 0, (x, y), 1)$ (see Remark 2.7) and $\mathrm{inv}_\kappa(0) = (3, 0, 1, 0, 1, 0, \infty)$. But X is stable-snc while Y is snc but not stable-snc at 0.

(2) Consider

$$X = (x = y = 0) \cup (w = z = 0) \quad \text{and} \quad Y = (x = y = 0) \cup (x + w^2 = y + wz = 0)$$

in $\mathbb{A}^4$, with $E = 0$. The X is stable-snc, while Y is not because the scheme-theoretic intersection of its components is not smooth at 0. X and Y each have minimal embedding dimension 4 and two components of dimension 2 at 0. The ideal of Y is $(x, y) \cap (x + w^2, y + wz) = (x^2 + xw^2, xy + yw^2, y^2 + ywz, yw - xz)$. Then $H_{X,0} = H_{Y,0}$, and $\mathrm{inv}_X(0) = \mathrm{inv}_Y(0) = (H, 0, 1, 0, 1, 0, 1, 0, \infty)$, where $H = H_{X,0}$.

Definition 3.4: We consider sequences $\Omega = (e, c_1, \ldots, c_m)$, where $e \in \mathbb{N}$ and $c_1 \geq c_2 \geq \cdots \geq c_m \geq 0$ are integers. Let $\Sigma_\Omega = \Sigma_\Omega(X)$ denote the set of points $a \in X$ where X has local embedding dimension $e_{X,a} = e$ and exactly m irreducible components of X of codimensions $c_1, \ldots, c_m$ in a minimal embedding variety. See also Definition 6.1.

The sequence of codimensions c_i is taken in decreasing order in this definition because we do not want Σ_Ω to depend on an ordering of the c_i (since $\mathrm{inv}_{c,s}$ does not depend on an ordering).

The following results deal with stable-snc singularities of the transforms (X_j, E_j) of (X, E) over a sequence (1.2) of inv_κ-admissible blowings-up. We are assuming that all irreducible components of X are smooth. For brevity of notation, we will write (X_j, E_j) simply as (X, E). See Section 2 above for the definition of the blocks of exceptional divisors $E^i(a)$ that are counted by the invariants $s_i(a)$.

Lemma 3.5 (Compare with [1, Lemma 3.1]): *Suppose that all irreducible components of X are smooth. Consider $(X, E) = (X_q, E_q)$, in some year q of a history* (1.2) *of* inv_κ*-admissible blowings-up. Let $a \in X$. If (X, E) is stable-snc at a, then $\mathrm{inv}_\kappa(a) = \mathrm{inv}_{c,s}$, for some $s = (s_1, \ldots, s_d)$, with $c = (c_1, \ldots, c_m)$, where $m = \kappa(a)$ and each c_k is the codimension of an irreducible component $X^{(k)}$ of X at a in a local minimal embedding variety for X (so that $a \in \Sigma_\Omega(X)$, where $\Omega = (e_{X,a}, c_1, \ldots, c_m)$).*

Proof. Suppose that X has m (smooth) irreducible components $X^{(1)}, \ldots, X^{(m)}$ at a (so that $\kappa_X(a) = m$), of codimensions $c_1, \ldots, c_m$, respectively, in a local minimal embedding variety Z of X at a. Assume (without loss of generality) that $c_1 = \max\{c_i\}$. Let $I^{(k)}$ denote the ideal of $X^{(k)}$ in $\mathcal{O}_Z$ at a, $k = 1, \ldots, m$.

As in §2.2, $\mathrm{inv}_\kappa(a) = (\kappa(a), s_1(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a))$, where $\underline{\mathcal{I}}^1 = \sum \underline{\mathcal{I}}^{(i)}|_{N^1}+$ boundary and $N^1 = X^{(1)}$.

Let $f_{k,l}$, $l = 1,\ldots,c_k$, denote generators of the ideal $I^{(k)}$ (with linearly independent gradients), $k = 1,\ldots,m$. Let u_1^j, $j = 1,\ldots,s_1(a)$, denote generators of the ideals of the components of $E^1(a)$. Then

$$\underline{\mathcal{I}}^1 = (Z,\, N^1 = X^{(1)},\, \mathcal{E}^1(a),\, \mathcal{I}^1 = (\{f_{k,l} :\, k \geq 2\},\, \{u_1^j\})|_{N^1},\, 1), \tag{3.2}$$

where $\mathcal{E}^1(a) = E(a) \setminus E^1(a)$ (cf. Section 2). The argument is now very similar to the proof of [1, Lemma 3.1].

We factor $\underline{\mathcal{I}}^1$ as the product $\mathcal{M}(\underline{\mathcal{I}}^1) \cdot \mathcal{R}(\underline{\mathcal{I}}^1)$ of its monomial and residual parts; in particular, $\mathcal{M}(\underline{\mathcal{I}}^1)$ is generated by a monomial m_1 in the components of $\mathcal{E}^1(a)$.

Since (X, E) is stable-snc at a, the generators of $\mathcal{I}^1$ in (3.2) are part of a regular coordinate system. It follows that $\mathcal{M}(\underline{\mathcal{I}}^1) = 1$ (since none of these generators define elements of $\mathcal{E}^1(a)$); i.e., all $\mu_{H,2}(a) = 0$. Since $\underline{\mathcal{I}}^1$ has maximal order, $(\mathrm{inv}_\kappa)_{3/2}(a) = (m, s_1, 1)$, and the companion ideal $\underline{\mathcal{J}}^1 = \underline{\mathcal{I}}^1$.

We can continue the computation of inv_κ, choosing the $f_{k,l}$ and the u_i^j successively as hypersurfaces of maximal contact to pass to the coefficient ideal plus boundary $\underline{\mathcal{I}}^p$, $p = 2, \ldots$. At each step, $\mathcal{M}(\underline{\mathcal{I}}^p) = 1$ (in particular, $\mu_{H,p}(a) = 0$ for every H), and $\underline{\mathcal{I}}^p$ is of maximal order, $= 1$. Therefore, $\nu_{p+1} = 1$ and $\underline{\mathcal{I}}^p$ equals the following companion ideal $\underline{\mathcal{J}}^p$. Once all $f_{k,l}$ and u_i^j have been used as hypersurfaces of maximal contact, we get coefficient ideal $= 0$. Therefore, $\mathrm{inv}_\kappa(a)$ has last entry $= \infty$ and r pairs before ∞. ∎

Lemma 3.6 (Compare with [1, Lemma 3.3]): *Again consider* $(X, E) = (X_q, E_q)$, *in some year* q *of a history* (1.2) *of* inv_κ*-admissible blowings-up, and let* $a \in X$. *Assume that* X *has* m *irreducible components* $X^{(1)}, \ldots, X^{(m)}$ *at* a *(all smooth). Let* f_h, $h = 1, \ldots, p$, *denote generators of the ideal of* $\bigcap_{k=1}^m X^{(k)}$ *in* $\mathcal{O}_N$ *at* a, *where* N *is a component* $X^{(k)}$ *of smallest dimension (say* $N = X^{(1)}$, *without loss of generality). Let* u_i^j, $j = 1, \ldots, s_i$, *denote generators of the ideals of the elements of* $E^i(a)|_N$, $i = 1, \ldots, d$, *and write* $s = (s_1, \ldots, s_d)$.

Assume that $\mathrm{inv}_\kappa(a) = \mathrm{inv}_{c,s}$, *with* $c = (c_1, \ldots, c_m)$. *Set*

$$r := |c| + |s| - \max\{c_i\}.$$

Then there is an injection $\{1, \ldots, r\} \to \{f_h, u_i^j\}$, *which we denote* $l \mapsto g_l$, *and* *a regular system of coordinates* $(x_1, \ldots, x_n)$ *for* N *at* a *(*$n \geq r$*), such that*

$$g_l = \xi_l + x_l \cdot \prod_{i=1}^{l-1} m_i, \quad l = 1, \ldots, r, \tag{3.3}$$

where each ξ_l *is in the ideal generated by* $(x_1, \ldots, x_{l-1})$ *and each* m_i *is a monomial in generators of the ideals of the elements* H *of* $\mathcal{E}^i(a)$*, each raised to the power* $\mu_{H,i+1}(a)$ *(cf. Section 2).*

Remark 3.7: Suppose that the irreducible components $X^{(k)}$ of X have codimensions c_k in a minimal embedding variety for X at a. Then we can take $\{f_h\} := \{f_{k,j}|_N\}_{k\geq 2}$, where the $f_{k,j}$, $j = 1, \ldots, c_k$ denote local generators of the ideal $I^{(k)}$ of $X^{(k)}$ in a minimal embedding variety for X at a. In this case, the mapping $l \mapsto g_l$ of the lemma is bijective.

Proof of Lemma 3.6. As in the proof of Lemma 3.5,

$$\mathrm{inv}_\kappa(a) = (\kappa(a), s_1(a), \mathrm{inv}_{\underline{\mathcal{I}}^1}(a)),$$

where $\underline{\mathcal{I}}^1 = \sum \underline{\mathcal{I}}^{(i)}|_{N^1}+$ boundary, $N^1 = N$ and

$$\underline{\mathcal{I}}^1 = (Z,\, N^1,\, \mathcal{E}^1(a),\, \mathcal{I}^1 = (\{f_h\},\, \{u_1^j\}),\, 1)$$

(with Z a local embedding variety). If $(\mathrm{inv}_\kappa)_{3/2}(a) = (m, s_1, 1)$, then there exists $g_1 \in \{f_h\} \cup \{u_1^j\}$ such that $x_1 := (m_1)^{-1} \cdot g_1|_{N^1} \in \mathcal{R}(\underline{\mathcal{I}}^1)$ has order 1 at a, and the companion ideal $\underline{\mathcal{J}}^1 = (Z, N^1, \mathcal{E}^1(a), \mathcal{R}(\underline{\mathcal{I}}^1), 1)$. We can take $N^2 := (x_1 = 0) \subset N^1$ as the next maximal contact subspace. Then the coefficient ideal plus boundary is

$$\underline{\mathcal{I}}^2 = (Z, N^2, \mathcal{E}^2(a) = \mathcal{E}^1(a) \setminus E^2(a), \big(\mathcal{R}(\underline{\mathcal{I}}^1) + (u_2^1, \ldots, u_2^{s_2})\big)\,|_{N^2}, 1).$$

We can again repeat the argument, as in [1, Sect. 3], and the process ends after r steps. ∎

Remark 3.8: In the proof above, we see that, if the truncated invariant

$$(\mathrm{inv}_\kappa)_{k+1/2}(a) = (\mathrm{inv}_{c,s})_{k+1/2},$$

where $0 \leq k < r = |c|+|s|-c_1$, then, for every $p \leq k+1$, the coefficient ideal plus boundary $\underline{\mathcal{I}}^p$ (or an equivalent marked ideal) has associated multiplicity $= 1$. Comparing with [1, Remark 3.6], note that a condition analogous to "$a \in \Sigma_p$" in the latter is not needed here because we are assuming all irreducible components of X at a are smooth.

THEOREM 3.9 (Characterization of stable-snc): *Consider* $(X, E) = (X_q, E_q)$*, in some year* q *of a history* (1.2) *of* inv_κ*-admissible blowings-up. Let* $a \in X$*, and let* $e = e_{X,a}$*. Assume that the irreducible components,* $X^{(k)}$*,* $k = 1, \ldots, m = \kappa(a)$ *of* X *at* a *are smooth and of dimensions* $e - c_k$*, respectively. Then* (X, E) *is stable-snc at* a *if and only if*

(1) $a \in \Sigma_\Omega(X)$*, where* $\Omega = (e, c_1, \ldots, c_m)$*;*
(2) κ*-inv*$(a) = \text{inv}_{c,s}$*, for some* $s = (s_1, \ldots, s_d)$*;*
(3) $\mu_{H,i+1}(a) = 0$*, for all* $i \geq 1$ *and all* $H \in \mathcal{E}^i(a)$*.*

Proof. "Only if" is immediate from Lemma 3.5. On the other hand, assume conditions (1), (2) and (3). By (3), (3.3) holds with all $m_i = 1$. Then, by Lemma 3.6, the scheme-theoretic intersection of the components of X and E at a is smooth, and (3.1) holds. So (X, E) is stable-snc. ∎

4. Cleaning

We recall the cleaning technique introduced in [6] and developed in [1, Section 4] under conditions that also apply here (in fact, in a more straightforward way).

Assume that all irreducible components of X are smooth. According to Theorem 3.9, if $a \in \Sigma_\Omega(X)$ and $\text{inv}_\kappa(a) = \text{inv}_{c,s}$, then (X, E) is stable-snc at a if and only if the invariants $\mu_{H,k+1}(a) = 0$, for every $k \geq 1$. In this section we study the cleaning blowings-up used to get the latter condition.

Cleaning blowings-up are not necessarily inv_κ-admissible. In the general cleaning algorithm of [6, Sect. 2], therefore, the invariant $\text{inv} = \text{inv}_X$ that is used is not defined in a natural way over a cleaning sequence, so that, after cleaning, we assume we are in year zero for the definition of the invariant. Over the particular cleaning sequences needed here, however, we can define a modified inv_κ which remains upper semicontinuous and infinitesimally upper semicontinuous, and show that maximal contact subspaces exist in every codimension involved; this is a consequence of Lemma 3.6 and Remark 3.8 (see Remarks 4.2).

Consider a point a in the locus $S := ((\text{inv}_\kappa)_k = (\text{inv}_{c,s})_k)$ for the truncated invariant, where $k \geq 1$ (in some year q of a history (1.2) of inv_κ-admissible blowings-up). In some neighbourhood of a, S is the cosupport of a marked ideal (a coefficient ideal plus boundary) $\underline{\mathcal{I}}^k = (\mathcal{I}^k, d^k) = (Z, N^k, \mathcal{E}^k(a), \mathcal{I}^k, d^k)$, where N^k is a maximal contact subspace of codimension $k-1$ in N^1 and $d^k = 1$

(see Remark 3.8). Recall that $\mathcal{E}^k(a) = E(a) \setminus E^1(a) \cup \cdots \cup E^k(a)$, where the block $E^k(a)$ defines the boundary.

The ideal $\mathcal{I}^k = \mathcal{M}(\underline{\mathcal{I}}^k) \cdot \mathcal{R}(\underline{\mathcal{I}}^k)$ (the product of its monomial and residual parts). The monomial part $\mathcal{M}(\underline{\mathcal{I}}^k)$ is the product of the ideals $\mathcal{I}_H|_{N^k}$ (where $H \in \mathcal{E}^k(a)$), each to the power $\mu_{H,k+1}(a)$ (since $d^k = 1$).

Let $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ denote the monomial marked ideal $(\mathcal{M}(\underline{\mathcal{I}}^k), d^k) = (\mathcal{M}(\underline{\mathcal{I}}^k), 1)$. Then $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}^k) \subset \operatorname{cosupp} \underline{\mathcal{I}}^k$ and any admissible sequence of blowings-up of $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ is admissible for $\underline{\mathcal{I}}^k$.

Definition 4.1: **Cleaning** of the locus $S = ((\mathrm{inv}_\kappa)_k = (\mathrm{inv}_{c,s})_k)$ means the sequence of blowings-up obtained from desingularization of the monomial marked ideal $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ (in a neighbourhood of any point of S) [5, Sect. 5, Step II, Case A], [6, Sect. 2].

The centres of the cleaning blowings-up are invariantly defined closed subspaces of $((\mathrm{inv}_\kappa)_k \geq (\mathrm{inv}_{c,s})_k)$. Definition 4.1 is simpler than the analogous definition [1, Def. 4.2] because of our assumption that all components of X are smooth.

Remarks 4.2: The blowings-up σ involved in desingularization of $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ are $(\mathrm{inv}_\kappa)_k$-admissible: Let C denote the centre of σ. Then C is snc with respect to E because, in the notation above, C lies inside every element of $E^1(a) \cup \cdots \cup E^k(a)$ and C is snc with respect to $\mathcal{E}^k(a)$. Since $C \subset S$, it follows that σ is $(\mathrm{inv}_\kappa)_k$-admissible. By Lemma 3.5, C contains no stable-snc points (since some $\mu_{H,k+1}(a) \neq 0$, for all $a \in C$).

Since $d_k = 1$, C is of the form $N^k \cap H$, for a single $H \in \mathcal{E}^k(a)$; i.e., C is of codimension 1 in N^k. Therefore, σ induces an isomorphism $(N^k)' \to N^k$, where $(N^k)'$ denotes the strict transform of N^k.

Lemma 4.3: *Assume that* $\mathrm{inv}_\kappa \leq \mathrm{inv}_{c,s}$ *on* $X = X_q$*, in some year* q *of a history* (1.2) *of* inv_κ*-admissible blowings-up. Consider the cleaning sequence for* (κ*-inv*$_k = (\mathrm{inv}_{c,s})_k$) *(Definition 4.1). Then, over the cleaning sequence, we can define maximal contact subspaces of every codimension involved, as well as (a modification of)* inv_κ *which remains both semicontinuous and infinitesimally semicontinuous.*

The proof is the same as that of [1, Lemma 3.20] (changing inv to inv_κ).

Remark 4.4: After cleaning the loci $((\mathrm{inv}_\kappa)_k = (\mathrm{inv}_{c,s})_k)$, for all k, we will apply further blowings-up to make (X, E) stable-snc on $(\mathrm{inv}_\kappa = \mathrm{inv}_{c,s})$ (see Section 5, Step 3). We will then continue to blow up with closed centres which lie in the complement of the stable-snc locus {stable-snc} (Section 5). The purpose of defining inv_κ over the cleaning sequences is to ensure that, in the complement of {stable-snc}, we will only have to consider values $\mathrm{inv}_{c',s'} < \mathrm{inv}_{c,s}$ in order to resolve all but {stable-snc} after finitely many steps. If, after cleaning $(\mathrm{inv}_\kappa = \mathrm{inv}_{c,s})$, we were to apply the resolution algorithm in the complement of {stable-snc}, beginning as if in year zero, we might introduce points where $\mathrm{inv}_\kappa = \mathrm{inv}_{c',s'} > \mathrm{inv}_{c,s}$.

5. Desingularization of a variety preserving stable-snc singularities

The purpose of this section is to give an algorithm for our main theorem in the case that $D = 0$. We prove the following result.

THEOREM 5.1: *Let X denote a reduced algebraic variety and let E be an snc divisor on X. Then there is a sequence of admissible smooth blowings-up* (1.2), *such that*

(1) *(X_t, E_t) has only stable-snc singularities;*
(2) *each blowing-up σ_{j+1} is an isomorphism over the locus of stable-snc points of (X_j, E_j).*

Proof. We will break the algorithm into three main steps, with the second and third to be iterated several times.

STEP 1. We first reduce to the case that all irreducible components of X are smooth, using Theorem 2.4.

Now let $\mathcal{S}$ denote the set of all special values $\mathrm{inv}_{c,s}$, $c = (c_1, \ldots, c_m)$, $s = (s_1, \ldots, s_d)$ (see Definition 3.2). Then $\mathcal{S}$ is totally ordered (lexicographically).

Consider the desingularization sequence determined by the invariant inv_κ, defined in §2.5.

STEP 2. We follow the desingularization algorithm determined by inv_κ (i.e., the sequence of blowings-up with successive centres given by the maximum locus of inv_κ) until the maximum of inv_κ is a value τ in $\mathcal{S}$ for the first time. We then blow up any irreducible component of the maximum locus $(\mathrm{inv}_\kappa = \tau)$ that contains no stable-snc points. The result is that (X, E) $(= (X_j, E_j)$, for some j) is generically stable-snc on every component of the locus $(\mathrm{inv}_\kappa = \tau)$. (The latter may now be empty.)

We now clean the locus $((\mathrm{inv}_\kappa)_k = (\tau)_k)$ of the truncated invariant, for every k, beginning with the largest k; see Section 4. The result of cleaning is that the invariants $\mu_{H,k+1} = 0$ on $(\mathrm{inv}_\kappa = \tau)$, for all $H \in E$ and $k \geq 1$. Recall that, for each k, the cleaning blowings-up are given by desingularization of a monomial marked ideal $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ with cosupport in $((\mathrm{inv}_\kappa)_k \geq (\tau)_k)$. The cleaning blowings-up may be nontrivial even in the case that $(\mathrm{inv}_\kappa = \tau) = \emptyset$, but are needed even in this case to guarantee functoriality.

Cleaning involves blowing up only points where $\mu_{H,k+1} > 0$, for some k, so never involves blowing up stable-snc points (by Theorem 3.9). After cleaning, we have the normal forms of Lemma 3.6 with all monomials $m_i = 1$.

Recall that the characterization of stable-snc points a given by Theorem 3.9 involves the the minimal embedding dimension $e_{X,a}$. After Step 2 above, it need not be true that e_X is constant on each irreducible component of the locus $(\mathrm{inv}_\kappa = \tau)$ (although the number of irreducible components of X is constant). The purpose of Step 3 following is to make e_X constant on components of the maximal locus, in order to apply Theorem 3.9.

Step 3. If the locus $T := (\mathrm{inv}_\kappa = \tau)$ is nonempty after Step 2, then T is the maximum locus of inv_κ, each irreducible component of T is generically stable-snc, and all $\mu_{H,k+1} = 0$ on T. We now apply the algorithm for simultaneous desingularization of the pair (X, T), as in §2.4; i.e., the sequence of blowings-up given by the maximum loci of the invariant inv_ι determined by $\iota := (H_X, H_T)$, with the lexicographic ordering of such pairs. Since T is smooth, the invariant inv_ι has the form $(\iota, s, \mathrm{inv}_{\underline{\mathcal{I}}^1})$, where $\underline{\mathcal{I}}^1$ is the marked ideal given by a presentation of the Hilbert–Samuel function H_X restricted to $N = T$, plus a boundary. We blow up following the algorithm until H_X and therefore the embedding dimension $e_{X,a}$ is constant on every component of T. The centres of all blowings-up involved lie in T (thus are inv_κ-admissible) and contain no stable-snc points; all $\mu_{H,k+1}$ remain zero on T.

After Step 3, every component of $T = (\mathrm{inv}_\kappa = \tau)$ lies in some Σ_Ω. By Theorem 3.9, (X, E) is stable-snc at every point of T, and therefore in some neighbourhood of T.

We can now iterate Steps 2 and 3 in the complement of T. All centres of blowing up involved are closed in X because they contain no stable-snc points and X is stable-snc in a neighbourhood of T. The process terminates after finitely many iterations of Steps 2 and 3 (see Remark 4.4), when (X, E) becomes stable-snc. ∎

Remarks 5.2: (1) The desingularization algorithm of Theorem 5.1 is functorial with respect to étale or smooth morphisms that preserve the number of irreducible components of X at every point; cf. [8, Sect. 9].

(2) If $a \in C_j$, where $C_j \subset X_j$ is the centre of the blowing-up σ_{j+1}, then the component of C_j at a lies in all irreducible components of X_j at a.

6. Characterization of stable-snc singularities of a triple

The remainder of the paper is devoted to an algorithmic proof our main theorem 1.16 for a general triple (X, D, E). We will begin by making (X, E) stable-snc, using Theorem 5.1. The remainder of the proof is by induction on the number of irreducible components of X, so we will henceforth assume that the components of X have a given ordering $X = X^{(1)} \cup \cdots \cup X^{(m)}$. Theorem 1.16 will be functorial with respect to triples (X, D, E) where the components of X have a fixed ordering.

The proof involves a characterization of stable-snc points (Proposition 6.7 below) that plays a role similar to that played by Theorem 3.9 in the proof of Theorem 5.1, but in the inductive setting needed here; in particular, Proposition 6.7 involves the assumption that (X, D, E) is stable-snc after dropping the last component $X^{(m)}$ of X together with the components of D that lie in $X^{(m)}$. Proposition 6.7 will be used after reducing the main problem to the case that (X, E) is stable-snc and D is a reduced divisor on X with no components in $\operatorname{Sing} X \cup \operatorname{Supp} E$. Proposition 6.7 treats points lying in at least two components of X and in the support of D. Points lying outside the support of D are already stable-snc by assumption, and points lying in only one component of X can be studied using Proposition 3.9.

The inductive proof of Theorem 1.16 begins with the case that X is smooth and irreducible. In this case, stable-snc means that D is snc. Snc points of a divisor can be characterized either using the desingularization invariant [1, Thm. 3.4] or (as a particular case of stable-snc) by Theorem 3.9 with

$c_1 = \cdots = c_{\kappa_X(a)} = 1$, or $c_1 = 0$ if $\kappa_X(a) = 1$; Theorem 1.16 in the case that X is smooth and irreducible follows from [1, Thm. 1.4] or from Theorem 5.1.

In the inductive setting of the proof of our main theorem, we will use a partition of the last component $X^{(m)}$ of X that is similar but not identical to the partition in Definition 3.4.

Definition 6.1: Consider $\Omega = (e, c)$, where $e \in \mathbb{N}$ and $c := (c_1, c_2, \ldots, c_n)$ with $n \leq m$ and $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Assume that (X, E) is stable-snc and that D has no components in $\operatorname{Sing} X \cup \operatorname{Supp} E$. Let $q \in \mathbb{N}$. We define $\Sigma_{\Omega,q} = \Sigma_{\Omega,q}(X, D) = \Sigma_{\Omega,q}(X, D, E)$ as the set of points $a \in X^{(m)}$ such that:

(1) there are precisely n different components $X^{(i_1)}, \ldots, X^{(i_n)}$ of X such that, for each j, either $X^{(i_j)} = X^{(m)}$ or $X^{(i_j)}$ contains a component of D at a;
(2) e is the minimal embedding dimension of $\bigcup_{j=1}^{n} X^{(i_j)}$ at a;
(3) $c_1, \ldots, c_n$ are the codimensions of $X^{(i_1)}, \ldots, X^{(i_n)}$, respectively, in a minimal embedding variety for $\bigcup_{j=1}^{n} X^{(i_j)}$ at a;
(4) q is the minimum number of components of D at a which lie in any one of the $X^{(i_j)}$.

As in Definition 3.4, we list the c_i in decreasing order so that the stratum $\Sigma_{\Omega,q}$ corresponds to a value of the Hilbert–Samuel function (Definition 6.3 below), which does not depend on an ordering of the c_i.

Example 6.2: Consider $X := X^{(1)} \cup X^{(2)} \cup X^{(3)} \subset \mathbb{A}^6$, where

$$X^{(1)} = (x_1 = x_2 = 0), \quad X^{(2)} = (x_4 = 0) \quad \text{and} \quad X^{(3)} = (x_3 = 0),$$

and let $D = (x_1 = x_2 = y_1 = 0) + (x_3 = y_1 y_2 = 0)$. Let $a = 0$. Then $a \in \Sigma_{(6,2,1),1}$. D has two components, one in each of $X^{(1)}$ and $X^{(3)}$. The latter have codimensions 2 and 1, respectively, in $\mathbb{A}^6$, which is a minimal embedding variety already for $X^{(1)} \cup X^{(3)}$.

Definition 6.3: Consider $\Omega = (e, c)$, with $c = (c_1, \ldots, c_n)$, and $q \in \mathbb{N}$, as in Definition 6.1. Assume that $|c| + q \leq e$, where $|c| = c_1 + \cdots + c_n$. We let $H_{\Omega,q}$ denote the Hilbert–Samuel function of the ideal

$$\bigcap_{i=1}^{n} (x_{i,1}, \ldots, x_{i,c_i}, y_1 \cdots y_q)$$

in the ring of formal power series $\mathbb{K}[\![x_{1,1},\dots,x_{n,c_n},y_1,\dots,y_{e-|c|}]\!]$. (See Section 7).

The $H_{\Omega,q}$ are precisely the values that the Hilbert–Samuel function of $\operatorname{Supp} D$ can take at stable-snc points.

See [8, Example 5.6] for an illustration of the kind of information provided by the Hilbert–Samuel function. The condition that the Hilbert–Samuel function of $\operatorname{Supp} D$ equal $H_{\Omega,q}$ at a point of $\Sigma_{\Omega,q}$ is necessary for stable-snc. But it is not sufficient, as shown by [8, Example 5.6]. Additional geometric data are needed; these will be given using an ideal sheaf that is an obstruction to stable-snc (Definition 6.5). This obstruction will be eliminated using "cleaning-type" blowings-up similar to those used in [8, Sect. 7] to eliminate an analogous obstruction; see Proposition 9.1.

Lemma 7.5 in the following section is used in the proof of Proposition 6.7, and provides some initial control over the divisor D at a point of $\Sigma_{\Omega,q}$ where X has ≥ 2 components and the Hilbert–Samuel function has the "correct" value $H_{\Omega,q}$.

Definition 6.4: Assume that no irreducible component of D lies in $\operatorname{Sing} X$. Set $X^i := X^{(1)} \cup \cdots \cup X^{(i)}$, $1 \leq i \leq m$. Let D_i denote the sum of all components of D lying in $X^{(i)}$; i.e., D_i is the divisorial part of the restriction of D to $X^{(i)}$. We will sometimes write $D_i = D|_{X^{(i)}}$. Set $D^i := \sum_{j=1}^{i} D_j$.

Definition 6.5: **Obstruction ideal.** Assume that X is stable-snc, and that no irreducible component of D lies in $\operatorname{Sing} X$. Let $J = J(X,D)$ denote the quotient ideal sheaf

$$J = J(X,D) := \bigcap_{1\leq i,j\leq m} [I_{D_i} + I_{X^{(j)}} : I_{D_j} + I_{X^{(i)}}],$$

where I_{D_i}, $I_{X^{(j)}}$, I_{D_j} and $I_{X^{(i)}}$ are the ideal sheaves of $\operatorname{Supp} D_i$, $X^{(j)}$, $\operatorname{Supp} D_j$ and $X^{(i)}$ (respectively) in $\mathcal{O}_X$.

Note that, at a point which does not lie in some component $X^{(i)}$ of X, all quotients involving $X^{(i)}$ in the intersection above are equal to $\mathcal{O}_X$ and can therefore be ignored.

An ideal sheaf defined in a similar way to $J(X,D)$ above was used in [8]. Definition 6.5 is more suitable here, and in fact also simplifies the argument in [8].

We consider decompositions $X = Y \cup T$, where Y and T are two closed subvarieties with no common components. The inductive characterization of stable-snc will be formulated using a 4-tuple of the form (Y, D, E, T), where $X = Y \cup T$, (X, E) is stable-snc, and D is a Weil divisor on Y such that $(Y, D, E|_Y)$ is stable-snc.

Definition 6.6: We say that (Y, D, E, T) is **stable-snc at** a if there exists a Weil divisor D_T on T such that $(Y \cup T, D + D_T, E)$ is stable-snc at a. The **transform** of (Y, D, E, T) by a sequence of admissible blowings-up for (X, E) is given by the transform of (X, D, E) as in Definition 1.15.

Proposition 6.7 (Inductive characterization of stable-snc): *Consider a triple (X, D, E) satisfying the hypotheses of Theorem 1.16 and let $X^{(i)}$, $i = 1, \ldots, m$, denote the irreducible components of X (ordered, as above). Assume $m \geq 2$. Let $a \in X^{m-1} \cap X^{(m)}$ (in the notation above). Then:*

1. *(X, D, E) is stable-snc at a if and only if both $(X^{m-1}, D^{m-1}, E, X^{(m)})$ and (X, D) are stable-snc at a.*
2. *Suppose that D is reduced, with no irreducible component in $\operatorname{Sing} X$. Assume that a belongs to at least two components of D, one in $X^{(m)}$ and the other in $X^{(i)}$, for some $i \neq m$. Then (X, D) is stable-snc at a if and only if*
   - (a) *$(X^{m-1}, D^{m-1}, 0, X^{(m)})$ is stable-snc at a;*
   - (b) *there exist Ω and q as in Definition 6.1, such that $a \in \Sigma_{\Omega,q}(X, D)$ and $H_{\operatorname{Supp} D,a} = H_{\Omega,q}$, where $H_{\operatorname{Supp} D,a}$ is the Hilbert–Samuel function of $\operatorname{Supp} D$ at a;*
   - (c) *$J_a = \mathcal{O}_{X,a}$.*

Proposition 6.7 will be proved at the end of Section 7. The inductive structure of this characterization of stable-snc is the main reason that the resulting resolution algorithm (Theorem 1.8) depends on an ordering of the components of X.

Remarks 6.8: Consider assertion (2) of the theorem. (1) If a lies in $X^{(m)}$ but in no other $X^{(i)}$, then of course (a) is vacuous and $J_a = \mathcal{O}_{X,a}$. In this case, Theorem 3.9 applied to $(X^{(m)}, \operatorname{Supp} D)$ replaces Proposition 6.7.

(2) We will use Proposition 6.7 to remove unwanted singularities at points lying in at least two components of X, by first blowing up to get either $a \notin X^{(m)}$,

or $a \in X^{(m)}$ satisfying (b), and then applying further blowings-up to get (c); see Section 9.1.

(3) Note that, assuming (a), J as given in Definition 6.5 coincides with the intersection for $i = 1, \ldots, m-1$ and $j = m$.

7. The Hilbert–Samuel function and stable simple normal crossings

Lemma 7.5 of this section plays an important part in our use of the Hilbert–Samuel function to characterize stable-snc points. See [8, Example 5.6] for an example that motivates the lemma. We begin with the definition of the Hilbert–Samuel function and its relationship with the diagram of initial exponents (cf. [2]). At the end of the section, we use Lemma 7.5 to prove the inductive characterization of stable-snc (Lemma 6.7).

Definition 7.1: Let A denote a Noetherian local ring A with maximal ideal $\mathfrak{m}$. The **Hilbert–Samuel function** $H_A \in \mathbb{N}^{\mathbb{N}}$ of A is defined by

$$H_A(k) := \text{length}\, \frac{A}{\mathfrak{m}^{k+1}}, \quad k \in \mathbb{N}.$$

If $I \subset A$ is an ideal, we sometimes write $H_I := H_{A/I}$. If X is an algebraic variety and $a \in X$ is a closed point, we define $H_{X,a} := H_{\mathcal{O}_{X,a}}$, where $\mathcal{O}_{X,a}$ denotes the local ring of X at a.

Definition 7.2: Let $F, G \in \mathbb{N}^{\mathbb{N}}$. We say that $F > G$ if $F(n) \geq G(n)$, for every n, and $F(m) > G(m)$, for some m. This relation induces a partial order on the set of all possible values for the Hilbert–Samuel functions of Noetherian local rings.

Note that $F \not\leq G$ if and only if either $F > G$ or F is incomparable to G.

Let $\widehat{A}$ denote the completion of A with respect to $\mathfrak{m}$. Then $H_A = H_{\widehat{A}}$ [13, §24.D]. If A is regular, then we can identify $\widehat{A}$ with a ring of formal power series, $\mathbb{K}[\![x]\!]$, where $x = (x_1, \ldots, x_n)$. Then

$$H_I(k) := \dim_{\mathbb{K}} \frac{\mathbb{K}[\![x]\!]}{I + \mathfrak{n}^{k+1}},$$

where $\mathfrak{n} := (x_1, \ldots, x_n)$ is the maximal ideal of $\mathbb{K}[\![x]\!]$.

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The lexicographic order of $(n+1)$-tuples, $(|\alpha|, \alpha_1, \ldots, \alpha_n)$ induces a total ordering of $\mathbb{N}^n$. Let $f \in \mathbb{K}[\![x]\!]$ and write $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$, where x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Define

$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : f_\alpha \neq 0\}$. The **initial exponent** $\exp(f)$ is defined as the smallest element of $\mathrm{supp}(f)$. If $\alpha = \exp(f)$, then $f_\alpha x^\alpha$ is called the **initial monomial** $\mathrm{mon}(f)$ of f.

Definition 7.3: Consider an ideal $I \subset K[\![x]\!]$. The **initial monomial ideal** $\mathrm{mon}(I)$ of I denotes the ideal generated by $\{\mathrm{mon}(f) : f \in I\}$. The **diagram of initial exponents** $\mathcal{N}(I) \subset \mathbb{N}^n$ is defined as

$$\mathcal{N}(I) := \{\exp(f) : f \in I \setminus \{0\}\}.$$

Clearly, $\mathcal{N}(I) + \mathbb{N}^n = \mathcal{N}(I)$. For any $\mathcal{N} \subset \mathbb{N}^n$ such that $\mathcal{N} = \mathcal{N} + \mathbb{N}^n$, there is a smallest set $\mathcal{V} \subset \mathcal{N}$ such that $\mathcal{N} = \mathcal{V} + \mathcal{N}$; moreover, $\mathcal{V}$ is finite. We call $\mathcal{V}$ the set of **vertices** of $\mathcal{N}$.

PROPOSITION 7.4: *For every $k \in \mathbb{N}$, $H_I(k) = H_{\mathrm{mon}(I)}(k)$ is the number of elements $\alpha \in \mathbb{N}^n$ such that $\alpha \notin \mathcal{N}(I)$ and $|\alpha| \leq k$.*

Proof. See [3, Corollary 3.20]. ∎

LEMMA 7.5: *Consider $a \in \Sigma_{\Omega,q}$, where $\Omega = (e,(c_1,\dots,c_m))$ and $m \geq 2$. Assume that X is embedded locally in a coordinate chart of a smooth variety Z of minimal dimension, with a system of coordinates $\{x_{i,j}\}_{1\leq i\leq m,\ 1\leq j\leq c_i}$, $\{y_i\}_{1\leq i\leq r}$, $\{w_i\}_{1\leq i\leq n-|c|-q}$. Assume $X = V(\bigcap_{i=1}^m (x_{i,1},\dots x_{i,c_i}))$. Suppose that D is a reduced divisor (so we view it as a subvariety), with no component in $\mathrm{Sing}\, X$, given at $a=0$ by an ideal I_D of the form*

$$I_D = \left[\bigcap_{i=1}^{m-1}(x_{i,1},\dots x_{i,c_i}) + (y_1\cdots y_r)\right] \cap (x_{m,1},\dots,x_{m,c_m},f). \tag{7.1}$$

(In particular, q is the minimum of r and the number of irreducible factors of $f|_{(x_{m,1}=\cdots=x_{m,c_m}=0)}$.)

Let H_D denote the Hilbert–Samuel function H_{I_D}. Then $H_D = H_{\Omega,q}$ (see Definition 6.3) if and only if we can choose f so that $\mathrm{ord}\, f = q$, $r = q$ and

$$f \in J := \bigcap_{i=1}^{m-1}(x_{i,1},\dots x_{i,c_i}) + (y_1\cdots y_r) + (x_{m,1},\dots,x_{m,c_m}).$$

Moreover, if either $\mathrm{ord}\, f > q$, $r > q$ or $f \notin J$, then $H_D \nleq H_{\Omega,q}$.

Remark 7.6: It follows immediately from the conclusion of the lemma that $H_D \nless H_{\Omega,q}$ at a point in $\Sigma_{\Omega,q}$.

Proof of Lemma 7.5. First we give a more precise description of the ideal I_D. Write $I_i := (x_{i,1}, \dots, x_{i,c_i})$, $i = 1, \dots, m$. Let $K \subset \{1, 2, \dots, m-1\} \times \{1, 2, \dots, r\}$ denote the set of all (i, j) such that $f \in I_m + I_i + (y_j)$. If $(i, j) \in K$, then any element of $I_m + (f)$ belongs to the ideal $I_m + I_i + (y_j)$. Set

$$G := \bigcap_{(i,j)\in K} (I_i + (y_j)) \quad \text{and} \quad H := \bigcap_{(i,j)\notin K} (I_i + (y_j))$$

(where the intersections are taken to be the local ring $\mathcal{O}_{Z,a}$ if the index set is empty); note that these are the prime decompositions. Then any element of $I_m + (f)$ belongs to $\bigcap_{(i,j)\in K}(I_m + I_i + (y_j)) = I_m + G$. Therefore we can take $f \in G$. Observe that we still have $f \notin I_i + (y_j)$ for $(i, j) \notin K$. By a computation the same as in [8, Proof of Lemma 5.7], replacing x_i, p in the latter by I_i, m (respectively) here, we get

$$I_D = I_m \cdot [H \cap G] + H \cdot (f). \tag{7.2}$$

The remainder of the proof is also quite similar to the hypersurface case treated in [8, Proof of Lemma 5.7], but we include it because it is not a direct translation as above. In particular, the diagrams of initial exponents here are more complicated.

We can pass to the completion of $\mathcal{O}_{Z,a}$ because this does not change the Hilbert–Samuel function, the order of f or ideal membership. So we assume we are working in a formal power series ring, where $\{x_{i,j}\}_{1\le i\le m,\ 1\le j\le c_i}$, $\{y_i\}_{1\le i\le r}$, $\{w_i\}_{1\le i\le n-|c|-q}$ are the indeterminates. For simplicity, we use the same notation for ideals and their generators before and after completion.

We can compute the Hilbert–Samuel function H_D using the diagram of initial exponents $\mathcal{N}(I_D)$. The latter should be compared to the diagram of the ideal $\bigcap_{i=1}^m I_i + (y_1 \cdots y_q)$, whose Hilbert–Samuel function is $H_{\Omega,q}$.

A. First, we show that $H_D \nleq H_{\Omega,q}$ in the following three cases:

Case 1. $H \neq (1)$ or $\operatorname{ord} f > q$. Then all elements of $H \cdot (f)$ have order $> q$. Moreover, all elements of

$$I_m \cdot [G \cap H] = I_m \cdot \left(\bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r) \right)$$

of order less than $q + 1$ have initial monomials in $\mathcal{N}(\bigcap_{i=1}^m I_i)$.

It follows that, if $H \neq (1)$ (i.e., $f \notin \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r)$) or if $\operatorname{ord} f > q$, then $H_D \nleq H_{\Omega,q}$. In fact, in $\mathcal{N}(I_D)$, below degree $q + 1$, we have at most the vertices of $\mathcal{N}(\bigcap_{i=1}^m I_i)$, while in $\mathcal{N}(\bigcap_{i=1}^m I_i + (y_1 \cdots y_q))$, below degree $q+1$, we have also the vertex corresponding to the monomial $y_1 \cdots y_q$.

Case 2. $H = (1)$ (i.e., $f \in \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r)$), $\operatorname{ord} f = q$ and $r > q$. Then $\operatorname{mon}(f) \in \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r)$, after perhaps adding an element of I_m to f. A simple computation shows that

$$\operatorname{mon}(I_D) = I_m \cdot \left(\bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r) \right) + (\operatorname{mon}(f)).$$

This follows from the fact that cancelling the initial monomial of f using elements of $I' := I_m \cdot (\bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r))$ leads to a function whose initial monomial is already in I'. In fact, $f \in \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r)$ but, since $\operatorname{ord}(f) = q$, then $\operatorname{mon}(f) \in \bigcap_{i=1}^{m-1} I_i$. This means that to eliminate the initial monomial of f, we multiply f by an element of I_m and then subtract an element of I'. This results again in an element of I', and therefore contributes no new vertices to $\mathcal{N}(I_D)$.

It follows that $H_D \not\leq H_{\Omega,q}$. In fact, since $\operatorname{mon}(f) \in \bigcap_{i=1}^{m-1} I_i$, there exists $b \in \bigcap_{i=1}^m I_i$ (any b that is not relatively prime to $\operatorname{mon}(f)$) such that there are points in $\mathcal{N}(I_D)$ that correspond to monomials that are multiples of both $\operatorname{mon}(f)$ and b, but not of $\operatorname{mon}(f) \cdot b$. This implies that, in degree $\deg(\operatorname{lcm}(\operatorname{mon}(f), b)) = q+1$, there are fewer vertices in $\mathcal{N}(I_D)$ than $\mathcal{N}(\bigcap_{i=1}^m I_i + (y_1 \cdots y_q))$; therefore $H_D(q+1) > H_{\Omega,q}(q+1)$.

B. Secondly, we show

$$H_D = H_{\Omega,q},$$

assuming that $H = (1)$ (i.e., $f \in \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_r)$), $\operatorname{ord} f = q$ and $r = q$. The first assumption implies that

$$I_D = \bigcap_{i=1}^m I_i + I_m \cdot (y_1 \cdots y_q) + (f). \tag{7.3}$$

Therefore, either $\operatorname{mon}(f) = y_1 y_2 \cdots y_q$ or $\operatorname{mon}(f) \in \bigcap_{i=1}^{m-1} I_i$.

In both cases, by the same argument as in Case 2 above,

$$\operatorname{mon}(I_D) = \bigcap_{i=1}^m I_i + I_m \cdot (y_1 \cdots y_q) + (\operatorname{mon}(f)).$$

We want to prove that $H_{\operatorname{mon}(I_D)} = H_{\Omega,q}$. If $\operatorname{mon}(f) = y_1 y_2 \cdots y_q$, then $H_{\operatorname{mon}(I_D)} = H_{\Omega,q}$, by the definition of $H_{\Omega,q}$. On the other hand, if $\operatorname{mon}(f) \in \bigcap_{i=1}^{m-1} I_i$, then the Hilbert–Samuel function of $I'' := \bigcap_{i=1}^m I_i + (\operatorname{mon}(f))$ is larger than $H_{\Omega,q}$ because, for each monomial b representing a vertex of $\mathcal{N}(\bigcap_{i=1}^m I_i)$ that is not relatively prime to $\mathrm{mon}(f)$, the monomials that are multiples of both $\mathrm{mon}(f)$ and b are not only those that are multiples of $\mathrm{mon}(f)\cdot b$. We will count the additional monomials (for each degree), and show that this number equals the number of monomials in $I_m\cdot(y_1\cdots y_q)$ that are not already in I''; i.e., the number of points of $\mathcal{N}(I''+I_m\cdot(y_1\cdots y_q))$ additional to those of $\mathcal{N}(I'')$.

Write a representative of a vertex of $\mathcal{N}(\bigcap_{i=1}^m I_i)$ that is not relatively prime to $\mathrm{mon}(f)$ as $ax_{m,i}b$, where $\mathrm{mon}(f)=ac$ and $x_{m,i}b, c$ are relatively prime. The monomials to be counted are of the form $ax_{m,i}bcM=\mathrm{mon}(f)x_{m,i}bM$, for some monomial $M\notin\bigcap_{i=1}^{m-1} I_i$. Now, $y_1\cdots y_q x_{m,i}M\in\mathrm{mon}(I_D)$ has the same degree as $ax_{m,i}bcM$, but does not lie in $\bigcup_{i=1}^{m-1} I_i+(\mathrm{mon}(f))$. This implies that, in each degree, $\mathcal{N}(I_D)$ and $\mathcal{N}(I''+I_m\cdot(y_1\cdots y_q))$ have the same number of points. Therefore $H_D=H_{\mathrm{mon}(I_D)}=H_{\Omega,q}$. This completes the proof of Lemma 7.5. ∎

Corollary 7.7: *In the setting of Lemma 7.5, if there exists q' such that $H_{\Omega,q'}\geq H_{\mathrm{Supp}\,D,a}$ at $a\in\Sigma_{\Omega,q}$, then $H_{\Omega,q'}\geq H_{\Omega,q}$. If, moreover, $q'=q$, then $H_{\Omega,q}=H_{\mathrm{Supp}\,D,a}$.*

Proof. As in the proof of Lemma 7.5, we pass to the completion of $\mathcal{O}_{Z,a}$. We have

$$I_D=\bigcap_{i=1}^m I_i+I_m\cdot(y_1\cdots y_r)+(f)\cdot H, \tag{7.4}$$

with H as in the proof of the lemma. Recall that $r\geq q$ and $\mathrm{ord}\,f\geq q$. In the right-hand side of (7.4), the first two terms are generated by monomials of degrees m and $r+1$, respectively, while the last term is an ideal of order at least $q+1$. We compare $\mathcal{N}(I_D)$ with $\mathcal{N}(I)$, where

$$I:=\bigcap_{i=1}^m I_i+(y_1\cdots y_{q'}) \tag{7.5}$$

and where we assume $H_I=H_{\Omega,q'}$. Then $\mathcal{N}(I)$ has the same vertices in degree m as $\mathcal{N}(\bigcap_{i=1}^m I_i)$, and these vertices are the same as those of $\mathcal{N}(I_D)$ in degree m. In addition, $\mathcal{N}(I)$ has a vertex in degree q'. Since $H_{\Omega,q'}\geq H_{\mathrm{Supp}\,D,a}$ we have

$$q'\geq\min(r+1,\mathrm{ord}((f)\cdot H)). \tag{7.6}$$

This implies that $H_{\Omega,q'}\geq H_{\Omega,q}$.

If, moreover, $q' = q$, then (7.6) implies that $H = (1)$ and $\mathrm{ord}(f) = q$. As at the end of the proof of the lemma, it follows that $H_{\mathrm{Supp}\, D,a} = H_{\Omega,q}$. ∎

Proof of Proposition 6.7. In (1), the "only if" direction is obvious. Suppose that (X, D) is stable-snc at a. Then (X, D, E) is stable-snc at a if and only if $D|_Z + E|_Z$, where Z denotes the intersection of the components of X at a, is an snc divisor on Z. Since (X, D) is stable-snc at a, the restriction of D to Z is the same as that of D^{m-1}. But, if $(X^{m-1}, D^{m-1}, E, X^{(m)})$ is stable-snc at a, then $D^{m-1}|_Z + E|_Z$ is an snc divisor.

For (2), first assume that (X, D) is stable-snc at a. Then (a) is obvious. The ideal of Supp D has the form $\bigcap_{i=1}^{m}(I_i + (y_1 \cdots y_q))$, where $I_i := (x_{i,1}, \ldots, x_{i,c_i})$, $i = 1, \ldots, m$, in suitable coordinates for a minimal embedding variety Z of X at $a = 0$ (recall that D is reduced). Then (b) follows and, for (c), we compute

$$J_a = \bigcap_{1\le i\ne j\le m} [(I_i + I_j + (y_1 \cdots y_q)) : (I_i + I_j + (y_1 \cdots y_q)] = \mathcal{O}_{X,a}.$$

Conversely, assume the conditions (a)–(c). By (a), there is a system of coordinates $\{x_{i,j}\}_{1\le i\le m,\ 1\le j\le c_i}$, $\{y_i\}_{1\le i\le q}$, $\{z_i\}_{1\le i\le n-|c|-q}$ for Z at a, in which $X^{(m)} = (x_{m,1} = \cdots = x_{m,c_m} = 0)$ and $\mathrm{Supp}\, D$ is defined by the ideal

$$I_D = (I_m + (f)) \cap \bigcap_{i=1}^{m-1} (I_i + (y_1 \cdots y_q)).$$

By (b) and Lemma 7.5, we can choose $f \in \bigcap_{i=1}^{m-1}(I_i + (y_1 \cdots y_q)) + I_m$, and therefore we can choose $f \in \bigcap_{i=1}^{m-1}(I_i + (y_1 \cdots y_q)) = \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_q)$. Write f in the form $f = g_1 + y_1 \cdots y_q g_2$, where $g_1 \in \bigcap_{i=1}^{m-1} I_i$. Then

$$\begin{aligned} J_a &= \bigcap_{i=1}^{m-1} [(I_m + I_i + (f)) : (I_m + I_i + y_1 \cdots y_q)] \\ &= \bigcap_{i=1}^{m-1} [(I_m + I_i + (y_1 \cdots y_q g_2)) : (I_m + I_i + (y_1 \cdots y_q))] \\ &= \bigcap_{i=1}^{m-1} (I_m + I_i + (g_2)). \end{aligned}$$

Since no component of D lies in $\mathrm{Sing}\, X$, then $g_2 \notin I_m + I_i$, $i = 1, \ldots, m-1$. Therefore, $J_a = I_m + (g_2) + \bigcap_{i=1}^{m-1} I_i$.

The condition $J_a = \mathcal{O}_{Y,a}$ means that g_2 is a unit. Then

$$\begin{aligned} I_D &= \left[\bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_q g_2)\right] \cap (I_m + (f)) \\ &= \left[\bigcap_{i=1}^{m-1} I_i + (g_1 + y_1 \cdots y_q g_2)\right] \cap (I_m + (f)) \\ &= \left[\bigcap_{i=1}^{m-1} I_i + (f)\right] \cap (I_m + (f)). \end{aligned}$$

Since no component of D lies in $\operatorname{Sing} X$, then $f \notin I_m + I_i$, for every $i = 1, \ldots, m-1$. Therefore, $I_D = \bigcap_{i=1}^m I_i + (f)$.

By Lemma 7.5, since $a \in \Sigma_{\Omega,q}$, $\operatorname{ord} f = q$. It follows that $f|_{V(I_m)}$ is a product $f_1 \cdots f_q$ of q irreducible factors each of order one. For each $i = 1, \ldots, q$, set $A_i := \{(j,k) : f_i \in I_j + (y_k)|_{V(I_m)},\ j \le m-1,\ k \le q\}$. Then $f_i \in \bigcap_{(j,k)\in A_i} (I_j + (y_k))|_{V(I_m)}$, where the intersection is understood to be the entire local ring if $A_i = \emptyset$. Note that $\bigcup_i A_i = \{(j,k) :\ j \le m-1,\ k \le q\}$, since $f \in \bigcap_{i=1}^{m-1} (I_i + (y_1 \cdots y_q))$.

We will extend each f_i to a regular function on Z (still denoted f_i) preserving the condition that $f_i \in \bigcap_{(j,k)\in A_i} (I_j + (y_k))$. In fact, $\bigcap_{(j,k)\in A_i} (I_j + (y_k))|_{V(I_m)}$ is generated by a finite set of monomials $\{m_r\}$ in the $x_{\alpha,\beta}|_{V(I_m)}$ and $y_k|_{V(I_m)}$. Then f_i is a combination $\sum m_r a_r$. So we can get an extension of f_i as desired, using arbitrary extensions of the a_r to regular functions on Z. This means we can assume that $f = f_1 \cdots f_q \in \bigcap_{i=1}^{m-1} I_i + (y_1 \cdots y_q)$ (using the extended f_i).

Since $f|_{V(\sum_{i=1}^{m-1} I_i)} = y_1 \cdots y_q g_2$, where g_2 is a unit, it follows that $f = y_1 \cdots y_q g_2 \bmod \sum_{i=1}^{m-1} I_i$, where g_2 is a unit. Since $I_D = \bigcap_{i=1}^m I_i + (f)$, it remains to check only that $\{x_{i,j}\}_{1\le i\le m,\ 1\le j\le c_i}, f_1, \ldots, f_q$ are part of a coordinate system. We can pass to the completion of $\mathcal{O}_{Z,a}$, which we identify with a ring of formal power series in variables including $\{x_{i,j}\}_{1\le i\le m,\ 1\le j\le c_i}, \{y_i\}_{1\le i\le q}$. It is enough to prove that the images of the f_i and $x_{i,j}$ in $\hat{m}/\hat{m}^2$ are linearly independent, where $\hat{m}$ is the maximal ideal of the completed local ring. If we put $x_{i,j} = 0$ for every (i,j) in the power series representing each f_i we get

$$(f_1 \cdots f_q)|_{V(\sum_{i=1}^m I_i)} = y_1 \cdots y_q.$$

This means that, after reordering the f_i, each $f_i|_{V(\sum_{i=1}^m I_i)} \in (y_i)$, and the desired conclusion follows. ∎

8. Algorithm for the main theorem

In this section we prove Theorem 1.16. The proof will depend on the results given in Sections 9 and 10 following. We divide the proof into several steps or subroutines each of which specifies certain blowings-up.

STEP 1. *Make* (X, E) *stable-snc.* This is an application of Theorem 5.1. The blowings-up involved preserve stable-snc singularities of (X, E) and therefore also of (X, D, E). As a result of Step 1, we can assume that (X, E) is stable-snc.

In the following steps, all blowings-up will be both admissible and snc with respect to X, to preserve the property that (X, E) is stable-snc.

STEP 2. *Remove irreducible components of* D *lying in* $\operatorname{Sing} X$ *or* $\operatorname{Supp} E$. Given a triple (X, D, E), consider the union $\mathcal{Z}$ of the supports of the (irreducible) components of D lying in $\operatorname{Sing} X \cup \operatorname{Supp} E$. Any such component is a component either of the intersection of two components of X, or of the intersection of a component of X and a component of E. Therefore, $\mathcal{Z}$ is snc, in general with components of different dimensions. Blowings-up as needed can simply be given by the usual desingularization of $\mathcal{Z}$, followed by blowing up the final strict transform.

The point is that, locally, there is a smooth ambient variety, with coordinates $(x_1, \ldots, x_p, \ldots, x_n)$ in which each component of $\mathcal{Z}$ is of the form $(x_{i_1} = \cdots = x_{i_k} = 0)$, $i_1 < \cdots < i_k \leq p$. Let C denote the set of irreducible components of intersections of arbitrary subsets of components of $\mathcal{Z}$. Elements of C are partially ordered by inclusion, and are snc with respect to X and E. Desingularization of $\mathcal{Z}$ involves blowing up elements of C starting with the smallest, until all components of $\mathcal{Z}$ are separated. Then blowing up the final (smooth) strict transform removes all components of $\mathcal{Z}$.

As a result of Step 2, we can assume that no component of D lies in $\operatorname{Sing} X$ or in $\operatorname{Supp} E$.

STEP 3. *Make* (X, D_{red}, E) *stable-snc* (i.e., transform (X, D, E) by the blowings-up needed to make (X, D_{red}, E) stable-snc). The algorithm for Step 3 is given following Step 4 and the paragraph on functoriality below.

We can therefore now assume that (X, D_{red}, E) is stable-snc and that D has no irreducible components in $\operatorname{Sing} X$ or $\operatorname{Supp} E$.

STEP 4. *Make* (X, D, E) *stable-snc.* A simple combinatorial argument for Step 4 will be given in Section 10. This completes the algorithm.

FUNCTORIALITY. (See also [8, Sect. 9].) The steps above involve several applications of the general desingularization algorithm. Beginning with a local étale invariant ι (e.g., the Hilbert–Samuel function), the centres of blowing up are determined by a corresponding étale invariant inv_ι defined recursively over a sequence of admissible blowings-up. The monomial marked ideals used in cleaning (Section 4) are étale-invariant. The obstruction ideal $J(X, D)$ (Section 6) is an invariant of étale morphisms preserving the number of irreducible components of X at every point (see Remark 9.6). The functoriality assertion of Theorem 1.16 follows because the blowing-up sequence given by the four steps above depends, at a given point, only on the preceding objects and the desingularization invariant, as well as the number of components of X and D, and their codimensions in a local minimal embedding variety.

The theorem is functorial for triples (X, D, E) with a given ordering of the components of X because Theorem 8.1 used in the following algorithm for Step 3 is proved by induction on the number of components; the corresponding algorithm resolves non-stable-snc singularities of (X, D_{red}, E) successively in the ordered components. Proposition 9.1 of the following section is used in the proof of Theorem 8.1, and involves the inductive characterization of stable-snc of Proposition 6.7.

ALGORITHM FOR STEP 3. The input is a triple (X, D, E), where (X, E) is stable-snc, D is reduced and no irreducible component of D lies in $\mathrm{Sing}\, X \cup \mathrm{Supp}\, E$. We will argue by induction on the number of components of X. Since D is reduced, we make no distinction between D and $\mathrm{Supp}\, D$. The algorithm for Step 3 is given in the proof of Theorem 8.1 below, applied to the 4-tuple $(X, D, E, \emptyset)$.

THEOREM 8.1: *Assume that* (X, D, E, Y) *is a* 4*-tuple as in Definition 6.6, such that* $(W := X \cup Y, E)$ *is stable-snc, and* D *is a reduced Weil divisor on* X *with no component in* $\mathrm{Sing}\, W \cup \mathrm{Supp}\, E$*. Then there is a morphism* $\tau : W' \to W$ *given by a composite of admissible smooth blowings-up whose centres are snc with respect to* W*, such that:*

(1) *Each blowing-up is an isomorphism over the stable-snc points of its target* 4*-tuple.*

(2) *The transform* (X', D', E', Y') *of* (X, D, E, Y) *by* τ *is everywhere stable-snc.*

Proof. The proof is by induction on the number of components m of X. We use the notation of Definitions 6.4 and 6.6.

Case $m = 1$. For $m = 1$ (i.e., $X = X^{(1)}$), we apply Theorem 5.1 to $(D \cup Y|_X, E|_X)$, and end up with $(D' \cup Y'|_{X'}, E'|_{X'})$ stable-snc. Since D is a divisor on X, then D' is a divisor on X', and we have (X', D', E', Y') stable-snc. All centres of blowing up involved are snc with respect to $W = X \cup Y$, by Remarks 5.2(2).

General case. The sequence of blowings-up will depend on the ordering of the components $X^{(i)}$ of X.

By induction, we can assume that $(X^{m-1}, D^{m-1}, E, X^{(m)} \cup Y)$ is stable-snc. We want to construct a sequence of admissible blowings-up after which the transform (X', D', E', Y') of (X, D, E, Y) is stable-snc. For this purpose, we only have to remove the unwanted singularities of D in $X^{(m)}$.

A. We will first reduce to the case that (X, D, E, Y) is stable-snc at every point of X^{m-1}. For this purpose, we use the partition of $X^{(m)}$ by the sets $\Sigma_{\Omega,q} = \Sigma_{\Omega,q}(X, D)$ (see Definition 6.1). Clearly, the $\Sigma_{\Omega,q}$ with $r_\Omega := r \geq 2$, where $\Omega = (e, c_1, \ldots, c_r)$, form a partition of $X^{(m)} \cap X^{m-1}$.

We use the ordering of the set of Hilbert–Samuel functions $H_{\Omega,q}$ (Definitions 7.2 and 6.3) to order the set of tuples (Ω, q) and thus the strata $\Sigma_{\Omega,q}$ (Definition 6.1).

Definition 8.2: We say that $(\Omega_1, q_1) \geq (\Omega_2, q_2)$ and also that $\Sigma_{\Omega_1,q_1} \geq \Sigma_{\Omega_2,q_2}$, if $(\Omega_1, H_{\Omega_1,q_1}) \geq (\Omega_2, H_{\Omega_2,q_2})$ in the lexicographic order, where we compare Ω_1, Ω_2 also lexicographically, and H_{Ω_1,q_1}, H_{Ω_2,q_2} by Definition 7.2.

The order above corresponds to that in which we will eliminate the non-stable-snc points from the strata $\Sigma_{\Omega,q}$, $r_\Omega \geq 2$.

Clearly for all Ω and q, the closure $\overline{\Sigma}_{\Omega,q}$ of $\Sigma_{\Omega,q}$ has the property

$$\overline{\Sigma}_{\Omega,q} \subset \bigcup_{(\Omega',q') \geq (\Omega,q)} \Sigma_{\Omega',q'}. \tag{8.1}$$

Definition 8.3: Let $\mathcal{M}$ denote the set of all possible values of (Ω, q), and let $\mathcal{M}(X, D) := \{(\Omega, q) \in \mathcal{M} : \emptyset \neq \Sigma_{\Omega,q}(X, D) \subset X^{(m)} \cap X^{m-1}\}$. Let $K(X, D)$ denote the set of maximal elements of $\mathcal{M}(X, D)$.

Note that $K(X,D)$ consists only of incomparable pairs (Ω,q), and that, after an admissible blowing-up σ, all points of $\sigma^{-1}(a)$, where $a\in\Sigma_{\Omega,q}$, lie in strata $\leq\Sigma_{\Omega,q}$.

We apply Proposition 9.1 of the following section to construct a morphism $X'\to X$ given by a sequence of admissible blowings-up such that (X',D',E',Y') is stable-snc on the strata $\Sigma_{\Omega,q}(X',D')$, where $(\Omega,q)\in K(X,D)$,

Let $U' := X'\setminus\bigcup_{(\Omega,q)\in K(X,D)}\Sigma_{\Omega,q}(X',D')$. By (8.1), U' is open. Clearly, $\mathcal{M}(U',D'|_{U'})=\mathcal{M}(X',D')\setminus K(X,D)$. The set $\mathrm{Fin}(\mathcal{M})$ of finite subsets of $\mathcal{M}$, ordered by inclusion, is a partially ordered set in which every nonempty subset has a minimal element. We can therefore assume by induction on $\mathrm{Fin}(\mathcal{M})$ that $(U',D'|_{U'},E',Y')$ is stable-snc at every point in X^{m-1}. The blowings-up involved have centres that are nowhere stable-snc and are, therefore, closed not only in U' but also in X'.

B. Under the assumption that (X,D,E,Y) is stable-snc at every point of X^{m-1}, we complete the proof as follows: Let $U=X^{(m)}\setminus X^{m-1}$. We apply Theorem 5.1 to $(D|_U\cup Y|_U,E)$ (regarding $D|_U\cup Y|_U$ as a subvariety of the smooth variety U), to get $(D'|_{U'}\cup Y'|_{U'},E')$ stable-snc. Since $D|_U$ is a divisor on U, then $D'|_{U'}$ is a divisor on U'. Therefore, $(U',D'|_{U'},E'|_{U'},Y'|_{U'})$ is stable-snc. The centres of blowing up involved contain no stable-snc points. Since (X,D,E,Y) is stable-snc at every point of $X\setminus U$ and the stable-snc locus is open, these centres are closed not only in U but also in X. ∎

9. Desingularization at the singular locus of X

In this section, we complete the proof of Theorem 8.1 by showing how to eliminate non-stable-snc singularities from the strata $\Sigma_{\Omega,q}$ with $r_\Omega\geq 2$. We recall that these strata consist of points belonging to at least 2 components of D which lie in different components of X. We will use the notation of Section 8.

Proposition 9.1: *Let (X,D,E,Y) denote a 4-tuple as in Definition 6.6, satisfying the hypotheses of Theorem 8.1. Assume that $(X^{m-1},D^{m-1},E,X^{(m)}\cup Y)$ is stable-snc. Then there is a sequence of admissible smooth blowings-up whose centres are snc with respect to $W=X\cup Y$, such that:*

(1) *each centre of blowing-up contains only non-stable-snc points;*

(2) *the transform* (X', D', E', Y') *of* (X, D, E, Y) *by the blowing-up sequence is stable-snc at all points of the strata* $\Sigma_{\Omega,q}(X', D')$, *where* $(\Omega, q) \in K(X, D)$.

The proof will involve several lemmas. We will use the assumptions of Proposition 9.1 throughout the section.

Let $a \in X$. There is a minimal smooth local embedding variety Z of X at a, with a system of coordinates $\{x_{i,j}\}_{1\le i\le m,\ 1\le j\le c_i}$, $\{y_k\}_{1\le k\le q}$, $\{w_l\}_{1\le l\le n-|c|-q}$, $|c| = c_1 + \cdots + c_m$, in which $a = 0$ and

$$X = X^{(1)} \cup \cdots \cup X^{(m)},$$
$$D = D_1 + \cdots + D_m,$$

where $X^{(i)} = (x_{i,1} = \cdots = x_{i,c_i} = 0)$, $i = 1, \ldots, m$, $D_i = (x_{i,1} = \cdots = x_{i,c_i} = y_1 \cdots y_q = 0)$, $i = 1, \ldots, m-1$, and $D_m = (x_{m,1} = \cdots = x_{m,c_m} = f = 0)$, for some $f \in \mathcal{O}_{Z,a}$. This notation will be used throughout the section.

9.1. Reduction of the obstruction ideal $J(X, D)$ to $\mathcal{O}_X$. Recall Definition 6.5 and Proposition 6.7.

Since $(X^{m-1}, D^{m-1}, 0, X^{(m)})$ is stable-snc, $\operatorname{cosupp} J \subset X^{(m)} \cap X^{m-1}$ (where $\operatorname{cosupp} J := \operatorname{supp} \mathcal{O}_X/J$). Let $W_1, \ldots, W_s$ denote the irreducible components of $(X^{m-1} \cup Y)|_{X^{(m)}}$, and let $D(1), D(2), \ldots, D(q)$ denote the restrictions to $X^{(m)} \cap X^{m-1}$ of the components of D_i, for any given $i = 1, \ldots, m-1$ (the definition is independent of such i). Let $\mathcal{H}_i \subset \mathcal{O}_{X^{(m)}}$ denote the ideal of $D(i) \subset X^{(m)}$, $i = 1, \ldots, q$, and let $\mathcal{K}_j \subset \mathcal{O}_{X^{(m)}}$ denote the ideal of W_j, $j = 1, \ldots, s$.

Consider the marked ideal

$$\underline{\mathcal{I}} := (X^{(m)}, X^{(m)}, E|_{X^{(m)}}, J + \mathcal{H} + \mathcal{K}, 1), \tag{9.1}$$

where $\mathcal{H} := \sum_{i=1}^{q} \mathcal{H}_i$ and $\mathcal{K} = \sum_{j=1}^{s} \mathcal{K}_j$. We can use desingularization of the marked ideal $\underline{\mathcal{I}}$ (treating $(\mathcal{H}+\mathcal{K}, 1)$ as a "boundary"; cf. Section 2) to desingularize $(J, 1)$ after perhaps moving the $D(i)$, $i = 1, \ldots, q$, and W_j, $j = 1, \ldots, s$, away from $\operatorname{cosupp}(J, 1)$. The blowings-up involved are admissible for (X, D, E, Y), and snc with respect to $W = X \cup Y$ (since the boundary includes $\mathcal{K}$). The final transform $J(X, D)' = \mathcal{O}_{X'}$. It is not necessarily true, however, that $J(X, D)' = J(X', D')$, so we do not necessarily have $J(X', D') = \mathcal{O}_{X'}$. Additional "cleaning" blowings-up (given by Lemma 9.5) will be needed.

Example 9.2: Consider $X = X^{(1)} + X^{(2)} = (x_1 = 0) \cup (x_2 = 0)$ and $D_1 + D_2 = (x_1 = y = 0) + (x_2 = x_1 + yzw = 0)$. Then $J(X, D) = (x_1, x_2, zw)$. The desingularization algorithm for J first blows up $(x_1 = x_2 = z = w = 0)$. In the z-chart, we get $X' = (x_1 x_2 = 0)$ and $D' = (x_1 = y = 0) + (x_2 = x_1 + yzw = 0)$. Then the desingularization of J is completed by blowing up $(x_1 = x_2 = w = 0)$. In the w-chart we have

$$X'' = (x_1 x_2 = 0) \quad \text{and} \quad D'' = (x_1 = y = 0) + (x_2 = x_1 + yz = 0).$$

Note that $J(X'', D'') = (x_1, x_2, z) \neq (1) = J(X, D)''$. Since $z = 0$ is now a component of the exceptional divisor, we can blow up with centre

$$X^{(1)} \cap X^{(2)} \cap (z = 0).$$

After this "cleaning" blowing-up, we have

$$\begin{aligned} X''' =& (x_1 x_2 = 0), \\ D''' =& (x_1 = y = 0) + (x_2 = x_1 + y) = (x_1 = x_1 + y = 0) + (x_2 = x_1 + y), \end{aligned}$$

and $J(X''', D''') = (1)$; in particular (X''', D''') is stable-snc.

LEMMA 9.3: *Consider the morphism $X' \to X$ given by a sequence of admissible blowings-up for* (9.1). *Then*

$$J(X', D') \subset J(X, D)'. \tag{9.2}$$

Moreover, if $J(X, D)' = \mathcal{O}_{X'}$ and $a' \in X'$, then

$$J(X', D')_{a'} = \bigcap_{1 \leq i \neq j \leq m} (I_i + I_j + (u^{\alpha_{i,j}})),$$

where the $u^{\alpha_{i,j}}$ are monomials in generators u_p of the ideals of the components of the exceptional divisor of $X' \to X$.

Remark 9.4: By (9.2), if $J(X, D)' \neq \mathcal{O}_{X'}$, then $J(X', D') \neq \mathcal{O}_{X'}$. Therefore, by Lemma 6.7, we never blow-up stable-snc points of the transforms of (X, D) while desingularizing $J(X, D)$.

Proof of Lemma 9.3. It is enough to prove the lemma for one of the "factors" $[I_{D_i} + I_{X^{(j)}} : I_{D_j} + I_{X^{(i)}}]$ of J. The proof is then the same as that of [8, Proof of Lemma 7.3], replacing x_i in the latter by I_i here. ∎

LEMMA 9.5: *Consider the transform (X', D', E', Y') of (X, D, E, Y) by the desingularization sequence for* (9.1) *above. Then:*

(1) *For every* (Ω, q), $\Sigma_{\Omega,q}(X', D')$ *lies in the inverse image of* $\Sigma_{\Omega,q}(X, D)$.

(2) *Let* $a' \in X'$. *Then*

$$J(X', D')_{a'} = \bigcap_{1 \le i \ne j \le m} (I_i + I_j + (u^{\alpha}_{i,j})),$$

where each I_i *denotes the ideal of the component* $X^{(i)\prime}$ *of* X', *and the* $u^{\alpha_{i,j}}$ *are monomials in generators* u_p *of the ideals of the components of* E'. *Thus the variety* $V(J(X', D'))$ *consists of certain components of intersections of pairs of components of* X' *and components of* E'.

(3) *After finitely many blowings-up of components of* $V(J(X', D'))$ *(and its successive transforms), the transform* (X'', D'') *of* (X, D) *satisfies* $J(X'', D'') = \mathcal{O}_{X''}$. *(For functoriality, the components to be blown up can be chosen according to the order on the components of* E*.)*

Proof. (1) has already been remarked in the previous section. (2) and (3) can be proved in the same way as the corresponding assertions of [8, Lemma 7.5] replacing x_i in the latter by I_i here, and multiples of x_i by linear combinations with coefficients in $\mathcal{O}_X$ of the $x_{i,j}$ here. (2) follows from the second assertion of Lemma 9.3 and, for (3), we can directly compute the effect of the blowings-up. ∎

Remark 9.6: The desingularization algorithm of Theorem 9.1 is functorial with respect to étale morphisms that preserve the number of irreducible components at every point, since J has an étale-invariant meaning and the algorithms involved in desingularizing J and in cleaning are controlled by étale invariants.

9.2. Simplification of $\operatorname{Supp} D$. In order to prove Proposition 9.1 above, we need to construct a blowing-up sequence that will allow us to decrease and control the Hilbert–Samuel function on the strata $\Sigma_{\Omega,q}$, where $(\Omega, q) \in K(X, D)$. We can use the desingularization of $\operatorname{Supp} D$ to decrease the Hilbert–Samuel function, but we will blow up only certain irreducible components of the centres prescribed by this desingularization, in a convenient way.

At every point $a \in X^{(m)}$, we introduce the invariant

$$\iota(a) = (e(a), c(a), H_{\operatorname{Supp} D, a}),$$

where $(e(a), c(a)) = (e, c)$ is defined as in Definition 6.1. The set of values of this invariant is partially ordered, lexicographically (using the partial ordering of the set of Hilbert–Samuel functions given by Definition 7.2).

Clearly, the invariant $\iota = ((e,c), H_{\operatorname{Supp} D})$ is upper-semi-continuous on $X^{(m)}$. Since (e,c) is constant on $\{x \in \operatorname{Supp} D : \ H_{\operatorname{Supp} D,x} = H_{\operatorname{Supp} D,a}\}$ near a (i.e., on the cosupport of a presentation of $H_{\operatorname{Supp} D}$ at a), a presentation of the Hilbert–Samuel function of $\operatorname{Supp} D$ at a is also a presentation of the invariant ι. In particular, we can extend ι to a desingularization invariant inv_ι.

The centres of blowing-up involved in the desingularization algorithm for inv_ι are locally the same as in the standard desingularization algorithm, corresponding to the invariant determined by $H_{\operatorname{Supp} D}$, but the use of ι instead of $H_{\operatorname{Supp} D}$ means that, globally, the centres may have components that are blown up in a different order.

Given $a \in X^{(m)}$, ι admits a presentation of the form $\underline{\mathcal{I}} = (X^{(m)}, X^{(m)}, 0, \mathcal{I}, d)$ at a. We will consider the desingularization invariant inv and desingularization algorithm determined by this presentation of ι, treating the restrictions to $X^{(m)}$ of the components of E and the remaining components of $W = X \cup Y$ as a "boundary" $\underline{\mathcal{B}}$ (even though the latter are not necessarily codimension one in $X^{(m)}$). In other words, we let $\underline{\mathcal{B}}$ denote the marked ideal $(X^{(m)}, X^{(m)}, 0, \mathcal{B}, 1)$, where $\mathcal{B}$ denotes the sum of the ideals on $X^{(m)}$ of the components of E and the components of $W \setminus X^{(m)}$, and we consider the desingularization algorithm given locally by desingularization of the marked ideal $\underline{\mathcal{I}} + \underline{\mathcal{B}}$. The effect of the algorithm is to decrease ι after perhaps moving the components of E and $W \setminus X^{(m)}$ away from $\operatorname{Supp} D$. The blowings-up involved are admissible for (X, D, E, Y) and snc with respect to W.

Proposition 9.7: *Given (X, D, E, Y) as in Proposition 9.1, there is a sequence of admissible blowings-up $(X', D', E', Y') \to (X, D, E, Y)$, with centres snc with respect to $W = X \cup Y$ and containing no stable-snc points, such that for every $\Sigma_{\Omega,q} \in K(X, D)$ and $a \in \Sigma_{\Omega,q}(X', D')$, $H_{\operatorname{Supp} D,a} = H_{\Omega,q}$.*

Lemma 9.8: *Let C be an irreducible smooth subvariety of* $\operatorname{Supp} D$. *Given (Ω, q), suppose that $\iota = (\Omega, H_{\Omega,q})$ at every point of C. If $C \cap \Sigma_{\Omega,q} \neq \emptyset$, then $C \subset \Sigma_{\Omega,q}$.*

Proof. Let $a \in C \cap \Sigma_{\Omega,q}$. Since $H_{\operatorname{Supp} D}$ is constant on C, a has a neighbourhood $U \subset C$ such that each point of U lies in precisely those components of D containing a. Therefore, $U \subset \Sigma_{\Omega,q}$. Since the closure of $\Sigma_{\Omega,q}$ lies in the union of the $\Sigma_{\Omega',q'}$ with $(\Omega', q') \geq (\Omega, q)$, any $b \in C \setminus U$ belongs to $\Sigma_{\Omega',q'}$, for some $(\Omega', q') \geq (\Omega, q)$. Moreover, $\Omega' = \Omega$, since ι is constant on C. Thus $H_{\operatorname{Supp} D,b} =$

$H_{\Omega,q} < H_{\Omega,q'}$. But, by Corollary 7.7, the Hilbert–Samuel function cannot be $< H_{\Omega,q'}$ on $\Sigma_{\Omega,q'}$. Therefore $b \in \Sigma_{\Omega,q}$. ∎

Proof of Proposition 9.7. We consider the desingularization algorithm preceding Proposition 9.7, but will blow up only certain components of the centres of blowing-up involved in the algorithm. The centres of blowing-up given by the algorithm are the maximum loci of inv. The maximum locus of inv includes points with all maximal values of ι. The maximum locus of inv can be written as a disjoint union $A \cup B$ in the following way: A is the union of those components of the maximum locus containing no stable-snc points, and B is the union of the remaining components. Thus B is the union of those components of the maximum locus of inv with generic point stable-snc. Each component of B has Hilbert–Samuel function $H_{\Omega,q}$, for some (Ω, q), and lies in the corresponding $\Sigma_{\Omega,q}$ by Lemma 9.8.

In each year j of the blowing-up history, write $A = A_j$, $B = B_j$. We will blow up with centre A_j only. Then inv decreases in the preimage of A_j. In the following year $j+1$, B_{j+1} may acquire new components in addition to those of B_j, but eventually $A_k = \emptyset$. So we reduce to the case that $A = \emptyset$.

Lemma 9.9: *Suppose $A = \emptyset$. If $(\Omega, q) \in K(X, D)$, then $H_{\operatorname{Supp} D,a} = H_{\Omega,q}$, for all $a \in \Sigma_{\Omega,q}$.*

Proof. Let $a \in \Sigma_{\Omega,q}$, where $(\Omega, q) \in K(X, D)$. Set $H = H_{\operatorname{Supp} D,a}$. Assume that $H \neq H_{\Omega,q}$. Recall that, for every $b \in B$, $H_{\operatorname{Supp} D,b} = H_{\Omega',q'}$ for some (Ω', q'), and $b \in \Sigma_{\Omega',q'}$. Therefore $a \notin B$, so that $\operatorname{inv}(a)$ is not maximal. Thus there exists $b \in B$ such that $\iota(b) = (\Omega', H_{\Omega',q'})$ and $(\Omega', H_{\Omega',q'}) \geq (\Omega, H)$, for some (Ω', q'), and $b \in \Sigma_{\Omega',q'}$. If $\Omega' > \Omega$ then $(\Omega', q') > (\Omega, q)$; this contradicts $(\Omega, q) \in K(X, D)$. If $\Omega' = \Omega$ then, by Corollary 7.7, $H_{\Omega',q'} \geq H_{\Omega,q}$. If $H_{\Omega',q'} > H_{\Omega,q}$, then $(\Omega', q') > (\Omega, q)$, again contradicting $(\Omega, q) \in K(X, D)$. If $H_{\Omega',q'} = H_{\Omega,q}$, then $H = H_{\Omega,q}$, by Corollary 7.7, as desired. ∎

Lemma 9.9 finishes the proof of Proposition 9.7. ∎

Proof of Proposition 9.1. We first reduce to the case $J = \mathcal{O}_X$, using Lemma 9.5. The proof then has two steps:

(1) We apply Proposition 9.7 to make $H_{\operatorname{Supp} D,a} = H_{\Omega,q}$, for all $a \in \Sigma_{\Omega,q}$ and all $(\Omega, q) \in K(X, D)$.
(2) We use Lemma 9.5 to reduce to $J = \mathcal{O}_X$.

The initial reduction to $J = \mathcal{O}_X$ is for the purpose of functoriality: The centres of blowing-up involved in desingularization of J may include points outside the strata of $K(X, D)$. Therefore, on an open set U outside the strata of $K(X, D)$, the centres of blowing-up from desingularization of J (in Step (2), for example) may play a role when applying Step (1) for $K(U, D|_U)$ (in the inductive step of Case B in the proof of Theorem 8.1).

After Step (1), $H_{\operatorname{Supp} D, a} = H_{\Omega, q}$, for all $a \in \Sigma_{\Omega,q}$ and all $(\Omega, q) \in K(X, D)$. Then, by Lemma 7.5, at each $a \in \Sigma_{\Omega,q}$, where $(\Omega, q) \in K(X, D)$, we have

$$I_{D_m} + I_{D^{m-1}} = I_{D^{m-1}} + I_m,$$

where I_{D_m}, $I_{D^{m-1}}$ and I_m are the ideals of D_m, D^{m-1} and $X^{(m)}$, respectively. In the notation of Lemma 7.5, $I_{D_m} = (x_{m,1}, \ldots, x_{m,c_m}, f)$, $I_{D^{m-1}} = \bigcap_{i=1}^{m-1}(x_{i,1}, \ldots, x_{i,c_i}) + (y_1 \cdots y_r)$ and $I_m = (x_{m,1}, \ldots, x_{m,c_m})$, and the lemma says that $f \in I_{D^{m-1}}$. This property is preserved by blowings-up as involved in Step (2). By Lemma 7.5, we also have $\operatorname{ord}(f) = \operatorname{ord}(\operatorname{Supp} D^{(1)}) = q$. This property is preserved by blowings-up with smooth centres in $\operatorname{Supp} D_m$ that are normal crossings to D^{m-1}; this is the case for the blowings-up from desingularization of J (see Section 9.1). Thus the properties above are preserved by Step (2).

We can therefore apply Theorem 3.9 to conclude that (X, D, E, Y) is stable-snc at every point of $\Sigma_{\Omega,q}$, for $(\Omega, q) \in K(X, D)$. ∎

10. The non-reduced case

The previous sections establish Theorem 1.16 in the case that D is reduced. In this section we describe the blowings-up necessary to establish the non-reduced case. In other words, we assume that (X, D_{red}, E) is stable-snc, and we prove Theorem 1.16 under this assumption.

The algorithm is a simple modification of that in [8, Section 8], to account for the fact that the components of X and therefore of D are not necessarily of the same dimension here. For this reason, we only give the modified algorithm and refer to [8] for the proof.

We define an equivalence relation on the components of D at a point of X.

Definition 10.1: Let $a \in X$ and let D_1, D_2 denote components of D at a. Assume that, for each $i = 1, 2$, $D_i \subset X^{(i)}$, where $X^{(i)}$ is a component of X of codimension c_i in a minimal local embedding variety Z of X at a. We say that D_1 and D_2 are **equivalent (at** a) if either $D_1 = D_2$ or the irreducible component of $D_1 \cap D_2$ at a has codimension $c_1 + c_2 + 1$ in Z.

Given $a \in X$, let $\kappa_X(a)$ denote the number of components of X at a, and let $q(a)$ denote the number of equivalence classes present in the set of components of D at a. Define $\iota : X \to \mathbb{N}^2$ by $\iota(a) := (\kappa_X(a), q(a))$. We give $\mathbb{N}^2$ the partial order where $(\kappa_1, q_1) \geq (\kappa_2, q_2)$ means that $\kappa_1 \geq \kappa_2$ and $q_1 \geq q_2$. Then ι is upper semi-continuous. Therefore, the maximal locus of ι is a closed set.

Each irreducible component Q of the maximal locus of ι consists of only stable-snc points or only non-stable-snc points, because all points of Q belong to the same irreducible components of D. We blow up with centre $C =$ the union of the components of the maximal locus of ι that contain only non-stable-snc points. In the preimage of C, ι decreases.

Let W be the union of the components of the maximal locus consisting of stable-snc points. The blowing-up above is an isomorphism on W, so (X', D') is stable-snc on $W' = W$, and therefore in a neighbourhood of W'. For this reason, the union of the components of the maximal locus of ι on $X' \setminus W'$ that contain only non-stable-snc points is closed in X'. Therefore, we can repeat the procedure on $X' \setminus W'$.

Clearly, $\mathbb{N}^2$ has no infinite decreasing sequences with respect to the order above. After the blowing-up above, the maximal values of ι on the non-stable-snc locus of (X, D) decrease. Therefore, after a finite number of iterations of the procedure above, the non-stable-snc locus becomes empty.

Remark 10.2: Suppose that (X, D_{red}) is stable-snc. Then the blowing-up sequence in this section is given simply by the desingularization algorithm for $\operatorname{Supp} D$, but blowing up only those components of the maximal locus of the invariant on the non-stable-snc locus.

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