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ON THE COHOMOLOGY OF TORI OVER LOCAL FIELDS WITH PERFECT RESIDUE FIELD

BY

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ABSTRACT

If T is an algebraic torus defined over a discretely valued field K with perfect residue field k, we relate the K-cohomology of T to the k-cohomology of certain objects associated to T. When k has cohomological dimension ≤ 1, our results have a particularly simple form and yield, more generally, isomorphisms between Borovoi's abelian K-cohomology of a reductive group G over K and the k-cohomology of a certain quotient of the algebraic fundamental group of G.

1. Introduction

Let A be a complete discrete valuation ring with field of fractions K and perfect residue field k. Let $\overline{k}$ and K^{sep} be fixed separable algebraic closures of k and K, respectively, and let g and $\mathcal{G}$ denote the corresponding absolute Galois groups.

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Further, let K^{nr} denote the maximal unramified extension of K in K^{sep} and let $I = \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$ be the inertia subgroup of $\mathcal{G}$. Then there exists a canonical isomorphism of groups $\mathcal{G}/I = g$. If M is a $\mathcal{G}$-module, M_I will denote the g-module of I-coinvariants of M. Now let T be a K-torus and let $X^*(T)$ (respectively, $X_*(T)$) denote the $\mathcal{G}$-module of characters (respectively, cocharacters) of T. Let $\mathcal{T}$ denote the Néron model of T over $S := \mathrm{Spec}\, A$ and let $i\colon \mathrm{Spec}\, k \to S$ be the canonical closed immersion. The **group of components of** $\mathcal{T}$, i.e., the (continuous) g-module $\phi(T)$ which corresponds to the étale k-sheaf $i^*(\mathcal{T}/\mathcal{T}^0)$, was described by Xarles in [31] in terms of $X^*(T)$. The description given in [31] is simple when $\phi(T)$ is either torsion or torsion-free, but this is not the case in general. When k is finite, a much simpler description of $\phi(T)$ was obtained by Bitan in [2, (3.1)], who showed the existence of an isomorphism of g-modules $\phi(T) \simeq X_*(T)_I$ for such k. In this paper we generalize Bitan's strikingly simple formula to the case of any perfect residue field k. That is, we prove

Theorem 1.1: *There exists a canonical isomorphism of g-modules*

$$\phi(T) \xrightarrow{\sim} X_*(T)_I.$$

We note that Bitan [2, (3.1)] obtained his formula by combining work of Kottwitz [19, §7.2] and of Haines and Rapoport [24, Appendix]. Since [24, Appendix] depends on Bruhat–Tits theory, so does the proof of [2, (3.1)]. Although Bitan's method can be extended to yield a proof of his formula for any perfect field k, in this paper we have chosen to generalize it by means of an explicit and functorial construction of the isomorphism of Theorem 1.1 which is independent of Bruhat–Tits theory.

The above theorem has a number of (immediate) consequences which shed new light on the present subject. For example, the theorem implies that the functor $\phi(-)$ transforms short exact sequences of K-tori into 6-term exact sequences of g-modules. See Proposition 3.6 for the precise statement.

In Section 4 we use Theorem 1.1 to relate the K-cohomology of T to the k-cohomology of $X_*(T)_I$. When k has cohomological dimension ≤ 1, our results have the following simple form.

Theorem 1.2 (=Theorem 4.5): *Assume that k has cohomological dimension ≤ 1. Then, for $r = 1$ and 2, there exist canonical isomorphisms of abelian groups*

$$H^r(K, T) \simeq H^r(k, X_*(T)_I).$$

If $r \geq 3$, the groups $H^r(K, T)$ vanish.

In Section 5 we generalize the above theorem from K-tori to arbitrary connected reductive algebraic K-groups G. More precisely, let $\pi_1(G)$ be the algebraic fundamental group of G. Then the following holds.

THEOREM 1.3 (=Theorem 5.1): *Assume that k has cohomological dimension ≤ 1 and let G be a connected reductive algebraic group over K. Then, for $r = 1$ and 2, there exist isomorphisms of abelian groups*

$$H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \simeq H^r(k, \pi_1(G)_I).$$

If $r \geq 3$, the groups $H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G)$ vanish.

COROLLARY 1.4 (=Corollary 5.2): *Assume that k has cohomological dimension ≤ 1 and let G be a connected reductive algebraic group over K. Then there exists a bijection of pointed sets*

$$H^1(K, G) \simeq H^1(k, \pi_1(G)_I).$$

In particular, $H^1(K, G)$ can be endowed with an abelian group structure.

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2. Preliminaries

2.1. THE BASIC SETTING. We keep the notation introduced above. If K is any field and T is a K-torus, let $\underline{X}^*(T) := \underline{\mathrm{Hom}}_K(T, \mathbb{G}_{m,K})$ be the étale K-sheaf of characters of T and set $X^*(T) = \underline{X}^*(T)(K^{\mathrm{sep}})$. Note that, since $\underline{X}^*(T)$ is locally constant, it is represented by a unique (commutative) étale K-group scheme. See [29, Proposition II.9.2.3, p. 153].

There exists a canonical isomorphism of étale K-sheaves

$$T = \underline{\mathrm{Hom}}_K(\underline{X}^*(T), \mathbb{G}_{m,K}). \tag{1}$$

See, for example, [5, Theorem 0.3.12, p. 11]. In particular, if L/K is any Galois subextension of K^{sep}/K which splits T, then there exists a canonical isomorphism of $\mathrm{Gal}(L/K)$-modules

$$T(L) = \mathrm{Hom}(X^*(T), L^*). \tag{2}$$

Now let $\underline{X}_*(T) := \underline{\mathrm{Hom}}_K(\mathbb{G}_{m,K}, T)$ be the étale K-sheaf of cocharacters of T and set $X_*(T) = \underline{X}_*(T)(K^{\mathrm{sep}})$. We will identify $\underline{X}_*(T)$ with $\underline{X}^*(T)^\vee := \underline{\mathrm{Hom}}_K(\underline{X}^*(T), \mathbb{Z}_K)$ and $X_*(T)$ with $X^*(T)^\vee = \mathrm{Hom}(X^*(T), \mathbb{Z})$.

Let S be a scheme and let $(\mathrm{Sch}/S)^{\sim}_{\text{ét}}$ be the category of sheaves of sets on the étale site over S. By [10, I, 1.1] (see also [29, Theorem II.3.1.2, p. 97]), the functor $h_S\colon (\mathrm{Sch}/S) \to (\mathrm{Sch}/S)^{\sim}_{\text{ét}}$, $Y \mapsto \mathrm{Hom}_S(-, Y)$, is fully faithful. If Y is an S-scheme, we will identify Y with $h_S(Y)$, i.e., with the étale sheaf that it represents. If $S = \mathrm{Spec}\, F$, where F is a field, then

$$h_F\colon (\mathrm{Sch}/F) \to (\mathrm{Sch}/F)^{\sim}_{\text{ét}}, \quad Y \mapsto \mathrm{Hom}_F(-, Y), \tag{3}$$

is an equivalence of categories [21, p. 54, last paragraph].

Now recall $S = \mathrm{Spec}\, A$ and $i\colon \mathrm{Spec}\, k \to S$. Let $j\colon \mathrm{Spec}\, K \to S$ be the canonical open immersion. The Néron model $\mathcal{T}$ of T over S is a smooth and separated S-group scheme which represents the sheaf j_*T on the étale (in fact, small smooth) site over S. See [4, Proposition 10.1.6, p. 292]. With one exception (namely, in Proposition (2.2)(i)), we will regard j_*T as an étale sheaf on S and identify it with (the étale sheaf on S represented by) $\mathcal{T}$. Thus we may write $j_*T = \mathcal{T}$. The identity component $\mathcal{T}^0$ of $\mathcal{T}$ is a smooth affine S-group scheme of finite type. See [20, Proposition 3, p. 18] and [10, $\mathrm{VI_B}$, Corollary 3.6]. Now set $\mathcal{T}_s = \mathcal{T} \times_S \mathrm{Spec}\, k$ and $\mathcal{T}^0_s = \mathcal{T}^0 \times_S \mathrm{Spec}\, k$. Then $\mathcal{T}^0_s$ is a smooth, connected and affine k-group scheme of finite type (see [18, Proposition 17.3.3(iii)], [17, Proposition 1.6.2(iii)] and [16, Proposition 6.3.4(iii), p. 304]). By [11, II, §5, no. 1, Proposition 1.8, p. 237], the étale k-group scheme $\pi_0(\mathcal{T}_s) := \mathcal{T}_s/\mathcal{T}^0_s$ has the following universal property: if E is an étale k-group scheme and $\mathcal{T}_s \to E$ is a homomorphism of k-group schemes, then there exists a unique homomorphism of k-group schemes $\pi_0(\mathcal{T}_s) \to E$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_s & \twoheadrightarrow & \pi_0(\mathcal{T}_s) \\ & \searrow & \downarrow \\ & & E. \end{array} \tag{4}$$

The k-group schemes $\mathcal{T}_s$ and $\mathcal{T}^0_s$ represent the étale k-sheaves $i^*\mathcal{T}$ and $i^*\mathcal{T}^0$, i.e., we have equalities of étale k-sheaves $i^*\mathcal{T} = \mathcal{T}_s$ and $i^*\mathcal{T}^0 = \mathcal{T}^0_s$. Then the étale k-sheaf $\phi(T) := i^*(\mathcal{T}/\mathcal{T}^0)$ is represented by $\pi_0(\mathcal{T}_s)$, i.e,

$$\phi(T) = \pi_0(\mathcal{T}_s).$$

We will often identify the étale sheaf $\phi(T)$ and the g-module $\phi(T)(\overline{k})=\pi_0(\mathcal{T}_s)(\overline{k})$ (see [29, Corollary II.2.2(i), p. 94]). By [5, Theorem 2.3.2, p. 51], $\phi(T)$ is a finitely generated g-module. Now, since $j^*(\mathcal{T}/\mathcal{T}^0)=0$, there exists a canonical isomorphism of étale sheaves $i_*\phi(T)=\mathcal{T}/\mathcal{T}^0$ (see [29, proof of Theorem II.8.1.2, p. 135]). Thus there exists a canonical exact sequence of étale sheaves on S

$$0\to\mathcal{T}^0\to\mathcal{T}\to i_*\phi(T)\to 0. \tag{5}$$

If $T=\mathbb{G}_{m,K}$, the preceding sequence is

$$0\to\mathbb{G}_{m,S}\to j_*\mathbb{G}_{m,K}\to i_*\mathbb{Z}_k\to 0, \tag{6}$$

where the right-hand nontrivial morphism is induced by the valuation $v\colon K^*\to\mathbb{Z}$. See [4, §10.1, Example 5, p. 291].

2.2. Group Cohomology. Let J be a finite group. We will write $|J|$ for its order and $\mathfrak{A}_J$ for the augmentation ideal of $\mathbb{Z}[J]$, i.e., $\mathfrak{A}_J$ is the kernel of the homomorphism $\mathbb{Z}[J]\to\mathbb{Z}$, $\Sigma\, n_\sigma\sigma\mapsto\Sigma\, n_\sigma$. If M is a finitely generated (left) J-module, $M_J:=M/\mathfrak{A}_JM$ is the largest quotient of M on which J acts trivially. Let $M^\vee=\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ be the linear dual of M. Then $M^\vee$ has a natural structure of J-module by [7, (1), p. 238] and $M^\vee=(M/M_{\rm tors})^\vee$ is either zero or $\mathbb{Z}$-free. We have (see [7, pp. 238–240])

$$(M^\vee)^J=(M_J)^\vee \tag{7}$$

and

$$M^{\vee\vee}=M/M_{\rm tors}. \tag{8}$$

In particular, $M^{\vee\vee}=M$ if M is $\mathbb{Z}$-free. The kernel of the canonical norm map

$$N\colon M\to M^J,\quad m\mapsto\sum_{\sigma\in J}\sigma m, \tag{9}$$

will be denoted by ${}_NM$. Now recall the definition of the Tate cohomology groups $\widehat{H}^r(J,M)$ for $r\in\mathbb{Z}$: we have $\widehat{H}^r(J,M)=H^r(J,M)$ if $r\geq 1$, $\widehat{H}^r(J,M)=H_{-r-1}(J,M)$ if $r\leq -2$ and

$$\widehat{H}^0(J,M)=M^J/NM,$$
$$\widehat{H}^{-1}(J,M)={}_NM/\mathfrak{A}_JM.$$

Note that, if J acts trivially on M and M is $\mathbb{Z}$-free, then ${}_NM=0$ and therefore $\widehat{H}^{-1}(J,M)=0$. Further, by [7, Chapter XII, Proposition 2.5, p. 236, and

Exercise 3, p. 263], $\widehat{H}^r(J,M)$ is a finite group which is annihilated by $|J|$ for every $r\in\mathbb{Z}$.

Next, if M is an abelian group, we will write $M^D=\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. If M is free and finitely generated, then $M^D=M^\vee\otimes_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}$ by [7, Chapter XII, beginning of §3, pp. 237–238].

Now assume that M is a $\mathbb{Z}$-free and finitely generated J-module. Since $M^D=M^\vee\otimes_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}$ as noted above, the short exact sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$ induces a short exact sequence of J-modules $0\to M^\vee\to M^\vee\otimes_{\mathbb{Z}}\mathbb{Q}\to M^D\to 0$. The latter sequence induces, in turn, a canonical isomorphism of abelian groups $\widehat{H}^{r-1}(J,M^D)=\widehat{H}^r(J,M^\vee)$ for every $r\in\mathbb{Z}$. On the other hand, by [7, Chapter XII, §6, Theorem 6.4, p. 249], $\widehat{H}^{r-1}(J,M^D)$ is canonically isomorphic to $\widehat{H}^{-r}(J,M)^D$. Thus, for every $r\in\mathbb{Z}$, there exists a canonical isomorphism of finite abelian groups

$$\widehat{H}^r(J,M)^D=\widehat{H}^{-r}(J,M^\vee). \tag{10}$$

Lemma 2.1: *Let M be a $\mathbb{Z}$-free and finitely generated J-module.*

(i) *There exists a canonical exact sequence of abelian groups*

$$0\to H^1(J,M)^D\to(M^\vee)_J\to(M^J)^\vee\to 0.$$

(ii) *If $M^J\neq M$, then M/M^J is $\mathbb{Z}$-free and $(M/M^J)^J=0$.*

Proof. We apply the snake lemma to the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{A}_J(M^\vee) & \longrightarrow & M^\vee & \longrightarrow & (M^\vee)_J & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & {}_N(M^\vee) & \longrightarrow & M^\vee & \longrightarrow & N(M^\vee) & \longrightarrow & 0
\end{array}$$

and use (10) to obtain an exact sequence

$$0\to H^1(J,M)^D\to(M^\vee)_J\to N(M^\vee)\to 0. \tag{11}$$

Now, since $N(M^\vee)$ is a subgroup of the $\mathbb{Z}$-free group $M^\vee$ and $H^1(J,M)^D$ is torsion, the latter sequence induces, by (7) and (8), canonical isomorphisms

$$N(M^\vee)=(M^\vee)_J/(M^\vee)_{J,\,\mathrm{tors}}=((M^\vee)_J)^{\vee\vee}=((M^{\vee\vee})^J)^\vee=(M^J)^\vee. \tag{12}$$

Assertion (i) is now clear.

To prove (ii), we first note that, since M^J is $\mathbb{Z}$-free, (12) yields a canonical isomorphism $N(M^\vee)^\vee=(M^J)^{\vee\vee}=M^J$. Therefore, taking the linear dual of the bottom row of the above diagram, we obtain an isomorphism

$$({}_N(M^\vee))^\vee = M/M^J. \tag{13}$$

Consequently, if $M^J \neq M$, then M/M^J is $\mathbb{Z}$-free. Further, it follows from (11) that, if M' is a $\mathbb{Z}$-free and finitely generated J-module such that $NM' = 0$, then M'_J is torsion. Thus $({}_N(M^\vee))_J$ is torsion. Now, by (7) and (13)

$$(M/M^J)^J = (({}_N(M^\vee))^\vee)^J = (({}_N(M^\vee))_J)^\vee = 0,$$

since $({}_N(M^\vee))_J$ is torsion. ∎

2.3. A Canonical Resolution of T. Let L be the minimal splitting field of T, i.e., L is the fixed field of the kernel of the canonical homomorphism $\mathcal{G} \to \mathrm{Aut}(X^*(T))$. We will write T_L for the (split) L-torus $T \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Let J be the inertia subgroup of $\mathrm{Gal}(L/K)$. Then I acts on the free and finitely generated $\mathbb{Z}$-module $X^*(T)$ through the finite quotient J and (9) is a map

$$N\colon X^*(T) \to X^*(T)^J. \tag{14}$$

Note that, since $H^1(I', X^*(T)) = \mathrm{Hom}(I', X^*(T)) = 0$ for any subgroup I' of I which acts trivially on $X^*(T)$ (since I' is torsion and $X^*(T)$ is torsion-free), the inflation-restriction exact sequence [25, VII, §6, Proposition 4, p. 117] shows that $H^1(J, X^*(T)) = H^1(I, X^*(T))$.

A K-torus T is said to have **multiplicative reduction** if the special fiber $\mathcal{T}_s^0$ of $\mathcal{T}^0$ is a k-torus. An equivalent condition is that I act trivially on $X^*(T)$, i.e., T splits over K^{nr}. If this is the case, the g-module of characters of $\mathcal{T}_s^0$ is $X^*(T)$ [23, Proposition-Definition 1.1, p. 462]. Further, there exists a canonical isomorphism of g-modules

$$\phi(T) \xrightarrow{\sim} X_*(T). \tag{15}$$

See [5, Theorem 1.1.2, p. 29] and recall the identification $X^*(T)^\vee = X_*(T)$.

Proposition 2.2: *Let $0 \to T_1 \to T_2 \to T_3 \to 0$ be an exact sequence of K-tori. Assume that the following conditions hold:*

(i) $\mathrm{R}^1 j_* T_1 = 0$ *for the smooth topology on S, and*
(ii) $\phi(T_1)$ *is torsion-free.*

Then the induced sequence of g-modules $0 \to \phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0$ is exact.

Proof. By (i), the sequence of Néron models $0 \to \mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3 \to 0$ is exact in the smooth topology. Thus, by [5, Theorem 2.3.1, p. 50], the induced sequence of g-modules $\phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0$ is exact. On the other hand, by [5, Theorem 2.3.4, p. 52], $\mathrm{Ker}[\phi(T_1) \to \phi(T_2)]$ is a finite g-submodule of $\phi(T_1)$, which is therefore zero by (ii). This completes the proof. ∎

Corollary 2.3: *Let $0 \to T_1 \to T_2 \to T_3 \to 0$ be an exact sequence of K-tori, where T_1 has multiplicative reduction. Then the induced sequence of g-modules $0 \to \phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0$ is exact.*

Proof. Since T_1 splits over K^{nr}, $\mathrm{R}^1 j_* T_1 = 0$ for the smooth topology on S by [5, Corollary 4.2.6, p. 82]. On the other hand, by (15), $\phi(T_1) \simeq X_*(T_1)$, which is torsion-free. The corollary is now immediate from the proposition. ∎

A K-torus T is said to have **unipotent reduction** if the special fiber $\mathcal{T}_s^0$ of $\mathcal{T}^0$ is a unipotent k-group scheme.

Lemma 2.4: *A K-torus T has unipotent reduction if, and only if, $X^*(T)^I = 0$.*

Proof. By [23, proof of Theorem 1.3], T has unipotent reduction if, and only if, T contains no nontrivial K-subtorus having multiplicative reduction, i.e., $X^*(T)$ admits no free quotient on which I (or, equivalently, J) acts trivially. Assume that the latter holds and recall the norm map (14). Since $NX^*(T)$ is a quotient of $X^*(T)$ with trivial J-action, we have $NX^*(T) = 0$. Thus $X^*(T)^J = X^*(T)^J / NX^*(T) = \widehat{H}^0(J, X^*(T))$ is a finite subgroup of the free group $X^*(T)$, i.e., $X^*(T)^I = X^*(T)^J = 0$. Conversely, assume that $X^*(T)^I = 0$ and let Y be an I-submodule of $X^*(T)$ such that I acts trivially on $X^*(T)/Y$. Then the I-cohomology sequence associated to $0 \to Y \to X^*(T) \to X^*(T)/Y \to 0$ shows that $X^*(T)/Y$ is isomorphic to a subgroup of the finite group $H^1(I, Y)$. In particular, it is not free. ∎

What follows is an elaboration of [31, Lemma 2.13].

Let T be any K-torus. The maximal quotient torus $T^{(m)}$ of T having multiplicative reduction is the K-torus with character module $X^*(T)^I$. On the other hand, the maximal subtorus $T_{(u)}$ of T having unipotent reduction is the K-torus with character module $X^*(T)/X^*(T)^I$. This follows from Lemmas 2.1(ii) and 2.4 together with the fact that, if Y is an I-submodule of $X^*(T)$ such that $X^*(T)/Y$ is free and $(X^*(T)/Y)^I = 0$, then $X^*(T)^I \subset Y$. Now the exact sequence of $\mathcal{G}$-modules $0 \to X^*(T)^I \to X^*(T) \to X^*(T)/X^*(T)^I \to 0$ induces an exact sequence of K-tori

$$0 \to T_{(u)} \to T \to T^{(m)} \to 0. \tag{16}$$

Recall now the minimal splitting field L of T. The norm map $N_{L/K}\colon L^* \to K^*$ induces an epimorphism of K-tori $R_{L/K}(T_L) \to T$ whose kernel is denoted by $R^{(1)}_{L/K}(T_L)$ and called the **norm one torus** associated to T. See [5, Theorems 0.4.4, p. 16, and 0.4.7, p. 18]. Thus there exist canonical exact sequences

$$0 \to R^{(1)}_{L/K}(T_L) \to R_{L/K}(T_L) \to T \to 0 \tag{17}$$

and

$$0 \to R^{(1)}_{L/K}(T_L)_{(u)} \to R^{(1)}_{L/K}(T_L) \to R^{(1)}_{L/K}(T_L)^{(m)} \to 0, \tag{18}$$

where (18) is the sequence (16) associated to the K-torus $R^{(1)}_{L/K}(T_L)$. Set

$$P = R^{(1)}_{L/K}(T_L)^{(m)} \tag{19}$$

and let Q be the pushout of the canonical morphisms $R^{(1)}_{L/K}(T_L) \hookrightarrow R_{L/K}(T_L)$ and $R^{(1)}_{L/K}(T_L) \twoheadrightarrow R^{(1)}_{L/K}(T_L)^{(m)} = P$ appearing in (17) and (18), respectively. Thus there exists a canonical exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{(1)}_{L/K}(T_L) & \longrightarrow & R_{L/K}(T_L) & \longrightarrow & T & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & T & \longrightarrow & 0,
\end{array}$$

where the top row is (17). By (18), the kernel of the left-hand vertical map in the above diagram equals $R^{(1)}_{L/K}(T_L)_{(u)}$. This immediately yields the formula

$$Q = R_{L/K}(T_L)/R^{(1)}_{L/K}(T_L)_{(u)}. \tag{20}$$

Thus there exists a canonical exact sequence of K-tori

$$0 \to P \to Q \to T \to 0, \tag{21}$$

which will be referred to as the **canonical resolution of T**. Since P has multiplicative reduction, the following lemma is an immediate consequence of Corollary 2.3.

LEMMA 2.5: *The canonical resolution* (21) *induces an exact sequence of* $\mathfrak{g}$-*modules*

$$0 \to \phi(P) \to \phi(Q) \to \phi(T) \to 0.$$

Lemma 2.6: *The canonical resolution* (21) *induces an exact sequence of g-modules*

$$0 \to X_*(P) \to X_*(Q)_I \to X_*(T)_I \to 0.$$

Proof. The J-homology sequence associated to the short exact sequence of J-modules $0 \to X_*(P) \to X_*(Q) \to X_*(T) \to 0$ corresponding to (21) is

$$\cdots \to H_1(J, X_*(Q)) \to H_1(J, X_*(T)) \to X_*(P)_I \to X_*(Q)_I \to X_*(T)_I \to 0.$$

Since $H_1(J, X_*(T))$ is torsion and $X_*(P)_I = X_*(P)$ is torsion-free, the lemma follows. ∎

Lemma 2.7: $H^1(I, X^*(Q)) = 0$.

Proof. By [5, Theorem 0.4.3, p. 14, and proof of Theorem 0.4.4, p. 16], there exists a canonical isomorphism of $\mathcal{G}$-modules $X^*(R_{L/K}(T_L)) = \mathbb{Z}^d[\mathrm{Gal}(L/K)]$, where d is the dimension of T. Thus (20) yields a canonical exact sequence of $\mathcal{G}$-modules

$$0 \to X^*(Q) \to \mathbb{Z}^d[\mathrm{Gal}(L/K)] \to X^*(R^{(1)}_{L/K}(T_L)_{(u)}) \to 0.$$

Since $X^*(R^{(1)}_{L/K}(T_L)_{(u)})^J = 0$ by Lemma 2.4, the J-cohomology sequence associated to the above short exact sequence yields an injection

$$H^1(I, X^*(Q)) = H^1(J, X^*(Q)) \hookrightarrow H^1(J, \mathbb{Z}^d[\mathrm{Gal}(L/K)]).$$

Finally, since $\mathbb{Z}^d[\mathrm{Gal}(L/K)]$ is a free (right) $\mathbb{Z}^d[J]$-module of finite rank, the latter cohomology group vanishes by Shapiro's lemma [30, Lemma 6.3.2, p. 171], which completes the proof. ∎

3. Proof of Theorem 1.1

By Lemma 2.1(i), there exists a canonical exact sequence of g-modules

$$(22)\qquad 0 \longrightarrow H^1(I, X^*(T))^D \longrightarrow X_*(T)_I \xrightarrow{q_T} (X^*(T)^I)^\vee \longrightarrow 0.$$

We will write $\underline{X}_*(T)_I$ and $\underline{X}^*(T)^I$, respectively, for the étale k-sheaves that correspond to the continuous g-modules $X_*(T)_I$ and $X^*(T)^I$ (see [29, Corollary II.2.2(i), p. 94]). Now recall the canonical immersions $i\colon \mathrm{Spec}\, k \to S$ and $j\colon \mathrm{Spec}\, K \to S$.

Lemma 3.1: *There exists a canonical exact sequence of étale sheaves on* S

$$0 \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), \mathbb{G}_{m,S}) \to \mathcal{T} \to i_*(\underline{X}^*(T)^I)^\vee \to 0.$$

Proof. Since $\underline{\mathrm{Ext}}^1_{S_{\text{ét}}}(j_*\underline{X}^*(T), \mathbb{G}_{m,S}) = 0$ by [5, Theorem B.3, p. 131], (6) induces an exact sequence of étale sheaves on S

$$0 \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), \mathbb{G}_{m,S}) \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{G}_{m,K}) \\ \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), i_*\mathbb{Z}_k) \to 0.$$

Now, since $\underline{X}^*(T) = j^*j_*\underline{X}^*(T)$ and $i^*j_*\underline{X}^*(T) = \underline{X}^*(T)^I$ by [29, Proposition II.8.1.1, p. 134] and [21, Example II.3.12, p. 75] (respectively), (1) and [21, Exercise II.3.22(a), p. 80] yield canonical isomorphisms of étale sheaves on S

$$\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{G}_{m,K}) = j_*\underline{\mathrm{Hom}}_K(j^*j_*\underline{X}^*(T), \mathbb{G}_{m,K}) = j_*T = \mathcal{T} \tag{23}$$

and

$$\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), i_*\mathbb{Z}_k) = i_*\underline{\mathrm{Hom}}_k(i^*j_*\underline{X}^*(T), \mathbb{Z}_k) = i_*(\underline{X}^*(T)^I)^\vee. \tag{24}$$

The lemma is now clear. ∎

Remark 3.1: The same argument that proves (23) yields a canonical isomorphism of étale sheaves on S

$$j_*\underline{X}_*(T) = \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{Z}_K).$$

Let

$$v_T\colon \mathcal{T} \to i_*(\underline{X}^*(T)^I)^\vee \tag{25}$$

be the epimorphism of étale sheaves which appears in the exact sequence of Lemma 3.1. If $T = \mathbb{G}_{m,K}$, then $v_T = v_{\mathbb{G}_{m,K}} : j_*\mathbb{G}_{m,K} \to i_*\mathbb{Z}_k$ is the morphism appearing in the exact sequence (6). For arbitrary T, and via the identifications (23) and (24), v_T is the morphism

$$\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), v_{\mathbb{G}_{m,K}})\colon \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{G}_{m,K}) \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), i_*\mathbb{Z}_k).$$

Now, since $(\underline{X}^*(T)^I)^\vee = i^*i_*(\underline{X}^*(T)^I)^\vee$ by [29, Proposition II.8.1.1, p. 134], the exact sequence of Lemma 3.1 induces an exact sequence of étale k-sheaves

$$0 \longrightarrow i^*\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), \mathbb{G}_{m,S}) \longrightarrow i^*\mathcal{T} \xrightarrow{i^*v_T} (\underline{X}^*(T)^I)^\vee \longrightarrow 0.$$

Since (3) is an equivalence, $i^*v_T\colon i^*\mathcal{T} \to (\underline{X}^*(T)^I)^\vee$ corresponds to a homomorphism of k-group schemes $\mathcal{T}_s \to E$, where E is the étale k-group scheme which represents $(\underline{X}^*(T)^I)^\vee$ (see [29, Proposition II.9.2.3, p. 153]). By diagram (4), the latter homomorphism factors (uniquely) through a homomorphism of k-group schemes $\pi_0(\mathcal{T}_s) \to E$. Thus there exists a commutative diagram of étale k-sheaves

$$\begin{array}{ccc} i^*\mathcal{T} & \twoheadrightarrow & \phi(T) \\ & {\scriptstyle i^*v_T} \searrow & \downarrow \\ & & (\underline{X}^*(T)^I)^\vee . \end{array} \tag{26}$$

Let

$$\alpha_T \colon \phi(T) \twoheadrightarrow (X^*(T)^I)^\vee \tag{27}$$

be the epimorphism of g-modules which corresponds to the vertical morphism in (26).

Proposition 3.2: *The map*

$$\alpha_T^\vee \colon X^*(T)^I \hookrightarrow \phi(T)^\vee$$

induced by (27) *is an isomorphism of g-modules.*

Proof. See [5, Theorem 5.1.6, p. 93] and note that the g-module $E(T) = \operatorname{Coker} \alpha_T^\vee$ which appears there vanishes if k is perfect, by [5, Theorem 5.3.8, p. 104]. ∎

Corollary 3.3: *The map* (27) *induces an isomorphism of g-modules*

$$\phi(T)/\phi(T)_{\mathrm{tors}} \xrightarrow{\sim} (X^*(T)^I)^\vee .$$

Proof. This is immediate from the proposition and (8). ∎

Lemma 3.4: *If $H^1(I, X^*(T)) = 0$, then there exists a canonical isomorphism of g-modules*

$$\beta_T \colon \phi(T) \xrightarrow{\sim} X_*(T)_I .$$

Proof. By [31, Proposition 2.7], $\phi(T)$ is torsion free. Thus, by Corollary 3.3, $\alpha_T \colon \phi(T) \to (X^*(T)^I)^\vee$ is an isomorphism. On the other hand, by (22), $q_T \colon X_*(T)_I \to (X^*(T)^I)^\vee$ is an isomorphism as well. Thus

$$\beta_T := q_T^{-1} \circ \alpha_T \colon \phi(T) \to X_*(T)_I$$

is the required isomorphism of g-modules. ∎

We note that, if T has multiplicative reduction (i.e., I acts trivially on $X^*(T)$), then the isomorphism of the lemma is the isomorphism (15). Further, since the isomorphism of the previous lemma is canonical (i.e., functorial in T), given a morphism of K-tori $T_1 \to T_2$ such that $H^1(I, X^*(T_1)) = H^1(I, X^*(T_2)) = 0$, the induced diagram

$$\begin{array}{ccc} \phi(T_1) & \longrightarrow & \phi(T_2) \\ \downarrow{\scriptstyle \beta_{T_1}} & & \downarrow{\scriptstyle \beta_{T_2}} \\ X_*(T_1)_I & \longrightarrow & X_*(T_2)_I \end{array} \tag{28}$$

commutes.

Now recall the canonical resolution (21)

$$0 \to P \to Q \to T \to 0,$$

where P and Q are given by (19) and (20), respectively. Since P has multiplicative reduction, we have $H^1(I, X^*(P)) = 0$. Further, $H^1(I, X^*(Q)) = 0$ by Lemma 2.7. Thus, by (28) and Lemmas 2.5, 2.6 and 3.4, there exists a canonical exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \phi(P) & \longrightarrow & \phi(Q) & \longrightarrow & \phi(T) & \longrightarrow & 0 \\ & & \simeq\downarrow{\scriptstyle \beta_P} & & \simeq\downarrow{\scriptstyle \beta_Q} & & & & \\ 0 & \longrightarrow & X_*(P)_I & \longrightarrow & X_*(Q)_I & \longrightarrow & X_*(T)_I & \longrightarrow & 0. \end{array} \tag{29}$$

It is now clear that there exists a unique isomorphism of g-modules

$$\beta_T \colon \phi(T) \xrightarrow{\sim} X_*(T)_I$$

such that the following diagram, derived from (29),

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \phi(P) & \longrightarrow & \phi(Q) & \longrightarrow & \phi(T) & \longrightarrow & 0 \\ & & \simeq\downarrow{\scriptstyle \beta_P} & & \simeq\downarrow{\scriptstyle \beta_Q} & & \simeq\downarrow{\scriptstyle \beta_T} & & \\ 0 & \longrightarrow & X_*(P)_I & \longrightarrow & X_*(Q)_I & \longrightarrow & X_*(T)_I & \longrightarrow & 0, \end{array}$$

commutes.

The isomorphism thus defined fits into a commutative diagram

$$\begin{array}{ccc} \phi(T) & & \\ \simeq \Big\downarrow \beta_T & \searrow^{\alpha_T} & \\ X_*(T)_I & \xrightarrow{q_T} & (X^*(T)^I)^\vee, \end{array}$$

where q_T and α_T are the epimorphisms given by (22) and (27), respectively.

The proof of Theorem 1.1 is now complete.

The following consequence of Theorem 1.1 was previously established in [31, Corollary 2.18].

Corollary 3.5: *There exists a canonical exact sequence of g-modules*

$$0 \to H^1(I, X^*(T))^D \to \phi(T) \to (X^*(T)^I)^\vee \to 0.$$

In particular, $\phi(T)_{\mathrm{tors}}$ *is canonically isomorphic to* $H^1(I, X^*(T))^D$.

Proof. This follows from Theorem 1.1 together with (22). ∎

Remark 3.2: By Lemma 2.4 and the corollary, T has unipotent reduction if, and only if, $\phi(T)$ is finite. If this is the case, then there exists a canonical isomorphism of finite g-modules $\phi(T) = H^1(I, X^*(T))^D$.

The following result clarifies the exactness properties of the functor $\phi(-)$.

Proposition 3.6: *Let* $0 \to T_1 \to T_2 \to T_3 \to 0$ *be an exact sequence of* K*-tori. Then the given sequence of* K*-tori induces an exact sequence of* g*-modules*

$$0 \to H^2(I, X^*(T_1))^D \to H^2(I, X^*(T_2))^D \to H^2(I, X^*(T_3))^D \\ \to \phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0.$$

Proof. The exactness of the sequence

$$0 \to H^2(I, X^*(T_1))^D \to H^2(I, X^*(T_2))^D \to H^2(I, X^*(T_3))^D,$$

which is induced by the short exact sequence of I-modules

$$0 \to X^*(T_3) \to X^*(T_2) \to X^*(T_1) \to 0,$$

follows from the fact that $H^3(I, X^*(T_3)) = 0$ since K^{nr} has cohomological dimension ≤ 1 [26, II, beginning of §4.3, p. 85]. On the other hand, by the right exactness of the functor (from $\mathcal{G}$-modules to g-modules) $M \mapsto M_I$ (see [30, Exercise 6.1.1(2), p. 160]), the short exact sequence of $\mathcal{G}$-modules

$0 \to X_*(T_1) \to X_*(T_2) \to X_*(T_3) \to 0$ induces an exact sequence of g-modules $X_*(T_1)_I \to X_*(T_2)_I \to X_*(T_3)_I \to 0$. By Theorem 1.1, the latter sequence can be identified with a sequence $\phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0$. Now the connecting homomorphism $H^2(I, X^*(T_3))^D \to \phi(T_1) = X_*(T_1)_I$ in the sequence of the proposition factors as

$$H^2(I, X^*(T_3))^D \to H^1(I, X^*(T_1))^D \hookrightarrow X_*(T_1)_I$$

(see (22)), and its kernel is therefore equal to the kernel of

$$H^2(I, X^*(T_3))^D \to H^1(I, X^*(T_1))^D,$$

i.e., to the image of $H^2(I, X^*(T_2))^D \to H^2(I, X^*(T_3))^D$. It remains only to check exactness at $\phi(T_1)$. By [5, Theorem 2.3.4, p. 52], the kernel of $\phi(T_1) \to \phi(T_2)$ is a finite g-module. Thus it agrees with

$$\mathrm{Ker}[\phi(T_1)_{\mathrm{tors}} \to \phi(T_2)_{\mathrm{tors}}] = \mathrm{Ker}[H^1(I, X^*(T_1))^D \to H^1(I, X^*(T_2))^D]$$

(see Corollary 3.5). Since the latter group agrees with the image of the homomorphism $H^2(I, X^*(T_3))^D \to H^1(I, X^*(T_1))^D$, the proof is complete. ∎

The following corollary of the proposition generalizes Lemma 2.3.

Corollary 3.7: *Let $0 \to T_1 \to T_2 \to T_3 \to 0$ be an exact sequence of K-tori. If $\phi(T_1)$ is torsion-free, then the induced sequence of g-modules*

$$0 \to \phi(T_1) \to \phi(T_2) \to \phi(T_3) \to 0$$

is exact.

Proof. It was shown in the above proof that the connecting homomorphism $H^2(I, X^*(T_3))^D \to \phi(T_1)$ which appears in the exact sequence of the proposition factors through $H^1(I, X^*(T_1))^D = \phi(T_1)_{\mathrm{tors}} = 0$. ∎

Remark 3.3: Let $0 \to T_1 \to T_2 \to T_3 \to 0$ be as in the corollary, i.e., $\phi(T_1)$ is torsion-free. Further, for $i = 1, 2$ and 3, let $\mathcal{T}_i$ denote the Néron model of T_i over S. By the corollary and [21, Theorem II.2.15, p. 63], the sequence $0 \to i_*\phi(T_1) \to i_*\phi(T_2) \to i_*\phi(T_3) \to 0$ is an exact sequence of étale sheaves on S. On the other hand, since $\mathrm{R}^1 j_* T_1 = 0$ for the étale topology on S (see [31, Lemma 2.3]), $0 \to \mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3 \to 0$ is an exact sequence of étale sheaves on S. Thus there exists a canonical exact commutative diagram of étale sheaves on S

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_1 & \longrightarrow & \mathcal{T}_2 & \longrightarrow & \mathcal{T}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*\phi(T_1) & \longrightarrow & i_*\phi(T_2) & \longrightarrow & i_*\phi(T_3) & \longrightarrow & 0. \end{array}$$

By (5), the above diagram yields an exact sequence $0 \to \mathcal{T}_1^0 \to \mathcal{T}_2^0 \to \mathcal{T}_3^0 \to 0$ of étale sheaves on S and therefore an exact sequence of (representable) étale k-sheaves

(30) $$0 \to i^*\mathcal{T}_1^0 \to i^*\mathcal{T}_2^0 \to i^*\mathcal{T}_3^0 \to 0.$$

Since (3) is an equivalence, the latter sequence corresponds to a sequence of smooth, affine, commutative and connected algebraic k-group schemes

(31) $$0 \to \mathcal{T}_{1,s}^0 \xrightarrow{f} \mathcal{T}_{2,s}^0 \xrightarrow{g} \mathcal{T}_{3,s}^0 \to 0.$$

The exactness of (30) implies that g is surjective and that f identifies $\mathcal{T}_{1,s}^0$ with $\operatorname{Ker} g := \mathcal{T}_{2,s}^0 \times_{\mathcal{T}_{3,s}^0} \operatorname{Spec} k$. See, for example, [1, Lemma 2.21]. Now [1, Lemma 2.24] shows that the sequence of representable presheaves on $\operatorname{Spec} k$ induced by (31) is an exact sequence of fppf and fpqc sheaves on $\operatorname{Spec} k$. In other words, (31) is exact for the étale, fppf and fpqc topologies on $\operatorname{Spec} k$.

4. The cohomology of tori

All cohomology groups below are taken with respect to the étale topology on the relevant scheme.

Let T be a K-torus and recall the minimal splitting field L of T, i.e., L is the fixed field of the kernel of the canonical homomorphism $\mathcal{G} \to \operatorname{Aut}(X^*(T))$. Let J be the inertia subgroup of $\operatorname{Gal}(L/K)$. The maximal subtorus $T_{(m)}$ of T having multiplicative reduction is the K-torus with character module $NX^*(T)$ (see [23, Proposition 1.2]). On the other hand, the maximal quotient torus $T^{(u)}$ of T having unipotent reduction is the K-torus with character module ${}_NX^*(T)$. Indeed, if Y is a $\mathcal{G}$-submodule of $X^*(T)$ such that $Y^I = Y^J = 0$, then $NY = 0$, i.e., $Y \subset {}_NX^*(T)$. Now the exact sequence of $\mathcal{G}$-modules

$$0 \to {}_NX^*(T) \to X^*(T) \to NX^*(T) \to 0$$

induces an exact sequence of K-tori

(32) $$0 \to T_{(m)} \to T \to T^{(u)} \to 0.$$

Let $\mathcal{T}_{(m)}$, $\mathcal{T}$ and $\mathcal{T}^{(u)}$ denote, respectively, the Néron models of $T_{(m)}, T$ and $T^{(u)}$ over S. Since $X^*(T_{(m)}) = NX^*(T)$ is torsion-free, Remark 3.3 yields a sequence of smooth, affine, commutative and connected algebraic k-group schemes

$$0 \to \mathcal{T}^0_{(m),s} \to \mathcal{T}^0_s \to (\mathcal{T}^{(u)})^0_s \to 0. \tag{33}$$

The latter sequence is exact for the étale, fppf and fpqc topologies on $\operatorname{Spec} k$. Set

$$\tau = \mathcal{T}^0_{(m),s}, \tag{34}$$

which is the unique maximal k-torus of $\mathcal{T}^0_s$. Note that the g-module of characters of τ is $NX^*(T)$. In particular, if T has multiplicative reduction, so that $J = 1$ and $NX^*(T) = X^*(T)$, the character module of τ is $X^*(T)$ regarded as a g-module.

Lemma 4.1: *For every $r \geq 1$, there exists a canonical isomorphism of abelian groups $H^r(k, \mathcal{T}^0_s) = H^r(k, \tau)$.*

Proof. Since k is perfect, the sequence (33) splits (see [10, XVII, Theorem 6.1.1]), i.e., there exists an isomorphism of k-group schemes $\mathcal{T}^0_s \simeq \tau \times U$, where $U = (\mathcal{T}^{(u)})^0_s$. Since U is algebraic, smooth, connected and unipotent, it has a composition series whose successive quotients are k-isomorphic to $\mathbb{G}_{a,k}$ (see [10, XVII, Corollary 4.1.3]). Thus, since $H^r(k, \mathbb{G}_a) = 0$ for every $r \geq 1$ by [25, Chapter X, §1, Proposition 1, p. 150], we have $H^r(k, U) = 0$ for every $r \geq 1$. The lemma is now clear. ∎

Now, since $\mathrm{R}^s j_* T = 0$ for the étale topology on S for all $s > 0$ by [31, Lemma 2.3], the Leray spectral sequence in étale cohomology

$$H^r(S, \mathrm{R}^s j_* T) \implies H^{r+s}(K, T)$$

yields isomorphisms $H^r(S, \mathcal{T}) = H^r(S, j_* T) = H^r(K, T)$ for every $r \geq 0$. On the other hand, $H^r(S, \mathcal{T}^0) = H^r(k, \mathcal{T}^0_s) = H^r(k, \tau)$ for every $r \geq 1$ by Lemma 4.1 and [15, Theorem 11.7]. Further, $H^r(S, i_*\phi(T)) = H^r(k, i^* i_* \phi(T)) = H^r(k, \phi(T))$ for every $r \geq 0$ by [22, Proposition II.1.1(b), p. 149]. Thus, by (5), there exists a canonical exact sequence of abelian groups

$$\cdots \to H^r(k, \tau) \to H^r(K, T) \to H^r(k, \phi(T)) \to \cdots, \tag{35}$$

where $r \geq 1$ and τ is the k-torus (34). By Theorem 1.1, the preceding sequence is canonically isomorphic to a sequence

$$\cdots \to H^r(k, \tau) \to H^r(K, T) \to H^r(k, X_*(T)_I) \to \cdots . \tag{36}$$

We now derive some consequences of (35) and (36).

Proposition 4.2: *Assume that T has unipotent reduction. Then, for every $r \geq 1$, there exists a canonical isomorphism of abelian groups*

$$H^r(K, T) = H^r(k, H^1(I, X^*(T))^D).$$

Proof. This is clear from (35) and Remark 3.2 since $\tau = 0$ in this case. ∎

The next proposition generalizes [21, Example III.2.22(c), p. 108] (at least when the ring A appearing there is complete).

Proposition 4.3: *Assume that T has multiplicative reduction. Then, for every $r \geq 1$, the sequence of abelian groups induced by* (36)

$$0 \to H^r(k, \tau) \to H^r(K, T) \to H^r(k, X_*(T)) \to 0$$

is split exact.

Proof. Since I acts trivially on $X_*(T)$, (25) is a morphism

$$v_T \colon \mathcal{T} \to i_*(\underline{X}^*(T)^I)^\vee = i_*\underline{X}_*(T)$$

and the map $H^r(K, T) \to H^r(k, X_*(T))$ appearing in the sequence of the proposition can be identified with the homomorphism

$$H^r(v_T) \colon H^r(S, \mathcal{T}) \to H^r(S, i_*\underline{X}_*(T))$$

induced by v_T. Recall also that, via the identifications (23) and (24), v_T can be identified with the morphism

$$\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), v_{\mathbb{G}_{m,K}}) \colon \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{G}_{m,K}) \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), i_*\mathbb{Z}_k),$$

where $v_{\mathbb{G}_{m,K}} : j_*\mathbb{G}_{m,K} \to i_*\mathbb{Z}_k$ is the morphism intervening in the exact sequence (6). Now choose a uniformizer $\pi \in A$, let $u \colon \mathbb{Z}_K \to \mathbb{G}_{m,K}$ be the homomorphism of K-group schemes which maps $1 \in \mathbb{Z}$ to $\pi \in \mathbb{G}_{m,K}(K) = K^*$ and let $u_{\mathbb{G}_{m,K}} : j_*\mathbb{Z}_K \to j_*\mathbb{G}_{m,K}$ be the morphism of étale sheaves on S induced by u. By Remark 3.1, there exists a canonical isomorphism (of étale sheaves on S) $j_*\underline{X}_*(T) = \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{Z}_K)$. Thus

$$\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), u_{\mathbb{G}_{m,K}}) \colon \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{Z}_K) \to \underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), j_*\mathbb{G}_{m,K})$$

can be identified with a morphism $u_T\colon j_*\underline{X}_*(T) \to \mathcal{T}$. Clearly, the morphism of étale sheaves on S

$$v_T \circ u_T\colon j_*\underline{X}_*(T) \to i_*\underline{X}_*(T)$$

can be identified with the morphism $\underline{\mathrm{Hom}}_S(j_*\underline{X}^*(T), v_{\mathbb{G}_{m,K}} \circ u_{\mathbb{G}_{m,K}})$, where $v_{\mathbb{G}_{m,K}} \circ u_{\mathbb{G}_{m,K}}\colon j_*\mathbb{Z}_K \to i_*\mathbb{Z}_k$. On the other hand, by [29, Proposition II.8.2.1, p. 142], for any étale sheaf $\mathcal{F}$ on S there exists a canonical exact exact sequence of étale sheaves on S

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0.$$

Setting $\mathcal{F} = j_*\underline{X}_*(T)$ above and observing that $j^*j_*\underline{X}_*(T) = \underline{X}_*(T)$ and $i^*j_*\underline{X}_*(T) = \underline{X}_*(T)$, we obtain a canonical exact sequence of étale sheaves on S

$$0 \longrightarrow j_!\underline{X}_*(T) \longrightarrow j_*\underline{X}_*(T) \xrightarrow{w_T} i_*\underline{X}_*(T) \longrightarrow 0.$$

It is immediate that $v_{\mathbb{G}_{m,K}} \circ u_{\mathbb{G}_{m,K}} = w_{\mathbb{G}_{m,K}}$, which implies that $v_T \circ u_T = w_T$ for any T. On the other hand, since $H^r(S, j_!\underline{X}_*(T)) = 0$ for every $r \geq 1$ by [22, Proposition II.1.1(a), p. 149], the map

$$H^r(w_T)\colon H^r(S, j_*\underline{X}_*(T)) \to H^r(S, i_*\underline{X}_*(T))$$

induced by w_T is an isomorphism for every $r \geq 1$. We conclude that

$$H^r(u_T) \circ H^r(w_T)^{-1}$$

is a section (i.e., right inverse) of $H^r(v_T)$, and this completes the proof. ∎

The following lemma is well-known but we were unable to find an appropriate reference.

Lemma 4.4: *Assume that k has cohomological dimension ≤ 1. If τ is a k-torus, then $H^r(k, \tau) = 0$ for every $r \geq 1$.*

Proof. Since k is perfect, k is a field of dimension ≤ 1 by [26, Chapter II, §3.1, Proposition 6(b), p. 78]. Thus, by [26, Chapter II, §3.1, Proposition 5(iii), p. 78], for any finite Galois extension l/k, l^* is a cohomologically trivial $\mathrm{Gal}(l/k)$-module. Let l be the minimal splitting field of τ. Then, since $X^*(\tau)$ is a free (and therefore projective) $\mathbb{Z}$-module, we have $\mathrm{Ext}^1_{\mathbb{Z}}(X^*(\tau), l^*) = 0$, whence $\tau(l) = \mathrm{Hom}(X^*(\tau), l^*)$ (see (2)) is cohomologically trivial as well by [25, Chapter IX, §5, Theorem 9, p. 145]. Finally, since $H^r(k, \tau)$ is the inductive limit of the groups $H^r(\mathrm{Gal}(l/k), \tau(l))$ as l/k ranges over the set of all finite Galois subextensions of $\overline{k}/k$, the proof is complete. ∎

Theorem 4.5: *Assume that k has cohomological dimension ≤ 1. Then:*

(i) *The sequence $0 \to \mathcal{T}^0(A) \to T(K) \to \phi(T)(k) \to 0$ is exact.*
(ii) *For $r = 1$ and 2, there exist canonical isomorphisms*

$$H^r(K, T) \simeq H^r(k, X_*(T)_I).$$

If $r \geq 3$, the groups $H^r(K, T)$ vanish.

Proof. The last assertion follows from (35) since $H^r(k, \tau) = H^r(k, \phi(T)) = 0$ for $r \geq 3$. Assertions (i) and (ii) are immediate from (35), (36) and Lemma 4.4. ∎

Remark 4.1: If k has cohomological dimension ≤ 1, then $H^r(k, H^1(I, X^*(T))^D)$ is trivial for $r \geq 2$ since $H^1(I, X^*(T))^D$ is a finite g-module. Thus, by (22) and assertion (ii) of the theorem,

$$H^2(K, T) = H^2(k, X_*(T)_I) = H^2(k, (X^*(T)^I)^\vee).$$

The preceding remark can be generalized as follows.

Proposition 4.6: *Assume that k has finite cohomological dimension $n \geq 1$. Then there exists a canonical isomorphism of divisible abelian groups*

$$H^{n+1}(K, T) = H^{n+1}(k, (X^*(T)^I)^\vee).$$

If $r \geq n + 2$, the groups $H^r(K, T)$ vanish.

Proof. The group on the right above is divisible by [27, Corollary 1, p. 55]. Now, since $\tau(\overline{k})$ is divisible, we have $H^r(k, \tau) = 0$ for every $r \geq n + 1$ by [27, Proposition 14, p. 54]. Thus (36) yields a canonical isomorphism of abelian groups $H^r(K, T) = H^r(k, X_*(T)_I)$ for every $r \geq n+1$. On the other hand, since $H^1(I, X^*(T))^D$ is finite, we have $H^r(k, H^1(I, X^*(T))^D) = 0$ for all $r \geq n + 1$ and (22) yields isomorphisms

$$H^r(k, X_*(T)_I) = H^r(k, (X^*(T)^I)^\vee)$$

for each $r \geq n + 1$. The latter group vanishes if $r \geq n + 2$ by [27, Proposition 14, p. 54], which completes the proof. ∎

5. Abelian cohomology of reductive groups

Assume that k has cohomological dimension ≤ 1. Recall that a K-torus F is called *flasque* if the $\mathcal{G}$-module $X_*(F)$ is H^1-trivial. See [8, Lemma 1(iv), p. 179]. By [9, Proposition-Definition 3.1, p. 88, and Proposition 2.2, p. 86], any connected reductive algebraic group G over K admits a **flasque resolution**, i.e., there exists a central extension

$$1 \to F \to H \to G \to 1, \tag{37}$$

where F is a flasque K-torus and H is a connected reductive algebraic over K such that the derived group H^{der} of H is simply connected and $R := H/H^{\mathrm{der}}$ is a quasi-trivial K-torus. The finitely generated (continuous) $\mathcal{G}$-module $\pi_1(G) := \mathrm{Coker}[X_*(F) \to X_*(R)]$ is independent (up to isomorphism) of the choice of resolution (37) and is called the **algebraic fundamental group of** G. See [9, Proposition-Definition 6.1, p. 102]. Recall from [13, Corollary 4.3] that, if $r \geq 1$ is an integer, the r**-th (flat) abelian cohomology group of** G may be defined as the flat hypercohomology group

$$H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G) := \mathbb{H}^r(K_{\mathrm{fppf}}, \pi_1(G) \otimes^{\mathbf{L}} \mathbb{G}_{m,K}).$$

These abelian groups are of interest because they can be related to the pointed Galois cohomology sets $H^r(K, G)$ for $r = 1$ and 2. See Corollary 5.2 below for the case $r = 1$.

Now, since R is quasi-trivial, we have $H^1(K, R) = 0$. Further, $H^r(K, F) = 0$ for every $r \geq 3$ by the last assertion of Theorem 4.5. Thus, by [13, Proposition 4.2], $H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G) = 0$ for every $r \geq 3$ and (37) induces an exact sequence of abelian groups

$$0 \to H^1_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \to H^2(K, F) \to H^2(K, R) \to H^2_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \to 0. \tag{38}$$

Now, by [9, Proposition 6.2, p. 102], there exists a canonical exact sequence of $\mathcal{G}$-modules

$$0 \to X_*(F) \to X_*(R) \to \pi_1(G) \to 0 \tag{39}$$

which induces a short exact sequence of g-modules

$$0 \to X_*(F)_I/M \to X_*(R)_I \to \pi_1(G)_I \to 0, \tag{40}$$

where M is a finite submodule of $X_*(F)_I$ which is isomorphic to a quotient of $H_1(J, \pi_1(G))$ (see the proof of Lemma 2.6). Since $H^r(k, M) = 0$ for every $r \geq 2$, we have $H^2(k, X_*(F)_I/M) = H^2(k, X_*(F)_I)$ and $H^r(k, X_*(F)_I/M) = 0$ for

$r \geq 3$ (see [27, Proposition 14, p. 54]). Further, $H^1(k, X_*(R)_I) = H^1(K, R) = 0$ by Theorem 4.5(ii). Thus (40) induces an exact sequence of abelian groups

$$H^1(k, \pi_1(G)_I) \hookrightarrow H^2(k, X_*(F)_I) \to H^2(k, X_*(R)_I) \twoheadrightarrow H^2(k, \pi_1(G)_I). \tag{41}$$

Now consider the exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^1_{\mathrm{ab}}(K_{\mathrm{fl}}, G) & \to & H^2(K, F) & \to & H^2(K, R) & \to & H^2_{\mathrm{ab}}(K_{\mathrm{fl}}, G) & \to 0 \\ & & & \downarrow \wr & & \downarrow \wr & & & \\ 0 \to & H^1(k, \pi_1(G)_I) & \to & H^2(k, X_*(F)_I) & \to & H^2(k, X_*(R)_I) & \to & H^2(k, \pi_1(G)_I) & \to 0 \end{array}$$

whose top and bottom rows are the sequences (38) and (41), respectively, and vertical maps are the isomorphisms of Theorem 4.5(ii). Clearly, the maps $H^2(K, F) \xrightarrow{\sim} H^2(k, X_*(F)_I)$ and $H^2(K, R) \xrightarrow{\sim} H^2(k, X_*(R)_I)$ induce isomorphisms

$$H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \xrightarrow{\sim} H^r(k, \pi_1(G)_I)$$

for $r = 1$ and 2. Thus the following holds.

Theorem 5.1: *Assume that k has cohomological dimension ≤ 1 and let G be a connected reductive algebraic group over K. Then the flasque resolution* (37) *induces isomorphisms of abelian groups*

$$H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \simeq H^r(k, \pi_1(G)_I)$$

for $r = 1$ and 2. If $r \geq 3$, the groups $H^r_{\mathrm{ab}}(K_{\mathrm{fl}}, G)$ vanish.

Corollary 5.2: *Assume that k has cohomological dimension ≤ 1. Then there exists a bijection of pointed sets*

$$H^1(K, G) \simeq H^1(k, \pi_1(G)_I).$$

In particular, $H^1(K, G)$ can be endowed with an abelian group structure.

Proof. Let $\widetilde{G}$ be the simply connected central cover of G^{der} (see [13, p. 1161] for the definition of $\widetilde{G}$). By [6, Theorem 4.7(ii), p. 697, and Remark 3.16(3), p. 695], the Galois cohomology set $H^1(K, \widetilde{G})$ is trivial. Further, by [12, VII, Theorem 3.1, p. 99], K is a field of Douai type in the sense of [14, Definition 5.2]. Thus, by [14, Theorem 5.8(i)], the first abelianization map $\mathrm{ab}^1 \colon H^1(K, G) \to H^1_{\mathrm{ab}}(K_{\mathrm{fl}}, G)$ is bijective. The corollary is now immediate from the theorem. ∎

Corollary 5.3: *Assume that k is quasi-finite. Then there exists a bijection of pointed sets $H^1(K, G) \simeq \pi_1(G)_{\mathcal{G},\mathrm{tors}}$.*

Proof. Since a quasi-finite field is perfect and of cohomological dimension ≤ 1 by [25, XIII, p. 190] and [27, III, §2, Corollary 3, p. 69], the corollary is immediate from the previous corollary and the next lemma. ∎

LEMMA 5.4: *Assume that k is quasi-finite and let M be a continuous g-module. Then there exists a canonical isomorphism of abelian groups $H^1(k, M) \simeq M_{g,\mathrm{tors}}$.*

Proof. Let σ be a free generator of g. By [25, XIII, §1, Proposition 1, p. 189], there exists a canonical isomorphism of abelian groups

$$H^1(k, M) = M'/(\sigma - 1)M,$$

where M' is the subgroup of M consisting of those $x \in M$ for which there exists an integer $n \geq 1$ such that $(1 + \sigma + \cdots + \sigma^{n-1})x = 0$. Thus it remains only to check that $M'/(\sigma-1)M$ is the full torsion subgroup of $M_g = M/(\sigma-1)M$. Let $x \in M$ be such that $mx = (\sigma-1)y$ for some positive integer m and some $y \in M$ and choose a positive integer r such that $\sigma^r x = x$ and $\sigma^r y = y$. The latter is possible since M is a continuous g-module [25, X, beginning of §3, p. 154]. Then

$$\begin{aligned}(1 + \sigma + \cdots &+ \sigma^{mr-1})x\\ &=(1 + \sigma + \cdots + \sigma^{r-1})(1 + \sigma^r + \cdots + \sigma^{(m-1)r})x\\ &=(1 + \sigma + \cdots + \sigma^{r-1})mx = (1 + \sigma + \cdots + \sigma^{r-1})(\sigma - 1)y\\ &=(\sigma^r - 1)y = 0.\end{aligned}$$

Thus $x \in M'$, thereby completing the proof. ∎

Remarks 5.1:

(a) When K is a finite extension of $\mathbb{Q}_p$, Corollary 5.3 is due to Borovoi. See [3, Corollary 5.5(i)].

(b) Assume that k has finite cohomological dimension $n \geq 1$ and let μ be the kernel of the canonical morphism $\widetilde{G} \to G$. Then μ is a finite and commutative K-group scheme. By [26, §II.4.3, Proposition 12, p. 85] and [28, Theorem 4, p. 593], $H^r(K_{\mathrm{fl}}, \mu) = 0$ for every $r \geq n+2$. Consequently, the exact sequence in [13, p. 1174, line 8] yields an isomorphism of abelian groups $H^{n+1}_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \simeq H^{n+1}(K, G^{\mathrm{tor}})$, where $G^{\mathrm{tor}} = G/G^{\mathrm{der}}$. Thus, by the proof of Proposition 4.6, there exists an isomorphism $H^{n+1}_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \simeq H^{n+1}(k, X_*(G^{\mathrm{tor}})_I)$. On the other hand, it follows from [9, Proposition 6.4, p. 104] that there exists an isomorphism of g-modules $X^*(G^{\rm tor})_I \simeq \pi_1(G)_I/M'$, where M' is a finite submodule of $\pi_1(G)_I$. Thus there exists an isomorphism of abelian groups

$$H^{n+1}_{\rm ab}(K_{\rm fl}, G) \simeq H^{n+1}(k, \pi_1(G)_I)$$

which generalizes the case $r = 2$ of Theorem 5.1.

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