

TIGHT LAGRANGIAN HOMOLOGY SPHERES IN COMPACT HOMOGENEOUS KÄHLER MANIFOLDS

BY

CLAUDIO GORODSKI*

*Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil
e-mail: gorodski@ime.usp.br*

AND

FABIO PODESTÀ

*Dipartimento di Matematica e Informatica “Ulisse Dini”, Università di Firenze
V.le Morgagni 67/A, 50100 Firenze, Italy
e-mail: podesta@unifi.it*

ABSTRACT

For any irreducible compact homogeneous Kähler manifold, we classify the compact tight Lagrangian submanifolds which have the \mathbb{Z}_2 -homology of a sphere.

1. Introduction

Let M be a homogeneous Kähler manifold. Following [Oh91], we call a compact Lagrangian submanifold L of M *globally tight* (resp. *locally tight* or simply *tight*) if the cardinality of the set $L \cap g \cdot L$ is equal to the sum of \mathbb{Z}_2 -Betti numbers of L , for every isometry g of M (resp. every isometry sufficiently close to the identity) such that the intersection is transversal.

It turns out that tightness has a bearing on the problem of Hamiltonian volume minimization. For instance, in [Oh91] it is proved that a tight Lagrangian

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submanifold of $\mathbb{C}P^n$ must be the totally real embedding of $\mathbb{R}P^n$; an argument of Kleiner and Oh shows that the standard $\mathbb{R}P^n$ in $\mathbb{C}P^n$ has the least volume among its Hamiltonian deformations; Iriyeh [Iri05] then notes that this gives uniqueness of the Hamiltonian volume minimization problem for Hamiltonian deformations of $\mathbb{R}P^n \subset \mathbb{C}P^n$ (similar results have been obtained for the product of equatorial circles $S^1 \times S^1 \subset S^2 \times S^2 = Q_2$ [IOS03, IS10]). More generally, real forms of Hermitian symmetric spaces have recently been proved to be globally tight [TT12]. The question of classification of tight Lagrangian submanifolds in Hermitian symmetric spaces was already posed in [Oh91] and remains open.

Herein we take a different standpoint in that we allow M to be an arbitrary compact homogeneous irreducible Kähler manifold but we considerably restrict the topology of L . A compact homogeneous Kähler manifold M is a Kähler manifold on which a compact connected Lie group of isometries acts transitively. A simply-connected compact homogeneous Kähler manifold M is also called a *Kählerian C-space*. In this case, it is known that M is a homogeneous space G/H where G is a compact semi-simple Lie group and H is the centralizer of a toral subgroup of G (in other words, it is a (*generalized*) *complex flag manifold*); moreover, M is irreducible if and only if G is a simple Lie group. Our main result is:

THEOREM 1: *Let $M = G/H$ be a simply-connected irreducible compact homogeneous Kähler manifold. Let L be a compact tight Lagrangian submanifold of M . Assume that L has the \mathbb{Z}_2 -homology of a sphere.*

Then L is an orbit of a compact subgroup of G , and M and L are given, respectively up to biholomorphic homothety and up to congruence, as follows:

- (a) M is a complex quadric $Q_n = \mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ ($n \geq 3$) and $L \cong S^n$ is its standard real form, orbit of a subgroup isomorphic to $\mathrm{SO}(n+1)$;
- (b) M is the twistor space $Z = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n-1))$ ($n \geq 3$) of the complex Grassmannian of 2-planes $\mathrm{Gr}_2(\mathbb{C}^{n+1})$ endowed with its standard Kähler–Einstein structure and $L \cong S^{2n-1}$ is an orbit of a subgroup isomorphic to $\mathrm{U}(n)$;
- (c) M is the full flag manifold $\mathrm{SU}(3)/\mathrm{T}^2$ endowed with its Kähler–Einstein homogeneous metric and $L \cong S^3$ is an orbit of a subgroup isomorphic to $\mathrm{U}(2)$;

- (d) $M = \mathrm{Sp}(n + 2)/\mathrm{U}(2) \times \mathrm{Sp}(n)$ ($n \geq 1$) and $L \cong S^{4n+3}$ is an orbit of a subgroup isomorphic to $\mathrm{Sp}(1) \times \mathrm{Sp}(n + 1)$;
- (e) $M = \mathrm{F}_4/\mathrm{T}^1 \cdot \mathrm{Spin}(7)$ and $L \cong S^{15}$ is an orbit of a subgroup isomorphic to $\mathrm{Spin}(9)$;
- (f) $M = \mathbb{C}P^1 = \mathrm{SU}(2)/\mathrm{T}^1$ and $L = \mathbb{R}P^1$ is an orbit of the subgroup T^1 .

Moreover, L coincides with a connected component of the fixed point set of an antiholomorphic isometric involution of M .

Remark 1: Note that the space $\mathrm{SU}(3)/\mathrm{T}^2$ admits only one invariant complex structure up to equivalence, while the spaces appearing in (d) and (e) carry only one invariant Kähler structure up to biholomorphism and homothety, so that it is not necessary to specify which structure we are considering. For further details we refer to §2.

In section 2, we briefly review some basic facts about homogeneous Kähler manifolds. The proof of Theorem 1 is scattered throughout sections 3, 4 and 5. In particular, in § 5 we also show that a real flag manifold can be always embedded as a tight real form of a suitable complexification given by a complex flag manifold; see Proposition 3.

Notation: For a compact Lie group, we denote its Lie algebra by the corresponding lowercase gothic letter. If a group G acts on a manifold M , for every $X \in \mathfrak{g}$ we denote by X^* the corresponding vector field on M induced by the G -action.

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2. Preliminary material

Let $M = G/H$ be a generalized flag manifold, where G is a compact connected semisimple Lie group and H is the centralizer of a toral subgroup of G . We shall recall the standard description of invariant Kähler structures on M (see, e.g., [BFR86, Ale97]).

Denote by p the basepoint and by \langle, \rangle the negative of the Cartan–Killing form of \mathfrak{g} . Then there is a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where \mathfrak{m} is the orthogonal complement of \mathfrak{h} , and we can as usual identify $\mathfrak{m} \cong T_p(G/H)$ via $X \mapsto X_p^*$. Since \mathfrak{h} is of maximal rank in \mathfrak{g} , there is a maximal Abelian

subalgebra \mathfrak{s} of \mathfrak{g} contained in \mathfrak{h} . Then the complexification $\mathfrak{s}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and we denote by Δ the corresponding root system. Each root space of $\mathfrak{g}^{\mathbb{C}}$ is either contained in $\mathfrak{h}^{\mathbb{C}}$ or in $\mathfrak{m}^{\mathbb{C}}$ and thus there is an associated partition $\Delta = \Delta_H \cup \Delta_M$. Define the real subspace

$$\mathfrak{t} = i \mathfrak{z}(\mathfrak{h}) \subset \mathfrak{s}^{\mathbb{C}}$$

where $\mathfrak{z}(\mathfrak{h})$ is the center of \mathfrak{h} . Every root $\alpha \in \Delta$ is real valued when restricted to \mathfrak{t} and the restriction $\alpha|_{\mathfrak{t}} \in \mathfrak{t}^*$ is called a T -root (note that the elements of Δ_H restrict to zero). Note that the set $\Delta_T \subset \mathfrak{t}^*$ of all T -roots is not a root system. Its significance for us lies in the fact that there is a natural bijective correspondence between the set of G -invariant complex structures on M and the set of T -chambers in \mathfrak{t} , where a T -chamber is a connected component of the regular set \mathfrak{t}_{reg} of \mathfrak{t} , namely, the complement of the union of the hyperplanes $\ker \lambda$ for $\lambda \in \Delta_T$. For later reference, we recall that in case $\dim \mathfrak{t} = 1$ or \mathfrak{h} is Abelian, any two G -invariant complex structures on M are biholomorphic [BH58, 13.8] (see also [Nis84, p. 57]).

On the other hand, there is a natural bijective correspondence between the set of G -invariant symplectic structures ω on M and the set of regular elements $\xi \in i \mathfrak{t}_{reg}$ given by

$$\omega_p(X_p^*, Y_p^*) = \langle [X, Y], \xi \rangle$$

where $X, Y \in \mathfrak{g}$. Finally the G -invariant Kähler metrics on M are all given by

$$g_p = \omega_p(\cdot, J\cdot)$$

where ω is the invariant symplectic structure associated to $\xi \in i \mathfrak{t}_{reg}$ and J is the invariant complex structure associated to the T -chamber containing $-i \xi$.

Henceforth we fix an invariant Kähler structure on M . Since \mathfrak{h} coincides with the centralizer of ξ in \mathfrak{g} , there is a canonical embedding of M into \mathfrak{g} as the adjoint orbit $\text{Ad}(G) \cdot \xi$ mapping p to ξ . Now for each $X \in \mathfrak{g}$, the vector field X^* on M is given by $X_q^* = [X, q]$, where $q \in M$, and it is easily seen that

$$\omega_q(X_q^*, Y_q^*) := \langle [X, Y], q \rangle,$$

showing that ω coincides with the Kirillov–Kostant–Souriau symplectic structure. It follows that X^* is Hamiltonian with corresponding potential function given by the height function $h_X(q) = \langle q, X \rangle$ for $q \in M$. Thus the moment map

$$\mu : M \rightarrow \mathfrak{g}^* \cong \mathfrak{g}, \quad \mu(q)(X) = h_X(q)$$

is just the inclusion (above we have identified \mathfrak{g} with its dual via the Cartan-Killing form).

For later reference, we quote the following result due to Onishchik [Oni94, p. 244].

THEOREM 2: *If G is a compact connected simple Lie group and acts effectively on $M = G/H$, then G coincides, up to covering, with the identity component of the full isometry group Q , with the following exceptions:*

- (a) $M = \mathbb{C}P^{2n+1}$ and $\mathfrak{g} = \mathfrak{sp}(n + 1)$, $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{sp}(n)$, $\mathfrak{q} = \mathfrak{su}(2n + 2)$;
- (b) $\mathfrak{g} = \mathfrak{so}(2n - 1)$, $\mathfrak{h} = \mathfrak{u}(n - 1)$, $\mathfrak{q} = \mathfrak{so}(2n)$, $n \geq 4$;
- (c) $M = Q_5$ and $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{h} = \mathfrak{u}(2)$, $\mathfrak{q} = \mathfrak{so}(7)$.

3. Lagrangian submanifolds

We keep the notation from the previous section and consider a compact Lagrangian submanifold L of M through the basepoint p . Denote by K the identity component of the stabilizer subgroup of L in G . Then K is a closed subgroup of G that acts effectively on L by the Lagrangian property of L . Define the linear map

$$(1) \quad \sigma : \mathfrak{g} \rightarrow \Gamma(\nu L), \quad X \mapsto (X^*|_L)^\perp,$$

where $\Gamma(\nu L)$ is the space of sections of the normal bundle νL and $(\)^\perp$ denotes the normal component to L . The map (1) is clearly K -equivariant; note that its kernel coincides with the Lie algebra \mathfrak{k} of K . Choose a reductive complement \mathfrak{k}^\perp and K -equivariantly identify \mathfrak{k}^\perp with the image V of σ . Now we can write

$$(2) \quad \mathfrak{g} = \mathfrak{k} + V.$$

LEMMA 1: *For $X \in \mathfrak{g}$, the critical points of the height function $h_X|_L$ are precisely the zeros of the vector field $\sigma(X) \in V$.*

Proof. A point $q \in L$ is a critical point of $h_X|_L$ if and only if X is perpendicular to T_qL . Given $v \in T_qL$, there exists $Y \in \mathfrak{g}$ such that $Y_q^* = v$ and then $\omega_q(X_q^*, v) = \langle [X, Y], q \rangle = \langle X, [Y, q] \rangle = \langle X, v \rangle$. Now q is a critical point of $h_X|_L$ if and only if $\omega_q(X_q^*, T_qL) = 0$. Since L is Lagrangian, this implies that $X_q^* \in T_qL$, as desired. ■

Let $\varphi : L \rightarrow V$ be the restriction of the orthogonal projection $\mathfrak{g} \rightarrow V$. We elaborate on an idea of Oh, use the previous lemma to translate the tight

Lagrangian property of L to the taut property of $\varphi(L)$, and apply known results to the latter.

PROPOSITION 1: *The map $\varphi : L \rightarrow V$ is a K -equivariant full embedding. Moreover, if L is tight Lagrangian in M then $\varphi(L)$ is a taut submanifold of the Euclidean space V ; the converse holds in the case G coincides, up to covering, with the identity component of the isometry group of M .*

Proof. Since (2) is a reductive decomposition, it is clear that φ is K -equivariant. The moment map of the restricted K -action on M is $\pi_{\mathfrak{k}} \circ \mu$, where $\pi_{\mathfrak{k}} : \mathfrak{g} \rightarrow \mathfrak{k}$ is the orthogonal projection; since L is a K -invariant Lagrangian submanifold, we have

$$\pi_{\mathfrak{k}} \circ \mu(L) = \eta,$$

where η is a constant central element of \mathfrak{k} . Denote by $\pi_V : \mathfrak{g} \rightarrow V$ the orthogonal projection. Then

$$(3) \quad \mu|_L = (\pi_{\mathfrak{k}} + \pi_V) \circ \mu|_L = \eta + \varphi.$$

Since $\mu|_L$ is the inclusion into \mathfrak{g} , this shows that φ is an embedding.

If $\varphi : L \rightarrow V$ is not full, then $\varphi(L)$ is contained in an affine hyperplane of V , namely, $\langle \varphi(q), \zeta \rangle = \langle q, \zeta \rangle$ is a constant for every $q \in L$ and some nonzero $\zeta \in V$. This implies that the height function h_ζ is constant on L and thus, by Lemma 1, $\sigma(\zeta)$ is everywhere zero, namely $\zeta \in \mathfrak{k}$, a contradiction to $\mathfrak{k} \cap V = \{0\}$. This proves that $\varphi : L \rightarrow V$ is full.

Recall that $\varphi : L \rightarrow V$ is by definition a tight embedding if and only if every height function $h_X|_{\varphi(L)}$ for $X \in V$ which is a Morse function is also perfect, i.e., has the minimum number of critical points allowed by the Morse inequalities [CC97]. By Lemma 1, this is equivalent to $X^*|_L$ having a number of zeros equal to the sum of \mathbb{Z}_2 -Betti numbers of L for generic $X \in V$. Since such X are the infinitesimal generators of one-parameter groups of isometries of M , the latter condition follows from the tightness of L in M , and is equivalent to it in case \mathfrak{g} coincides with the Lie algebra of all isometries of M .

Finally, note that M is a G -orbit so it is contained in a round sphere of \mathfrak{g} . By (3) also $\varphi(L)$ is contained in a round sphere of V . A submanifold contained in a round sphere in a Euclidean space is tight if and only if all distance functions are perfect Morse functions, namely, if and only if it is taut [CC97]. Furthermore, in this situation the set of critical points of a distance function will also occur as the set of critical points of a height function, and vice versa. ■

Remark 2: It follows from (3) that $L \subset V$ if \mathfrak{k} is centerless.

Remark 3: If K is a symmetric subgroup of G , then its orbits in V are taut submanifolds (see, e.g., [GT03]). Thus it easily follows from Proposition 1 that real forms of Hermitian symmetric spaces of compact type are locally tight; we omit the details. Note that this result already follows from the work of Takeuchi and Kobayashi [TK68]. Moreover, it has been recently proved by Tanaka and Tasaki that those real forms are indeed globally tight [TT12].

Recall that the Chern–Lashof theorem [CL57] implies that a taut and substantial smooth embedding of a \mathbb{Z}_2 -homology sphere into an Euclidean sphere must be round and have codimension one (see also [NR72]). Hence:

COROLLARY 1: *If L is a compact tight Lagrangian \mathbb{Z}_2 -homology sphere, then $\varphi(L)$ is a codimension one round sphere in V . In particular*

$$\dim V = \dim L + 1.$$

4. The Ohnita–Gotoh formula

Keep the notation from the previous two sections and assume for the moment that L is an arbitrary compact Lagrangian submanifold of M . We introduce the subspace \mathfrak{l} of \mathfrak{m} corresponding to T_pL . The G -invariant Riemannian metric on M corresponds to an $\text{Ad}(H)$ -invariant inner product in \mathfrak{m} ; let \mathfrak{l}^\perp be the orthogonal complement of \mathfrak{l} in \mathfrak{m} . Also, denote the normalizer subalgebra of \mathfrak{l} in \mathfrak{h} by \mathfrak{n} .

The following proposition elaborates on results by Ohnita [Ohn87] and Gotoh [Got99].

PROPOSITION 2: *We have*

$$(4) \quad \dim V \geq \dim L + \dim \mathfrak{h} - \dim \mathfrak{n}.$$

Moreover, if equality holds then L is homogeneous under the action of K and $\mathfrak{n} \subset \mathfrak{k} \cap \mathfrak{h}$.

Proof. Throughout we identify $T_pL \cong \mathfrak{l}$ and $\nu_pL \cong \mathfrak{l}^\perp$ whenever clear from context. We consider the diagram

$$(5) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Psi = \Psi_1 \oplus \Psi_2} & \mathfrak{l}^\perp \oplus \text{Hom}(\mathfrak{l}, \mathfrak{l}^\perp) \\ \downarrow \sigma & & \downarrow \cong \\ V & \xrightarrow{\Phi_p} & \nu_pL \oplus \text{Hom}(T_pL, \nu_pL) \end{array}$$

where $\Psi_1 : \mathfrak{g} \rightarrow \mathfrak{l}^\perp$ is the projection with respect to the vector space direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{l} + \mathfrak{l}^\perp$, the map $\Psi_2 : \mathfrak{g} \rightarrow \text{Hom}(\mathfrak{l}, \mathfrak{l}^\perp)$ is given by

$$\Psi_2(X)(Y) = (\nabla_{Y^*} X_m^*)|_p^\perp + [X_{\mathfrak{h}}, Y]^\perp - B(X_{\mathfrak{l}}, Y),$$

where $B : \mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}^\perp$ is the second fundamental form of L in M at p , and

$$\Phi_p(\eta) = (\eta_p, \nabla^\perp \eta|_p)$$

for $X \in \mathfrak{g}$, $Y \in \mathfrak{l}$ and $\eta = (X^*|_L)^\perp \in V$.

The commutativity of diagram (5) follows from $\sigma(X)|_p = (X_p^*)^\perp$ and

$$(6) \quad (\nabla_{Y^*} X_m^*)^\perp|_p + [X_{\mathfrak{h}}, Y]^\perp - B(X_{\mathfrak{l}}, Y) = \nabla_{Y^*}^\perp \sigma(X)|_p$$

for $X \in \mathfrak{g}$ and $Y \in \mathfrak{l}$. In turn, we check (6) as follows:

$$\begin{aligned} (\nabla_{Y^*} X_m^*)^\perp|_p &= (\nabla_{Y^*} X^*)^\perp|_p - (\nabla_{Y^*} X_{\mathfrak{h}}^*)^\perp|_p \\ &= (\nabla_{Y^*} (X^*)^\top)^\perp|_p + (\nabla_{Y^*} (X^*)^\perp)^\perp|_p - (\nabla_{Y^*} X_{\mathfrak{h}}^*)^\perp|_p \\ &= B(X_{\mathfrak{l}}, Y) + \nabla_{Y^*}^\perp \sigma(X)|_p - (\nabla_{Y^*} X_{\mathfrak{h}}^*)^\perp|_p. \end{aligned}$$

Finally, the result follows from the formula

$$(\nabla_{W^*} U^*)|_p = [U, W]_p^*$$

for $U \in \mathfrak{h}$ and $W \in \mathfrak{m}$, which is easily proved using (7.27) in [Bes87] and the fact that $\text{ad}(U) \in \text{End}(\mathfrak{m})$ is skew-symmetric with respect to the metric in \mathfrak{m} .

It follows from the commutativity of diagram (5) and the surjectivity of σ that $\text{im}(\Psi) = \text{im}(\Phi_p)$ and thus

$$(7) \quad \dim V \geq \dim \text{im}(\Phi_p) = \dim \text{im}(\Psi).$$

It is obvious that $\Psi(\mathfrak{l}^\perp) \cap \Psi(\mathfrak{h} + \mathfrak{l}) = \{0\}$. Therefore

$$\begin{aligned}
 \dim \operatorname{im}(\Psi) &= \dim \Psi(\mathfrak{l}^\perp) + \dim \Psi(\mathfrak{h} + \mathfrak{l}) \\
 &\geq \dim \mathfrak{l}^\perp + \dim \Psi(\mathfrak{h}) \\
 (8) \qquad &= \dim L + \dim \mathfrak{h} - \dim \ker(\Psi|_{\mathfrak{h}}) \\
 &= \dim L + \dim \mathfrak{h} - \dim \mathfrak{n},
 \end{aligned}$$

proving (4).

In the case of equality in (4), we follow [Got99]. Use (7) and (8) to see that Φ_p is injective and $\Psi(\mathfrak{h} + \mathfrak{l}) = \Psi(\mathfrak{h})$. Now for given $v \in T_p L$ we can find $X \in \mathfrak{l}$ and $Y \in \mathfrak{h}$ with $X_p^* = v$ and $\Psi(Y) = -\Psi(X)$. Therefore $\sigma(X + Y) = 0$, namely, $X + Y \in \mathfrak{k}$. This proves that K acts transitively on L . Moreover, $\mathfrak{n} = \ker(\Psi|_{\mathfrak{h}}) \subset \ker \sigma = \mathfrak{k}$. ■

COROLLARY 2: *If L is a compact tight Lagrangian \mathbb{Z}_2 -homology sphere, then L is homogeneous under K and $\mathfrak{n} = \mathfrak{k} \cap \mathfrak{h}$ is the isotropy subalgebra of \mathfrak{k} at p and a codimension one ideal of \mathfrak{h} . Moreover, $(\mathfrak{g}, \mathfrak{k})$ is either a symmetric pair of rank one or $(\mathfrak{g}_2, \mathfrak{su}(3))$ or $(\mathfrak{so}(7), \mathfrak{g}_2)$.*

Proof. We see that $\mathfrak{n} \subsetneq \mathfrak{h}$ by noting that $\xi \notin \mathfrak{n}$. Indeed if $[\xi, \mathfrak{l}] \subseteq \mathfrak{l}$ then

$$(9) \qquad 0 = \omega_p(T_p L, T_p L) = \langle [\xi, \mathfrak{l}], \mathfrak{l} \rangle,$$

which implies $[\xi, \mathfrak{l}] = \{0\}$, contradicting the facts that the centralizer of ξ in \mathfrak{g} is \mathfrak{h} , and $\mathfrak{h} \cap \mathfrak{l} = \{0\}$.

Further, it follows from Corollary 1 and Proposition 2 that $\dim \mathfrak{n} = \dim \mathfrak{h} - 1$, L is K -homogeneous and $\mathfrak{n} \subset \mathfrak{k} \cap \mathfrak{h} = \mathfrak{k}_p$. The reverse inclusion $\mathfrak{k}_p \subset \mathfrak{n}$ is obvious and therefore $\mathfrak{n} = \mathfrak{k} \cap \mathfrak{h}$.

It also follows from Corollary 1 that K acts on V with cohomogeneity one and the last claim follows (see, e.g., [HPTT94, 3.12]). ■

5. End of the proof of Theorem 1

First we explain a standard construction which allows one to construct compact tight Lagrangian submanifolds in suitable complex flag manifolds.

Let K be a connected *symmetric* subgroup of a compact connected semisimple Lie group G and consider the decomposition $\mathfrak{g} = \mathfrak{k} + V$ into eigenspaces of the involution. Any orbit of K on V , say $L = \operatorname{Ad}(K) \cdot \xi$ for some $\xi \in V$, is called a (*generalized*) *real flag manifold*. There is a natural “complexification” of L ,

namely, we next show that the adjoint orbit $M = \text{Ad}(G) \cdot \xi$ is a complex flag manifold containing L as the connected component of the fixed point set of an anti-holomorphic involutive isometry.

Since G is compact, it embeds into its complexification $G^{\mathbb{C}}$ as a maximal compact subgroup. The Lie algebra of $G^{\mathbb{C}}$ is $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ and admits $\mathfrak{g}_0 = \mathfrak{k} + iV$ as a non-compact real form; let G_0 denote the corresponding connected subgroup of $G^{\mathbb{C}}$ and τ the associated conjugation of $G^{\mathbb{C}}$ over G_0 .

Fix a maximal Abelian subalgebra \mathfrak{a} of iV containing $a := i\xi$ and consider the restricted root decomposition $\mathfrak{g}_0 = Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \sum_{\lambda \in \Sigma} \mathfrak{g}_{0,\lambda}$ where $Z_{\mathfrak{k}}(\mathfrak{a})$ denotes the centralizer of \mathfrak{a} in \mathfrak{k} . Choose a positive restricted root system $\Sigma^+ \subset \Sigma$ so that $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition, where $\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_{0,\lambda}$. As a homogeneous space, $L = K/Z_K(a)$, where $Z_K(a)$ is the centralizer of a in K and its Lie algebra is

$$Z_{\mathfrak{k}}(a) = Z_{\mathfrak{k}}(\mathfrak{a}) + \sum_{\substack{\lambda \in \Sigma^+ \\ \lambda(\mathfrak{a})=0}} \mathfrak{k}_{\lambda}, \quad \text{where } \mathfrak{k}_{\lambda} = (\mathfrak{g}_{0,\lambda} + \mathfrak{g}_{0,-\lambda}) \cap \mathfrak{k}.$$

It turns out that G_0 acts on L . To see that, recall that a minimal parabolic subalgebra of \mathfrak{g}_0 is any subalgebra conjugated to $\mathfrak{p}_{0,min} = Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \mathfrak{n}$, and a parabolic subalgebra of \mathfrak{g}_0 is any subalgebra containing a minimal parabolic subalgebra (see, e.g., [War72, §1.2.3 and 1.2.4]). Now

$$\mathfrak{p}_0 := Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \mathfrak{n} + \sum_{\substack{\lambda \in \Sigma^+ \\ \lambda(\mathfrak{a})=0}} \mathfrak{g}_{0,-\lambda}$$

is a parabolic subalgebra of \mathfrak{g}_0 . The normalizer P_0 of \mathfrak{p}_0 in G_0 is called a parabolic subgroup. Let Θ be the set of simple restricted roots λ satisfying $\lambda(a) = 0$, and let \mathfrak{a}_{Θ} be the subspace of \mathfrak{a} that Θ annihilates. By Theorem 1.2.4.8 in [War72], $P_0 = M_{\Theta}AN$ where $A = \exp(\mathfrak{a})$, $N = \exp(\mathfrak{n})$ and M_{Θ} is the centralizer $Z_K(\mathfrak{a}_{\Theta})$ of \mathfrak{a}_{Θ} in K (loc. cit., p. 73). Note that a is a generic element in \mathfrak{a}_{Θ} , so $Z_K(\mathfrak{a}_{\Theta}) = Z_K(a)$. In particular, $K \cap P_0 = M_{\Theta} = Z_K(a)$. The group K acts by left translations on G_0/P_0 with an orbit that is open (by counting dimensions, since $\dim \mathfrak{k}_{\lambda} = \dim \mathfrak{g}_{\lambda}$) and closed (by compactness of K), so $K/Z_K(\xi) = K/Z_K(a) = K/K \cap P_0 = G_0/P_0$. This realizes the real flag L as a G_0 -homogeneous space.

On the other hand, $G^{\mathbb{C}}$ acts on M . Indeed, let $\mathfrak{t}_{\mathfrak{k}}$ be a maximal Abelian subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{s} = \mathfrak{t}_{\mathfrak{k}} + \mathfrak{a}$ is a Cartan subalgebra of \mathfrak{g}_0 , and $\mathfrak{s}^{\mathbb{C}}$ is a

Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with root system Δ and root decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

The roots are real valued on $\mathfrak{s}_{\mathbb{R}} = it_{\mathfrak{t}} + \mathfrak{a}$, and we take a lexicographic order that takes \mathfrak{a} before $it_{\mathfrak{t}}$. The point is that a restricted root of the form $\lambda = \alpha|_{\mathfrak{a}}$ for $\alpha \in \Delta$ is positive if and only if $\alpha \in \Delta^+$. Now

$$\begin{aligned} \mathfrak{p} &:= \mathfrak{p}_0 \otimes \mathbb{C} \\ &= Z_{\mathfrak{t}}(\mathfrak{a})^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}} + \mathfrak{n}^{\mathbb{C}} + \sum_{\substack{\lambda \in \Sigma^+ \\ \lambda|_{\mathfrak{a}}=0}} \mathfrak{g}_{0,-\lambda} \otimes \mathbb{C} \\ &= \left(\mathfrak{t}_{\mathfrak{t}}^{\mathbb{C}} + \sum_{\alpha|_{\mathfrak{a}}=0} \mathfrak{g}_{\alpha}^{\mathbb{C}} \right) + \mathfrak{a}^{\mathbb{C}} + \sum_{\substack{\alpha \in \Delta^+ \\ \alpha|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_{\alpha}^{\mathbb{C}} + \sum_{\substack{\alpha \in \Delta^+ \\ \alpha|_{\mathfrak{a}} \neq 0, \alpha(\mathfrak{a})=0}} \mathfrak{g}_{-\alpha}^{\mathbb{C}} \\ &= \underbrace{\mathfrak{s}^{\mathbb{C}} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}^{\mathbb{C}}}_{\text{Borel subalgebra}} + \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(\mathfrak{a})=0}} \mathfrak{g}_{-\alpha}^{\mathbb{C}} \end{aligned}$$

is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$, since it contains a Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The normalizer P of \mathfrak{p} in $G^{\mathbb{C}}$ is called a parabolic subgroup of G . It is closed and connected. As in the real case above, it follows (even easier) that $G \cap P = Z_G(\xi)$ (cf. [Wol69, Corollary 2.7]) and $G/Z_G(\xi) = G/G \cap P = G^{\mathbb{C}}/P$. This realizes the complex flag M as a $G^{\mathbb{C}}$ -homogeneous space.

The involution τ of $G^{\mathbb{C}}$ stabilizes P so induces an involution $\bar{\tau}$ of M whose connected component through the basepoint coincides with L [Wol69, Theorem 3.6].

Consider the Kirillov–Kostant–Souriau invariant symplectic form on M defined at ξ by

$$\omega_{\xi}(X, Y) = \langle [X, Y], \xi \rangle$$

for $X, Y \in T_{\xi}M$. Since τ preserves the Killing form of \mathfrak{g} and maps ξ to $-\xi$, we see that $\bar{\tau}$ is antisymplectic and thus L is a Lagrangian submanifold of M . Note that $\bar{\tau}$ is also antiholomorphic with respect to the complex structure J on M given by the T -chamber containing ξ , and hence it is an isometry with respect to the Kähler metric g induced by ω and J .

The real flag manifold L is called a *real form* of the complex flag manifold M endowed with the Kähler metric g . In case L is already a complex flag manifold viewed as a real flag manifold (namely, G_0 is a complex semisimple Lie group

viewed as a real Lie group), M can be identified with $L \times \bar{L}$, where \bar{L} is equipped with the opposite complex structure and L sits in M as the diagonal.

Remark 4: Note that we can start with any real flag manifold and “complexify” it to a complex flag manifold. Conversely, if we start with a complex flag $G^{\mathbb{C}}/Q$ and fix a real form G_0 , it is not always true that there is a G_0 -orbit in $G^{\mathbb{C}}/Q$ which is a real form; it is not true, for instance, for a full flag $G^{\mathbb{C}}/B$ where $G^{\mathbb{C}}$ is a complex semisimple Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and B is a Borel subgroup such that its Lie algebra contains the complexification of a maximally split Cartan subalgebra $\mathfrak{t}_{\mathbb{R}} + \mathfrak{a}$ of a real form \mathfrak{g}_0 , and \mathfrak{a} contains no regular element of $\mathfrak{g}^{\mathbb{C}}$ [Wol69, p. 1139]. On the other hand, if G_0 is the Cartan normal real form, we can always find a G_0 -orbit in $G^{\mathbb{C}}/Q$ which is a real form.

As far as we know, real forms of complex flags manifolds other than Hermitian symmetric spaces have not been explicitly classified (see [Wol69, Theorem 3.6], though). Compare the next result with Remark 3.

PROPOSITION 3: *Consider the real form L of the complex flag manifold M constructed above. Then L is a tight Lagrangian submanifold of M .*

Proof. We need only prove the tightness. A symmetric space of compact type splits into the direct product of irreducible symmetric spaces of compact type, and the linear isotropy representation splits accordingly, so we may assume G/K is irreducible.

Suppose first G/K is type of I, namely, G is simple. Since K is a symmetric subgroup of G , its orbits in V are taut submanifolds [GT03]. If the pair $(\mathfrak{g}, \mathfrak{h})$ is not listed in Theorem 2, Proposition 1 already implies that L is tight. Otherwise, the center of \mathfrak{h} is one-dimensional and therefore $M = G/H$ admits precisely one invariant complex structure up to biholomorphism and one invariant compatible Kähler metric up to homothety. The complex manifold $M = G/H = G^{\mathbb{C}}/P$ can also be written as $M = Q/H'$, where Q denotes the connected group of all biholomorphisms of M and the space Q/H' is a Hermitian symmetric space. The symmetric metric \tilde{g} on Q/H' (unique up to homothety) is also G -invariant and is a scalar multiple of g . This means that the submanifold L is a real form of M also with respect to the symmetric metric \tilde{g} , and therefore it is tight (see Remark 3).

In case G/K is of type II, $M=L \times \bar{L}$ (see above) and the proof is analogous. ■

The case of interest for us is that in which $(\mathfrak{g}, \mathfrak{k})$ has rank one. Here L is a sphere, umbilic in V , for $\xi \neq 0$.

Conversely, we now proceed to the proof of Theorem 1 and examine to which extent the above construction supplies the possible examples. So let $M = G/H$ and L be as in the statement of the theorem and write $\mathfrak{g} = \mathfrak{k} + V$ as in section 3. We view M as the adjoint orbit through $\xi \in \mathfrak{g}$ and assume $\xi \in L$. Corollary 2 says that L is homogeneous under K , $\mathfrak{h} = \mathfrak{k}_\xi \oplus \mathfrak{u}(1)$ and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair of rank one or one of two other pairs.

Note that if \mathfrak{k} is centerless, then we must have $\xi \in V$ by Remark 2; if, in addition, the dimension of the center of \mathfrak{h} is one, then ξ lies in the $\mathfrak{u}(1)$ -summand of \mathfrak{h} and the Kähler structure on M is unique, up to biholomorphic homothety. This is the situation for the symmetric pairs $(\mathfrak{so}(n+2), \mathfrak{so}(n+1))$ ($n \geq 3$), $(\mathfrak{sp}(n+2), \mathfrak{sp}(1) \oplus \mathfrak{sp}(n+1))$ ($n \geq 1$) and $(\mathfrak{f}_4, \mathfrak{so}(9))$ (in these cases $(\mathfrak{g}, \mathfrak{h})$ is not listed in Theorem 2), for which the construction above yields the examples described in parts (a), (d) and (e) in the statement of the theorem.

In case $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{su}(3))$, the center of $\mathfrak{su}(3)$ is zero and that of $\mathfrak{h} = \mathfrak{u}(2)$ is one-dimensional, so we also must have $\xi \in V$. Since $\mathfrak{su}(3) \subset \mathfrak{g}_2$ is spanned by the long roots of \mathfrak{g}_2 , \mathfrak{h} is the centralizer of a short root and thus $M = G_2/U(2)$ is again the quadric $Q_5 = SO(7)/SO(2) \times SO(5)$ and $L \cong SU(3)/SU(2) \cong S^5$.

In case $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}(7), \mathfrak{g}_2)$ again $\xi \in V$, so $\mathfrak{h} \cong \mathfrak{u}(3)$ is the centralizer of a short root of $\mathfrak{so}(7)$. It is known that $M = SO(7)/U(3) \cong SO(8)/U(4)$ and an outer automorphism τ of $\mathfrak{so}(8)$ induces a diffeomorphism between M and, again, the quadric $Q_6 = SO(8)/SO(2) \times SO(6)$. Since G_2 acts with cohomogeneity one on $SO(7)/U(3)$ [Kol02, p. 573], G_2 and $SO(7)$ share the same orbits in Q_6 and L is congruent to the standard real form S^6 .

The last case we need to consider is $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(n+1), \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n)))$. This case is somewhat more involved because the center of \mathfrak{k} is non-trivial and $\mathfrak{h} \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(n-1)$ has a center of dimension bigger than one. We distinguish between two cases, namely, $n \geq 3$ and $n = 2$ (the case $n = 1$ is trivial and corresponds to the item (f) in Theorem 1).

(a) $n \geq 3$. The flag manifold M is known to admit precisely two inequivalent $SU(n+1)$ -invariant complex structures (see, e.g., [Nis84]). We note that the tangent space $T_\xi M$ splits under the isotropy representation of

$$H = S(U(1) \times U(1) \times U(n-1))$$

as the sum $\bigoplus_{i=1}^3 V_i$ where V_i are mutually inequivalent complex H -submodules with $\dim_{\mathbb{C}} V_1 = 1$ and $\dim_{\mathbb{C}} V_2 = \dim_{\mathbb{C}} V_3 = n - 1$. It is relatively easy to apply the machinery of section 2 to show that the two inequivalent invariant complex structures J_0, J_1 on M can be described as follows (compare [BH58, §13.9]): if $(u, z, w) \in \bigoplus_{i=1}^3 V_i$, then $J_\alpha(u, z, w) = (iu, iz, (-1)^\alpha iw)$ for $\alpha = 0, 1$.

We now consider the standard $SU(n + 1)$ -equivariant fibration $\pi : M \rightarrow W$, where W denotes the complex Grassmannian

$$\text{Gr}_2(\mathbb{C}^{n+1}) = SU(n + 1)/S(U(2) \times U(n - 1)).$$

It is clear that the projection π is holomorphic when we endow M with the complex structure J_0 and W with its standard complex structure J_W as a Hermitian symmetric space. On the other hand, the space W is also a homogeneous quaternion-Kähler manifold, a so-called Wolf space, endowed with an invariant quaternion Kähler structure Q , namely a rank three subbundle $Q \subset \text{End}(TW)$ locally spanned by three local complex structures $\{I, J, K\}$ with $IJK = -\text{Id}$. It is well known that J_W is not even a local section of Q (see, e.g., [Bes87, 14.53(b)]) and therefore (M, J_1) is biholomorphic to the twistor space Z of W .

LEMMA 2: *If M has an invariant Kähler structure (g, J) admitting a compact tight Lagrangian \mathbb{Z}_2 -homology sphere L , then J is equivalent to J_1 and g is Kähler-Einstein, i.e., (M, g, J) is biholomorphically homothetic to the twistor space of W .*

Proof. We know there is a subgroup $K \subset SU(n + 1)$ isomorphic to $U(n)$ which acts transitively on L . We claim that there exists $v \in \mathbb{C}^{n+1}$, $v \neq 0$, such that $K = \{g \in SU(n + 1) \mid gv \in \mathbb{C}^*v\}$. Indeed we first note that K acts reducibly on \mathbb{C}^{n+1} because otherwise the center of $K \subset SU(n + 1)$ would act as a multiple of the identity by Schur’s Lemma and therefore it would be finite. The claim now follows from the fact that any irreducible representation of $SU(n)$ has dimension at least n .

Note that the semisimple part of H is contained in K , and non-trivial because $n \geq 3$. Therefore $v \in \text{Span}\{e_1, e_2\}$, where $\{e_1, \dots, e_{n+1}\}$ is the canonical basis of \mathbb{C}^{n+1} . Moreover, H is not contained in K , so $v \notin \mathbb{C}e_1$ and $v \notin \mathbb{C}e_2$. This implies that

$$H \cap K = \{(z, z, A) \in S(U(1) \times U(1) \times U(n - 1))\}.$$

The isotropy representation of H restricted to $H \cap K$ preserves the subspaces V_2, V_3 endowed with the invariant complex structure J and it is of real type on $V_2 \oplus V_3$. We can suppose that J is either J_0 or J_1 and therefore (V_2, J) and (V_3, J) are either equivalent or dual to each other as $H \cap K$ -modules. Since the center of $H \cap K$ acts as a scalar multiple of the identity on $V_2 \oplus V_3$ and it is of real type, we see that (V_2, J) and (V_3, J) are dual to each other and thus $J = J_1$.

The Kähler metric g induces a J -Hermitian scalar product on $\bigoplus_{i=1}^3 V_i$ such that this is an orthogonal decomposition. We denote by g_i the restriction of g on V_i for $i = 1, 2, 3$. As $H \cap K$ -modules, we can write $V_3 = V_2^*$ and $g_3 = \alpha \cdot g_2^*$, where $\alpha \in \mathbb{R}^+$ and g_2^* is the Hermitian metric induced by g_2 on V_2^* . We may assume that the $H \cap K$ -invariant real form ℓ of $V_2 \oplus V_2^*$ is given by $\ell = \{(v, v^*) \mid v \in V_2\}$, where $v^* \in V_2^*$ is the dual of $v \in V_2$ with respect to g_2 . Writing the condition that ℓ is Lagrangian relative to the metric g , we immediately see that $\alpha = 1$. This completely determines the metric by the Kähler condition (see, e.g., [WG68, Theorem 9.4(2)]) and it turns out that the projection $\pi : (M, g) \rightarrow W$ is a Riemannian submersion when we choose a suitable multiple of the symmetric metric on W . This means that the metric g is (up to a multiple) the standard Kähler–Einstein metric on the twistor space (see, e.g., [Bes87, 14.80]). ■

Now Lemma 2 shows that the standard construction described above provides the twistor space Z with a tight Lagrangian sphere $L \cong U(n)/U(n - 1) \cong S^{2n-1}$. It is interesting to remark that L coincides with a *natural lift* in Z of a totally complex projective space $\mathbb{C}P^{n-1} \subset W$ [ET05].

(b) $n = 2$. The flag manifold M is the full flag manifold $SU(3)/T^2$ which admits only one invariant complex structure J , up to equivalence. The standard construction shows that M , endowed with a suitable invariant Kähler metric g , admits a tight Lagrangian sphere $L \cong S^3$ which is an orbit of a subgroup $K \cong U(2)$ of $SU(3)$ and a connected component of the fixed point set of an antiholomorphic isometry τ . Arguments like those in the proof of Lemma 2 show that g is Kähler–Einstein.

We are left with proving that the tight Lagrangian spheres are unique up to the G -action. This is clear for the real form S^n in the quadric Q_n , so we will focus on the remaining cases. If L' is another tight Lagrangian sphere, we know that L' is homogeneous under the action of a subgroup K' with (G, K')

a symmetric pair of rank one. This implies that K' is unique up to conjugation and we can suppose that $(G, K') = (G, K)$, where (G, K) is one of the standard pairs $(\mathrm{SU}(n+1), \mathrm{U}(n))$, $(\mathrm{Sp}(n+1), \mathrm{Sp}(1) \times \mathrm{Sp}(n))$ or $(\mathrm{F}_4, \mathrm{Spin}(9))$, respectively. Therefore it is enough to show that the standard subgroup K has a unique orbit which is a Lagrangian sphere, and this follows from [BG08, Theorem 1.2].

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